Counting Abelian Orbifolds

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Introduction

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- How many orbifolds of C^3 ?

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- Class of backgrounds in string theory

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- Class of backgrounds in string theory
- Class of finite theories in 3+1 dimensions with a cubic superpotential
- Brane Tilings with hexagonal tiles

Motivation

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- M2 branes probing CY4

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- How Many Abelian CY4?

Simple orbifolds one per each N

$$\mathbb{C}^2/\Gamma_N$$

$$\Gamma_N = \mathbb{Z}_N$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \omega^{(a_1,a_2)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \ e^{\frac{i2\pi a_1}{N}} \\ z_2 \ e^{\frac{i2\pi a_2}{N}} \end{pmatrix}$$

$$a_1 + a_2 = 0 \pmod N$$

Higher dimensions more than 1 per N

$$\mathbb{C}^3/\Gamma_N$$

$$n_1 n_2 = N$$

$$\Gamma_N = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$$

$$\omega^{(\{a_i\},\{b_i\})} \; : \; \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \omega^{(\{a_i\},\{b_i\})} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \; e^{i2\pi(\frac{a_1}{n_1}+\frac{b_1}{n_2})} \\ z_2 \; e^{i2\pi(\frac{a_2}{n_1}+\frac{b_2}{n_2})} \\ z_3 \; e^{i2\pi(\frac{a_3}{n_1}+\frac{b_3}{n_2})} \end{pmatrix}$$

action ambiguity

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- For D=3, N=5, (1,1,3) is the same as (2,2,1) and the same as (3,3,4), same as (4,4,2)

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- Count these solutions only once

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- (1,a,-1-a)?

D=3 Orbifolds

- Are all orbifold actions of the form
- (1,a,-1-a)?
- First case it is not: N=30 with orbifold action (2,3,25)

Product groups higher dimensions

$$\mathbb{C}^4/\Gamma_N$$

$$\Gamma_N = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \subset SU(4)$$

$$\mathbb{C}^D/\Gamma_N$$

$$\Gamma_N = \otimes_{j=1}^{D-1} \mathbb{Z}_{n_j} \subset SU(D) \qquad N = \prod_{j=1}^{D-1} n_j$$

Results
orbifold actions

$\mathbb{C}^D/\Gamma_N$							
		D					
		2	3	4	5	6	7
N	1	1	1	1	1	1	1
	2	1	1	2	2	3	3
	3	1	2	3	4	6	7
	4	1	3	7	10	17	23
	5	1	2	5	8	13	19
	6	1	3	10	19	40	65
	7	1	3	7	13	27	46
	8	1	5	20	45	106	
	9	1	4	14	33	72	
	10	1	4	18	47	127	
	11	1	3	11	30	79	
	12	1	8	41	129	391	
	13	1	4	15	43	129	
	14	1	5	28	96	321	
	15	1	6	31	108		
	16	1	9	58	224		
	17	1	4	21	78		
	18	1	8	60	264		
	19	1	5	25	102		
	20	1	10	77	357		
	21	1	8	49	226		
	22	1	7	54	277		
	23	1	5	33	163		
	24	1	15	144	813		
	25	1	7	50	260		
	26	1	8	72	425		
	27	1	9	75	436		
	28	1	13	123	780		
	29	1	6	49	297		
	30	1	14	158	1092		
	31	1	7	55			

How to Compute these numbers?

Similar problems

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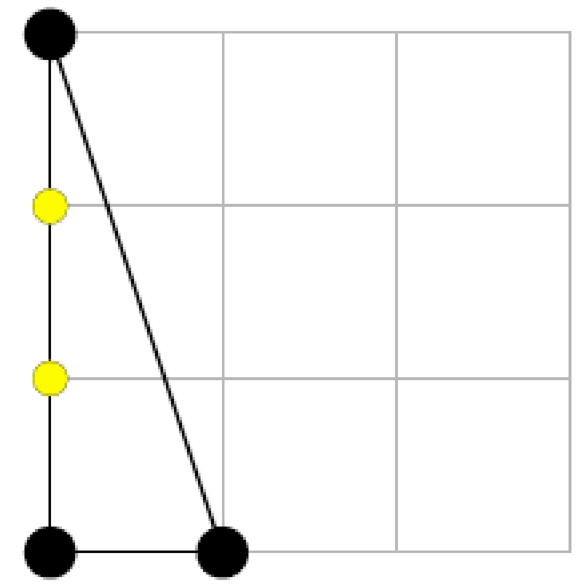
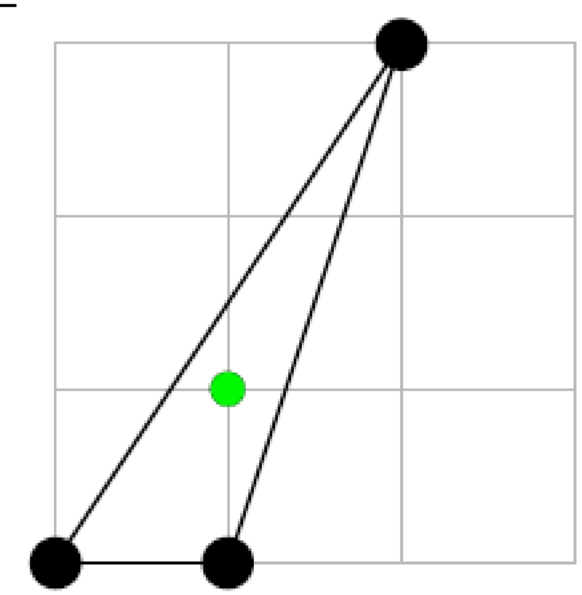
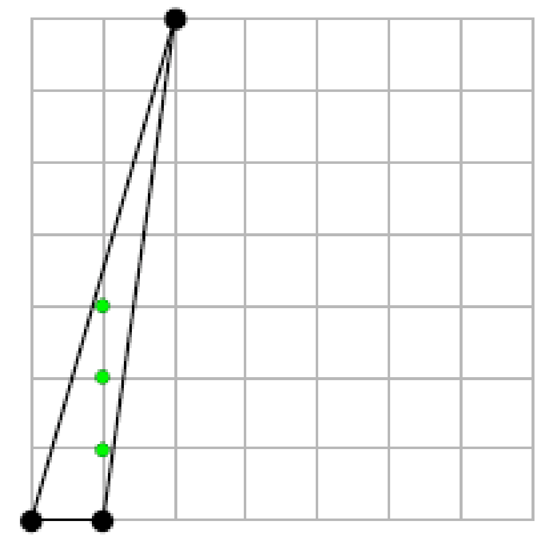
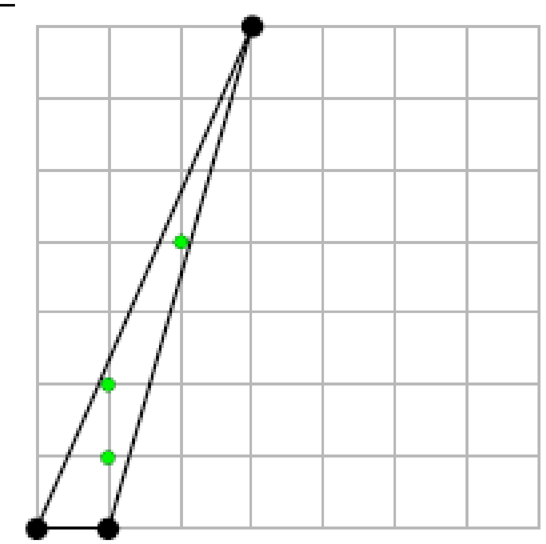
- Brane Tilings - hexagonal tilings; N hex.
- Hexagonal sub-lattices of index N
- lattice tetrahedra of volume N
- lattice simplices of hyper-volume N

