The role of double trace deformations in AdS/CMT

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Based on Faulkner, Roberts, G.H.
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Double trace deformations can

- Implement spontaneous symmetry breaking
- Provide a new way to construct holographic superconductors
- Provide a knob to tune the critical temperature of holographic superconductors
- Lead to new quantum critical points with nontrivial critical exponents

Consider the following action:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - (\nabla\phi)^2 - 2V(\phi)\right]$$

where

$$V(\phi) = -3 + \frac{1}{2}m^2\phi^2 + \ldots$$

Need $m^2 > m_{BF}^2 = -9/4$ for a stable ground state.

Start with asymptotically (globally) AdS solutions

$$ds^2 = r^2(-dt^2 + d\Omega) + \frac{dr^2}{r^2}$$

Asymptotically, the scalar field is

$$\phi = \frac{\alpha}{r^{\Delta_-}} + \frac{\beta}{r^{\Delta_+}}$$

where $$\Delta_\pm = 3/2 \pm \sqrt{9/4 + m^2}$$

Usually normalizability requires $\alpha = 0$, but if

$$m^2_{BF} < m^2 < m^2_{BF} + 1$$

both modes are normalizable and one has a choice of boundary conditions.

If O is the operator dual to ϕ,

$\alpha = 0$ => O has dimension Δ_+ and $\langle O \rangle = \beta$
(standard quantization)

$\beta = 0$ => O has dimension Δ_- and $\langle O \rangle = \alpha$
(alternative quantization)

More generally, one can set $\beta = W'(\alpha)$ for any $W(\alpha)$

Double trace deformations

In alternative quantization, $\Delta < 3/2$, and one can modify the action by

$$S \to S - \int d^3x \, \kappa \, O^\dagger O$$

κ has dimension $3 - 2\Delta > 0$ so this is a relevant coupling.

On gravity side it corresponds to boundary conditions (Witten; Berkooz, Sever and Shomer)

$$\beta = \kappa\alpha$$

Theory with κ < 0 can still have a stable ground state. (Faulkner, Roberts, G.H., 2010)

In the new ground state, <O> is nonzero. This is a classic example of spontaneous symmetry breaking.

For real O, you break a Z_2 symmetry, but the argument can be extended to complex O. Then you break a U(1) symmetry.

We first review an earlier result and then extend it.
(Assume $m^2 = -2$, so $\Delta_- = 1$, $\Delta_+ = 2$.)

Theorem (Amsel, Hertog, Hollands, Marolf, 2007):
If $V(\phi)$ admits a suitable superpotential, and $W(\alpha)$ is bounded from below, then the total energy is bounded from below.

Outline of proof: Let $P(\phi)$ satisfy

$$V(\phi) = 2\left(\frac{dP}{d\phi}\right)^2 - 3P^2$$

Near $\phi = 0$, a solution is $P(\phi) = 1 + \phi^2/4 + O(\phi^4)$

Following Witten and Townsend, let

$$\hat{\nabla}_\mu \Psi = \nabla_\mu \Psi + \frac{1}{2} P(\phi) \Gamma_\mu \Psi$$

Given a spacelike surface Σ with boundary C, let Ψ be a solution to Witten's equation: $\Gamma^i \hat{\nabla}_i \Psi = 0$ such that $-\bar{\Psi}\Gamma^\mu \Psi$ approaches $\partial/\partial t$ asymptotically.

Let $B_{\mu\nu} = \bar{\Psi}\Gamma_{[\mu}\Gamma_\nu \Gamma_{\rho]} \hat{\nabla}^\rho \Psi + h.c.$ (Nester)

Then the spinor charge $Q = \int_C {}^*B$

satisfies Q ≥ 0.

In asymptotically flat spacetime, Q is the total energy E. But in AdS, with general boundary conditions:

$$E = Q + \oint [W(\alpha) + \alpha\beta] + \lim_{r\to\infty} \oint \left[\frac{1}{2} r\alpha^2 - 2r^3(P-1)\right]$$

Using the solution for P and the asymptotic form of ϕ

$$E = Q + \oint W$$

So $\qquad E \geq 4\pi \ \inf \ W$

One can prove an even stronger positive energy theorem (Faulkner, Roberts, G.H., 2010):

The equation for the superpotential:

$$V(\phi) = 2\left(\frac{dP}{d\phi}\right)^2 - 3P^2$$

admits a one parameter family of solutions for small ϕ (also noticed by Papadimitriou, 2007):

$$P_s(\phi) = 1 + \frac{1}{4}\phi^2 - \frac{s}{6}|\phi|^3 + O(\phi^4)$$

Repeating the above argument with this P(ϕ) yields

$$E \geq \oint \left[W(\alpha) + \frac{s}{3}|\alpha|^3 \right]$$

So the energy remains bounded from below even for, e.g., $W = (\kappa/2)\alpha^2$ with $\kappa < 0$, corresponding to double trace deformations with negative coefficient!

Of course, this assumes that solutions $P_s(\phi)$ exist for all ϕ. This depends on $V(\phi)$, but typically they do up to a critical value s_c. Thus

$$E \geq 4\pi \ \inf \left[W(\alpha) + \frac{1}{3} s_c |\alpha|^3 \right]$$

Existence of superpotentials

The equation for P can be written:

$$P'(\phi) = \sqrt{\frac{3P^2}{2} + \frac{V(\phi)}{2}}$$

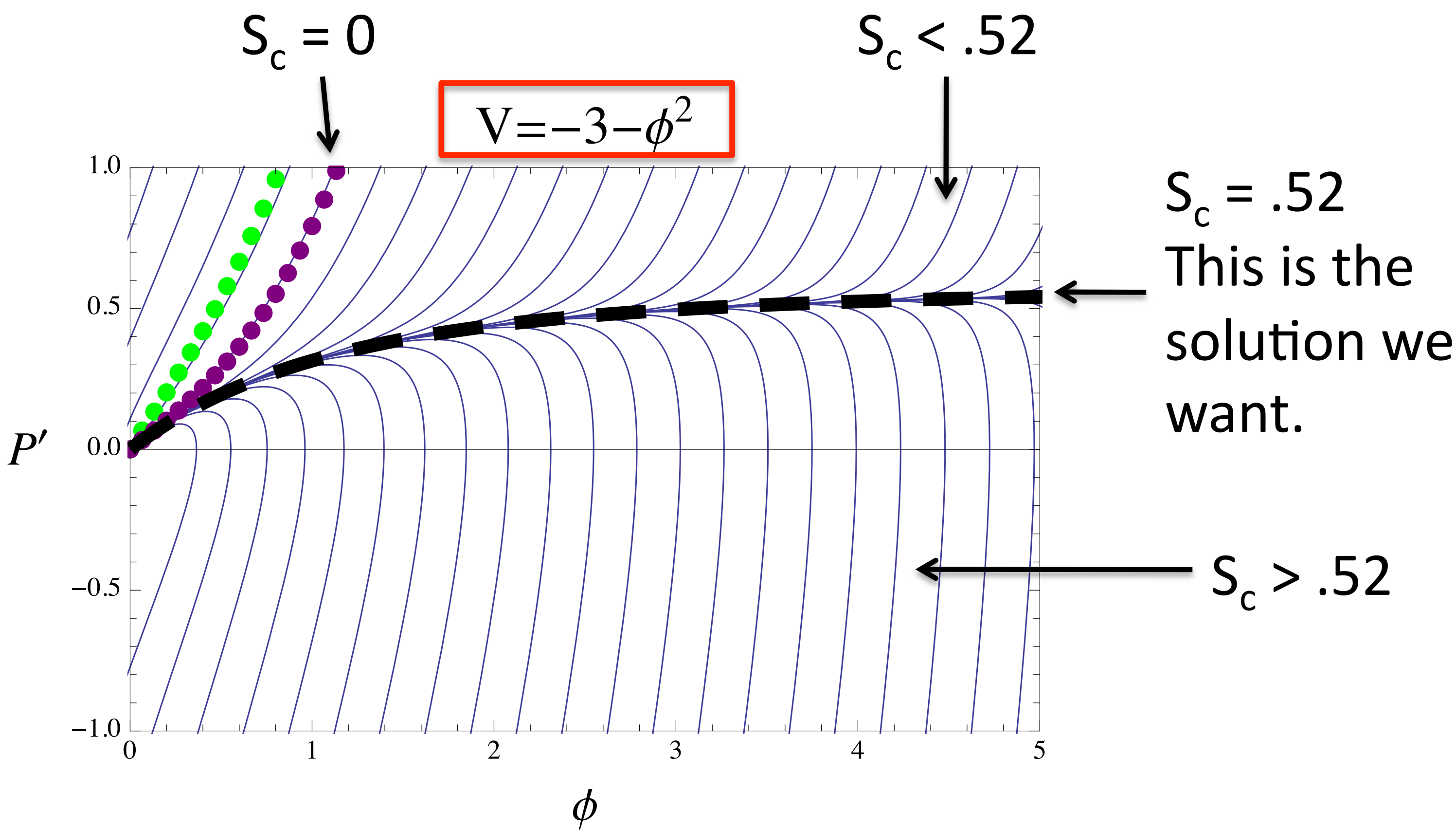
Clearly, a solution fails to exist when the argument of the square root becomes negative. Initially,

$$P'(\phi) = \frac{1}{2}[\phi - s\phi^2]$$

Since $$V' = [4P'' - 6P]P'$$
one expects V' = 0 if P' = 0, but this is usually not the case. Instead, $P' \propto \pm(\phi_1 - \phi)^{1/2}$

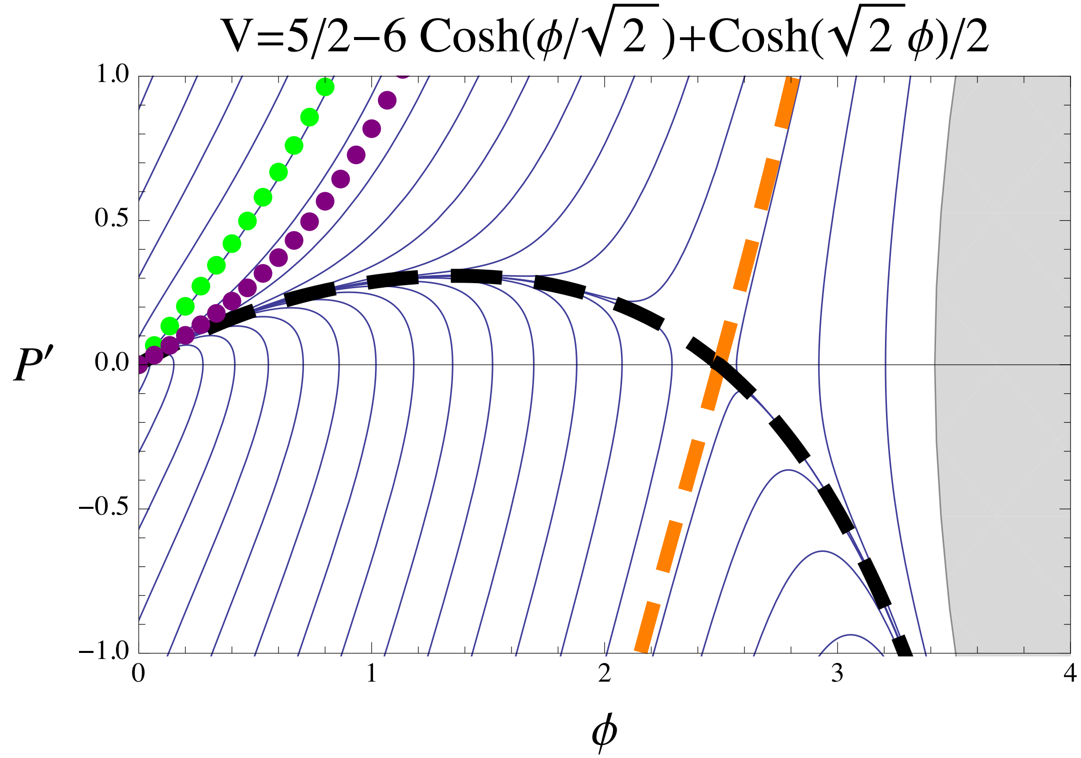
The two branches of solutions meet at $\phi = \phi_1$ and P does not exist for $\phi > \phi_1$.

