NNLO Jet Cross Sections by Subtraction.

Gábor Somogyi
DESY Zeuthen
HP2.3rd

in collaboration with U. Aglietti, P. Bolzoni, V. Del Duca, C. Duhr, S.-O. Moch, Z. Trócsányi

HELMHOLTZ ASSOCIATION

Motivation

Why jets at NNLO?

Hadronic jets occur frequently in final states of high energy particle collisions.

Because of large production cross sections, jet observables can be measured with high statistical accuracy; can be ideal for precision studies.

Examples include measurements of:

- α_{s} **from jet rates and event shapes in** $e^+e^- \to$ **jets;**
- **gluon PDFs and** α_{s} **from** $2+1$ **jet production in DIS;**
- **PDFs in single jet inclusive,** $V+$ **jet in** pp **(or** $p\bar{p}$**) collisions.**

Often, relevant observables measured with accuracy of a few % or better.

Theoretical predictions with same level of accuracy necessary. This usually requires NNLO corrections.

What is a subtraction scheme?

We know that IR singularities cancel according to the KLN theorem between real and virtual quantum corrections at the same order in perturbation theory, for sufficiently inclusive (IR safe) observables.

Example (simple residuum subtraction)

$$\boxed{\sigma = \int_0^1 \mathrm{d}\sigma^{\mathrm{R}}(x) + \sigma^{\mathrm{V}}\,,}$$

where

$$\mathrm{d}\sigma^{\mathrm{R}}(x) = x^{-1-\epsilon} S(x)\,,$$
$$S(0) = S_0 < \infty\,,$$
$$\sigma^{\mathrm{V}} = S_0/\epsilon + F\,.$$

Define the counterterm $\mathrm{d}\sigma^{\mathrm{R,A}}(x) = x^{-1-\epsilon} S_0$**. Then**

$$\begin{aligned}
\sigma &= \int_0^1 \left[\mathrm{d}\sigma^{\mathrm{R}}(x) - \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} + \left[\sigma^{\mathrm{V}} + \int_0^1 \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} \\
&= \int_0^1 \left[\frac{S(x)-S_0}{x^{1+\epsilon}}\right]_{\epsilon=0} + \left[\frac{S_0}{\epsilon} + F - \frac{S_0}{\epsilon}\right]_{\epsilon=0} \\
&= \int_0^1 \frac{S(x)-S_0}{x} + F
\end{aligned}$$

The last integral is finite, computable with standard numerical methods.

In a rigorous mathematical sense, the cancellation of both kinematical singularities and ϵ-poles must be local. I.e. the counterterm must have the following general properties

- **must match the singularity structure of the real emission cross section pointwise, in d dimensions**
- **its integrated form must be combined with the virtual cross section explicitly, before phase space integration; ϵ-poles must cancel point by point**

In a rigorous mathematical sense, the cancellation of both kinematical singularities and ϵ-poles must be local. I.e. the counterterm must have the following general properties

- **must match the singularity structure of the real emission cross section pointwise, in d dimensions**
- **its integrated form must be combined with the virtual cross section explicitly, before phase space integration; ϵ-poles must cancel point by point**

The construction should be universal (i.e. process and observable independent)

- **to avoid tedious adaptation to every specific problem**
- **the integration of counterterms can be performed once and for all**
- **the IR limits of QCD (squared) matrix elements are universal, so a general construction should be possible**

In a rigorous mathematical sense, the cancellation of both kinematical singularities and ϵ-poles must be local. I.e. the counterterm must have the following general properties

- **must match the singularity structure of the real emission cross section pointwise, in d dimensions**
- **its integrated form must be combined with the virtual cross section explicitly, before phase space integration; ϵ-poles must cancel point by point**

The construction should be universal (i.e. process and observable independent)

- **to avoid tedious adaptation to every specific problem**
- **the integration of counterterms can be performed once and for all**
- **the IR limits of QCD (squared) matrix elements are universal, so a general construction should be possible**

Different specific choices of the counterterm correspond to different IR subtraction schemes (CS dipole, FKS, antenna,. . .).

