Generalized double-logarithmic large-x resummation

Andreas Vogt (University of Liverpool)

mainly with G. Soar, A. Almasy (UoL), S. Moch (DESY), J. Vermaseren (NIKHEF)

- **Hard lepton-hadron processes in higher-order perturbative QCD**
  Large-x / large-N splitting functions P_{ik} and coefficient functions $C_{a,i}$
- **$\ln^n(1-x)$ behaviour of DIS, SIA and non-singlet DY physical kernels**
  All-order predictions for $C_{a,\mathrm{ns}}$, fourth-order $\ln^{6,5,4}(1-x)$ of P_{ik}

MV, arXiv: 0902.2342, 0909.2124; SMVV, 0912.0369

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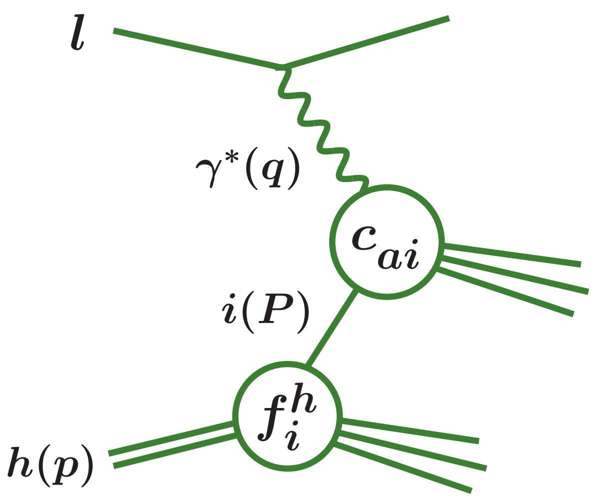
- **Hard lepton-hadron processes in higher-order perturbative QCD**
  Large-x / large-N splitting functions P_{ik} and coefficient functions $C_{a,i}$
- $\ln^n(1-x)$ **behaviour of DIS, SIA and non-singlet DY physical kernels**
  All-order predictions for $C_{a,\mathrm{ns}}$, fourth-order $\ln^{6,5,4}(1-x)$ of P_{ik}, $C_{L,\mathrm{g}}$
- **Iteration of (next-to) leading-log unfactorized $1/N$ structure functions**
  LL resummation of off-diagonal splitting and coefficient functions
- **General D-dimensional structure of large-x DIS and SIA amplitudes**
  Verification and extension to higher logarithmic accuracy for DIS/SIA

MV, arXiv: 0902.2342, 0909.2124; SMVV, 0912.0369; A.V., 1005.1606; ASV, 1010.nnnn

Hard lepton-hadron processes in pQCD (I)

Inclusive deep-inelastic scattering (DIS), semi-incl. l^+l^- annihilation (SIA)



Left $\longrightarrow$ right: DIS, q spacelike, $Q^2 = -q^2$

$P = \xi p\,,\ f_i^h =$ **parton distributions**

Top $\longrightarrow$ bottom: l^+l^-, q timelike, $Q^2 = q^2$

$p = \xi P\,,$ **fragmentation distributions**

Drell-Yan (DY) l^+l^- production: bottom $\longrightarrow$ top, 2nd hadron from right ($\{\ldots\}$)

Structure functions/normalized cross sections F_a: coefficient functions

$$F_a(x,Q^2) \; = \; \Big[C_{a,i\{j\}}(\alpha_s(\mu^2),\mu^2/Q^2) \otimes f_i^h(\mu^2)\{\otimes f_j^{h'}(\mu^2)\} \Big](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables: $x = Q^2/(2p\cdot q)$ in DIS etc. μ: renorm./mass-fact. scale

Hard lepton-hadron processes in pQCD (II)

Parton/fragmentation distributions f_i : (renorm. group) evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) \;=\; \left[P_{ik}^{(S,T)}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

$\otimes$ **= Mellin convolution. Initial conditions: fits to reference observables**

Expansion in α_s: splitting functions P, coefficient fct's c_a of observables

$$P \;=\; \alpha_s\, P^{(0)} + \alpha_s^2\, P^{(1)} + \alpha_s^3\, P^{(2)} + \alpha_s^4\, P^{(3)} + \ldots$$

$$C_a \;=\; \underbrace{\alpha_s^{n_a}\left[c_a^{(0)} + \alpha_s\, c_a^{(1)}\right.}\!\! + \alpha_s^2\, c_a^{(2)} + \alpha_s^3\, c_a^{(3)} + \ldots \Big]$$

NLO: first real prediction of size of cross sections

NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate of pQCD predictions
N^{3}LO: for high precision (α_s from DIS), slow convergence (Higgs in $pp/p\bar{p}$)

The 2010 frontier: α_s^4/α_s^3 for DIS/SIA (+ DY) **Baikov, Chetyrkin; MV, . . .**

$\overline{\text{MS}}$ splitting functions at large x / large N

Mellin trf. $f(N) = \int_0^1 dx\, (x^{N-1}\{-1\})\, f(x)_{\{+\}}$**: M-convolutions → products**

$$\frac{\ln^n(1-x)}{(1-x)_+} \overset{\text{M}}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \ldots\,, \quad \ln^n(1-x) \overset{\text{M}}{=} \frac{(-1)^n}{N} \ln^n N + \ldots$$

Diagonal splitting functions: no higher-order enhancement at N^0, N^{-1}

$$P^{(l-1)}_{\mathrm{qq/gg}}(N) = A^{(l)}_{\mathrm{q/g}} \ln N + B^{(l)}_{\mathrm{q/g}} + C^{(l)}_{\mathrm{q/g}} \frac{1}{N} \ln N + \ldots\,, \quad A_{\mathrm{g}} = C_A/C_F\, A_{\mathrm{q}}$$