If V' = 0 when P' = 0, then $P' \propto (\phi_1 - \phi)$ and the solution exists for $\phi > \phi_1$.



For a purely quadratic potential with $m^2 = -2$,

$s_c = .52$



For the consistent truncation of supergravity used by Gauntlett, Sonner and Wiseman (2009)

$s_c = .56$

Candidate ground states

Expect the ground state to be static and spherically symmetric. Look for solutions of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega, \qquad \phi = \phi(r),$$

The equations of motion give three ODE's for f(r), g(r), and ϕ (r). Solutions are solitons.

The general asymptotic solution is

$$\begin{aligned}
\phi(r) &= \alpha/r + \beta/r^2 + \ldots \\
g(r) &= r^2 + (1 + \alpha^2/2) - M_0/r + \ldots \\
f(r) &= r^2 + 1 - (M_0 + 4\alpha\beta/3)/r + \ldots
\end{aligned}$$

There are three undetermined parameters: α, β, M_0

Regularity at the origin requires

$$\begin{aligned} \phi &= \phi_0 + \frac{V'(\phi_0)}{6} r^2 + \ldots \\ g &= 1 - \frac{V(\phi_0)}{3} r^2 + \ldots \\ f &= f_0 - f_0 \frac{V(\phi_0)}{3} r^2 + \ldots \end{aligned}$$

But f_0 is fixed by requiring $f = r^2 + \ldots$ asymptotically.
So the only free parameter is ϕ_0.

This one parameter family of solitons define a curve $\beta_0(\alpha)$.

For small α, one can determine the curve $\beta_0(\alpha)$ by solving the linearized equation for ϕ in global AdS. The solution is $\phi = \tan^{-1}(r)$, so for small α:

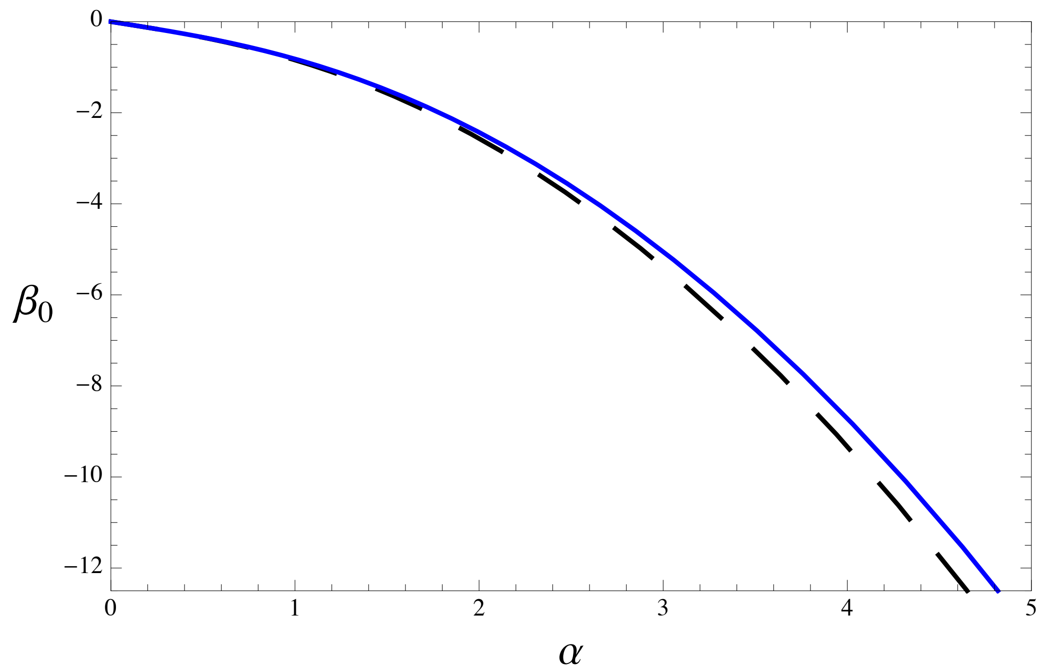
$$\beta_0(\alpha) = -\frac{2}{\pi}\alpha$$

For large α, one can show

$$\boxed{\beta_0(\alpha) = -s_c\,\alpha^2}$$

These solutions are related to planar solutions in which there is a scaling symmetry $r \to \lambda r$. Thus

$$\phi = \frac{\alpha/\lambda}{r} + \frac{\beta/\lambda^2}{r^2}$$



The values of (α,β) realized by solitons.

Blue line is for $V = -3 - \phi^2$

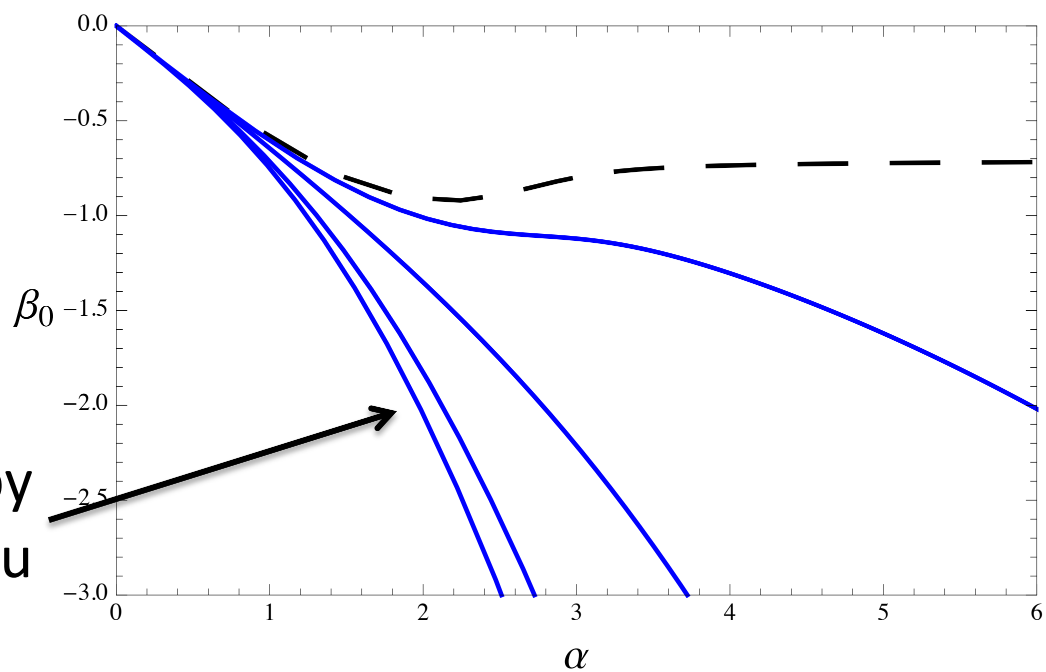
Dashed line is for the potential in Gauntlett et al.:

$$V = 5/2 - 6\cosh(\phi/\sqrt{2}) + \cosh(\sqrt{2}\phi)/2$$

Another potential coming from a consistent truncation of supergravity is (Duff and Liu, 1999)

$$V = -\cosh(a\phi) - 2\cosh(ab\phi)$$

with a = $[2/(1 + 2b^2)]^{1/2}$. Below are soliton curves for b = 1, .5, .25, .1, 0 (dashed). Note: $s_c = 0$ for b = 0.



Found analytically by Papadimitriou

Total energy

Given a boundary condition, $\beta = W'(\alpha)$, the total energy of these solitons is

$$E = 4\pi(M_0 + \alpha\beta + W)$$

Let $$W_0(\alpha) = -\int_0^\alpha \beta_0(\tilde{\alpha})d\tilde{\alpha}$$

and set $\mathcal{V} = 4\pi(W + W_0)$. This is an effective potential since $\mathcal{V}_{,\alpha} = 0 \Rightarrow \beta = \beta_0$.

There are solitons at each extrema of $\mathcal{V}$.

The energy of the soliton is just the value of $\mathcal{V}$ at its extrema:

Suppose we choose $\beta(\alpha) = \beta_0(\alpha)$. Then all solitons are allowed. But static solutions are extrema of the energy, so all solitons have the same energy. This includes $\alpha = \beta = 0$, so

$$M_0 + \alpha\beta - W_0 = 0$$

Therefore, for general boundary condition $\beta = W'(\alpha)$

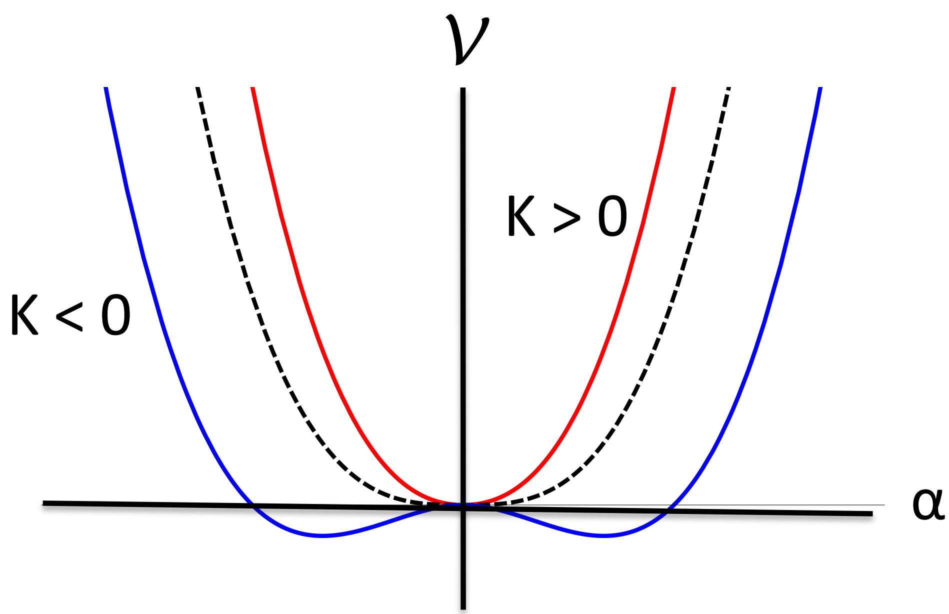
$$E = 4\pi(M_0 + \alpha\beta + W) = 4\pi(W + W_0) = \mathcal{V}$$

(Hertog and G.H., 2004)

Application to double trace deformations

Consider planar solutions

$$\mathcal{V} = 2\pi[\kappa\alpha^2 + (2s_c/3)\,|\alpha|^3]$$



At the minimum
$\alpha = \langle O \rangle = -\,\kappa/s_c$

For real ϕ, you break a Z_2 symmetry. But you can do the same thing for a complex ϕ. This now breaks a U(1) symmetry and provides a new way to construct holographic superconductors with zero net charge density.

Previous constructions were based on instabilities associated with the near horizon geometry of an extremal charged black hole in AdS.

Gravity Dual of a Superconductor

(Hartnoll, Herzog, and G.H., 2008)

Gravity	Superconductor
Black hole	Temperature
Charged scalar field	Condensate

Need to find a black hole that has scalar hair at low temperatures, but no hair at high temperatures.

Gubser (2008) argued that a charged scalar field around a charged black hole would have the desired property. Consider

$$\mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |\partial\Psi - iqA\Psi|^2 - m^2|\Psi|^2$$

For an electrically charged black hole, the effective mass of Ψ is

$$m^2_{eff} = m^2 + q^2 g^{tt} A_t^2$$

But the last term is negative. This causes scalar hair at low temperature.

There is another source of instability: nearly extremal charged AdS black holes are unstable to forming neutral scalar hair.

An extremal AdS black hole has a near horizon geometry $AdS_2 \times R^2$. The Breitenlohner-Freedman (BF) bound for AdS_{d+1} is $m^2_{BF} = - d^2/4$. Our scalar can be above the BF bound for AdS_4, but below the bound for AdS_2.