Look at Toric Diagrams

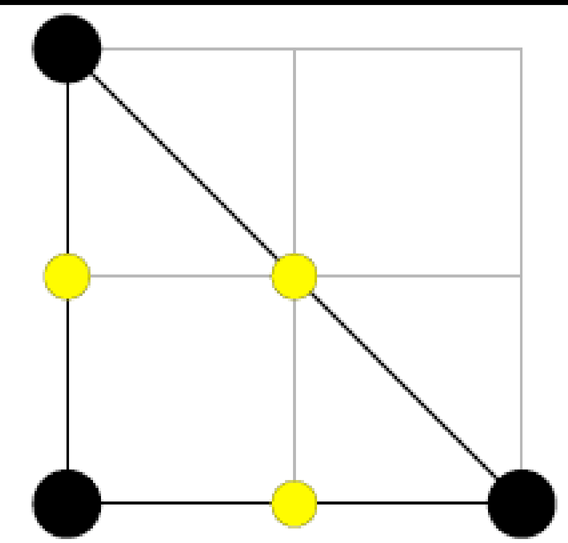
More than 1 in higher dimensions

(3.1)	3	$\mathbb{C}^3/\mathbb{Z}_3$	$\begin{pmatrix} (0,1,2) \\ (0,0,0) \end{pmatrix}$	
(3.2)	3	$\mathbb{C}^3/\mathbb{Z}_3$	$\begin{pmatrix} (1,1,1) \\ (0,0,0) \end{pmatrix}$	

notation ambiguity in higher dimensions

(7.2)	7	$\mathbb{C}^3/\mathbb{Z}_7$	$\begin{pmatrix} (1,1,5) \\ (0,0,0) \end{pmatrix}$		3
(7.3)	7	$\mathbb{C}^3/\mathbb{Z}_7$	$\begin{pmatrix} (1,2,4) \\ (0,0,0) \end{pmatrix}$		2

Product groups

(4.3)	4	$\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} (1,0,1) \\ (0,1,1) \end{pmatrix}$	

Two dimensions

Two dimensions

- One way to construct lines of length N

Count toric diagrams precisely once?

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- For a toric diagram with k vertices v_i act with a matrix M v_i
- M upper diagonal matrix with integer entries further restricted
- det M = N
- For D=3: $N = n_1 n_2$; $M_{12} < n_2$

Hermite Normal Form on 3 vertices

The unit area triangle has corner points

$$v_1 = (0,0)\ ,\ \ v_2 = (1,0)\ ,\ \ v_3 = (0,1)$$

$$M \;=\; \begin{pmatrix} n_1 & c \\ 0 & n_2 \end{pmatrix}$$

$$D(N) \;=\; \{\{v_1, v_2, v_3\} = \{(0,0), (n_1,0), (c,n_2)\}\ \mid N = n_1 n_2\ ,\ 0 \le c \le n_2 - 1\ ,\ c \in \mathbb{Z}\}$$

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- 2d sub-lattices of index N

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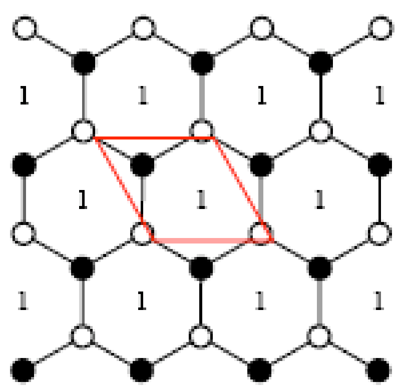
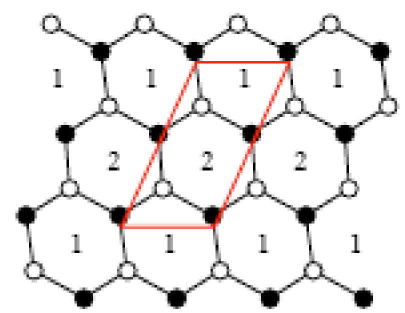
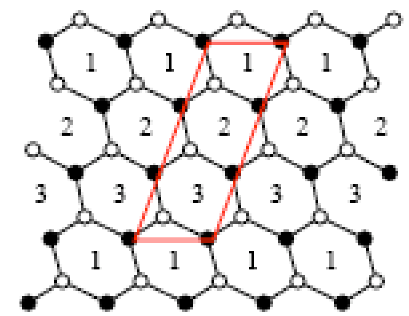
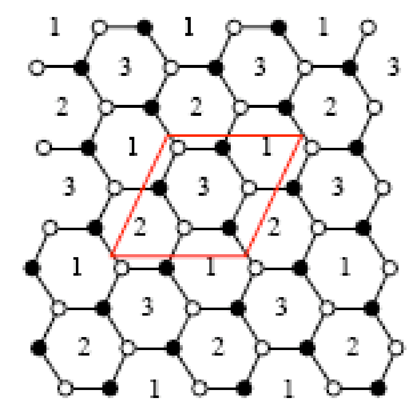
- Abelian orbifolds: specify HNF
- However, some toric diagrams are symmetric under a finite discrete group
- Makes some HNF's equivalent
- Use Polya Enumeration Theorem!