Why a new subtraction scheme at NNLO?

- **Dipole subtraction** (Catani, Seymour)

  - ✔ fully local counterterms
  - ✔ explicit expressions including colour for a general process
  - ✘ faces fundamental difficulties when going to NNLO

- **Antenna subtraction** (Gehrmann-De Ridder, Gehrmann, Glover; Weinzierl)
- $q_\perp$ **subtraction** (Catani, Grazzini; Cieri, Ferrera, de Florian)
- **Sector decomposition** (Binoth, Heinrich; Anastasiou, Melnikov, Petriello)
- **This scheme** (Del Duca, GS, Trócsányi)

Why a new subtraction scheme at NNLO?

- **Dipole subtraction** (Catani, Seymour)
- **Antenna subtraction** (Gehrmann-De Ridder, Gehrmann, Glover; Weinzierl)

  ✔ successfully applied to $e^+e^- \to 3$ jets

  ✔ complete analytical integration of antennae tractable

  ✘ counterterms not fully local

  ✘ cannot constrain subtractions near singular regions

- $q_\perp$ **subtraction** (Catani, Grazzini; Cieri, Ferrera, de Florian)
- **Sector decomposition** (Binoth, Heinrich; Anastasiou, Melnikov, Petriello)
- **This scheme** (Del Duca, GS, Trócsányi)

Why a new subtraction scheme at NNLO?

- **Dipole subtraction** (Catani, Seymour)
- **Antenna subtraction** (Gehrmann-De Ridder, Gehrmann, Glover; Weinzierl)
- $q_\perp$ **subtraction** (Catani, Grazzini; Cieri, Ferrera, de Florian)

  ✔ exploits universal behaviour of $q_\perp$ distribution at small $q_\perp$

  ✔ numerically efficient implementation possible

  ✘ applicable only to the production of colourless final states in hadron collisions

- **Sector decomposition** (Binoth, Heinrich; Anastasiou, Melnikov, Petriello)
- **This scheme** (Del Duca, GS, Trócsányi)

Why a new subtraction scheme at NNLO?

- **Dipole subtraction** (Catani, Seymour)
- **Antenna subtraction** (Gehrmann-De Ridder, Gehrmann, Glover; Weinzierl)
- $q_\perp$ **subtraction** (Catani, Grazzini; Cieri, Ferrera, de Florian)
- **Sector decomposition** (Binoth, Heinrich; Anastasiou, Melnikov, Petriello)

  ✔ dispenses with the subtraction method, but conceptually very simple

  ✔ first method to yield physical cross sections

  ✘ cancellation of ϵ-poles numerical

  ✘ can it handle complicated final states?

- **This scheme** (Del Duca, GS, Trócsányi)

Why a new subtraction scheme at NNLO?

- **Dipole subtraction** (Catani, Seymour)
- **Antenna subtraction** (Gehrmann-De Ridder, Gehrmann, Glover; Weinzierl)
- $q_\perp$ **subtraction** (Catani, Grazzini; Cieri, Ferrera, de Florian)
- **Sector decomposition** (Binoth, Heinrich; Anastasiou, Melnikov, Petriello)
- **This scheme** (Del Duca, GS, Trócsányi)

✔ fully local counterterms (efficiency, mathematical rigour)

✔ explicit expressions including colour (colour space notation of dipole subtraction used)

✔ very algorithmic construction (in principle valid at N^{n}LO)

✔ option to constrain subtraction near singular regions (efficiency, important check)

✘ analytical integration of counterterms requires computing many new high dimensional integrals, but can be done once and for all

Subtraction at NNLO

What is needed to define a subtraction scheme?