General argument for instability

(Denef and Hartnoll, 2009)

Consider a scalar field with mass m and charge q in the near horizon geometry of an extremal Reissner-Nordstrom AdS black hole. Get a wave equation in AdS_2 with effective mass

$$m_{eff}^2 = \frac{m^2 - 2q^2}{6}$$

The extremal RN AdS black hole is unstable when this is below -1/4, the BF bound for AdS_2. The condition for instability is

$$\boxed{m^2 - 2q^2 < -3/2}$$

Hairy black holes

Look for static, homogeneous solutions:

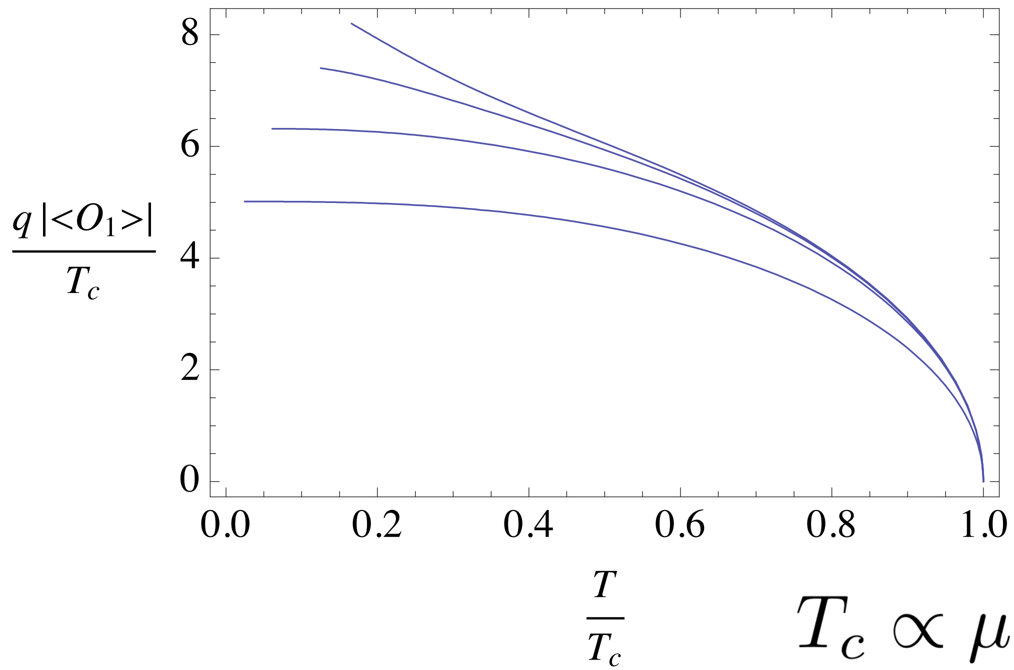
$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{g(r)} + r^2\left(dx^2 + dy^2\right)$$

$$A = \phi(r)dt, \quad \Psi = \psi(r)$$

Get four coupled nonlinear ODE's. At the horizon, $r = r_0$, g and Φ vanish, χ is constant. Asymptotically, metric approaches AdS_4 and

$$\phi(r) = \mu - \frac{\rho}{r}$$

Condensate (hair) as a function of T



$$T_c \propto \mu$$

Curves correspond to q = 1, 3, 6, 12

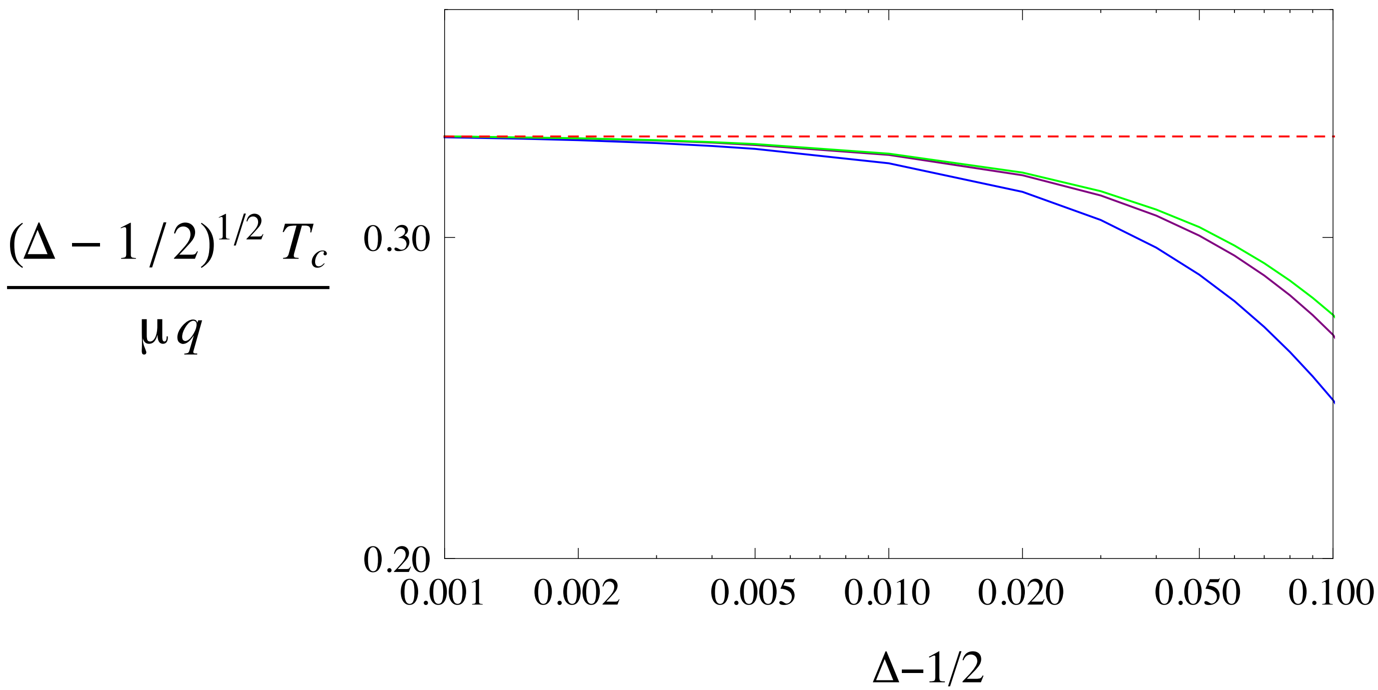
(from Hartnoll, Herzog, G.H., 2008)

The condition $m^2 - 2q^2 < -3/2$ is sufficient to cause an instability, but it is not necessary.