. . . ; Korchemsky (89); Dokshitzer, Marchesini, Salam (05)

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...; Korchemsky (89); Dokshitzer, Marchesini, Salam (05)

Off-diagonal: double-log behaviour, colour structure with $C_{AF} = C_A - C_F$

$$C_F^{-1}P^{(l)}_{\mathrm{gq}}\,/\,n_f^{-1}P^{(l)}_{\mathrm{qg}} = \tfrac{1}{N}\ln^{2l}N\;\#\,C^l_{AF}$$
$$+\,\tfrac{1}{N}\ln^{2l-1}N\,(\#\,C_{AF} + \#\,C_F + \#\,n_f)\,C^{l-1}_{AF} + \ldots$$

Double logs $\ln^n N$, $l+1 \le n \le 2l$ **vanish for** $C_F = C_A$ **(→ SUSY case)**

Aim: obtain, at least, these (next-to) leading terms to all orders l **in** α_{s}

$\overline{\text{MS}}$ coefficient functions at large x / large N

'Diagonal' [$\mathcal{O}(1)$] coeff. fct's for $F_{2,3,\phi}$ in DIS, $F_{T,A,\phi}$ in SIA, $F_{\rm DY} = \frac{1}{\sigma_0}\frac{d\sigma_{\rm q\bar{q}}}{dQ^2}$

$$C^{(l)}_{2,{\rm q}/\phi,{\rm g}/\ldots} \;=\; \#\, \ln^{2l} N + \ldots + N^{-1}(\#\, \ln^{2l-1} N + \ldots) \,+\, \ldots$$

N^0 parts: threshold exponentiation   Sterman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log (N^{3}LL) accuracy - mod. $A^{(4)}$

$\Rightarrow$ highest seven (DIS), six (SIA, DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05) (+ more papers, esp. using SCET, from 2006), SIA: Blümlein, Ravindran (06); MV (09)

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'Off-diagonal' [$\mathcal{O}(\alpha_s)$] quantities: leading N^{-1} double logarithms

$$C^{(l)}_{\phi,\mathrm{q}/2,\mathrm{g}/\ldots} \;=\; N^{-1}(\#\;\ln^{2l-1} N + \#\;\ln^{2l-2} N + \;\ldots) \;+\; \ldots$$

Longitudinal DIS/SIA structure functions [recall: l = order in α_s – 1]

$$C^{(l)}_{\mathrm{L,q}} \;=\; N^{-1}(\#\;\ln^{2l} N + \;\ldots) \;+\; \ldots\;, \quad C^{(l)}_{\mathrm{L,g}} \;=\; N^{-2}(\#\;\ln^{2l} N + \;\ldots) \;+\; \ldots$$

Aim: predict highest N^{-1} [N^{-2} for $C_{\mathrm{L,g}}$] double logarithms to all orders

Non-singlet and singlet physical kernels

Eliminate parton densities from scaling violations of observables ($\mu = Q$)

$$\frac{dF}{d\ln Q^2} \;=\; KF \;\equiv\; \sum_{l=0} {a_{\rm s}}^{l+1} K_l F \;=\; \frac{d\,C}{d\ln Q^2}\, q \;+\; CP\, q$$

$$= \; \Big(\beta(a_{\rm s})\, \frac{dC}{da_{\rm s}} C^{-1} \;+\; [C,P]\, C^{-1} + P \Big) F$$

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Singlet: a) $F = (F_2, F_\phi)$ with large-$m_{\rm top}$ Higgs-exchange DIS

Furmanski, Petronzio (81); ...

Coefficient functions for F_ϕ to order α_s^2/α_s^3

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni; SMVV (09)

b) $F = (F_2, \widehat{F}_L)$ with $\widehat{F}_L = F_L / a_s c_{L,q}^{(0)}$ Catani (96); Blümlein et al. (00)

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NNLO/N^{3}LO: all physical kernels K above single-log enhanced at large N

Conjecture: double-log contributions also vanish at all higher orders in α_s

Non-singlet evolution kernels and predictions

DIS$/$SIA $a \neq L$ leading-logarithmic kernels, with $p_{\rm qq}(x) = 2/(1-x)_+ - 1 - x$

$$K_{a,0}(x) \;=\; 2\, C_F\, p_{\rm qq}(x)$$

$$K_{a,1}(x) \;=\; \ln(1-x)\, p_{\rm qq}(x)\; \left[-2\, C_F \beta_0 \;\mp\; 8\, C_F^2 \;\ln x\right]$$

$$K_{a,2}(x) \;=\; \ln^2(1-x)\, p_{\rm qq}(x) \left[\;\; 2\, C_F \beta_0^2 \;\pm\; 12\, C_F^2\, \beta_0 \;\ln x + \mathcal{O}(\ln^2 x)\right]$$

$$K_{a,3}(x) \;=\; \ln^3(1-x)\, p_{\rm qq}(x) \left[-2\, C_F \beta_0^3 \mp 44/3\; C_F^2\, \beta_0^2 \;\ln x + \mathcal{O}(\ln^2 x)\right]$$

$$K_{a,4}(x) \;=\; \ln^4(1-x)\, p_{\rm qq}(x) \left[\;\; 2\, C_F \beta_0^4 \;\pm\; \xi_{K_4} C_F^2\, \beta_0^3 \;\ln x + \mathcal{O}(\ln^2 x)\right]$$

First term: leading large n_f, all orders via C_2 of Mankiewicz, Maul, Stein (97)

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Conjecture $\Rightarrow$ coefficients of highest three logs from fourth order in $\alpha_{\rm s}$,
$\ln^{7,6,5}(1-x)$ at order $\alpha_{\rm s}^4$ for $F_{1,2,3}$ in DIS and $F_{\rm T,I,A}$ in SIA etc

Leading terms: $K_1 = K_2$, $K_{\rm T} = K_{\rm I}$ [total (‘integrated’) fragmentation fct.]