Symmetries

n	1	2	3	3
brane tiling				
geometry	$\mathbb{C}^3$	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$	$\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$	$\mathbb{C}^3/\mathbb{Z}_3$

Burnside's Lemma
Polya Enumeration

Lemma. *Let G be a group of permutations of the set X. The number $N(G)$ of* orbits *of G is given by the average over G of the sizes of the fixed sets:*

$$N(G) = \frac{1}{|G|} \sum_{g \in G} |F_g| \; ; \qquad F_g = \{ x \in X \mid g(x) = x \} \; . \tag{3.7}$$

$$f^L(G) = \frac{1}{|G|} \sum_{g \in G} f_g^L(n) \; ; \qquad f_g^L(n) = |\{ x \in X_n \mid g(x) = x \}|$$

Cycle Index

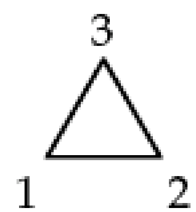
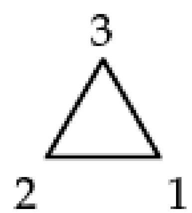
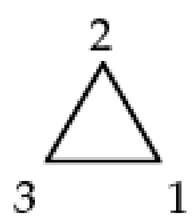
	g	α_1	α_2	α_3	$c(\alpha_i)$	ζ
	1	3	-	-	1	x_1^3
	(12)	1	1	-	3	$x_1 x_2$
	(123)	-	-	1	2	x_3

Table 2: Cycle index for the symmetric group S_3. $Z_{S_3} = \frac{1}{6}\left(x_1^3 + 3x_1x_2 + 2x_3\right)$.

Cycle Index

$$Z_{S_D} = \frac{1}{D}\sum_{r=1}^{D} x_r \; Z_{S_{D-r}} \; . \tag{4.12}$$

D	Orbifold	Cycle Index
1	$\mathbb{C}$	$Z_{S_1} = x_1$
2	$\mathbb{C}^2/\Gamma_N$	$Z_{S_2} = \frac{1}{2}(x_1^2 + x_2)$
3	$\mathbb{C}^3/\Gamma_N$	$Z_{S_3} = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)$
4	$\mathbb{C}^4/\Gamma_N$	$Z_{S_4} = \frac{1}{24}(x_1^4 + 6x_1^2x_2 + 3x_2^2 + 8x_1x_3 + 6x_4)$
5	$\mathbb{C}^5/\Gamma_N$	$Z_{S_5} = \frac{1}{120}(x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 20x_1^2x_3 + 20x_2x_3 + 30x_1x_4 + 24x_5)$
6	$\mathbb{C}^6/\Gamma_N$	$Z_{S_6} = \frac{1}{720}(x_1^6 + 15x_1^4x_2 + 45x_1^2x_2^2 + 15x_2^3 + 40x_1^3x_3 + 120x_1x_2x_3 + 40x_3^2 + 90x_1^2x_4 + 90x_2x_4 + 144x_1x_5 + 120x_6)$
7	$\mathbb{C}^7/\Gamma_N$	$Z_{S_7} = \frac{1}{5040}(x_1^7 + 21x_1^5x_2 + 105x_1^3x_2^2 + 105x_1x_2^3 + 70x_1^4x_3 + 420x_1^2x_2x_3 + 210x_2^2x_3 + 280x_1x_3^2 + 210x_1^3x_4 + 630x_1x_2x_4 + 420x_3x_4 + 504x_1^2x_5 + 504x_2x_5 + 840x_1x_6 + 720x_7)$

4 different sequences

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- Toric diagrams with area N - HNF's

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- Subset invariant under reflections

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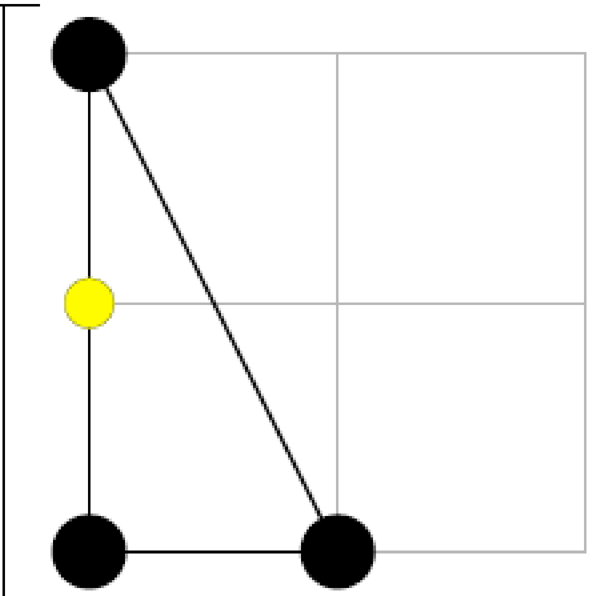
- inqeuvalent orbifolds:

4 different sequences

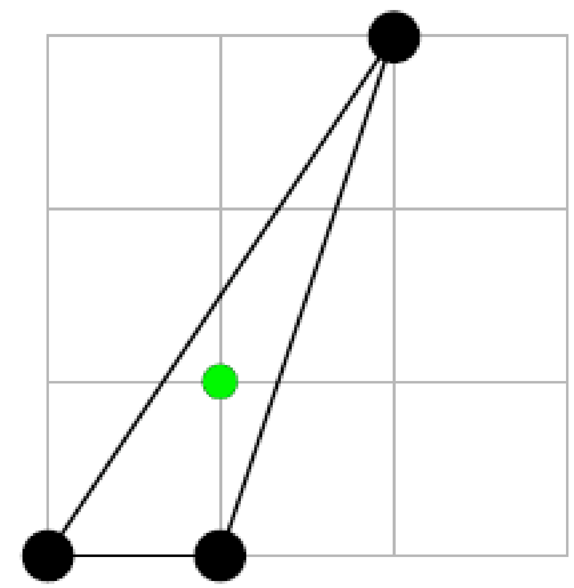
- Toric diagrams with area N - HNF's
- Subset invariant under reflections
- Subset invariant under rotations
- inqeuvalent orbifolds:
- Average according the Cycle Index

