3-algebras and $(2, 0)$ Supersymmetry

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Introduction and Motivation

String Theory offers a powerful and compelling framework to discuss gravity and gauge interactions in a unified and quantum manner.

However String Theory, as generally understood, is only really defined as a set of perturbative 'Feynman' rules.

- No nonperturbative definition
- 5 different sets of such 'rules'

A crucial ingredient of String Theory are Dp-branes [Polchinski]

- extended objects with p spatial dimensions
- end points of open strings
- quantum dynamics determined by $(p+1)$-dimensional Yang-Mills Gauge theory: nonabelian structure on parallel D-branes.

Introduction and Motivation

However there is strong (overwhelming?) evidence for a single complete unifying theory: M-theory

- Strong coupling limit of type IIA: $R_{11} = g_s l_s$
- weak curvature effective action is 11D supergravity [Cremmer, Julia, Sherk]
- no microscopic description/definition
- no strings: just 2-branes and 5-branes

Formally the M-theory/type IIA duality implies that

- M2-branes: strongly coupled (IR) limit of D2-branes (3D Super-Yang-Mills)
- M5-branes strongly coupled (UV) limit of D4-branes (5D Super-Yang-Mills)

Introduction and Motivation

The past few years has seen a great deal of progress in our understanding of M2-branes and in particular a description in terms of Lagrangian field theories

- Novel Chern-Simons-Matter CFT's in 3D with large amounts of supersymmetry ($N = 8, 6, ...$) ([BL][G], [ABJM],...).
- describe multiple M2-branes in flat space or orbifolds thereof.

One novel feature of these theories is that the amount of supersymmetry is determined by the gauge group, e.g.:

- $N = 8$: $SU(2) \times SU(2)$
- $N = 6$: $U(n) \times U(m)$, $Sp(n) \times U(1)$.
- ...

Introduction and Motivation

A key concept in the construction of M2-brane Lagrangians is a 3-algebra:

- Vector space V with basis T^a and a linear triple product

$$[T^A, T^B, T^C] = f^{ABC}{}_D T^D$$

- Fields take values in V; $X^I = X^I_A T^A$, $\Psi = \Psi_A T^A$

The 3-algebra generates a Lie-algebra action on the fields X^I:

$$X^I \to \Lambda_{AB}[X^I, T^A, T^B]$$

provided that the triple product satisfies a quadratic 'fundamental' identity (generalization of Jacobi).

Introduction and Motivation

An alternative definition of 3-algebras is that they are simply Lie algebras $Lie(G)$ with metric $(\ ,\ )$ along with a representation V with a gauge invariant inner-product $\langle\ ,\ \rangle$:

- Triple product arises from the Faulkner map: $\varphi : V \times V \to Lie(G)$

$$\begin{aligned}(\varphi(T^A, T^B), g) &= \langle g(T^A), T^B\rangle \\ [T^A, T^B, T^C] &= \varphi(T^A, T^B)(T^C)\end{aligned}$$

The Lagrangians are completely specified by $f^{ABC}{}_D$; symmetries of triple product determine the susy and gauge group:

- $N = 8$: $[T^A, T^B, T^C]$ is totally anti-symmetric
- $N = 6$: $[T^A, T^B; T_C] = -[T^B, T^A; T_C]$ and complex anti-linear in T_C

Introduction and Motivation

M-theory also possesses M5-branes.

- Parallel M5-branes lead to a strongly coupled 6D CFT
- Such a theory would have great powers:
  - e.g. manifest S-duality of $D = 4$, $N = 4$ super-Yang-Mills
  - recent work on $D = 4$ gauge theory [Gaiotto]

Very little is known about such a theory and it seems much, much harder than M2-branes (see below)

We will try to construct 6D theories with $(2, 0)$ supersymmetry.