Example: If $q = 0$ and $-3/2 < m^2 < m^2_{BF} + 1$, theory with $\beta = 0$ boundary condition is still unstable at low temperature.

Explanation: Even with $m^2 >$ BF bound for AdS_2, there are still unstable modes. These are ruled out by the AdS_4 boundary conditions if $\alpha = 0$, but not if $\beta = 0$.

In fact, T_c diverges as $\Delta_- \rightarrow ½$.



The curves are q = .1, .25, .5

T_c diverges like $\mu q / (\Delta - ½)^{1/2}$

Including a double trace deformation provides a new source of instability even for zero charge density.

With $\kappa < 0$, Schwarzschild AdS is unstable to forming scalar hair at low temperature.

Critical temperature: One can analytically find a static homogeneous mode of a massive scalar field in (planar) Schwarzschild AdS in terms of hypergeometric functions.

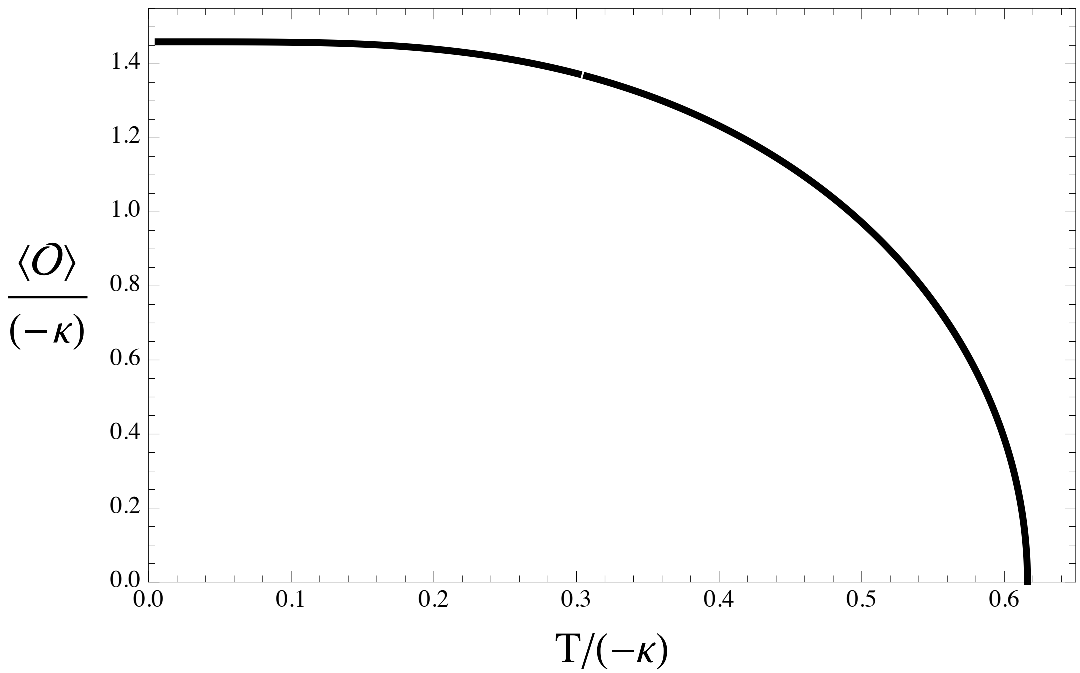
Impose regularity at the horizon and read off κ as a function of T. For $m^2 = -2$ in AdS_4,

$$T_c = -.62\ \kappa$$

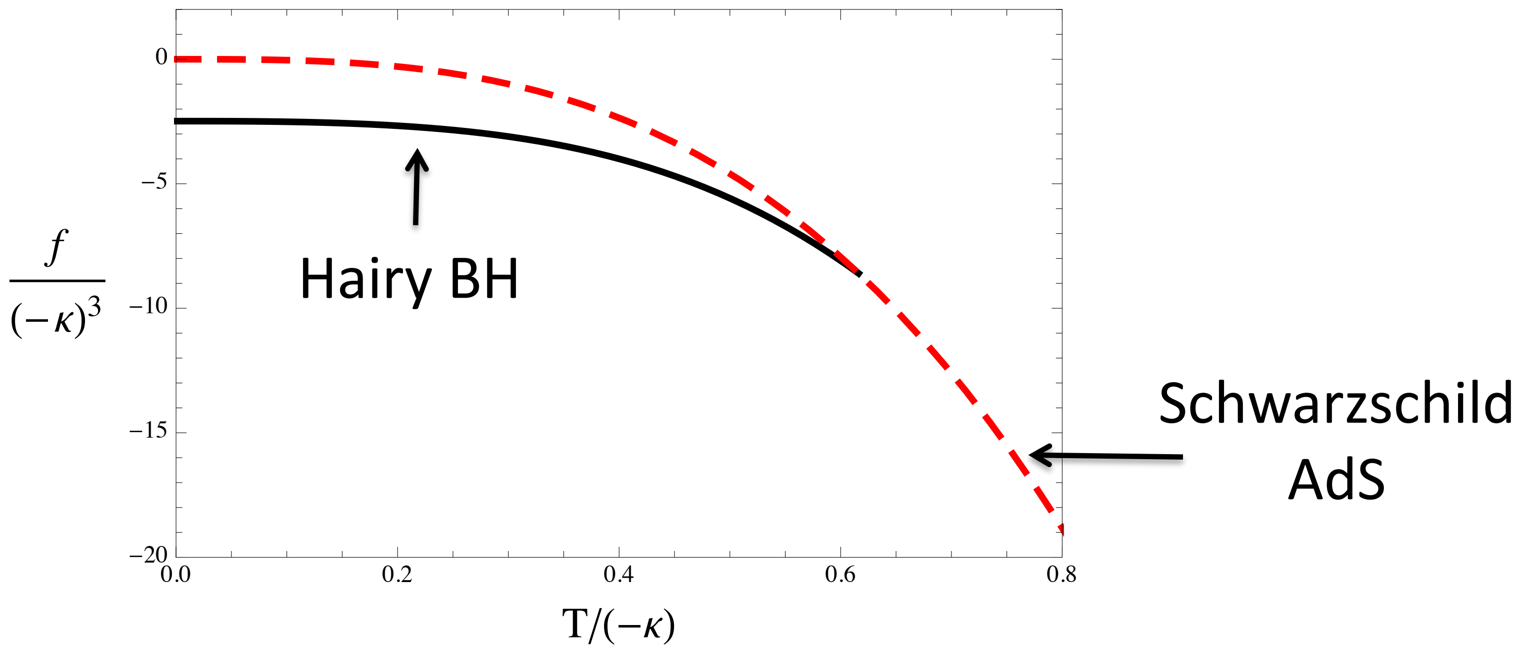
This was for dimension one operator. For general dimension Δ_-:

$$\langle O \rangle \propto (-\kappa)^{\Delta_-/(3-2\Delta_-)}, \quad T_c \propto (-\kappa)^{1/(3-2\Delta_-)}$$