Examples
reflection; no rotation

$\mathbb{C}^3/\mathbb{Z}_2$	$\begin{pmatrix} (0,1,1) \\ (0,0,0) \end{pmatrix}$	

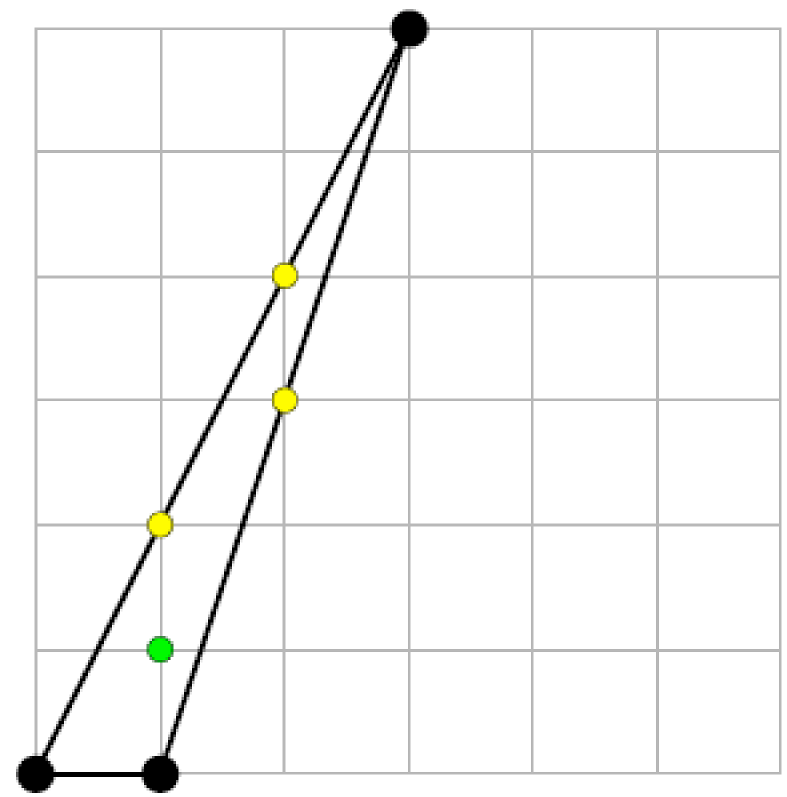
$$\frac{1}{6}3 + \frac{1}{2}1 + \frac{1}{3}0 = 1$$

Examples
Reflection; Rotation

(3.2)	3	$\mathbb{C}^3/\mathbb{Z}_3$	$\begin{pmatrix} (1,1,1) \\ (0,0,0) \end{pmatrix}$		1

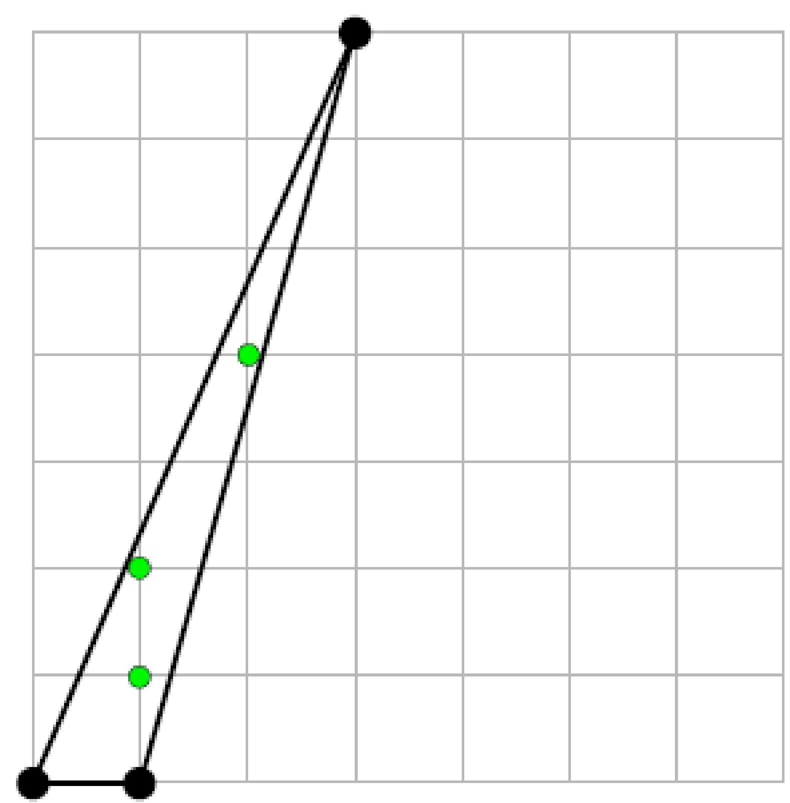
$$\frac{1}{6}1 + \frac{1}{2}1 + \frac{1}{3}1 = 1$$

No reflection
No rotation

(6.3)	6	$\mathbb{C}^3/\mathbb{Z}_6$	$\begin{pmatrix} (1,2,3) \\ (0,0,0) \end{pmatrix}$		6

$$\frac{1}{6}6 + \frac{1}{2}0 + \frac{1}{3}0 = 1$$

Example
No reflection; Rotation

(7.3)	7	$\mathbb{C}^3/\mathbb{Z}_7$	$\begin{pmatrix}(1,2,4)\\(0,0,0)\end{pmatrix}$		2

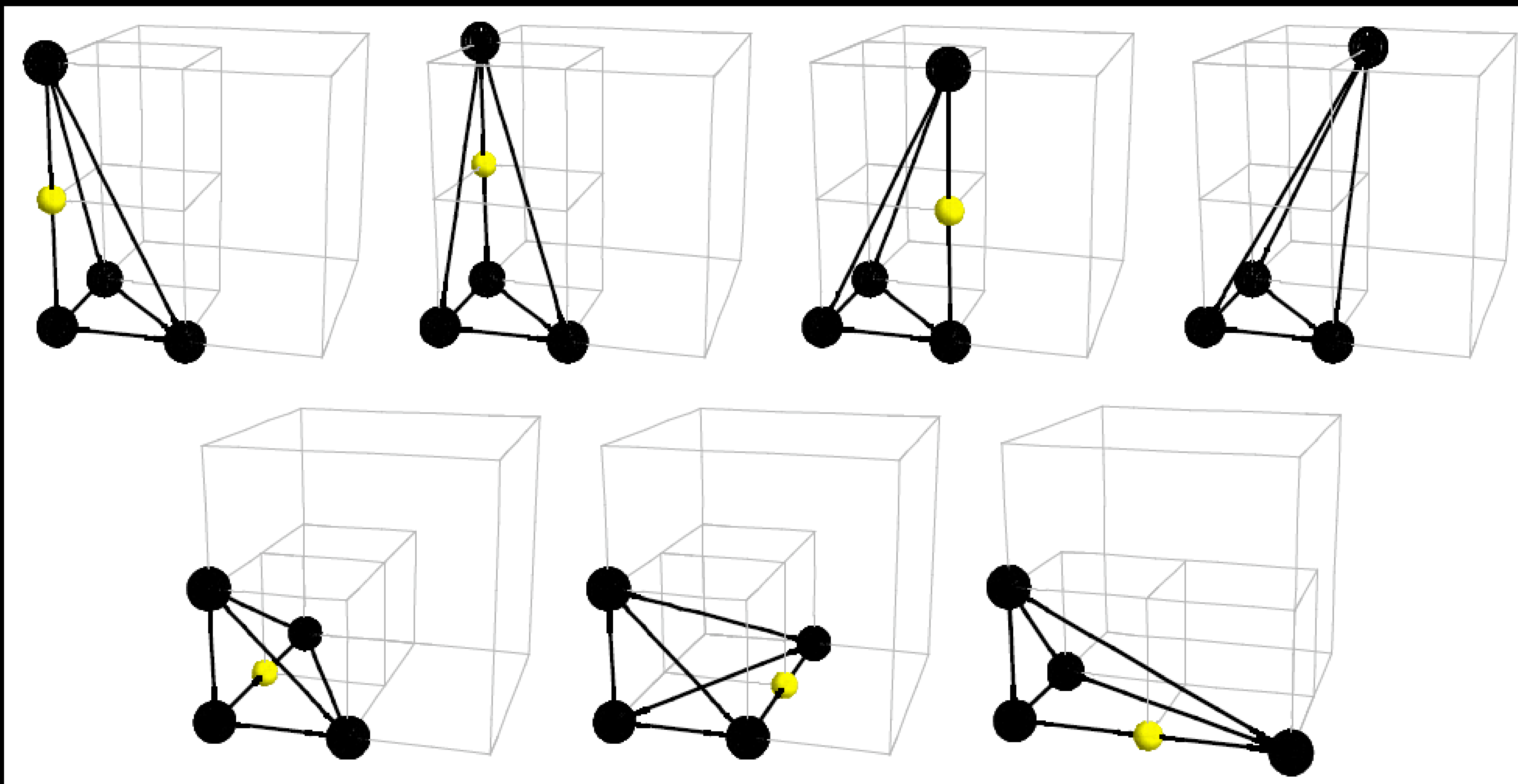
$$\frac{1}{6}2+\frac{1}{2}0+\frac{1}{3}2=1$$