- 3-algebras arise quite naturally
- Non-abelian dynamics is constrained to 5D
- Suggests a first step is to look for a $(2, 0)$ reformulation of D4-branes

PLAN:

- Introduction and Motivation (that's this!)
- The M5-brane
- $(2, 0)$ supersymmetry in $D = 6$
- Conclusions and Comments

M5-branes

The worldvolume of a parallel stack of M5-branes preserves 16 supersymmetries and $1+5$ dimensional Poincare symmetry along with an $SO(5)$ R-symmetry

$$SO(1,10) \to SO(1,5) \times SO(5) \qquad \mathbf{32} \to \mathbf{16}$$

In particular the preserved supersymmetries satisfy $\Gamma_{012345}\epsilon = \epsilon$ and this leads to $(2,0)$ supersymmetry in $D=6$ with Goldstinos zero modes

$$\Gamma_{012345}\Psi = -\Psi$$

and 5 scalars

$$X^I$$

The remaining Bosonic degrees of freedom arise from a self-dual tensor

$$H_{\mu\nu\lambda} = \frac{1}{3!}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}H^{\rho\sigma\tau}$$

M5-branes

From the type IIA perspective the M5-brane arises as the strong coupling (UV) limit of D4-branes.

- An extra spatial dimension arises (but the same R-symmetry).

The effective theory of n D4-branes is 5D maximally supersymmetric $U(n)$ Yang Mills

- naively non-renormalizable!
- M-theory implies that there is a UV completion given by the M5-brane: 6D CFT!
- Since no interacting 6D CFT is known and the 5D theory is non-renormalizable it is a case of the blind leading the blind, ie. no definition is available at either end.
  - although there is a matrix theory attempt [Aharony, Berkooz,Kachru, Seiberg,Silverstein]

M5-branes

The appearance of an extra spatial dimension is curious, and analogous to the type IIA to M-theory lift.

Where are the KK momentum modes?

- in type IIA an 11D KK mode appears as a D0-brane
- D0-branes appear in the D4-brane as instanton soliton states:

$$m \propto \frac{1}{g_{YM}^2} = \frac{1}{R_{11}}$$

- So the instantons of the 5D Yang-Mills theory have the interpretation as KK momentum of the 6D CFT on S^1

$$m \to 0 \qquad \text{as} \qquad R_{11} \to \infty \Longleftrightarrow g_{YM} \to \infty$$

M5-branes

This has several odd features:

- 6D momentum modes are not local with respect to other (charged) momentum modes
- Where are the KK modes in the Higgs phase (separated D4's)?
- Instanton moduli space is non-compact: continuous spectrum

Finally the entropy of D4-branes scales as n^2 whereas that of M5-branes like n^3.

On the other hand the D4-brane should already know about 6D of the form $\mathbf{R}^5 \times S^1$

$(2,0)$ supersymmetry in $D=6$

First consider the free abelian theory [Howe,Sezgin,West].

At linearized level the susy variations are

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta\Psi &= \Gamma^\mu\Gamma^I\partial_\mu X^I\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon \\ \delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi \ , \end{aligned}$$

and the equations of motion are those of free fields with $dH=0$ (and hence $dH = d\star H = 0$).

Reduction to the D4-brane theory sets $\partial_5 = 0$ and

$$F_{\mu\nu} = H_{\mu\nu 5}$$

More generally, in the non-linear version, one finds H satisfies a non-linear self-duality which upon reduction gives

$$dF = 0 \qquad d\star\left(\frac{F}{\sqrt{1+F^2}}\right) = 0$$

i.e. DBI

$(2,0)$ supersymmetry in $D = 6$

We wish to generalise this algebra to nonabelian fields with

$$D_\mu X^I_A = \partial_\mu X^I_A - \tilde{A}_\mu^B{}_A X^I_B$$

Upon reduction we expect Yang-Mills susy:

$$\begin{aligned}
\delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\
\delta\Psi &= \Gamma^\alpha\Gamma^I D_\alpha X^I\epsilon + \frac{1}{2}\Gamma^{\alpha\beta}\Gamma^5 F_{\alpha\beta}\epsilon - \frac{i}{2}[X^I, X^J]\Gamma^{IJ}\Gamma^5\epsilon \\
\delta A_\alpha &= i\bar{\epsilon}\Gamma_\alpha\Gamma_5\Psi \ ,
\end{aligned}$$