To define a subtraction scheme, three problems must be addressed

1. **Matching of limits: the known IR factorization formulae must be written in such a way, that the overlapping soft/collinear singularities can be disentangled in order to avoid multiple subtraction.**

$$\mathbf{A}_1|\mathcal{M}_{m+1}^{(0)}|^2 = \sum_i \left[\sum_{i\neq r} \frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i\neq r} \mathbf{C}_{ir}\mathbf{S}_r\right]|\mathcal{M}_{m+1}^{(0)}|^2$$

2. **Extension over PS: the IR factorization formulae valid in the strict soft/collinear limits have to be defined over the full PS. This requires the introduction of appropriate mappings of momenta that respect factorization and the (delicate) cancellation of IR singularities**

$$\{p\}_{m+1} \xrightarrow{r} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+1}(\{p\}_{m+1};Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m;Q)[\mathrm{d}p_{1,m}]$$

$$\{p\}_{m+2} \xrightarrow{r,s} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+2}(\{p\}_{m+2};Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m;Q)[\mathrm{d}p_{2,m}]$$

3. **Integration: the counterterms have to be integrated over the phase space of the unresolved parton(s).**

Specific issues at NNLO

- **Matching is cumbersome if done in a brute force way. However, an efficient solution that works at any order in PT is known.**
- **Extension is very delicate. Among other constraints, the counterterms for singly-unresolved real emission must have universal IR limits, which is not guaranteed by QCD factorization.**
- **Choosing the counterterms such that integration is (relatively) easy generally conflicts with the delicate cancellations in the various limits.**

The NNLO cross section

Consider the NNLO correction to a generic *m*-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}\,.$$

The NNLO cross section

Consider the NNLO correction to a generic *m*-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \,.$$

Doubly-real

- $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2}$
- **Tree MEs with $m+2$-parton kinematics**
- **kin. singularities as one or two partons unresolved: up to $\mathrm{O}(\epsilon^{-4})$ poles from PS integration**
- **no explicit ϵ poles**

The NNLO cross section

Consider the NNLO correction to a generic m-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} + \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{RV}} J_{m+1} + \int_{m} \mathrm{d}\sigma_{m}^{\mathrm{VV}} J_{m} \,.$$

Doubly-real

- $\mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2}$
- **Tree MEs with $m+2$-parton kinematics**
- **kin. singularities as one or two partons unresolved: up to $\mathrm{O}(\epsilon^{-4})$ poles from PS integration**
- **no explicit ϵ poles**

Real-virtual

- $\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} J_{m+1}$
- **One-loop MEs with $m+1$-parton kinematics**
- **kin. singularities as one parton unresolved: up to $\mathrm{O}(\epsilon^{-2})$ poles from PS integration**
- **explicit ϵ poles up to $\mathrm{O}(\epsilon^{-2})$**

The NNLO cross section

Consider the NNLO correction to a generic *m*-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \,.$$

Doubly-real

- $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2}$
- **Tree MEs with $m+2$-parton kinematics**
- **kin. singularities as one or two partons unresolved: up to $\mathrm{O}(\epsilon^{-4})$ poles from PS integration**
- **no explicit ϵ poles**

Real-virtual

- $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1}$
- **One-loop MEs with $m+1$-parton kinematics**
- **kin. singularities as one parton unresolved: up to $\mathrm{O}(\epsilon^{-2})$ poles from PS integration**
- **explicit ϵ poles up to $\mathrm{O}(\epsilon^{-2})$**

Doubly-virtual

- $\mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$
- **One- and two-loop MEs with m-parton kinematics**
- **kin. singularities screened by jet function: PS integration finite**
- **explicit ϵ poles up to $\mathrm{O}(\epsilon^{-4})$**

The NNLO counterterms

Rewrite the NNLO correction as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} + \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} + \int_{m} \mathrm{d}\sigma_{m}^{\mathrm{NNLO}}$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} J_m - \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} J_m \right] \right\}$$