The hexagonal sequences

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f^{\triangle}_{x_1^3}$	1	3	4	7	6	12	8	15	13	18	12	28	14	24	24	31
$f^{\triangle}_{x_1x_2}$	1	1	2	3	2	2	2	5	3	2	2	6	2	2	4	7
$f^{\triangle}_{x_3}$	1	0	1	1	0	0	2	0	1	0	0	1	2	0	0	1
$f^{\triangle}$	1	1	2	3	2	3	3	5	4	4	3	8	4	5	6	9

Table 3: Number of sublattices of index n for the hexagonal lattice, classified by the cycles of the symmetric group S_3. According to the cycle index decomposition, $f^{\triangle} = \frac{1}{6}\left(f^{\triangle}_{x_1^3} + 3f^{\triangle}_{x_1x_2} + 2f^{\triangle}_{x_3}\right)$.

Example; HNF's
D=4, N=2



number of generic 2d sublattices

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- 1,3,4,7,6,12,8,15,13..

Scatter Plot
Triangle Toric Diagrams

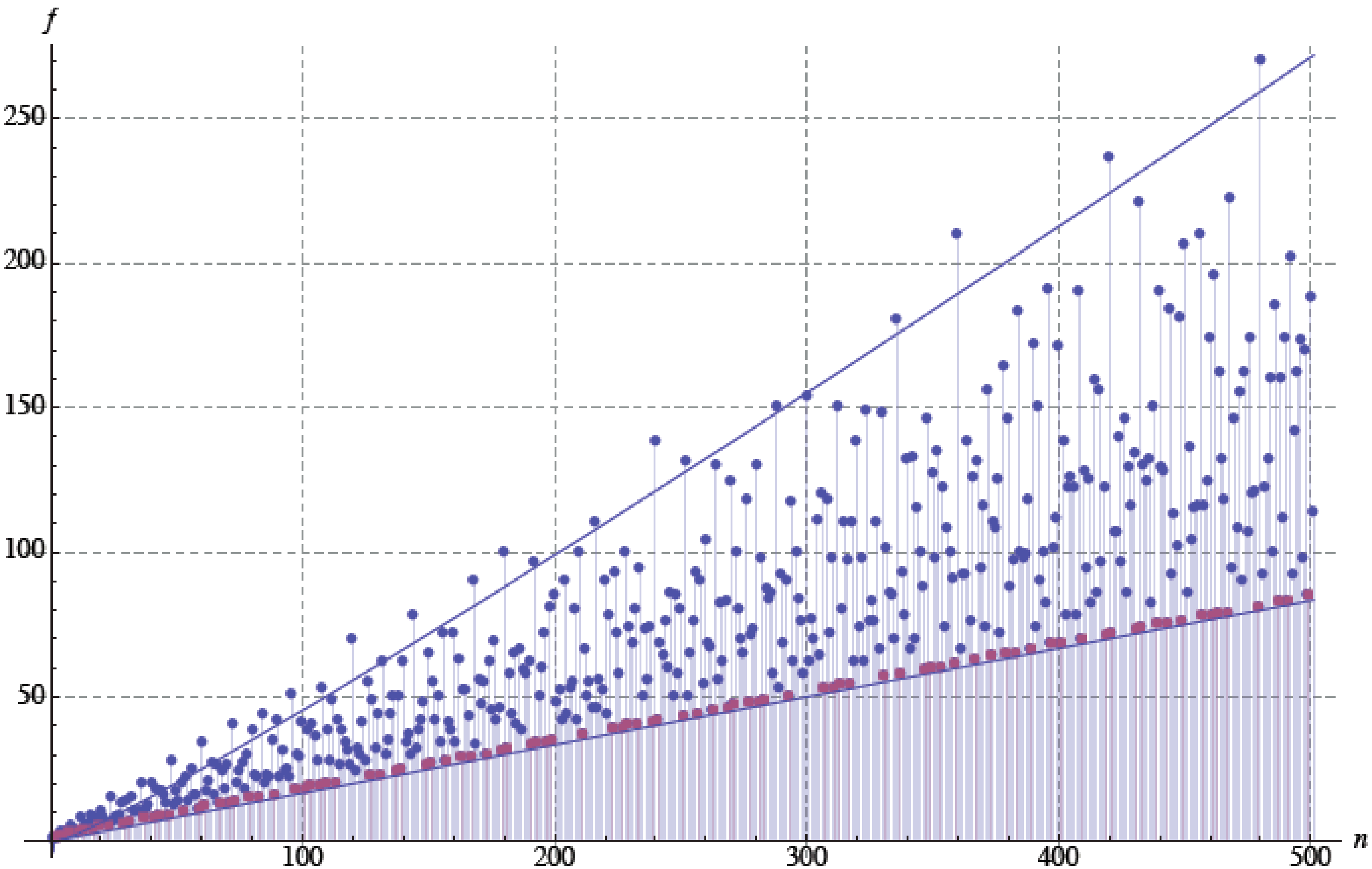

Figure 1: Scatter plot of the sequence $f^{\triangle}$ for a hexagonal lattice. Prime numbers are emphasized in red. The two lines correspond to $n/6$ and $e^{\gamma} n \log\log n/6$.

growth
3+1d theories

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- Divisor function

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growth
3+1d theories

- Divisor function
- Order of group of symmetries
- $G=S_3$, $|G|=6$ for $D=3$

$$f(n) \sim \frac{\sigma(n)}{|G|}, \qquad \text{for } n \gg 1$$

Results
computer code

$\mathbb{C}^D/\Gamma_N$						
	D					
	2	3	4	5	6	7
N = 1	1	1	1	1	1	1
2	1	1	2	2	3	3
3	1	2	3	4	6	7
4	1	3	7	10	17	23
5	1	2	5	8	13	19
6	1	3	10	19	40	65
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Multiplicative sequences

Definition. *A sequence f is multiplicative if*

$$f(nm) = f(n)f(m), \quad \text{when } (n,m) = 1,$$

where (n,m) denotes the greatest common divisor *between n and m.*

Dirichlet Convolution

Definition. *The* Dirichlet convolution *of two sequences f and g is the sequence h defined by*

$$f(n) = (g * h)(n) = \sum_{m|n} g(m)\, h(\frac{n}{m}),$$

where the notation $m|n$ means that the sum runs over all the divisors m of n.