$\Rightarrow$ also three logarithms for space- and timelike F_L: $\ln^{6,5,4}(1-x)$ at $\alpha_{\rm s}^4$ etc

Alternative derivation: physical kernels for F_L, agreement non-trivial check

All-order resummation of the $1/N$ terms (I)

For $F_{1,2,3}$, $F_{\mathrm{T,I,A}}$ and F_{DY}, up to terms of order $1/N^2$, with $L \equiv \ln N$

$$C_a(N) - C_a\Big|_{N^0 L^k} = \frac{1}{N}\left(\left[d^{(1)}_{a,1} L + d^{(1)}_{a,0}\right] a_{\mathrm{s}} + \left[\widetilde{d}^{(2)}_{a,1} L + d^{(2)}_{a,0}\right] a_{\mathrm{s}}^2 + \ldots\right)$$

$$\exp\left\{L h_1(a_{\mathrm{s}} L) + h_2(a_{\mathrm{s}} L) + a_{\mathrm{s}} h_3(a_{\mathrm{s}} L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_{\mathrm{s}} L) \equiv \sum_{n=1} h_{kn} (a_{\mathrm{s}} L)^n$

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Coefficients for DIS/SIA (upper/lower sign) relative to $N^0 L^k$ resummation

$$h_{1k} \;=\; g_{1k}$$

$g_{lk} =$ coefficients in soft-gluon exponentiation

$$h_{21} \;=\; g_{21} + \frac{1}{2}\beta_0 \pm 6\, C_F$$
$$h_{22} \;=\; g_{22} + \frac{5}{24}\beta_0^2 \pm \frac{17}{9}\beta_0\, C_F - 18\, C_F^2$$
$$h_{23} \;=\; g_{23} + \frac{1}{8}\beta_0^3 \pm \left(\frac{\xi_{\mathrm{K}_4}}{8} - \frac{53}{18}\right)\beta_0^2 C_F - \frac{34}{3}\beta_0\, C_F^2 \pm 72\, C_F^3$$

First term of h_3 also known, but non-universal within DIS and SIA ($\Leftrightarrow F_L$)

All-order resummation of the $1/N$ terms (II)

For space-like (-) and time-like (+) structure/fragmentation functions F_L

$$C_L^{(\pm)}(N) \;=\; N^{-1}(d_1^{(\pm)} a_s + d_2^{(\pm)} a_s^2 + \ldots)\;\exp\{L h_1(a_s L) + h_2(a_s L) + \ldots\}$$

with

$$h_{11} \;=\; 2\,C_F \;\;,\;\; h_{12} \;=\; \frac{2}{3}\,\beta_0\,C_F \;\;,\;\; h_{13} \;=\; \frac{1}{3}\,\beta_0^2\,C_F$$

$$h_{21} \;=\; \beta_0 \,+\, 4\,\gamma_e C_F \,-\, C_F \,+\, (4 - 4\,\zeta_2)(C_A - 2C_F)$$

$$h_{22} \;=\; \underbrace{\frac{1}{2}\,(\,\beta_0\,h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} \;-\; \underbrace{8\,(C_A - 2C_F)^2(1 - 3\,\zeta_2 + \zeta_3 + \zeta_2^2\,)}_{\text{Who ordered THIS?}}$$

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For space-like (-) and time-like (+) structure / fragmentation functions F_L

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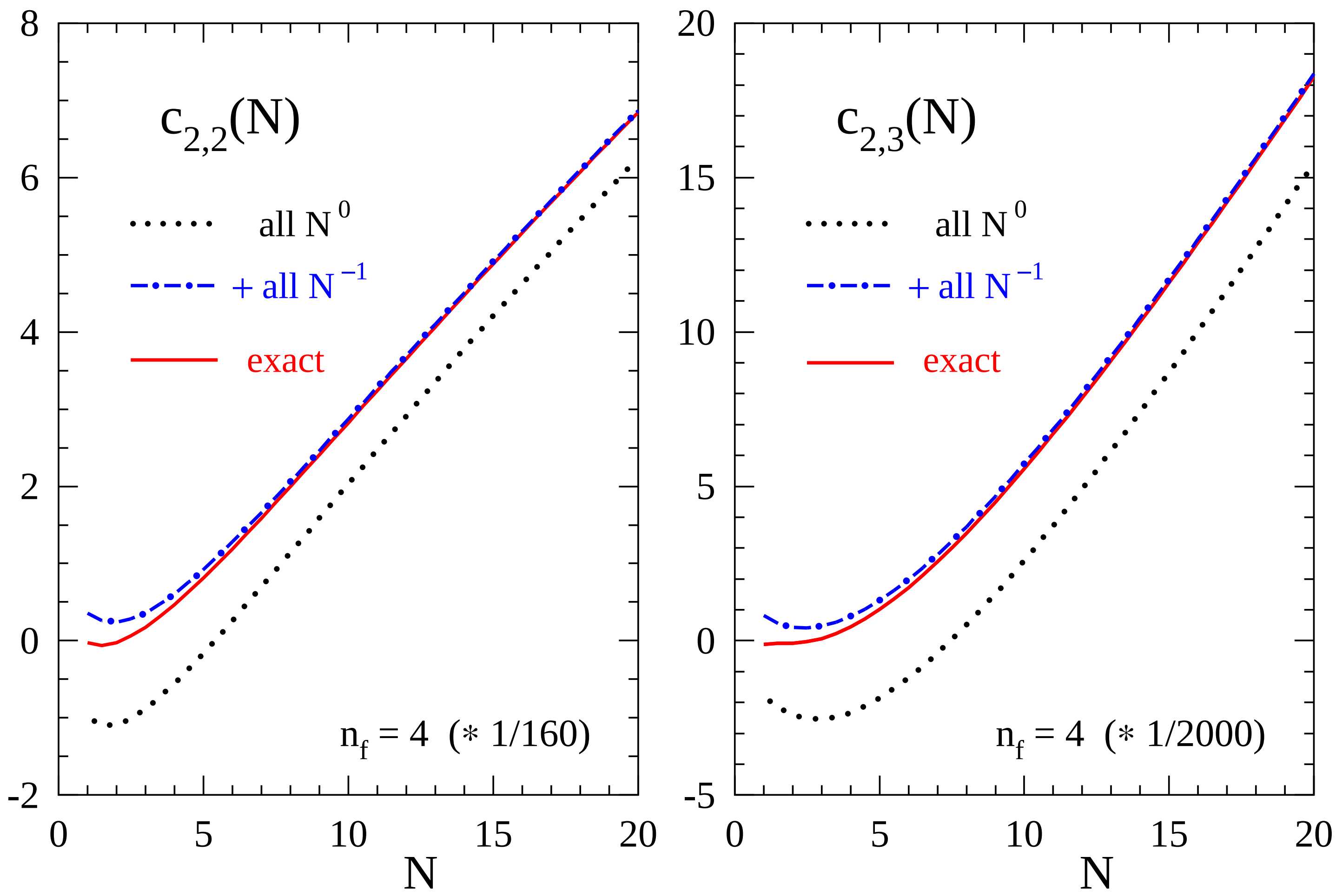
Remarks / questions