$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right] J_{m+1} - \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_{m} \left\{ \mathrm{d}\sigma_{m}^{\mathrm{VV}} + \int_2 \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} \right] + \int_1 \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

The NNLO counterterms

Rewrite the NNLO correction as

$$
\begin{aligned}
\sigma^{\mathrm{NNLO}} &= \int_{m+2} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m} \\
&= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\} \\
&+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\} \\
&+ \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m
\end{aligned}
$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ **regularizes the doubly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$

The NNLO counterterms

Rewrite the NNLO correction as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m}$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left[\boxed{\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right] \right\}$$

$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \boxed{\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$+ \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2}$ **regularizes the doubly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\boxed{\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$

The NNLO counterterms

Rewrite the NNLO correction as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m}$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ **regularizes the doubly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ **accounts for the overlap of** $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ **and** $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$

The NNLO counterterms

Rewrite the NNLO correction as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} + \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} + \int_{m} \mathrm{d}\sigma_{m}^{\mathrm{NNLO}}$$

$$= \int_{m+2} \Big\{ \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} J_m - \Big[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} J_m \Big] \Big\}$$

$$+ \int_{m+1} \Big\{ \Big[\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \Big] J_{m+1} - \Big[\boxed{\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}}} + \Big(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \Big)^{\mathrm{A_1}} \Big] J_m \Big\}$$

$$+ \int_{m} \Big\{ \mathrm{d}\sigma_{m}^{\mathrm{VV}} + \int_2 \Big[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} \Big] + \int_1 \Big[\boxed{\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}}} + \Big(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \Big)^{\mathrm{A_1}} \Big] \Big\} J_m$$

1. $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}}$ **regularizes the doubly-unresolved limits of** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR}}$
2. $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR}}$
3. $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}}$ **accounts for the overlap of** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}}$ **and** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}}$
4. $\boxed{\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}}}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma_{m+1}^{\mathrm{RV}}$

The NNLO counterterms

Rewrite the NNLO correction as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} + \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} + \int_{m} \mathrm{d}\sigma_{m}^{\mathrm{NNLO}}$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} J_m - \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} J_m \right] \right\}$$

$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right] J_{m+1} - \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_{m} \left\{ \mathrm{d}\sigma_{m}^{\mathrm{VV}} + \int_2 \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} \right] + \int_1 \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}}$ **regularizes the doubly-unresolved limits of** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR}}$
2. $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR}}$
3. $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}}$ **accounts for the overlap of** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}}$ **and** $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}}$
4. $\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma_{m+1}^{\mathrm{RV}}$
5. $\left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}}\right)^{\mathrm{A_1}}$ **regularizes the singly-unresolved limit of** $\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}}$

The NNLO counterterms

Rewrite the NNLO correction as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m}$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ **regularizes the doubly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ **accounts for the overlap of** $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ **and** $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$
4. $\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}$ **regularizes the singly-unresolved limits of** $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$
5. $\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}\right)^{\mathrm{A_1}}$ **regularizes the singly-unresolved limit of** $\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$

General features

- **The counterterms are based on IR limit formulae.**
- **The counterterms are given completely explicitly for any process without coloured particles in the initial state. (The extension to hadronic processes is known explicitly to NLO.)**
- **The counterterms are fully local in colour $\otimes$ spin space: no need to consider the colour decomposition of real emission matrix elements; azimuthal correlations correctly taken into account in gluon splitting; can check explicitly that the ratio of the sum of counterterms to the real emission cross section tends to unity in any IR limit.**
- **It is straightforward to constrain subtractions to near singular regions: in any given PS point only a (small) subset of all subtraction terms needs to be explicitly evaluated during PS integration. Large gain in efficiency and strong check.**