For $T < T_c$, one can find the hairy black holes and the curves of the condensate look similar to the previous case. For the potential in Gauntlett et al:



As before, the free energy is lower for the hairy black holes, showing that they are stable.

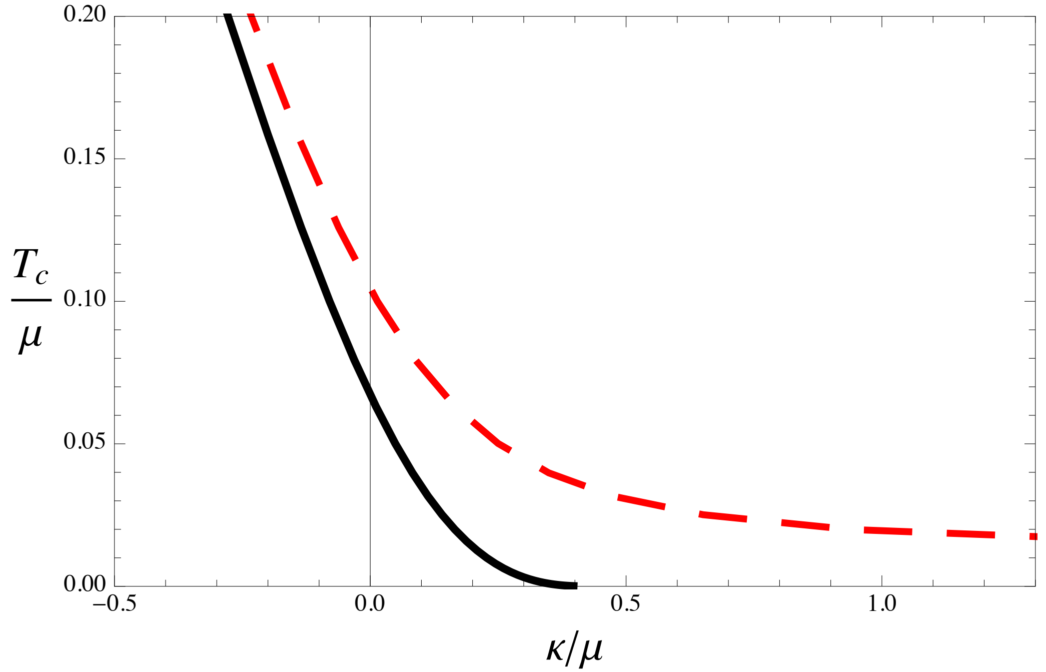


The main advantage of the new holographic superconductors is that the DC conductivity in the normal phase is finite.

Previously, the normal phase had nonzero charge density. This can be boosted to yield a nonzero current with no applied electric field, i.e., infinite DC conductivity.

(Superconductivity was seen as a change in the coefficient of $\delta(\omega)$ in the frequency dependent conductivity.)

Even with $\mu \neq 0$, adding a term like κO^2 (with $\kappa > 0$) makes it harder for O to condense. This gives a new way to tune the critical temperature.



For some m,q, can cause $T_c = 0$ creating a *quantum critical point* at $\kappa = \kappa_c$ (with $\mu = 1$).

What are the properties of this quantum critical point?

For $\kappa > \kappa_c$, $T = 0$ solution is extreme RN AdS with near horizon region $AdS_2 \times R^2$. The scalar has an effective mass

$$m_{eff}^2 = \frac{m^2 - 2q^2}{6}$$

So solutions are

$$\frac{v}{(r - r_+)^{\delta_-}} + \frac{w}{(r - r_+)^{\delta_+}}$$

where

$$\delta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + m_{\rm eff}^2}$$

By analogy with the asymptotic AdS_4 region, we define $\kappa_{IR} = w/v$. Then one can show

$$\kappa_{IR} \propto \kappa - \kappa_c$$

The instability for $\kappa < \kappa_c$ can be viewed as turning on a negative double trace deformation in the IR CFT.

For $\kappa < \kappa_c$ the IR does not include AdS_2, but for κ close to κ_c there is a large intermediate region which is approximately $AdS_2 \times R^2$. The critical exponents depend on δ_- in this region.

Dimension of operator dual to scalar in CFT_1

Critical exponents

For ¼ < δ_- < ½

$$\langle \mathcal{O} \rangle \propto (-\kappa_{IR})^{\delta_-/(1-2\delta_-)}\,, \quad T_c \propto (-\kappa_{IR})^{1/(1-2\delta_-)}$$

This is just the IR analog of what we had before.