$(2,0)$ supersymmetry in $D = 6$

Thus we need a term in $\delta\Psi$ that is quadratic in X^I and which has a single Γ_μ:

- Invent a field C^μ_A

$$[X^I, X^J]\Gamma^{IJ}\Gamma^5\epsilon \to [X^I, X^J, C_\mu]\Gamma^{IJ}\Gamma^\mu$$

So again a 3-algebra begins to arise:

- Note that $[X^I, X^J, C^\mu] = f^{ABC}{}_D X^I_A X^J_B C^\mu_C T^D$ is not necessarily totally antisymmetric - yet.

We expect to recover 5D SYM when $C^\mu \propto \delta^\mu_5$

$(2,0)$ supersymmetry in $D=6$

After starting with a suitably general anstaz we find closure of the susy algebra implies

$$\begin{aligned}
\delta X^I_A &= i\bar{\epsilon}\Gamma^I\Psi_A \\
\delta\Psi_A &= \Gamma^\mu\Gamma^I D_\mu X^I_A\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}_A\epsilon - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C X^J_D f^{CDB}{}_A\epsilon \\
\delta H_{\mu\nu\lambda\, A} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A + i\bar{\epsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D f^{CDB}{}_A \\
\delta\tilde{A}_\mu{}^B{}_A &= i\bar{\epsilon}\Gamma_{\mu\lambda}C^\lambda_C\Psi_D f^{CDB}{}_A \\
\delta C^\mu_A &= 0
\end{aligned}$$

where $f^{ABC}{}_D$ are totally anti-symmetric structure constants of the $N=8$ 3-algebra (possibly Lorentzian).

Has $(2,0)$ supersymmetry, SO(5) R-symmetry and scale symmetry (C^μ_A has dimensions of length)

$(2,0)$ supersymmetry in $D=6$

The algebra closes with the on-shell conditions

$$
\begin{aligned}
0 &= D^2 X^I_A - \frac{i}{2}\bar{\Psi}_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A - C^\nu_B C_{\nu G} X^J_C X^J_E X^I_F f^{EFG}{}_D f^{CDB}{}_A \\
0 &= D_{[\mu} H_{\nu\lambda\rho]\ A} + \frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma_B X^I_C D^\tau X^I_D f^{CDB}{}_A + \frac{i}{8}\epsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma_B \bar{\Psi}_C \Gamma^\tau \Psi_D \\
0 &= \Gamma^\mu D_\mu \Psi_A + X^I_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A \\
0 &= \tilde{F}_{\mu\nu}{}^B{}_A - C^\lambda_C H_{\mu\nu\lambda\ D} f^{CDB}{}_A \\
0 &= D_\mu C^\nu_A = C^\mu_C C^\nu_D f^{BCD}{}_A \\
0 &= C^\rho_C D_\rho X^I_D f^{CDB}{}_A = C^\rho_C D_\rho \Psi_D f^{CDB}{}_A = C^\rho_C D_\rho H_{\mu\nu\lambda\ A} f^{CDB}{}_A \ ,
\end{aligned}
$$

Thus C^μ_A picks out a fixed direction in space and in the 3-algebra,

- w.l.o.g $C^\mu_A = g^2_{YM}\delta^\mu_5\delta^0_A$
- The non-Abelian $(A \neq 0)$ momentum modes parallel to C^μ must vanish.
- So we obtain a non-abelian 5D Yang-Mills multiplet $(A \neq 0)$ along with free 6D tensor multiplets $(A = 0)$

$(2,0)$ supersymmetry in $D = 6$

Note that we haven't mentioned $B_{\mu\nu}$:

$$H_{\mu\nu\lambda A} = D_\mu B_{\nu\lambda A} + D_\nu B_{\lambda\mu A} + D_\lambda B_{\mu\nu A}$$

- This implies $DH = F \wedge H$
- We can't solve the equation of motion $DH = \mathit{sources}$ without losing degrees of freedom.
  - Not compatible with supersymmetry