Integrating the counterterms

Integrated counterterms

Counterterm	Types of integrals	Done
$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$	**tree level singly-unresolved**	✔
$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$	**one-loop singly-unresolved**	✔
$\int_1 (\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1})^{\mathrm{A}_1}$	**tree level iterated singly-unresolved (1)**	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$	**tree level iterated singly-unresolved (2)**	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$	**tree level iterated doubly-unresolved**	✘

Integrated counterterms

Counterterm	Types of integrals	Done
$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$	tree level singly-unresolved	✔
$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$	one-loop singly-unresolved	✔
$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}$	tree level iterated singly-unresolved (1)	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$	**tree level iterated singly-unresolved (2)**	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$	tree level iterated doubly-unresolved	✘

Phase space integrals - an example

Example (abelian soft-double soft counterterm)

Among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$ we have the abelian soft-double counterterm

$$\left(\mathcal{S}_t\mathcal{S}_{rt}^{(0)}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,j,k,l} \frac{1}{8}\mathcal{S}_{\hat{i}\hat{k}}(\hat{r})\mathcal{S}_{jl}(t)|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times (1-y_{tQ})^{d_0'-m(1-\epsilon)}(1-y_{\hat{r}Q})^{d_0'-m(1-\epsilon)}\Theta(y_0-y_{tQ})\Theta(y_0-y_{\hat{r}Q})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by an iterated mapping, and leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathrm{S}_t} \{\hat{p}\}_{m+1} \xrightarrow{\mathrm{S}_{\hat{r}}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\};Q)[\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]$$

We must then compute

$$\int [\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]\mathcal{S}_t\mathcal{S}_{rt}^{(0)} \equiv \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i,k,j,l} [\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$

where $[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl} \equiv [\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}(p_i,p_k,p_j,p_l,\epsilon,y_0,d_0')$ is a kinematics dependent function.

Example (abelian soft-double soft integral)

For simplicity, consider the terms in the sum where $j = i$ and $l = k$: $[\mathrm{S}_t \mathrm{S}^{(0)}_{rt}]_{ikik}$. Kinematical dependence is through $\cos\chi_{ik} = \measuredangle(p_i, p_k)$, we set $\cos\chi_{ik} = 1 - 2Y_{ik,Q}$.

Using angles and energies in some specific Lorentz frame to parametrize the factorized phase space measures, $[\mathrm{d}\widehat{p}_{1,m}]$ and $[\mathrm{d}p_{1,m+1}]$, we find that $[\mathrm{S}_t \mathrm{S}^{(0)}_{rt}]_{ikik}$ is proportional to

$$\mathcal{I}^{(11)}_{\mathcal{S}}(Y_{ik,Q};\epsilon,y_0,d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\frac{B_{y_0}(-2\epsilon,d_0'+1)}{\epsilon} Y_{ik,Q}\int_0^{y_0}\mathrm{d}y\, y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$
$$\times\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\Big[f(\vartheta,\varphi;0)\Big]^{-1}\Big[f(\vartheta,\varphi;Y_{ik,Q})\Big]^{-1}$$
$$\times\Big[Y(y,\vartheta,\varphi;Y_{ik,Q})\Big]^{-\epsilon}{}_2F_1\big(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\big)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1 - 2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi - (1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}$$

Example (abelian soft-double soft integral)

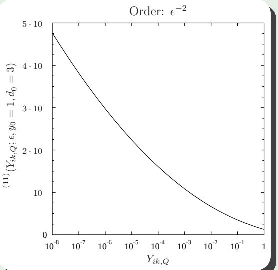
For this particular integral, we find

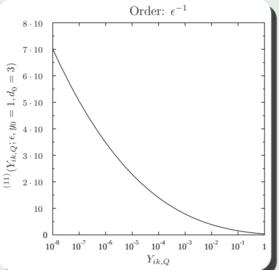
$$\mathcal{I}_{\mathcal{S}}^{(11)}(Y_{ik,Q};\epsilon,y_0,d_0') = \frac{1}{\epsilon^4} - 2\Big[\ln(Y_{ik,Q}) + \Sigma(y_0,D_0') + \Sigma(y_0,D_0'-1)\Big]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})$$