For 0 < δ_- < ¼, there are relevant higher multi-trace deformations. If the phase transition remains second order:

$$\langle \mathcal{O} \rangle \propto (-\kappa_{IR})^{1/2}\,, \quad T_c \propto (-\kappa_{IR})^{1/(1-2\delta_-)}$$

For δ_- < 0, we have mean field behavior:

$$\langle \mathcal{O} \rangle \propto (-\kappa_{IR})^{1/2}, \quad T_c \propto (-\kappa_{IR})$$

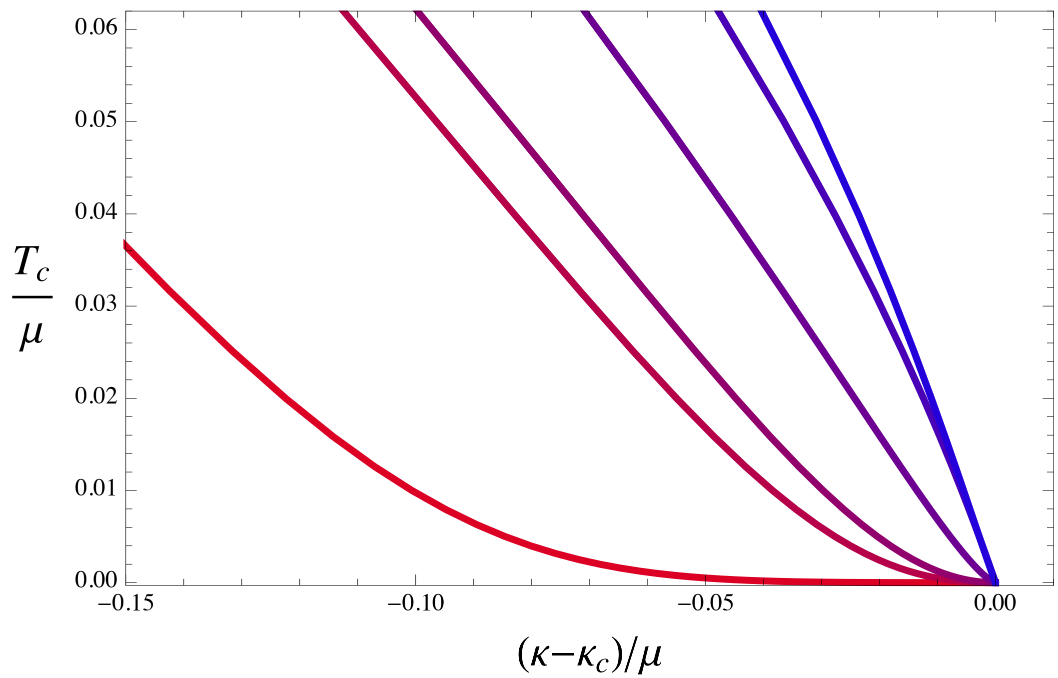
From the pole in the two point function <OO> at the critical point, we find a gapless mode satisfying

$$\omega \sim |\vec{p}|^z$$

with dynamical critical exponent $$z = \frac{2}{1 - 2\delta_-}$$

(z=2 for q ≠ 0 and δ_- < 0, z=1 for q=0 and δ_- < -1/2)

T_c close to κ_c for different values of δ_-

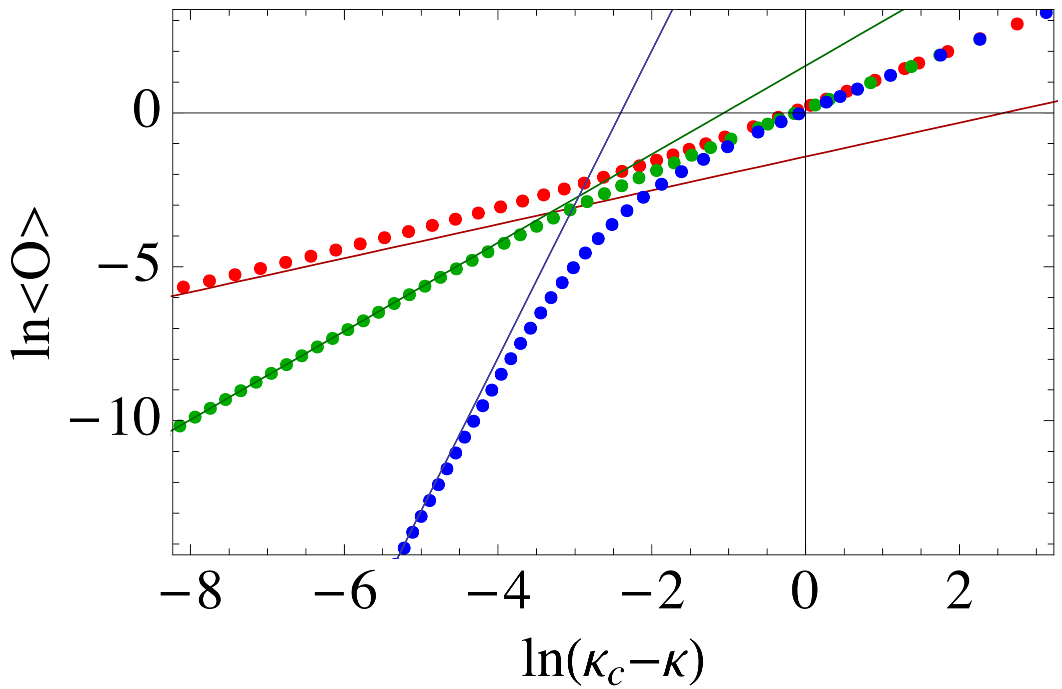


From left to right:

δ_- = 0.45, 0.30, 0.26, 0.15, 0, -0.15

Agrees with our prediction

<O> close to κ_c for different values of δ_-



δ_- = 0.45, 0.37, 0.26

Lines show our prediction: $\langle \mathcal{O} \rangle \propto (-\kappa_{IR})^{\delta_-/(1-2\delta_-)}$

Agrees to within 5%.

Asymptotically AdS_2 solution

As $\kappa \to \kappa_c$, the intermediate AdS_2 region becomes arbitrarily large. There are two critical solutions:

1) Usual extremal RN AdS which keeps the UV asymptotic AdS_4 region.

2) A new asymptotically $AdS_2 \times R^2$ solution whose IR region depends on details of V. If V has another extremum, can approach AdS_4 in IR. Get RG flow from AdS_2 to AdS_4 – the opposite of the usual case!

Sources of BH instability at low temperature

Charged BH:

- m_{eff} is below the BF bound for AdS_2
- m_{eff} is above the BF bound for AdS_2 but unstable modes are allowed by alternative boundary conditions in AdS_4

Neutral BH:

- Have boundary conditions corresponding to a double trace perturbation with $\kappa < 0$

Double trace deformations can

- Implement spontaneous symmetry breaking
- Provide a new way to construct holographic superconductors
- Provide a knob to tune the critical temperature of holographic superconductors
- Lead to new quantum critical points with nontrivial critical exponents