FROM WEAK TO STRONG COUPLING IN ABJM THEORY

Marcos Mariño
University of Geneva

Based on [M.M.-Putrov, 0912.1458]
[Drukker-M.M.-Putrov, 1007.1453]
[in progress]

Two well-known virtues of large *N* string/gauge theory dualities:

- The *large radius limit* of string theory is dual to the *strong coupling regime* in the gauge theory

$$\frac{R}{\ell_s} \gg 1 \leftrightarrow \lambda \gg 1$$

- The *genus expansion* of the string theory can be in principle mapped to the *1/N* expansion of the gauge theory

These virtues have their counterparts:

- It is hard to test the duality, since one has to do calculations at strong ‘t Hooft coupling in the gauge theory. More ambitiously, one would like to have results *interpolating between weak and strong coupling*

- It is hard to obtain information beyond the planar limit, even in the gauge theory side.

In this talk I will report on some recent progress on these problems in ABJM theory and its string dual.

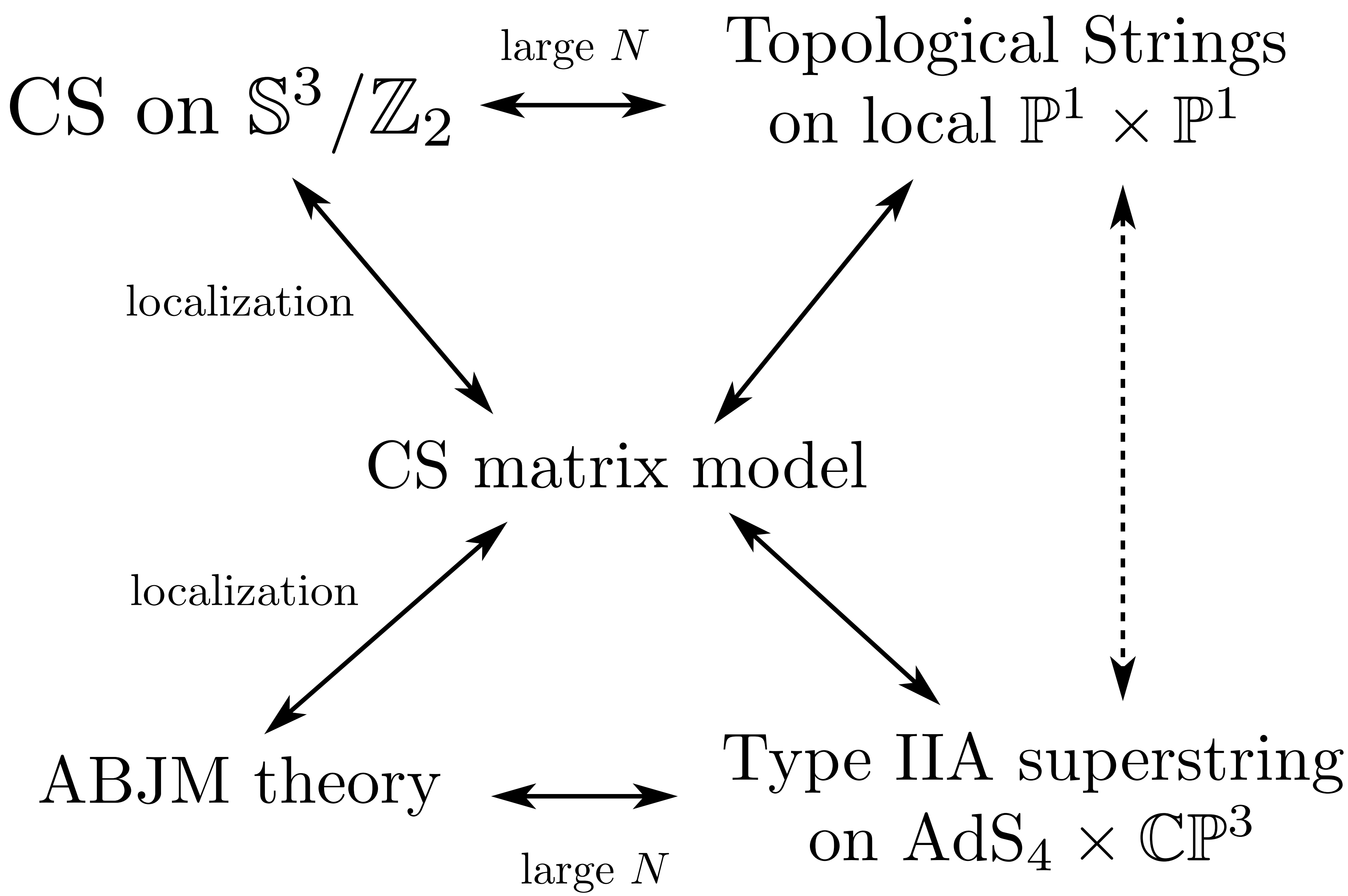
In particular, I will present *exact results (interpolating functions)* for the planar 1/2 BPS Wilson loop vev and for the planar free energy on the thee-sphere.