- **Less predictive than $N^0 L^k$ exponentiation: nothing new, but A_2, in g_{22}**
- **NLL exponentiation – complete $h_2(a_s L)$ – could be feasible for $F_{a\neq L}$**
- **NNLL exponentiation for $F_{1,2,3}$ etc, NLL for F_L: possible at all?**

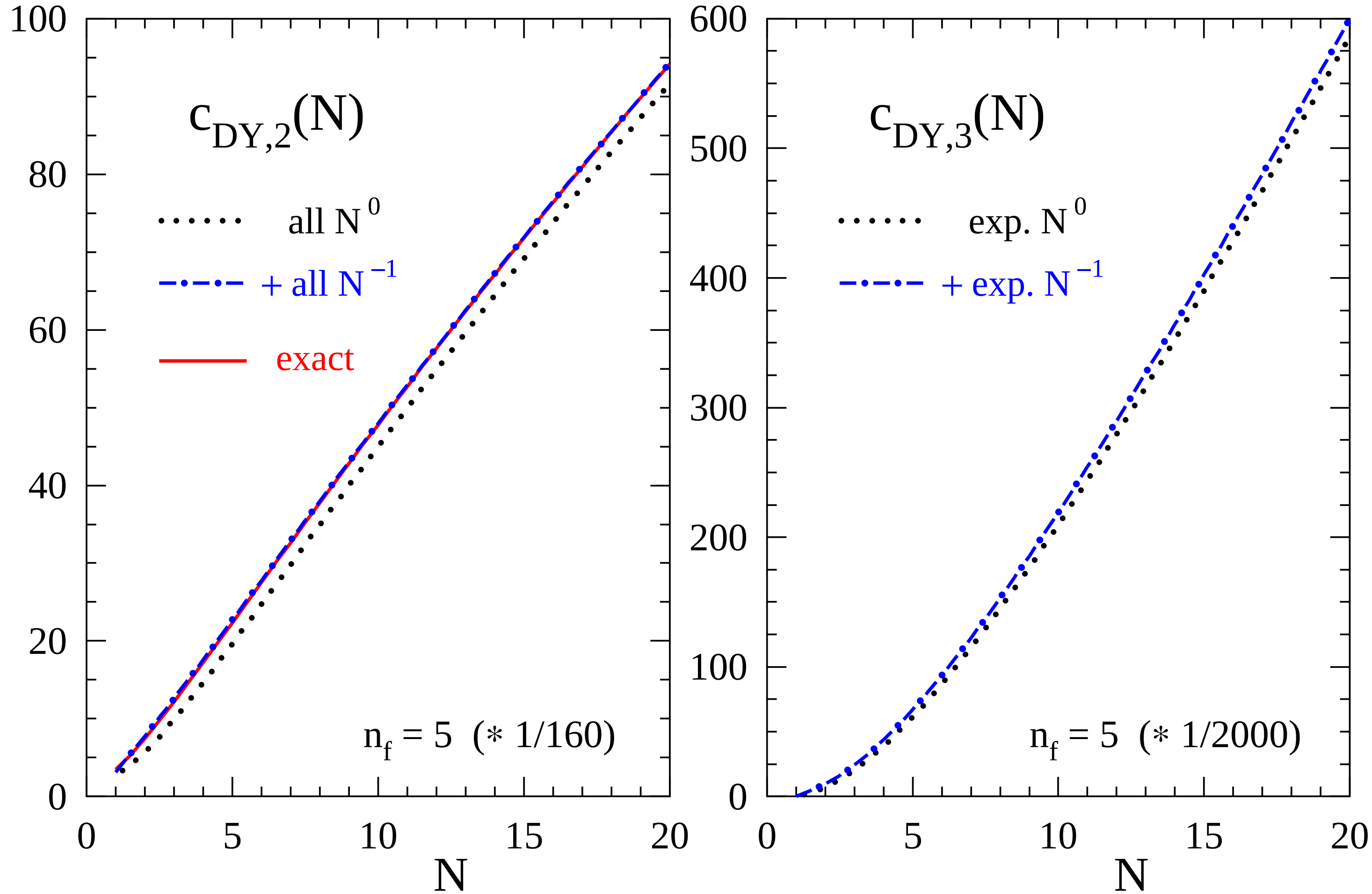
Second- and third-order C_2 in DIS in N-space