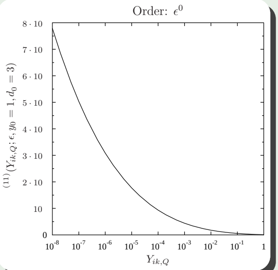
where $D_0' = d_0'|_{\epsilon=0}$ and the dependence on the cut parameters enters through

$$\Sigma(z,N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

Higher order expansion coefficients can be computed numerically ($y_0 = 1$, $D_0' = 3$)







Phase space integrals - methods

Several different methods to compute the integrals have been explored

- use of IBPs to reduce to master integrals + solution of MIs by differential equations
- use of MB representations to extract pole structure + summation of nested series
- use of sector decomposition

Phase space integrals - methods

Method	Analytical	Numerical
IBP	✔ Singly-unresolved integrals ✘ Bottleneck is the proliferation of denominators	✔ By evaluating full analytical results ✘ No numbers without full analytical results
MB	✔ Iterated singly-unresolved integrals ✘ Bottleneck is the evaluation of sums	✔ Direct numerical evaluation of MB integrals possible ✔ Fast and accurate
SD	✔ Easy to automatize ✘ Except for lowest order poles, possible only in principle	✘ Numerical behaviour is generally worse than MB method (speed, accuracy)

Analytical and numerical evaluation of the integrated counterterms

AS A MATTER OF PRINCIPLE

- The rigorous proof of cancellation of IR poles requires that all integrated counterterms are computed analytically (at least up to the pole parts).
- Analytical forms are fast and accurate compared to numerical ones.

HOWEVER

- Analytical forms show (in all cases where they are available) that the integrated counterterms are smooth functions of the kinematic variables.

HENCE

- For practical purposes, numerical forms of the integrated counterterms are sufficient. Final results can be conveniently given by interpolating tables computed once and for all or approximating functions. Thus, an efficient implementation is possible even in cases where the full analytical calculation is not feasible or practical (e.g. finite parts of integrated counterterms).

Results

Structure of the integrated counterterm

After summing over unresolved flavours ("counting of symmetry factors"), the integrated iterated singly-unresolved counterterm is a product of an insertion operator times the Born cross section

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon)$$

The insertion operator has the following structure in colour $\otimes$ flavour space

$$\mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \sum_i \Bigg[\mathrm{C}_{12,f_i}^{(0)} \mathbf{T}_i^2 + \sum_k \mathrm{C}_{12,f_i f_k}^{(0)} \mathbf{T}_k^2\Bigg] \mathbf{T}_i^2$$
$$+ \sum_{j,l} \Bigg[\mathrm{S}_{12}^{(0),(j,l)} C_\mathrm{A} + \sum_i \mathrm{CS}_{12,f_i}^{(0),(j,l)} \mathbf{T}_i^2\Bigg] \mathbf{T}_j \mathbf{T}_l$$
$$+ \sum_{i,k,j,l} \mathrm{S}_{12}^{(0),(i,k)(j,l)} \{\mathbf{T}_i \mathbf{T}_k, \mathbf{T}_j \mathbf{T}_l\} \Bigg\}$$

Here the $\mathrm{C}_{12,f_i}^{(0)}$, $\mathrm{C}_{12,f_i f_k}^{(0)}$, $\mathrm{S}_{12}^{(0),(j,l)}$, $\mathrm{CS}_{12,f_i}^{(0),(j,l)}$ and $\mathrm{S}_{12}^{(0),(i,k)(j,l)}$ functions depend on ϵ (having poles up to $\mathrm{O}(\epsilon^{-4})$) and kinematics (also on the factorized PS cut parameters).