The strong coupling limit is in perfect agreement with the AdS dual, and in particular provides the first quantitative test of the $N^{3/2}$ behaviour of the M2 brane theory

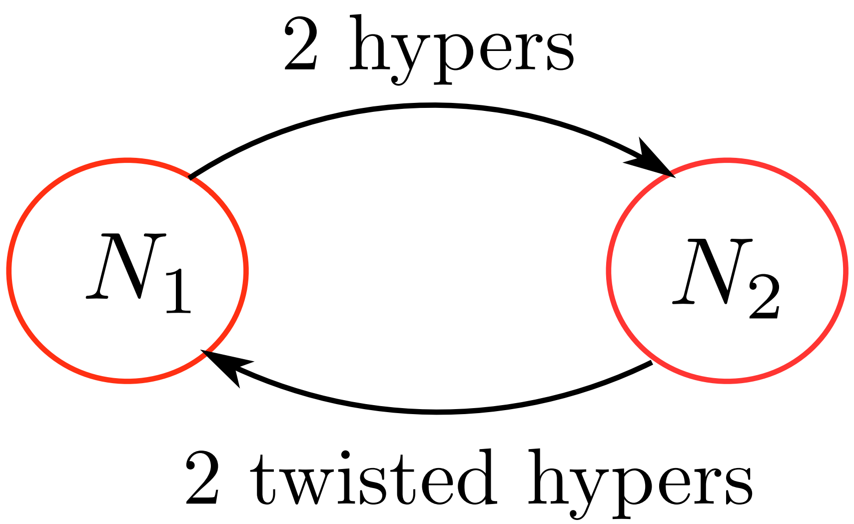
Moreover, I will show that it is possible to calculate explicitly the free energy *for all genera* (very much like in non-critical string theory).

This makes possible to address some *nonperturbative aspects of the genus expansion* in a quantitative way (large order behavior, Borel summability, spacetime instantons...)

We will rely on the following “chain of dualities”, which relates a sector of ABJM theory to a topological gauge/string theory via a matrix model:



A B J M theory



$$U(N_1)_k \times U(N_2)_{-k}$$

CS theories + 4 hypers C in the bifundamental; related to supergroup $U(N_1|N_2)$ theory via [Gaiotto-Witten]

two 't Hooft couplings $\lambda_i = \frac{N_i}{k}$

This is a 3d SCFT which (conjecturally) describes $\min(N_1, N_2)$ M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, with $|N_1 - N_2|$ fractional branes

Note: "ABJM slice" refers to $\lambda_1 = \lambda_2 = \lambda$

Gravity dual

M-theory on $\mathrm{AdS}_4 \times \mathbb{S}^7/\mathbb{Z}_k$ $\xrightarrow{\text{Hopf reduction}}$ type IIA theory/$\mathrm{AdS}_4 \times \mathbb{P}^3$

$$\mathrm{d}s^2 = \frac{L^2}{4\ell_s^2}\left(\mathrm{d}s^2_{\mathrm{AdS}_4} + 4\mathrm{d}s^2_{\mathbb{CP}^3}\right)$$

Gauge/gravity dictionary:

$$\left(\frac{L}{\ell_s}\right)^2 = \left(32\pi^2\hat{\lambda}\right)^{1/2}$$

$$g_{\mathrm{st}} = \frac{1}{k}\left(32\pi^2\hat{\lambda}\right)^{1/4}$$

$$B = \lambda_1 - \lambda_2 + \frac{1}{2}$$

Warning! shifted charges

$$\hat{\lambda} = \lambda_1 - \frac{1}{2}\left(B^2 - \frac{1}{4}\right) - \frac{1}{24}$$

[Bergman-Hirano, Aharony et al.]

Wilson loops

1/2 BPS Wilson loops constructed by [Drukker-Trancanelli]. They exploit the hidden supergroup structure

$$W^{1/2}_{\mathcal{R}} = \mathrm{sTr}_{\mathcal{R}} \mathrm{P} \exp \left[\mathrm{i} \int \begin{pmatrix} A_1 \cdot \dot{x} + \cdots & \\ & -A_2 \cdot \dot{x} + \cdots \end{pmatrix} \right]$$

rep $U(N_1|N_2)$ circle $U(N_1)$ connection $U(N_2)$ connection

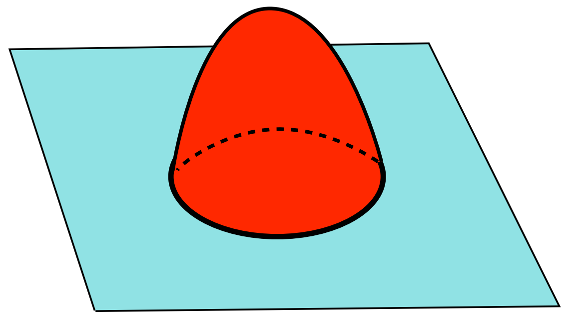
There are also circular 1/6 BPS Wilson loops. They involve only one gauge connection, but they know about the other node through the bifundamentals

$$W^{1/6}_{R} = \mathrm{Tr}_{R} \mathrm{P} \exp \left[\mathrm{i} \int \left(A_1 \cdot \dot{x} + |\dot{x}| C\overline{C} \right) \right]$$

Two string/gravity predictions

1) 1/2 BPS Wilson loop from fundamental string

$$\langle W_{\Box}^{1/2} \rangle_{\rm planar} \sim \mathrm{e}^{\pi\sqrt{2\hat{\lambda}}}$$



2) The planar free energy of the Euclidean theory on $\mathbb{S}^3$ should be given by the (regularized) Euclidean Einstein-Hilbert action on AdS4

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sinh^2(\rho)\, \mathrm{d}\Omega^2_{\mathbb{S}^3}\,,$$

$$-F(N,k) \approx S_{\mathrm{AdS}_4} = \frac{\pi}{2G_N} = \frac{\pi\sqrt{2}}{3} k^2 \hat{\lambda}^{3/2}, \quad \hat{\lambda} \gg 1,\ g_{\rm st} \ll 1$$

[Emparan-Johnson-Myers] using *universal counterterms*

Nonzero and probing the 3/2 growth!