N^{-1} **terms relevant over full range shown,** $\mathcal{O}(N^{-2})$ **sizeable only at** $N < 5$

Sum of $N^{-1} \ln^k N$ **looks almost constant: half of maximum only at** $N \simeq 150$

Second- and third-order $C_{\rm DY}$ in N-space



Exp. N^0: all logs, exp. N^{-1}: 3 of 5 logs – $\xi_{\rm DY_3}$ numerically insignificant

N^{-1} contributions small down to even lower moments than in the SIA case

Singlet results: α_s^4 splitting function $P_{qg}^{(3)}(x)$

3-loop coefficient functions + single-log $K^{(3)}_{2\phi,\phi 2}$ $\Rightarrow$ predictions for $P^{(3)}_{qg,gq}$

$C_{AF} \equiv C_A - C_F$

$$
\begin{aligned}
P_{qg}^{(3)}(x) \;=\;& \ln^6(1-x) \cdot 0 \\
&+ \ln^5(1-x)\left[\frac{22}{27}C_{AF}^3 n_f - \frac{14}{27}C_{AF}^2 C_F n_f + \frac{4}{27}C_{AF}^2 n_f^2\right] \\
&+ \ln^4(1-x)\left[\left(\frac{293}{27} - \frac{80}{9}\zeta_2\right)C_{AF}^3 n_f + \left(\frac{4477}{16} - 8\zeta_2\right)C_{AF}^2 C_F n_f \right. \\
&\left. - \frac{13}{81}C_{AF}C_F^2 n_f - \frac{116}{81}C_{AF}^2 n_f^2 + \frac{17}{81}C_{AF}C_F n_f^2 - \frac{4}{81}C_{AF} n_f^3\right] \\
&+ \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}
$$

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&+ \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}
$$

- Vanishing of the coefficient of the leading term at order α_s^4 :
  accidental (??) cancellation of contributions, for all four splitting fct's
- Remaining terms vanish in the supersymmetric case $C_A = C_F (= n_f)$
  Nontrivial check: same as for $P_{qg}^{(2)}$, not obvious from above construction

This prediction + single-logarithmic $K_{2L}^{(3)}$ $\Rightarrow$ $(1-x)\ln^{6,4,3}(1-x)$ of $c_{L,g}^{(3)}$

Threshold logarithms before factorization (I)

Unfactorized partonic structure functions in $D = 4 - 2\varepsilon$ dimensions

$$T_{a,j} = \widetilde{C}_{a,i}\, Z_{ij} \;, \quad -\gamma \equiv P = \frac{dZ}{d\ln Q^2}\, Z^{-1} \,, \quad \frac{da_s}{d\ln Q^2} = -\varepsilon a_s + \beta_{D=4}$$

a_s^n: $\varepsilon^{-n} \ldots \varepsilon^{-2}$: lower-order terms, ε^{-1}: n-loop splitting functions + ...,
ε^0: n-loop coefficient fct's + ..., ε^k, $0 < k < l$: required for order $n + l$

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ε^0: n-loop coefficient fct's + ..., ε^k, $0 < k < l$: required for order $n+l$

N^0 and N^{-1} transition functions Z to next-to-leading log (NLL) accuracy

$$Z\Big|_{a_s^n} = \frac{1}{\varepsilon^n}\frac{\gamma_0^{n-1}}{n!}\left[\gamma_0 - \frac{\beta_0}{2}n(n-1)\right] + \sum_{l=1}^{n-1}\frac{1}{\varepsilon^{n-l}}\sum_{k=1}^{n-l-1}\gamma_0^{n-l-k-1}\gamma_l\gamma_0^k\,\frac{(l+k)!}{n!\,l!}$$

$$-\frac{\beta_0}{2}\sum_{l=1}^{n-2}\frac{1}{\varepsilon^{n-l}}\sum_{k=1}^{n-l-2}\gamma_0^{n-l-k-2}\gamma_l\gamma_0^k\,\frac{(l+k)!}{n!\,l!}(n(n-1) - l(l+k+1))$$

$+$ NNLL contributions (explicit expressions) $+\ \ldots$

ε^{-n+l} off-diagonal entries: contributions up to $N^{-1}\ln^{n+l-1} N$

Diagonal cases: γ_0 only for N^0 part, second term with $l=1$ for N^{-1} NLL

Threshold logarithms before factorization (II)

D-dimensional coefficient functions $\widetilde{C}_a$: finite for $\varepsilon \to 0$

$$\widetilde{C}_{a,i} = 1_{\text{(diagonal cases)}} + \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} a_s^n \, \varepsilon^l \, c_{a,i}^{(n,l)}$$

$c_{a,i}^{(n,l)}$: l additional factors $\ln N$ relative to $c_{a,i}^{(n,0)} \equiv c_{a,i}^{(n)}$ discussed above

Full N^{m}LO calc. of $T_{a,j}$: highest $m+1$ powers of ε^{-1} to all orders in α_s