$(2,0)$ supersymmetry in $D = 6$

But we could also consider a null reduction, $x^\mu = (u, v, x^i)$:

$$C^\mu_A = g^2_{YM}\delta^\mu_v\delta^0_A$$

The resulting equations are $(f^{ab}{}_c = f^{0ab}{}_c)$

$$\begin{aligned}
0 &= D^2X^I_a - \frac{ig}{2}\bar{\Psi}_c\Gamma_v\Gamma^I\Psi_d f^{cd}{}_a \\
0 &= \Gamma^\mu D_\mu\Psi_a + g^2_{YM}X^I_c\Gamma_v\Gamma^I\Psi_d f^{cd}{}_a \\
0 &= D_{[\mu}H_{\nu\lambda\rho]\ a} - \frac{g^2_{YM}}{4}\epsilon_{\mu\nu\lambda\rho\tau v}X^I_c D^\tau X^I_d f^{cd}{}_a - \frac{ig^2_{YM}}{8}\epsilon_{\mu\nu\lambda\rho\tau v}\bar{\Psi}_c\Gamma^\tau\Psi_d f^{cd} \\
0 &= \tilde{F}_{\mu\nu}{}^b{}_a - g^2_{YM}H_{\mu\nu v\ d}f^{db}{}_a
\end{aligned}$$

with $D_v = 0$

$(2,0)$ supersymmetry in $D = 6$

Curious variation of Yang-Mills:

- 16 supersymmetries and an $SO(5)$ R-symmetry
- No potential for the scalars
- M5-brane wrapped on a null circle?

BPS states are light-like Dyonic Instantons [Tong, NL]

$$F_{ij} = (\star F)_{ij} \qquad F_{ui} = D_i X^6$$

Exist and are smooth even though $< X^6 > \neq 0$.

Conclusions and Speculations

We have discussed some needs and oddities of the M5-brane theory

We looked for interacting $(2, 0)$ theories in 6D

- Found a system in terms of 3-algebras
- However the non-Abelian dynamics is restricted to 5D
- Also obtained a null reduction
  - Novel interacting system with 16 susys, $SO(5)$ R-symmetry and no potential: M5 with vanishing null momentum.

The M5-brane is a rich and mysterious as ever.

But hopefully some progress can be made towards defining the theory and exploring its properties.

Conclusions and Speculations

The structure we used has appeared in the mathematical literature: 2-Lie algebra (2-category):

$$\begin{aligned} & Lie(G), Lie(H) \\ & i : Lie(H) \rightarrow Lie(G) \\ & R(G) : Lie(H) \rightarrow Lie(H) \quad \text{a Rep. of } Lie(G) \end{aligned}$$

Note that this is also just the data of a 3-algebra

Fields:

$$\begin{aligned} A &: 1- \text{ form valued in } Lie(G) \\ B &: 2- \text{ form valued in } Lie(H) \end{aligned}$$

Conclusions and Speculations

Field strengths:

$$\begin{aligned} \mathcal{F} &= dA + A \wedge A - i(B) \\ H &= DB = dB + R(A) \wedge B \end{aligned}$$

H and $\mathcal{F}$ are gauge covariant as normal.

There is also a curious shift symmetry

$$\begin{aligned} A &\rightarrow A + i(\eta) \qquad B \rightarrow B + D\eta + \eta \wedge \eta \\ \mathcal{F} &\rightarrow \mathcal{F} \qquad H \rightarrow H + R(\mathcal{F}) \wedge \eta \end{aligned} \tag{1}$$

Can we use this to remove A degrees of freedom?

Conclusions and Speculations

How much does 5D SYM already know about 6D physics?[NL,Papagergakis] in progress

- Dyonic Instantons are KK towers of charged states
- Where are the KK towers of uncharged states?

If all the KK modes can be accounted for in 5D SYM by instantons then no new degrees of freedom need arise in the UV theory

- Then the M5-brane is just strongly coupled 5D SYM
    - Our system is then a triumph (?!) since the M5-brane shouldn't have both momentum and non-abelian instanton-like states.
- UV is a CFT suggests 5D SYM is finite