Exact interpolation from a matrix model

A similar prediction: 1/2 BPS Wilson loop in N=4 SYM. The string prediction at strong coupling was obtained from an exact interpolating function: [Ericksson-Semenoff-Zarembo, Drukker-Gross]

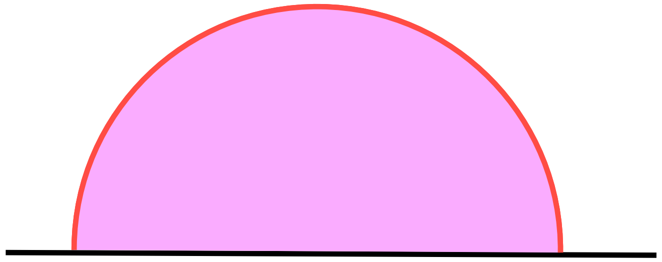
$$\frac{1}{N}\langle W_\square\rangle_{\text{planar}} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \quad \begin{cases} \sim \mathrm{e}^{\sqrt{\lambda}} & \lambda \gg 1 \\ 1 + \dfrac{\lambda}{8} + \cdots & \lambda \ll 1 \end{cases}$$

$$\lambda = g_{\rm YM}^2 N$$

Rationale: the path integral calculating of the vev of the Wilson loop reduces to a *Gaussian matrix model*

$$\langle W_R\rangle = \frac{1}{Z}\int \mathrm{d}M\, \mathrm{e}^{-\frac{2N}{\lambda}\operatorname{Tr} M^2}\operatorname{Tr}_R \mathrm{e}^M$$

This is the *simplest matrix model*, and the planar density of eigenvalues is the famous Wigner semicircle distribution



$$\rho(z) = \frac{2}{\pi\lambda}\sqrt{\lambda - z^2}$$

$$\frac{1}{N}\langle W_{\Box}\rangle_{\text{planar}} = \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \rho(z)\mathrm{e}^{z}\mathrm{d}z$$

One can also compute $1/N$ corrections systematically

This conjecture was finally proved by using *localization techniques* [Pestun].

Reduction to a matrix model in ABJM

Localization techniques were extended to the ABJM theory in a beautiful paper by [Kapustin-Willett-Yaakov]. The partition function on $\mathbb{S}^3$ is given by the following matrix integral:

contribution CS gauge fields

$$Z_{\rm ABJM}(N_1, N_2, g_{\rm top})$$
$$= \frac{1}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{{\rm d}\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{{\rm d}\nu_j}{2\pi} \frac{\prod_{i<j} \left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2 \left(2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right)^2}{\prod_{i,j}\left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^2} {\rm e}^{-\frac{1}{2g_{\rm top}}\left(\sum_i \mu_i^2 - \sum_j \nu_j^2\right)}$$

contribution 4 hypers

$$g_{\rm top} = \frac{2\pi {\rm i}}{k}$$

We “just” need the planar solution, but *exact* in the ‘t Hooft parameters, in order to go to strong coupling

Relation to Chern-Simons matrix models

Shortcut: relate this to the *lens space CS matrix model* [M.M. building on Lawrence-Rozansky] [AKMV, Halmagyi-Yasnov]

U(N) (pure) CS theory on $\mathbb{S}^3$:

$$Z_{\mathbb{S}^3}(N, g_{\rm top}) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{i<j} \left(2 \sinh \left(\frac{\mu_i - \mu_j}{2} \right) \right)^2 \mathrm{e}^{-\frac{1}{2 g_{\rm top}} \sum_i \mu_i^2}$$

can be rederived with SUSY localization [Kapustin et al.]

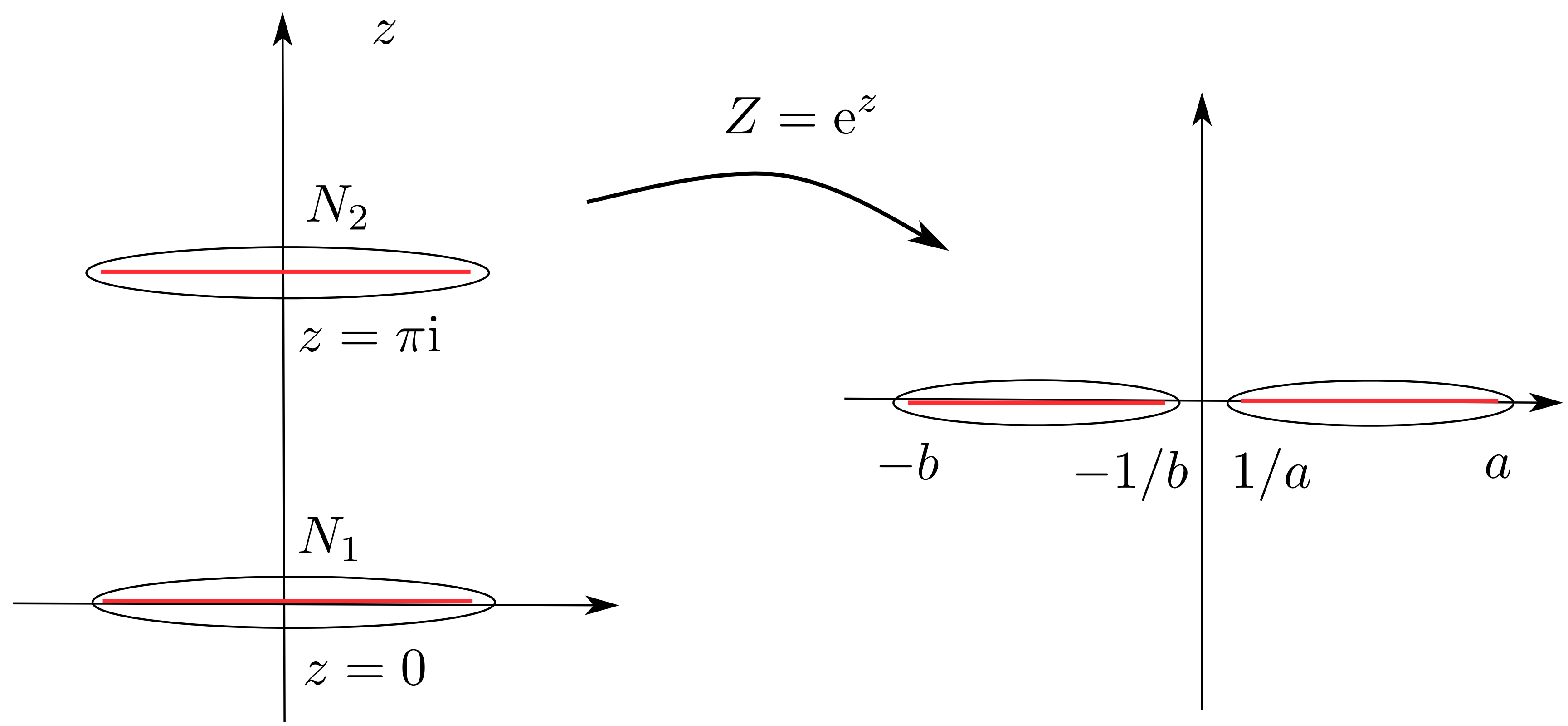
U(N) (pure) CS theory on L(2,1)= $\mathbb{S}^3/\mathbb{Z}_2$