The insertion operator – examples

Example ($e^+e^- \to 2j$)

The Born matrix element is $|\mathcal{M}_2^{(0)}(1_q, 2_{\bar{q}})|^2$. Colour and kinematics is trivial

$$\mathbf{T}_1^2 = \mathbf{T}_2^2 = -\mathbf{T}_1\mathbf{T}_2 = C_\mathrm{F}\,, \qquad y_{12} = \frac{2p_1 \cdot p_2}{Q^2} = 1$$

We find the insertion operator

$$\mathbf{I}_{12}^{(0)}(p_1, p_2; \epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \left\{\frac{2C_\mathrm{F}(3C_\mathrm{F} - C_\mathrm{A})}{\epsilon^4} + \frac{C_\mathrm{F}}{6}\Big[20C_\mathrm{A} + 81C_\mathrm{F} - 4T_\mathrm{R} n_\mathrm{f} \right.$$
$$\left. + 12(3C_\mathrm{A} - 2C_\mathrm{F})\Sigma(y_0, D_0') + 12(2C_\mathrm{A} - C_\mathrm{F})\Sigma(y_0, D_0' - 1)\Big]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})\right\}$$

Notice the dependence on the factorized PS cut parameters y_0 and D_0' through

$$\Sigma(z, N) = \ln z - \textstyle\sum_{k=1}^N \frac{1-(1-z)^k}{k}$$

which should cancel between the various integrated counterterms in the full doubly-virtual contribution.

Example ($e^+e^- \to 2j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{colour}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,2j}^{(\text{Col},i)} + \text{O}(\epsilon^1)$$

Kinematical dependence would enter through $y_{12} = 2p_1 \cdot p_2/Q^2$, but $y_{12} = 1$, hence no PS dependence

Preliminary

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	$\frac{76}{3}$	32.09	−87.90	−554.5
$C_A C_F$	−2	$-\frac{27}{2}$	−52.40	−150.7	−339.5
$C_F T_R n_f$	0	−1	−6.332	−17.65	1.013

The PS cut parameters are $\alpha_0 = y_0 = 1$, $d_0 = d_0' = 3$.

Example ($e^+e^- \to 3j$)

The Born matrix element is $|\mathcal{M}_3^{(0)}(1_q, 2_{\bar{q}}, 3_g)|^2$. Colour is still trivial

$$\mathbf{T}_1^2 = \mathbf{T}_2^2 = C_\mathrm{F}\,, \quad \mathbf{T}_3^2 = C_\mathrm{A}\,, \quad \mathbf{T}_1\mathbf{T}_2 = \frac{C_\mathrm{A} - 2C_\mathrm{F}}{2}\,, \quad \mathbf{T}_1\mathbf{T}_3 = \mathbf{T}_2\mathbf{T}_3 = -\frac{C_\mathrm{A}}{2}$$

We find the insertion operator

$$\begin{aligned}\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ &\frac{C_\mathrm{A}^2 + 2C_\mathrm{A}C_\mathrm{F} + 6C_\mathrm{F}^2}{\epsilon^4} + \Bigg[\frac{11C_\mathrm{A}^2}{2} + \frac{50C_\mathrm{A}C_\mathrm{F}}{3} \\ &+ 12C_\mathrm{F}^2 - \frac{C_\mathrm{A}T_\mathrm{R}n_\mathrm{f}}{3} - \frac{C_\mathrm{A}^2T_\mathrm{R}n_\mathrm{f}}{C_\mathrm{F}} - 4C_\mathrm{F}T_\mathrm{R}n_\mathrm{f} + \left(\frac{5C_\mathrm{A}^2}{2} - C_\mathrm{A}C_\mathrm{F} - 8C_\mathrm{F}^2\right)\ln y_{12} \\ &- \frac{C_\mathrm{A}(5C_\mathrm{A} + 8C_\mathrm{F})}{2}(\ln y_{13} + \ln y_{23}) + (C_\mathrm{A}^2 + 6C_\mathrm{A}2C_\mathrm{F} - 4C_\mathrm{F}^2)\Sigma(y_0, D_0') \\ &+ 4C_\mathrm{F}(C_\mathrm{A} - C_\mathrm{F})\Sigma(y_0, D_0' - 1)\Bigg]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})\Bigg\}\end{aligned}$$

Again depends on PS cut parameters through $\Sigma(y_0, D_0' - 1)$ and $\Sigma(y_0, D_0')$.