Extension to all powers of ε: all-order resummation of highest $m+1$ logs

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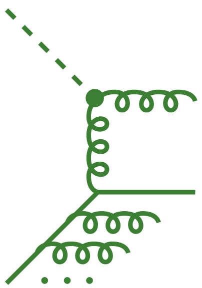
Example: Leading-log (LL) $1/N$ terms of $T_{\phi,q}^{(n)}$ and $T_{2,g}^{(n)}$, with $L \equiv \ln N$

$$\frac{1}{C_F} T_{\phi,q}^{(n)} = \frac{1}{n_f} T_{2,g}^{(n)} = \frac{L^{n-1}}{N\varepsilon^n} \sum_{k=0}^{\infty} (\varepsilon L)^k \mathcal{L}_{n,k} \left(C_F^{n-1} + C_F^{n-2} C_A + \ldots + C_A^{n-1} \right)$$

to all orders in ε (calc. + D-dim. structure), with same coefficients $\mathcal{L}_{n,k}$

$\Rightarrow$ all-order relation for one colour structure of either amplitude sufficient

All-order off-diagonal leading-log amplitudes



$$T^{(n)}_{\phi,q}\Big|_{C_F\ \mathrm{only}} \overset{\mathrm{LL}}{=} \frac{1}{n}\, T^{(1)}_{\phi,q}\, \underbrace{T^{(n-1)}_{2,q}}_{\frac{1}{(n-1)!}\,(T^{(1)}_{2,q})^{n-1}} \overset{\mathrm{LL}}{=} \frac{1}{n!}\, T^{(1)}_{\phi,q}\, (T^{(1)}_{2,q})^{n-1}$$

Three-loop diagram calculation + $P^{(3)}_{gq} \overset{\mathrm{LL}}{=} 0$ + general mass factorization:

first four powers in ε known at any order. Rest $\rightarrow$ higher-order predictions

$$T_{\phi,q}\Big|_{C_F\ \mathrm{only}} \overset{\mathrm{LL}}{=} T^{(1)}_{\phi,q}\, \frac{\exp(a_s T^{(1)}_{2,q}) - 1}{T^{(1)}_{2,q}}$$

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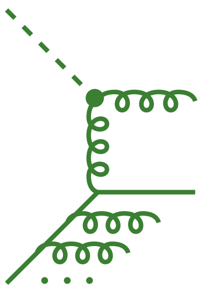

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Exact D-dimensional leading-log expressions for the one-loop amplitudes

$$T^{(1)}_{\phi,\mathrm{q}} \overset{\mathrm{LL}}{=} -2C_F\, \frac{1}{\varepsilon}(1-x)^{-\varepsilon} \overset{\mathrm{M}}{=} -\frac{2C_F}{N}\, \frac{1}{\varepsilon} \exp(\varepsilon \ln N)$$

$$T^{(1)}_{2,\mathrm{q}} \overset{\mathrm{LL}}{=} -4C_F\, \frac{1}{\varepsilon}(1-x)^{-1-\varepsilon} + \text{virtual} \overset{\mathrm{M}}{=} 4C_F\, \frac{1}{\varepsilon^2}(\exp(\varepsilon \ln N) - 1)$$

$\Rightarrow$ **leading-log expression for $T_{\phi,\mathrm{q}}$ and $T_{2,\mathrm{g}}$ completely determined**

Leading-log splitting and coefficient functions

Expansions and iterative mass factorization to 'any' order [done in FORM]

$\Rightarrow$ All-order expressions for LL off-diagonal splitting and coefficient fct's

$$P_{\rm qg}^{\rm LL}(N,\alpha_{\rm s}) \;=\; \frac{n_f}{N}\,\frac{\alpha_{\rm s}}{2\pi}\,\sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,\tilde{a}_{\rm s}^{\;n}\;, \quad \tilde{a}_{\rm s} \,=\, \frac{\alpha_{\rm s}}{\pi}\,(C_A - C_F)\,\ln^2 N$$

Bernoulli numbers B_n: zero for odd $n \geq 3 \;\Rightarrow\; P_{\rm gq}^{(3)}(N) \overset{\rm LL}{=} 0$ not accidental

$$B_0 = 1,\; B_1 = -\frac{1}{2},\; B_2 = \frac{1}{6},\; B_4 = -\frac{1}{30},\; B_6 = \frac{1}{42},\; \ldots,\; B_{12} = -\frac{691}{2730},\; \ldots$$

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$$C_{2,\rm g}^{\rm LL} \;=\; \frac{1}{2N\ln N}\,\frac{n_f}{C_A - C_F}\,\left\{\exp(2C_F\,a_{\rm s}\ln^2 N)\,\mathcal{B}_0(\tilde{a}_{\rm s}) - \exp(2C_A\,a_{\rm s}\ln^2 N)\right\}$$

$\exp(\ldots)$: LL soft-gluon exponentials   Parisi; Curci, Greco; Amati et al. (80)

$$\mathcal{B}_0(x) \;=\; \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,x^n$$

$P_{\rm gq}^{\rm LL}, C_{\phi,\rm q}^{\rm LL}$: same functions but with $C_F \leftrightarrow C_A$ (also in $\tilde{a}_{\rm s}$), then $n_f \rightarrow C_F$

First properties of the new $\mathcal{B}$-functions

Relation between even-n Bernoulli numbers and the Riemann ζ-function

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\,\zeta_{2n}\left(\frac{x}{2\pi}\right)^{2n}$$

$\mathcal{B}_0(2\pi i)$ **numerically known (Wolfram MathWorld, Sloane's A093721), no closed form**