$$Z_{L(2,1)}(N, g_{\rm top}) = \underbrace{\sum_{N_1+N_2=N}}_{\text{sum over flat connections}} Z_{L(2,1)}(N_1, N_2, g_{\rm top})$$

$$Z_{L(2,1)}(N_1, N_2, g_{\rm top}) = \frac{1}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{{\rm d}\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{{\rm d}\nu_j}{2\pi} \prod_{i<j} \left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2 \left(2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right)^2$$
$$\times \qquad \prod_{i,j} \left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^2 {\rm e}^{-\frac{1}{2g_{\rm top}}\left(\sum_i \mu_i^2 + \sum_j \nu_j^2\right)}$$

This is a *two-cut* model with two 't Hooft parameters

$$t_i = g_{\rm top} N_i$$



Superficially similar to the matrix model describing ABJM...

1/N expansion $$F(N_1, N_2, g_{\rm top}) = \sum_{g\geq 0} g_{\rm top}^{2g-2} F_g(t_1, t_2)$$

analytic functions

Fact [M.M.-Putrov]: the ABJM MM is the *supermatrix version* of the *L(2,1)* MM. They are related by analytic continuation:

$$N_2 \to -N_2$$

i.e. $$\underbrace{t_1}_{\text{CS matrix model}} = 2\pi \underbrace{\lambda_1}_{\text{ABJM theory}}, \quad t_2 = -2\pi\lambda_2$$

1/N expansion of the lens space matrix model gives the 1/N expansion of ABJM free energy on the sphere

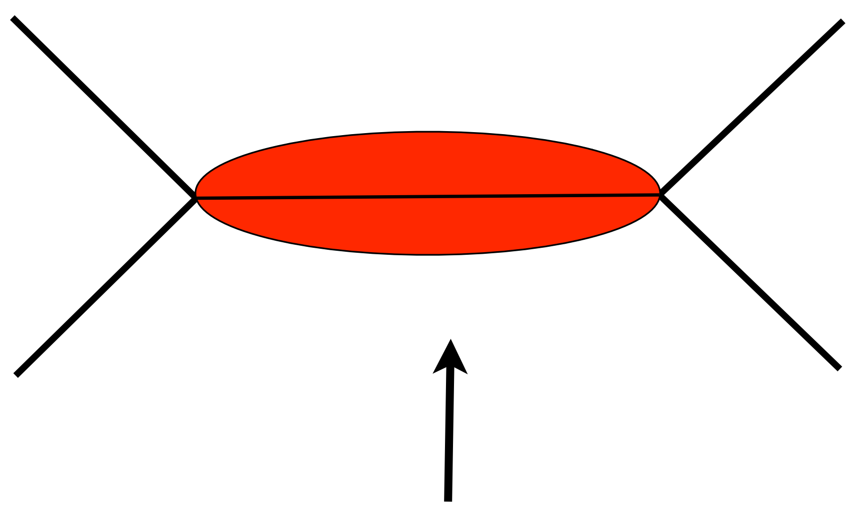
Topological string large *N* dual

CS theory on *L(p,1)* has a large *N* topological string dual [AKMV]. The genus g free energies (for a fixed, generic flat connection) are equal to the genus g free energies of a topological string theory on a toric CY manifold

$$F_g^{\mathrm{CS}}(t_i = g_s N_i) = F_g^{\mathrm{TS}}(t_i = \mathrm{moduli})$$

For *p=1* (i.e. *M*=$\mathbb{S}^3$) this is the original Gopakumar-Vafa large *N* duality. The CY target is the *resolved conifold*

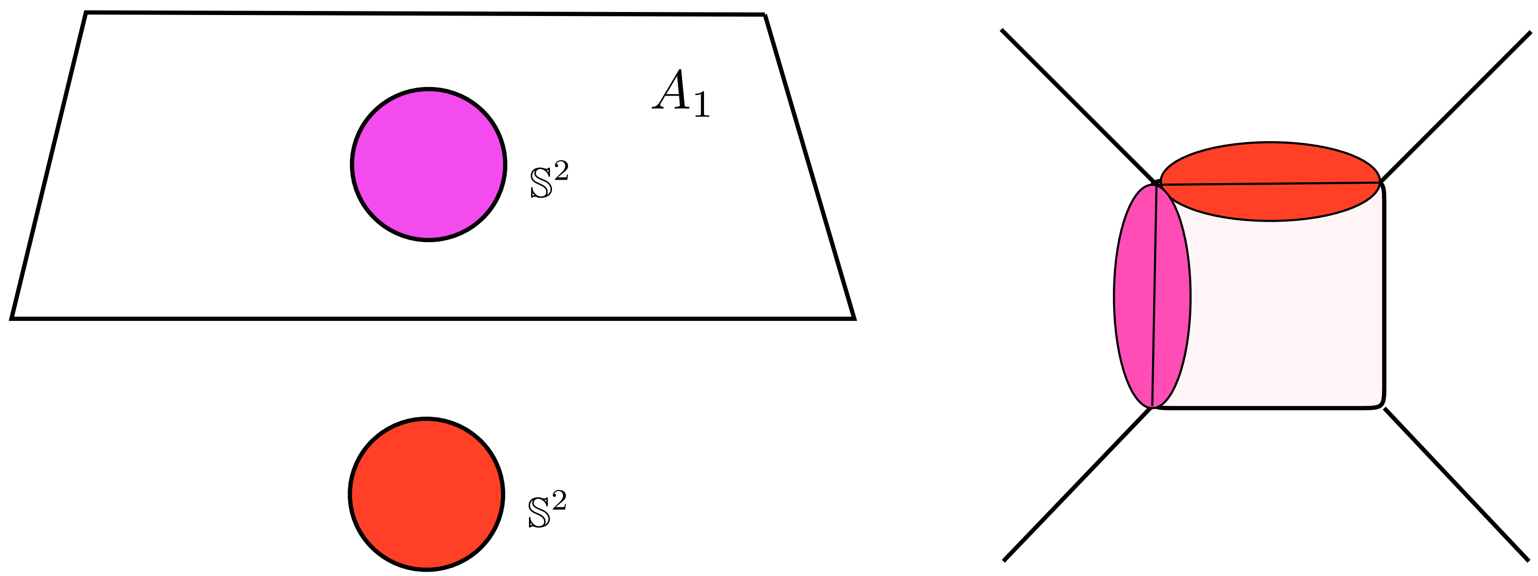
$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{S}^2$$