Example ($e^+e^- \to 3j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{colour}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

Kinematical dependence enters through $y_{ij} = 2p_i \cdot p_j / Q^2$, $i, j = 1, 2, 3$. **E.g. choose**

$$y_{12} = 0.333333, \qquad y_{13} = 0.333333, \qquad y_{23} = 0.333333$$

Preliminary

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	34.12	82.98	34.59	−543.8
$C_A C_F$	2	9.721	1.209	−142.2	−696.6
C_A^2	1	6.497	12.80	15.87	−47.92
$C_F T_R n_f$	0	$-\frac{13}{3}$	−32.40	−127.9	−355.2
$C_A T_R n_f$	0	$-\frac{3}{2}$	−12.01	−46.90	−104.1

The PS cut parameters are $\alpha_0 = y_0 = 1$, $d_0 = d_0' = 3$.

Example ($e^+e^- \to 3j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_s}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0}\sum_{\text{colour}}\frac{\text{Col}}{\epsilon^i}\mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1,p_2,p_3) + \text{O}(\epsilon^1)$$

Kinematical dependence enters through $y_{ij} = 2p_i \cdot p_j/Q^2$, $i,j = 1,2,3$. E.g. choose

$$y_{12} = 0.238667, \qquad y_{13} = 0.758153, \qquad y_{23} = 0.003180$$

Preliminary

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	36.79	106.0	120.6	-431.0
$C_\text{A}\,C_\text{F}$	2	25.38	143.6	537.3	1505
C_A^2	1	15.24	119.5	660.5	2902
$C_\text{F}\,T_\text{R}\,n_\text{f}$	0	$-\frac{13}{3}$	-31.30	-121.7	-346.0
$C_\text{A}\,T_\text{R}\,n_\text{f}$	0	$-\frac{3}{2}$	-17.72	-109.1	-470.9

The PS cut parameters are $\alpha_0 = y_0 = 1$, $d_0 = d_0' = 3$.

Example ($e^+e^- \to 3j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{colour}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

Kinematical dependence enters through $y_{ij} = 2p_i \cdot p_j / Q^2$, $i, j = 1, 2, 3$. **E.g. choose**

$$y_{12} = 0.937044, \qquad y_{13} = 0.024207, \qquad y_{23} = 0.038749$$

Preliminary

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	25.85	34.59	-84.25	-566.8
$C_A C_F$	2	27.79	136.8	330.6	46.20
C_A^2	1	21.02	195.4	1174	5355
$C_F T_R n_f$	0	$-\frac{13}{3}$	-57.59	-405.2	-2120
$C_A T_R n_f$	0	$-\frac{3}{2}$	-24.07	-194.7	-1083

The PS cut parameters are $\alpha_0 = y_0 = 1$, $d_0 = d_0' = 3$.

Conclusions

Conclusions

- ✔ **We have set up a general subtraction scheme for computing NNLO jet cross sections, for processes with no coloured particles in the initial state.**
- ✔ **We have investigated various methods to compute the integrated counterterms.**
- ✔ **We used the MB method to perform the integration of the iterated singly-unresolved counterterm, discussed in this talk. The SD method was used to provide independent checks.**
- ✔ **The integration of all singly-unresolved counterterms is finished. The iterated singly-unresolved counterterm is essentially finished.**
- ✘ **The integration of the doubly-unresolved counterterm is feasible with our methods, and is work in progress.**