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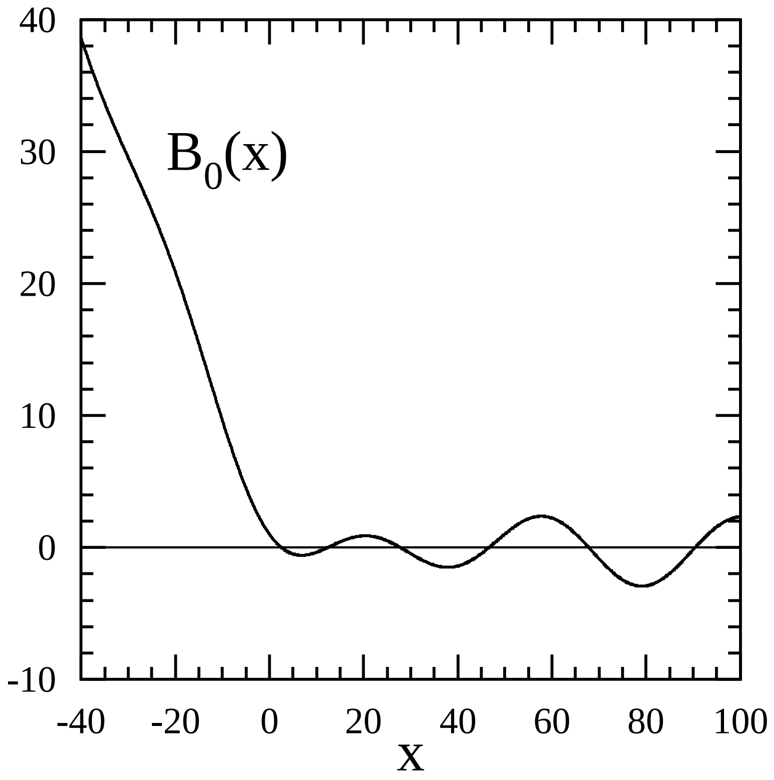

Further $\mathcal{B}$-functions for later use

$$\mathcal{B}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!}\,x^n$$

$$\mathcal{B}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!}\,x^n$$

Relations to $\mathcal{B}_0(x)$

$$\frac{d^k}{dx^k}\left(x^k\mathcal{B}_k\right) = \mathcal{B}_0\,, \quad x^k\frac{d^k}{dx^k}\,\mathcal{B}_0 = \mathcal{B}_{-k}$$

Next-to-leading logarithmic iteration for $T^{(n)}_{\phi,\mathrm{q}}$

Ansatz for $T^{(n)}_{\phi,\mathrm{q}}$ in terms of first-order quantity and diagonal amplitudes

$$T^{(n)}_{\phi,\mathrm{q}} \overset{\mathrm{NL}}{=} \frac{1}{n} T^{(1)}_{\phi,\mathrm{q}} \left\{ \sum_{i=0}^{n-1} T^{(i)}_{\phi,\mathrm{q}} T^{(n-i-1)}_{2,\mathrm{q}} f(n,i) - \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} T^{(i)}_{\phi,\mathrm{q}} T^{(n-i-2)}_{2,\mathrm{q}} g(n,i) \right\}$$

All-order agreement with known highest four powers of ε^{-1} for

$$f(n,i) = \binom{n-1}{i}^{-1} \left[1 + \varepsilon \cdot \left(\text{known simple function of } n \text{ and } i \right) \right]$$

$$g(n,i) = \binom{n}{i+1}^{-1}$$

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Ansatz for $T_{\phi,q}^{(n)}$ in terms of first-order quantity and diagonal amplitudes

$$T_{\phi,q}^{(n)} \stackrel{\mathrm{NL}}{=} \frac{1}{n} T_{\phi,q}^{(1)} \left\{ \sum_{i=0}^{n-1} T_{\phi,q}^{(i)} T_{2,q}^{(n-i-1)} f(n,i) - \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} T_{\phi,q}^{(i)} T_{2,q}^{(n-i-2)} g(n,i) \right\}$$

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Soft-gluon exponentiation: also $T_{\phi,g}^{(n)}$ and $T_{2,q}^{(n)}$ known at all powers of ε

$\Rightarrow$ next-to-leading logarithmic expression for $T_{\phi,q}$ completely predicted

Mass factorization $\Rightarrow$ P_{gq}^{NLL}, $c_{\phi,q}^{\mathrm{NLL}}$ to all orders. P_{qg}^{NLL}, $c_{2,g}^{\mathrm{NLL}}$ analogous

Extension of this approach to higher logarithmic accuracy problematic

D-dim. structure of unfactorized observables

Maximal phase space for deep-inelastic scattering/semi-incl. annihilation

$\mathbf{NLO:}$	$2 \to 2 \;/\; 1 \to 1+2$	$(1-x)^{-1-\varepsilon}\; x^{\cdots} \int_0^1$	one other variable
$\mathbf{N^2LO:}$	$2 \to 3 \;/\; 1 \to 1+3$	$(1-x)^{-1-2\varepsilon}\; x^{\cdots} \int_0^1$	four other variables
$\mathbf{N^3LO:}$	$2 \to 4 \;/\; 1 \to 1+4$	$(1-x)^{-1-3\varepsilon}\; x^{\cdots} \int_0^1$	seven other variables

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N^2LO: Matsuura, van Neerven (88), Rijken, vN (95), $N^{n\geq 3}$LO, indirectly: MV[V] (05)