(single) ‘t Hooft parameter= (complexified) area of two-sphere

Topological string large N dual 2

For p=2 the CY target is local $\mathbb{P}^1 \times \mathbb{P}^1$. It has two complexified Kahler moduli T_1, T_2 measuring the sizes of the two-spheres



This implies that *we can obtain the planar free energy of ABJM theory by using special geometry for this CY!*

or equivalently, mirror symmetry!

recap:

Topological Strings
on local $\mathbb{P}^1 \times \mathbb{P}^1$

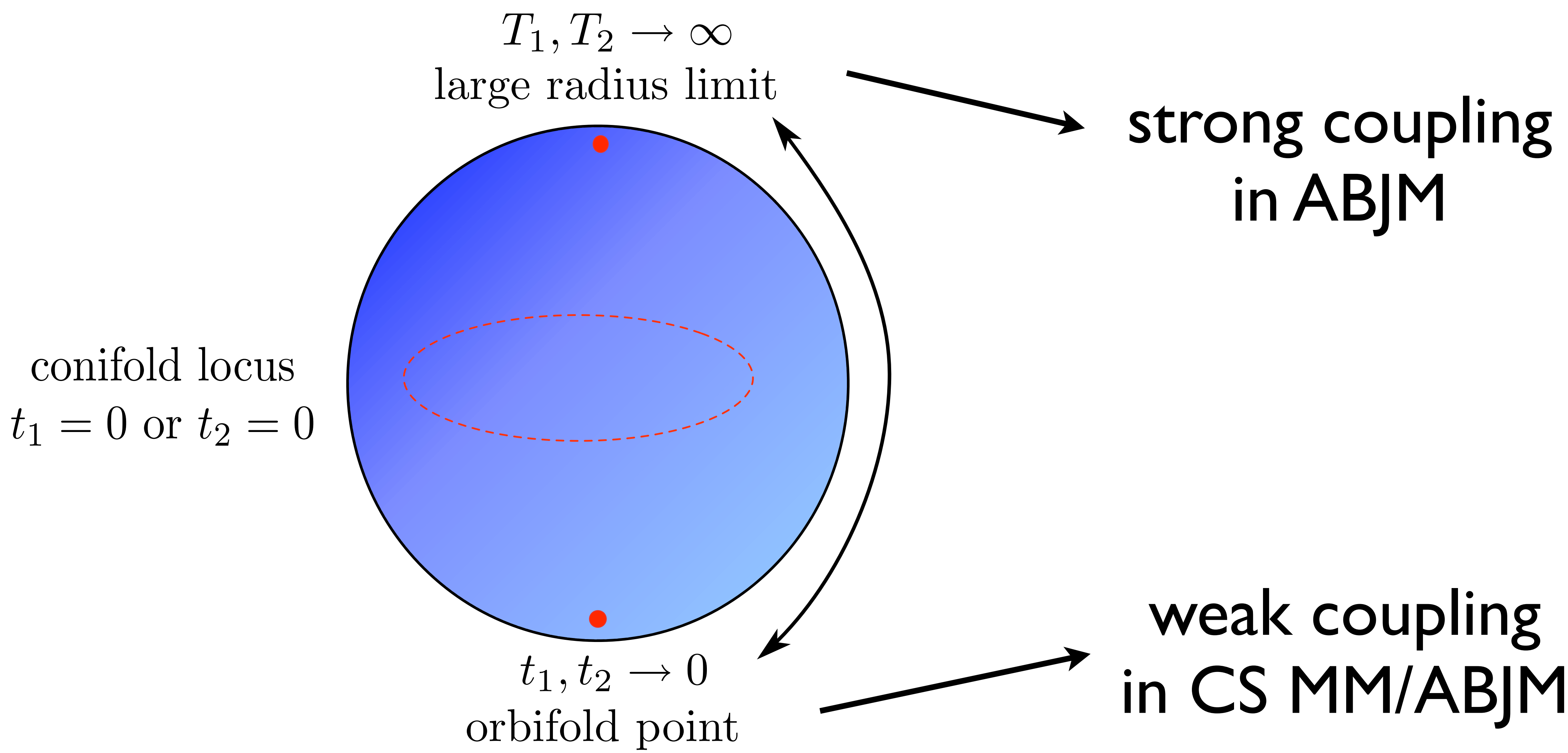
CS matrix model

ABJM theory

solvable! We want to compute the genus zero free energy. This is just the *prepotential* of the mirror manifold- a standard calculation in special geometry