Let f, g and h be such that

$$f = g * h.$$

The power series for h reads:

$$F(t) = \sum_{n=1}^{\infty} f(n) n^n = \sum_{n=1}^{\infty} \sum_{m|n} g(m)\, h(\frac{n}{m})\, t^n = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} g(m)\, h(k)\, t^{mk}$$

generic 2d toric diagrams - HNF's

Example (△). *In the case of the bipartite hexagonal lattice we find:*

1. *The sequence* $f^{\triangle}_{x_1^3} = \{1, 3, 4, 7, 6, 12, 8, 15, \dots\}$ *corresponds to the identity permutation* x_1^3 *and it is given by Equation (3.10) with* $d = 2$*. It can also be written as the convolution*

$$f^{\triangle}_{x_1^3} = \mathrm{u} * \mathrm{N}\,, \tag{5.10}$$

where

$$\mathrm{N}(n) = \{1, 2, 3, \dots\}\ . \tag{5.11}$$

$$\mathrm{u}(n) = \{1, 1, 1, \dots\}$$

reflection invariant

2. *The sequence* $f^{\triangle}_{x_1x_2} = \{1, 1, 2, 3, 2, 2, 2, 5, \dots\}$ *can be written as the convolution of a periodic sequence of period* 4 *and the unit:*

$$f^{\triangle}_{x_1x_2} = \{1, 0, 1, 2, 1, 0, 1, 2, 1, \dots\} * \mathrm{u} \,. \qquad (5.12)$$

rotation invariant

3. *The last sequence* $f_{x_3}^{\triangle} = \{1, 0, 1, 1, 0, 0, 2, 0, \dots\}$ *also has the form of the convolution of the unity with a periodic sequence of period* 3*:*

$$f_{x_3}^{\triangle} = \{1, -1, 0, 1, -1, 0, 1, -1, 0, \dots\} * \mathrm{u} \,. \tag{5.14}$$

Dirichlet Series

1. the formal power series (partition function)

$$F(t) = \sum_{n=1}^{\infty} f(n) t^n \, ;$$

2. the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \, .$$

Useful for Multiplicative Sequences

Multiplicative Sequences

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right)$$

Partition Functions Hexagonal Tilings

1. *The sequence* $f^{\triangle}_{x_1^3}$ *is decomposed as* $f^{\triangle}_{x_1^3} = \mathrm{u} * \mathrm{N}$*, hence the corresponding power seri*
erated by

$$G^{\triangle}_{x_1^3}(t) = \sum_{k=1}^{\infty} k\, t^k = \frac{1+t^3}{(1-t)\,(1-t^2)} - 1\,,$$

and the Dirichlet series reads

$$F^{\triangle}_{x_1^3}(s) = \zeta(s)\zeta(s-1)\,.$$

2. *The sequence* $f^{\triangle}_{x_1 x_2} = \{\,1, -1, 0, 2\,\} * \mathrm{u} * \mathrm{u}$ *gives*

$$G^{\triangle}_{x_1 x_2}(t) = \frac{1+t^3}{(1-t)\,(1+t^2)} - 1\,,$$

$$F^{\triangle}_{x_1 x_2}(s) = \left(1 - 2^{-s} + 2^{1-2s}\right)\zeta(s)^2\,.$$

3. *The sequence* $f^{\triangle}_{x_3} = \chi_{3,2} * \mathrm{u}$ *gives*

$$G^{\triangle}_{x_3}(t) = \frac{(1+t)\,(1-t^2)}{1-t^3} - 1\,,$$

Hexagonal Tilings partition functions

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$$1 + f(t) = \frac{1}{(1-t)(1+t^2)(1-t^3)}$$

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$$1 + f(t) = \frac{1}{(1-t)(1+t^2)(1-t^3)}$$

$$g(t) = \sum_{k=1}^{\infty} f(t^k)$$

$$\sum_{n=1}^{\infty} \left(\frac{n}{6} + \frac{1}{2} \left(\cos\left(\frac{n\pi}{2}\right) + 1 \right) + \frac{2\sin\left(\frac{2n\pi}{3}\right)}{3\sqrt{3}} \right) t^n$$

Hexagonal Tilings partition functions

$$1 + f(t) = \frac{1}{(1-t)(1+t^2)(1-t^3)}$$

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$$\sum_{n=1}^{\infty} \left(\frac{n}{6} + \frac{1}{2} \left(\cos\left(\frac{n\pi}{2} \right) + 1 \right) + \frac{2\sin\left(\frac{2n\pi}{3}\right)}{3\sqrt{3}} \right) t^n$$

$$F^{\triangle}(s) = \frac{\zeta(s)}{6} \left(\zeta(s-1) + 3\left(1 - 2^{-s} + 2^{1-2s}\right) \zeta(s) + 2L(s, \chi_{3,2}) \right)$$

Convolution preserves multiplicativity; is additive on primes

Use basic sequences

The Dirichlet character $\chi_{k,m}$ of modulo k and index m is defined under the conditions

$$\begin{aligned} \chi_{k,m}(1) &= 1 \\ \chi_{k,m}(a) &= \chi_{k,m}(a+k) \\ \chi_{k,m}(a)\chi_{k,m}(b) &= \chi_{k,m}(ab) \\ \chi_{k,m}(a) &= 0 \quad \text{if} \quad \gcd(k,a) \neq 1 \quad . \end{aligned} \tag{4.7}$$