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Purely real contributions to unfactorized structure functions

$$T_{a,j}^{(n)\mathrm{R}} = (1-x)^{-1-n\varepsilon} \sum_{\xi=0} (1-x)^{\xi} \, \frac{1}{\varepsilon^{2n-1}} \left\{ R_{a,j,\xi}^{(n)\mathrm{LL}} + \varepsilon R_{a,j,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Mixed contributions ($2 \rightarrow r+1$ with $n-r$ loops in DIS)

$$T_{a,j}^{(n)\mathrm{M}} = \sum_{l=r}^{n} (1-x)^{-1-l\varepsilon} \sum_{\xi=0} (1-x)^{\xi} \, \frac{1}{\varepsilon^{2n-1}} \left\{ M_{a,j,l,\xi}^{(n)\mathrm{LL}} + \varepsilon M_{a,j,l,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Purely virtual part (diagonal cases, $\eta = 0$ present): $\gamma^* qq$, Hgg form factors

$$T_{a,j}^{(n)\mathrm{V}} = \delta(1-x) \, \frac{1}{\varepsilon^{2n}} \left\{ V_{a,j}^{(n)\mathrm{LL}} + \varepsilon V_{a,j}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Resulting resummation of large-x double logs

KLN cancellation between purely real, mixed and purely virtual contributions

$$T_{a,j}^{(n)} = T_{a,j}^{(n)\mathrm{R}} + T_{a,j}^{(n)\mathrm{M}} \Big(+ T_{a,j}^{(n)\mathrm{V}} \Big) = \frac{1}{\varepsilon^n} \Big\{ T_{a,j}^{(n)0} + \varepsilon T_{a,j}^{(n)1} + \ldots \Big\}$$

$\Rightarrow$ **Up to $n-1$ relations between the coeff's of $(1-x)^{-1-l\varepsilon}$, $l = 1, \ldots, n$**

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Log expansion: N^{k}LL higher-order coefficients completely fixed, if first $k+1$ powers of ε known to all orders – provided by N^{k}LO calculation, see above

Present situation: (a) N^{3}LO for non-singlet $F_{a \neq L}$ in DIS – recall DMS (05)
(b) N^{2}LO for SIA, non-singlet F_{L} in DIS, and singlet DIS

$\Rightarrow$ **resummation of the (a) four and (b) three highest $N^{-1} \ln^{k} N$ terms to all orders in α_{s}: consistent with, and extending, our previous results**

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Soft-gluon exponentiation of the $(1-x)^{-1}/N^0$ diagonal coefficient functions:

$(1-x)^{-1-\varepsilon}, \ldots, (1-x)^{-1-(n-1)\varepsilon}$ at order n: products of lower-order quantities

$\Rightarrow$ N^{n}LO [+$A^{(n+1)}$] $\rightarrow$ N^{n}LL exponentiation; $2n[+1]$ highest logs predicted

Selection of some new results

NS cases: $K_{a,4}(x)$ of p.7 confirmed with $\xi_{K_4} = 100/3$: fourth log for $c_{a,\rm ns}^{(n\geq 4)}$

Off-diagonal splitting functions

$$\tilde{a}_{\rm s} = \frac{\alpha_{\rm s}}{\pi}\,(C_A - C_F)\ln^2 N$$

$$N P_{\rm qg}^{\rm NL}(N,\alpha_{\rm s}) \;=\; 2a_{\rm s} n_f\, \mathcal{B}_0(\tilde{a}_{\rm s}) + a_{\rm s}^2 \ln N\, n_f \Big\{ (6C_F - \beta_0)\Big(\frac{2}{\tilde{a}_{\rm s}}\mathcal{B}_{-1}(\tilde{a}_{\rm s}) + \mathcal{B}_1(\tilde{a}_{\rm s})\Big) + \frac{\beta_0}{\tilde{a}_{\rm s}}\,\mathcal{B}_{-2}(\tilde{a}_{\rm s}) \Big\}$$

$$N P_{\rm gq}^{\rm NL}(N,\alpha_{\rm s}) \;=\; 2a_{\rm s} C_F\, \mathcal{B}_0(-\tilde{a}_{\rm s}) \;+\; a_{\rm s}^2 \ln N\, C_F \Big\{ (12C_F - 6\beta_0)\,\frac{1}{\tilde{a}_{\rm s}}\,\mathcal{B}_{-1}(-\tilde{a}_{\rm s}) - \frac{\beta_0}{\tilde{a}_{\rm s}}\,\mathcal{B}_{-2}(-\tilde{a}_{\rm s}) + (14C_F - 8C_A - \beta_0)\,\mathcal{B}_1(-\tilde{a}_{\rm s}) \Big\}$$

Gluon contribution to F_L – 'non-singlet' $C_F = 0$ part done before **MV (09)**

$$N^2 c_{L,\rm g}^{\rm NL}(N,\alpha_{\rm s}) \;=\; 8a_{\rm s} n_f\, \exp(2C_A a_{\rm s}\ln^2 N) \;+\; 4a_{\rm s} C_F N C_{2,\rm g}^{\rm LL}(N,\alpha_{\rm s}) + 16 a_{\rm s}^2 \ln N\, n_f \Big\{ 4C_A - C_F + \frac{1}{3} a_{\rm s} \ln^2 N\, C_A \beta_0 \Big\} \exp(2C_A a_{\rm s} \ln^2 N)$$

NNLL contributions known to 'any' order, but (mostly) no closed expressions

Summary and outlook

- Non-singlet physical kernels for nine observables in DIS, SIA and DY:
  single-log large-x enhancement at NNLO$/$N^{3}LO to all orders in $1-x$
  All-order conjecture $\Rightarrow$ leading three (DY: two) logs of higher-order C_a
- Singlet kernels for (F_2, F_ϕ) and (F_2, F_L) in DIS also single-logarithmic
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- Complementary approach: Laenen, Magnea, Stavenga, White (from 08)
- Limited phenomenol. relevance now: assess relevance of NS $1/N$ terms
- Near/mid future: combine with other results, esp. fixed-N calculations
  (close to) feasible now: 4-loop sum rules Baikov, Chetyrkin, Kühn (10)