Moduli space and special geometry

A key subtlety is that there are many possible prepotentials, as in Seiberg-Witten theory. The reason is that the moduli space of this CY is nontrivial, and it has *special points* (or loci). At each special point in moduli space there is a preferred choice of local coordinates and prepotential



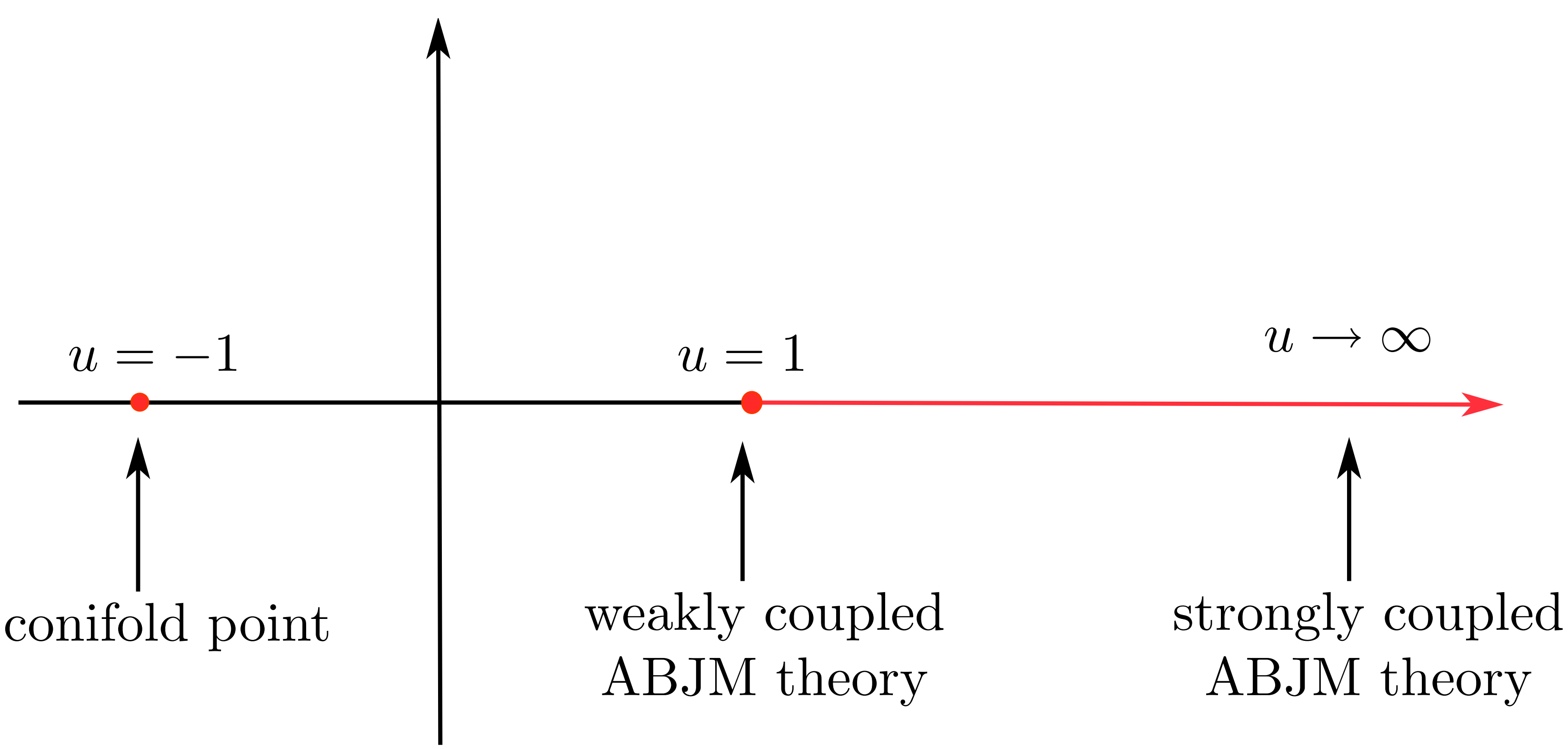
One can show that, for each choice of *B* field, the moduli space is a copy of the *u*-plane of Seiberg-Witten theory

global coordinates: $\kappa, \; B$ $\qquad$ $u = -\cos(2\pi B) + \frac{\kappa^2}{8}$

modulus

$$B = \frac{1}{2}$$

(ABJM slice)



$\kappa = \lambda = 0$ $\qquad$ $\kappa, \; \lambda \gg 1$

Special geometry and analytic continuation

The local coordinates and the derivatives of the local prepotential w.r.t. them are called the *periods*. They are defined in a neighborhood of the special points

orbifold period vector

$$\Pi_{\mathrm{weak}} = \Pi_{\mathrm{orbifold}} = (\lambda_i, \partial_{\lambda_i} F_0^{\mathrm{orb}})$$

$$\Big\downarrow \text{ analytic continuation}$$

$$\Pi_{\mathrm{strong}} = M \Pi_{\mathrm{LR}}, \quad \Pi_{\mathrm{LR}} = (T_i, \partial_{T_i} F_0^{\mathrm{LR}})$$

large radius period vector

(linear) symplectic transformation

cf. "phases" and small distances in CY manifolds [Witten, Aspinwall, Greene, Morrison]