The Dirichlet characters up to modulo 10 used in this work are

$\chi_{1,1} = \mathsf{u}$

$\chi_{2,1} = \{1, 0, \dots\}$

$\chi_{3,1} = \{1, 1, 0, \dots\}$

$\chi_{3,2} = \{1, -1, 0, \dots\}$

$\chi_{4,1} = \{1, 0, 1, 0, \dots\}$

$\chi_{4,2} = \{1, 0, -1, 0, \dots\}$

$\chi_{5,1} = \{1, 1, 1, 1, 0, \dots\}$

$\chi_{5,2} = \{1, i, -i, -1, 0, \dots\}$

$\chi_{5,3} = \{1, -1, -1, 1, 0, \dots\}$

$\chi_{5,4} = \{1, -i, i, -1, 0, \dots\}$

$\chi_{6,1} = \{1, 0, 0, 0, 1, 0 \dots\}$

$\chi_{6,2} = \{1, 0, 0, 0, -1, 0 \dots\}$

$\chi_{7,1} = \{1, 1, 1, 1, 1, 1, 0 \dots\}$

$\chi_{7,2} = \{1, -\omega, \omega^2, \omega^2, -\omega, 1, 0 \dots\}$

$\chi_{7,3} = \{1, \omega^2, \omega, -\omega, -\omega^2, -1, 0 \dots\}$

$\chi_{7,4} = \{1, 1, -1, 1, -1, -1, 0 \dots\}$

$\chi_{7,5} = \{1, -\omega, -\omega^2, \omega^2, \omega, -1, 0 \dots\}$

$\chi_{7,6} = \{1, \omega^2, -\omega, -\omega, \omega^2, 1, 0 \dots\}$

$\chi_{8,1} = \{1, 0, 1, 0, 1, 0, 1, 0, \dots\}$

$\chi_{8,2} = \{1, 0, 1, 0, -1, 0, -1, 0, \dots\}$

$\chi_{8,3} = \{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$

$\chi_{8,4} = \{1, 0, -1, 0, -1, 0, 1, 0, \dots\}$

$\chi_{9,1} = \{1, 1, 0, 1, 1, 0, 1, 1, 0, \dots\}$

$\chi_{9,2} = \{1, \omega, 0, \omega^2, -\omega^2, 0, -\omega, -1, 0, \dots\}$

$\chi_{9,3} = \{1, \omega^2, 0, -\omega, -\omega, 0, \omega^2, 1, 0, \dots\}$

$\chi_{9,4} = \{1, -1, 0, 1, -1, 0, 1, -1, 0, \dots\}$

$\chi_{9,5} = \{1, -\omega, 0, \omega^2, \omega^2, 0, -\omega, 1, 0, \dots\}$

$\chi_{9,6} = \{1, -\omega^2, 0, -\omega, \omega, 0, \omega^2, -1, 0, \dots\}$

$\chi_{10,1} = \{1, 0, 1, 0, 0, 0, 1, 0, 1, 0, \dots\}$

$\chi_{10,2} = \{1, 0, i, 0, 0, 0, -i, 0, -1, 0, \dots\}$

$\chi_{10,3} = \{1, 0, -1, 0, 0, 0, -1, 0, 1, 0, \dots\}$

$\chi_{10,4} = \{1, 0, -i, 0, 0, 0, i, 0, -1, 0, \dots\}$

C^4 sequences

$\mathbb{C}^4/\Gamma_N$										
N	1	2	3	4	5	6	7	8	9	10
$g_{x_1^4}$	1	7	13	35	31	91	57	155	130	217
$g_{x_1^2 x_2}$	1	3	5	11	7	15	9	31	18	21
$g_{x_2^2}$	1	3	5	11	7	15	9	31	18	21
$g_{x_1 x_3}$	1	1	1	2	1	1	3	2	4	1
g_{x_4}	1	1	1	3	3	1	1	5	2	3
$g^{D=4}$	1	2	3	7	5	10	7	20	14	18
N	11	12	13	14	15	16	17	18	19	20
$g_{x_1^4}$	133	455	183	399	403	651	307	910	381	1085
$g_{x_1^2 x_2}$	13	55	15	27	35	75	19	54	21	77
$g_{x_2^2}$	13	55	15	27	35	75	19	54	21	77
$g_{x_1 x_3}$	1	2	3	3	1	3	1	4	3	2
g_{x_4}	1	3	3	1	3	7	3	2	1	9
$g^{D=4}$	11	41	15	28	31	58	21	60	25	77

C^5 sequences

$\mathbb{C}^5/\Gamma_N$

N	1	2	3	4	5	6	7	8	9	10
$g_{x_1^5}$	1	15	40	155	156	600	400	1395	1210	2340
$g_{x_1^3 x_2}$	1	7	14	43	32	98	58	219	144	224
$g_{x_1 x_2^2}$	1	3	8	19	12	24	16	75	42	36
$g_{x_1^2 x_3}$	1	3	4	8	6	12	10	18	22	18
$g_{x_2 x_3}$	1	1	2	4	2	2	4	6	6	2
$g_{x_1 x_4}$	1	1	2	3	4	2	2	7	4	4
g_{x_5}	1	0	0	0	1	0	0	0	0	0
$g^{D=5}$	1	2	4	10	8	19	13	45	33	47

N	11	12	13	14	15	16	17	18	19	20
$g_{x_1^5}$	1464	6200	2380	6000	6240	11811	5220	18150	7240	24180
$g_{x_1^3 x_2}$	134	602	184	406	448	995	308	1008	382	1376
$g_{x_1 x_2^2}$	24	152	28	48	96	251	36	126	40	228
$g_{x_1^2 x_3}$	12	32	16	30	24	39	18	66	22	48
$g_{x_2 x_3}$	2	8	4	4	4	11	2	6	4	8
$g_{x_1 x_4}$	2	6	4	2	8	19	4	4	2	12
g_{x_5}	4	0	0	0	0	1	0	0	0	0
$g^{D=5}$	30	129	43	96	108	226	78	264	102	357

N	21	22	23	24	25	26	27	28	29	30
$g_{x_1^5}$	16000	21960	12720	55800	20306	35700	33880	62000	25260	93600
$g_{x_1^3 x_2}$	812	938	554	3066	838	1288	1354	2494	872	3136
$g_{x_1 x_2^2}$	128	72	48	600	98	84	184	304	60	288
$g_{x_1^2 x_3}$	40	36	24	72	32	48	85	80	30	72

$$P_{\mathfrak{g}_{x_1^4}}(p) = 1 + p + p^2$$

$$P_{\mathfrak{g}_{x_1^2 x_2}}(p) = P_{\mathfrak{g}_{x_2^2}}(p) = \begin{cases} 3 & \text{if } p = 2 \\ p+2 & \text{if } p \neq 2 \end{cases}$$

$$P_{\mathfrak{g}_{x_1 x_3}}(p) = \begin{cases} 3 & \text{if } p = 1 \mod 3 \\ 1 & \text{if } p = 0, 2 \mod 3 \end{cases}$$

$$P_{\mathfrak{g}_{x_4}}(p) = \begin{cases} 3 & \text{if } p = 1 \mod 4 \\ 1 & \text{if } p = 2, 3 \mod 4 \end{cases}$$