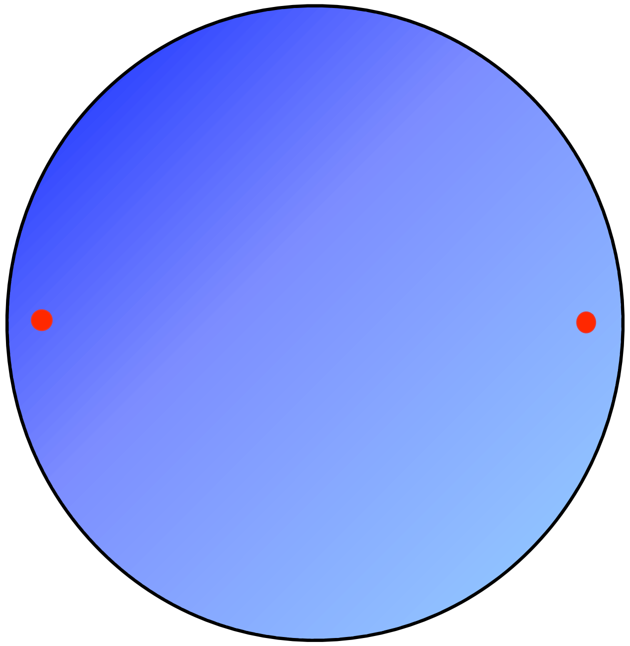
$N^{3/2}$ on the back of an envelope

We stay on the ABJM slice for simplicity

orbifold periods fixed by weak coupling [AKMV]

$$\lambda \sim \kappa$$

$$\partial_\lambda F_0^{\rm orb} \sim \kappa \log(\kappa)$$



standard periods at large radius:

$$T \sim \log(\kappa)$$

$$\partial_T F_0^{\rm LR} \sim \log^2(\kappa)$$

analytic continuation from the orbifold point to large radius is S-duality:

$$\lambda \sim \partial_T F_0^{\rm LR}$$

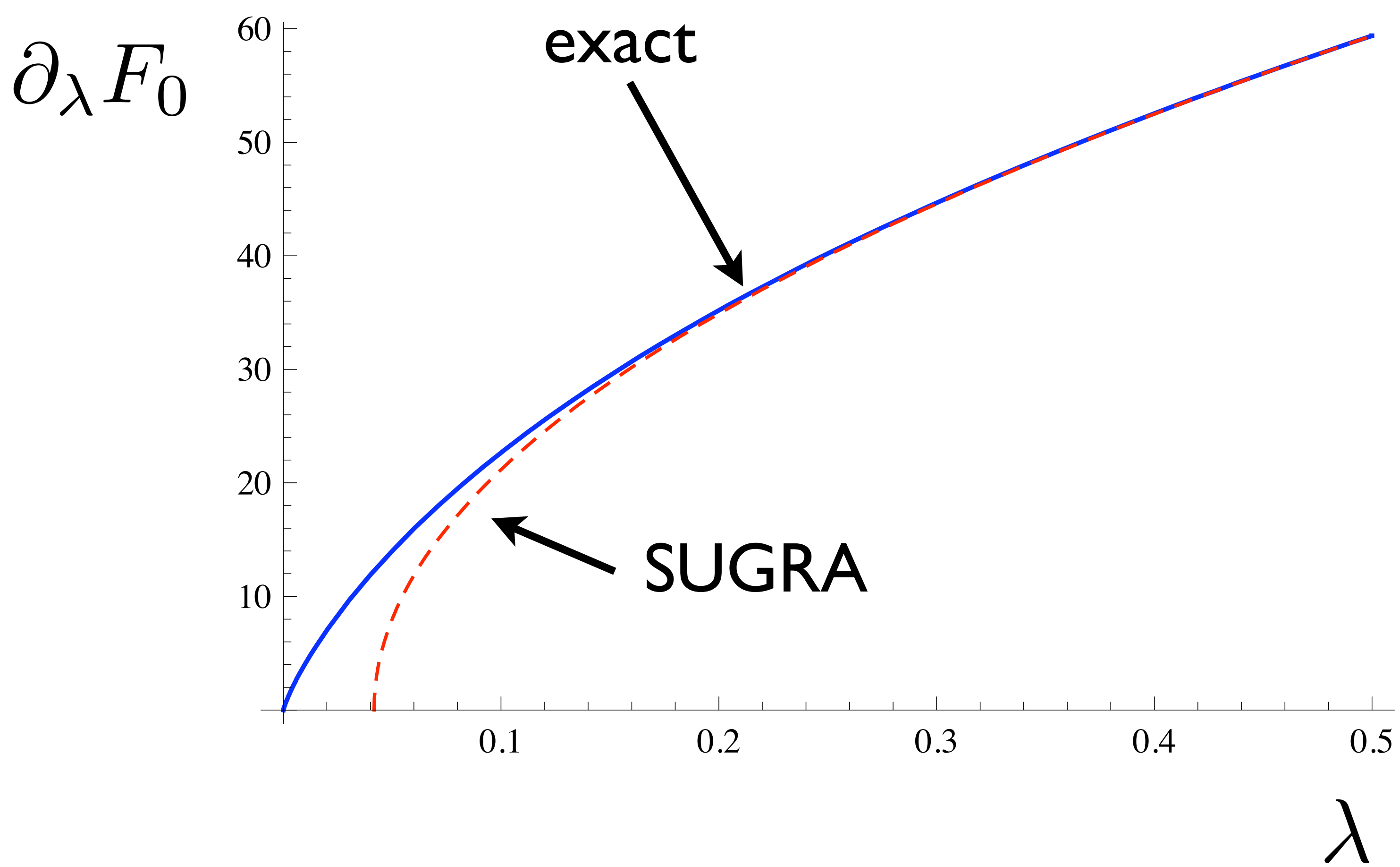
$$\partial_\lambda F_0^{\rm orb} \sim T$$

$$\boxed{\Rightarrow F_0^{\rm orb}(\lambda) \sim \lambda^{3/2}, \quad \lambda \gg 1}$$

In the ABJM slice we can write very explicit interpolating functions:

$$\lambda(\kappa) = \frac{\kappa}{8\pi}\ {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2}{16}\right)$$

$$\partial_\lambda F_0(\lambda) = \frac{\kappa}{4} G^{2,3}_{3,3}\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & -\frac{1}{2} \end{matrix} \middle| -\frac{\kappa^2}{16}\right) + 4\pi^3 \mathrm{i}\lambda$$