$$P_{\mathfrak{g}_{x_1^3}}(p) = 1 + p$$

$$P_{\mathfrak{g}_{x_1^3}}(p) = \begin{cases} 1 & \text{if } p = 2 \\ 2 & \text{if } p \neq 2 \end{cases}$$

$$P_{\mathfrak{g}_{x_3}}(p) = \begin{cases} 2 & \text{if } p = 1 \mod 3 \\ 0 & \text{if } p = 2 \mod 3 \\ 1 & \text{if } p = 0 \mod 3 \end{cases}$$

$$P_{\mathbf{g}_{x_1^5}}(p) = 1 + p + p^2 + p^3$$

$$P_{\mathbf{g}_{x_1^3 x_2}}(p) = \begin{cases} 7 & \text{if } p = 2 \\ p^2 + p + 2 & \text{if } p \neq 2 \end{cases}$$

$$P_{\mathbf{g}_{x_1 x_2^2}}(p) = \begin{cases} 3 & \text{if } p = 2 \\ 2p + 2 & \text{if } p \neq 2 \end{cases}$$

$$P_{\mathbf{g}_{x_1^2 x_3}}(p) = \begin{cases} p + 3 & \text{if } p = 1 \mod 3 \\ p + 1 & \text{if } p = 2 \mod 3 \\ 4 & \text{if } p = 3 \end{cases}$$

$$P_{\mathbf{g}_{x_2 x_3}}(p) = \begin{cases} 2 & \text{if } p = 2 \mod 3 \\ 4 & \text{if } p = 1 \mod 3 \\ 2 & \text{if } p = 3 \\ 1 & \text{if } p = 2 \end{cases}$$

$$P_{\mathbf{g}_{x_1 x_4}}(p) = \begin{cases} 4 & \text{if } p = 1 \mod 4 \\ 2 & \text{if } p = 3 \mod 4 \\ 1 & \text{if } p = 2 \end{cases}$$

$$P_{\mathbf{g}_{x_5}}(p) = \begin{cases} 1 & \text{if } p = 1, 5 \\ 4 & \text{if } p = 1 \mod 5 \\ 0 & \text{if } p = 2, 3, 4 \mod 5 \end{cases}$$

Series Convolutions

$\mathbb{C}^2/\Gamma_N$	
x_1^2	u
x_2	u
$\mathbb{C}^3/\Gamma_N$	
x_1^3	$\mathsf{u} * \mathsf{N}$
$x_1 x_2$	$\mathsf{u} * \mathsf{u} * \{1, -1, 0, 2\}$
x_3	$\mathsf{u} * \chi_{3,2}$
$\mathbb{C}^4/\Gamma_N$	
x_1^4	$\mathsf{u} * \mathsf{N} * \mathsf{N}^2$
$x_1^2 x_2$	$\mathsf{u} * \mathsf{u} * \mathsf{N} * \{1, -1, 0, 4\}$
x_2^2	$\mathsf{u} * \mathsf{u} * \mathsf{N} * \{1, -1, 0, 4\}$
$x_1 x_3$	$\mathsf{u} * \mathsf{u} * \chi_{3,2} * \{1, 0, -1, 0, 0, 0, 0, 0, 3\}$
x_4	$\mathsf{u} * \mathsf{u} * \chi_{4,2} * \{1, -1, 0, 2\}$
$\mathbb{C}^5/\Gamma_N$	
x_1^5	$\mathsf{u} * \mathsf{N} * \mathsf{N}^2 * \mathsf{N}^3$
$x_1^3 x_2$	$\mathsf{u} * \mathsf{u} * \mathsf{N} * \mathsf{N}^2 * \{1, -1, 0, 8\}$
$x_1 x_2^2$	$\mathsf{u} * \mathsf{u} * \mathsf{N} * \mathsf{N} * (t - 3t^2 + 14t^4 - 12t^8 + 16t^{16})$
$x_1^2 x_3$	$\mathsf{u} * \mathsf{u} * \mathsf{N} * \chi_{3,2} * \{1, 0, -1, 0, 0, 0, 0, 0, 9\}$
$x_2 x_3$	$\mathsf{u} * \mathsf{u} * \mathsf{u} * \chi_{3,2} * \{1, -1, 0, 2\} * \{1, 0, -1, 0, 0, 0, 0, 0, 3\}$
$x_1 x_4$	$\mathsf{u} * \mathsf{u} * \mathsf{u} * \chi_{4,2} * (t - 2t^2 + 3t^4 + 6t^{16} - 8t^{32})$
x_5	$\mathsf{u} * \chi_{5,2} * \chi_{5,3} * \chi_{5,4}$

Table 6: Summary of series convolutions for orbifolds of $\mathbb{C}^2$, $\mathbb{C}^3$, $\mathbb{C}^4$ and $\mathbb{C}^5$.

Generating Functions

Generating Functions for $\mathbb{C}^4$. As presented in [28], the generating function for the symmetry series of $\mathbb{C}^4$ are

$$
\begin{aligned}
f_{x_1^4}(t) &= \sum_{n,m=1}^{\infty} nm^2 t^{mn} \quad , \\
f_{x_1^2 x_2}(t) &= \sum_{n,m=1}^{\infty} m\left(t^{mn} - t^{2mn} + 4t^{4mn}\right) \quad , \\
f_{x_2^2}(t) &= \sum_{n,m=1}^{\infty} m\left(t^{mn} - t^{2mn} + 4t^{4mn}\right) \quad , \\
f_{x_1 x_3}(t) &= \frac{1}{2}\left[\sum_{n,m=-\infty}^{\infty} t^{n^2+4m^2} - 1\right] \\
f_{x_4}(t) &= \frac{1}{2}\left[\sum_{n,m=-\infty}^{\infty} t^{n^2+mn+7m^2} - 1\right] \quad ,
\end{aligned}
\tag{5.4}
$$

$$
g^{D=4}(t) \;=\; \frac{1}{24}\left(g_{x_1^4}(t) + 6g_{x_1^3 x_2}(t) + 3g_{x_2^2}(t) + 8g_{x_1x_3}(t) + 6g_{x_4}(t) \quad , \right)
$$

with $g_{x^\alpha} = \sum_{k=1}^{\infty} f_{x^\alpha}(t^k)$.

Summary

Summary

- Counting Abelian Orbifolds of C^D

Summary

- Counting Abelian Orbifolds of C^D
- Reveal a rich structure

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- Number Theory

Summary

- Counting Abelian Orbifolds of C^D
- Reveal a rich structure
- Methods from Crystalography
- Number Theory
- Good for orbifolds of any toric diagram, in any dimension

Summary

Summary

- Can study statistical aspects of such backgrounds with the explicit generating functions

Summary

- Can study statistical aspects of such backgrounds with the explicit generating functions
- Growth like $n^{D-2}/|G|$ for large n

Thank you!