B field and worldsheet instantons

We can now add the B-field. The orbifold periods read, after analytic continuation to strong coupling:

$$\lambda_1(\kappa, B) = \frac{1}{2}\left(B^2 - \frac{1}{4}\right) + \frac{1}{24} + \frac{\log^2 \kappa}{2\pi^2} + f\left(\frac{1}{\kappa^2}, \cos(2\pi B)\right)$$

we reproduce the shifts!

series

$$\boxed{F_0(\hat{\lambda}, B) = \frac{4\pi^3\sqrt{2}}{3}\hat{\lambda}^{3/2} - \pi^3 \mathrm{i}(\lambda_1^2 - \lambda_2^2) + \mathcal{O}\left(\mathrm{e}^{-2\pi\sqrt{2\hat{\lambda}} \pm 2\pi \mathrm{i} B}\right)}$$

after multiplying by $g_{\rm top}^{-2}$ it matches the AdS4 calculation !

phase of the partition function

calculable series of *worldsheet instantons* on $\mathbb{CP}^1 \subset \mathbb{CP}^3$

[Sorokin et al.]

Back to Wilson loops

$$\langle W_{\Box}^{1/2} \rangle = \mathrm{e}^{\pi \mathrm{i} B} \frac{\kappa(\hat{\lambda}, B)}{2} \approx \mathrm{e}^{\pi \sqrt{2\hat{\lambda}}} \left(1 + \mathcal{O}\left(\mathrm{e}^{-2\pi\sqrt{2\hat{\lambda}} \pm 2\pi \mathrm{i} B} \right) \right)$$

B=0: [M.M.-Putrov]

AdS prediction

worldsheet instanton corrections

This corresponds to a (topological) disk string amplitude in the topological string picture

One can refine this computation to obtain vevs for 1/6 BPS Wilson loops [M.M.-Putrov] and for 1/2 BPS "giant" Wilson loops [Drukker-M.M.-Putrov]

Beyond the planar approximation

It turns out that one can compute the full *1/N* expansion of the free energy in a systematic (and efficient!) way, at least in the ABJM slice

Mirror symmetry at higher genus is encoded in the BCOV holomorphic anomaly equations. Schematically,

$$\partial_{\bar{t}} F_g(t,\bar{t}) = \text{functional of } F_{g'<g}(t,\bar{t})$$

Direct integration [Klemm+Huang, M.M., ...] : formulate them in terms of modular forms and impose boundary conditions at special points in moduli space. In local CY they are fully integrable

$$F_2 = \frac{1}{432bd^2}\left(-\frac{5}{3}E_2^3 + 3bE_2^2 - 2E_4E_2\right) + \frac{16b^3+15db^2+21d^2b+2d^3}{12960bd^2}$$

Upgrading the matrix models of non-critical strings: we have an integrable structure encoding a *1/N* matrix model expansion, similar to the Painleve-type nonlinear ODEs

We can now address some nonperturbative issues *in the string coupling constant* by looking at the large genus behavior

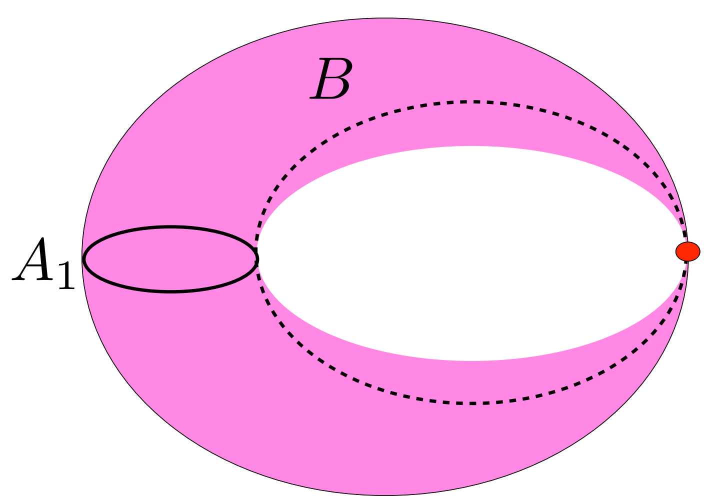
$$F_g(\lambda) \sim (2g)!(A_{\rm st}(\lambda))^{-2g}, \quad \lambda > \frac{1}{2}$$

[cf. Shenker]

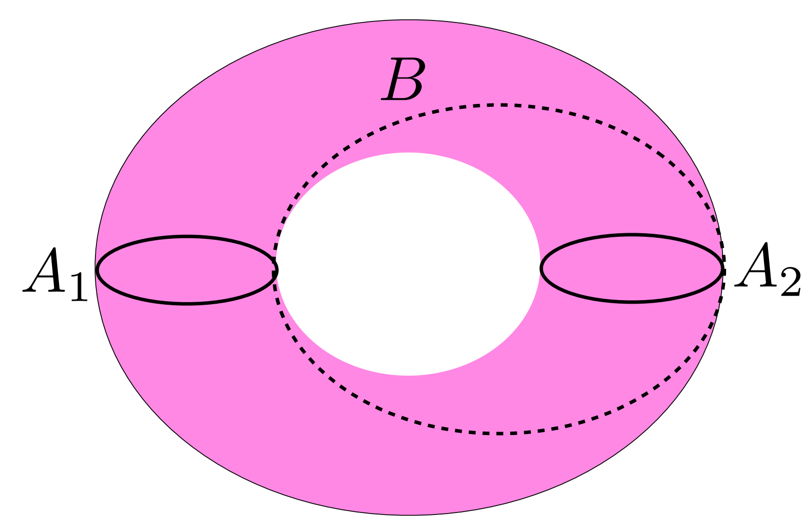
$$A_{\rm st}(\lambda) \propto \frac{1}{\pi}\partial_\lambda F_0(\lambda) + \pi^2 \mathrm{i}$$

(complex) eigenvalue tunneling

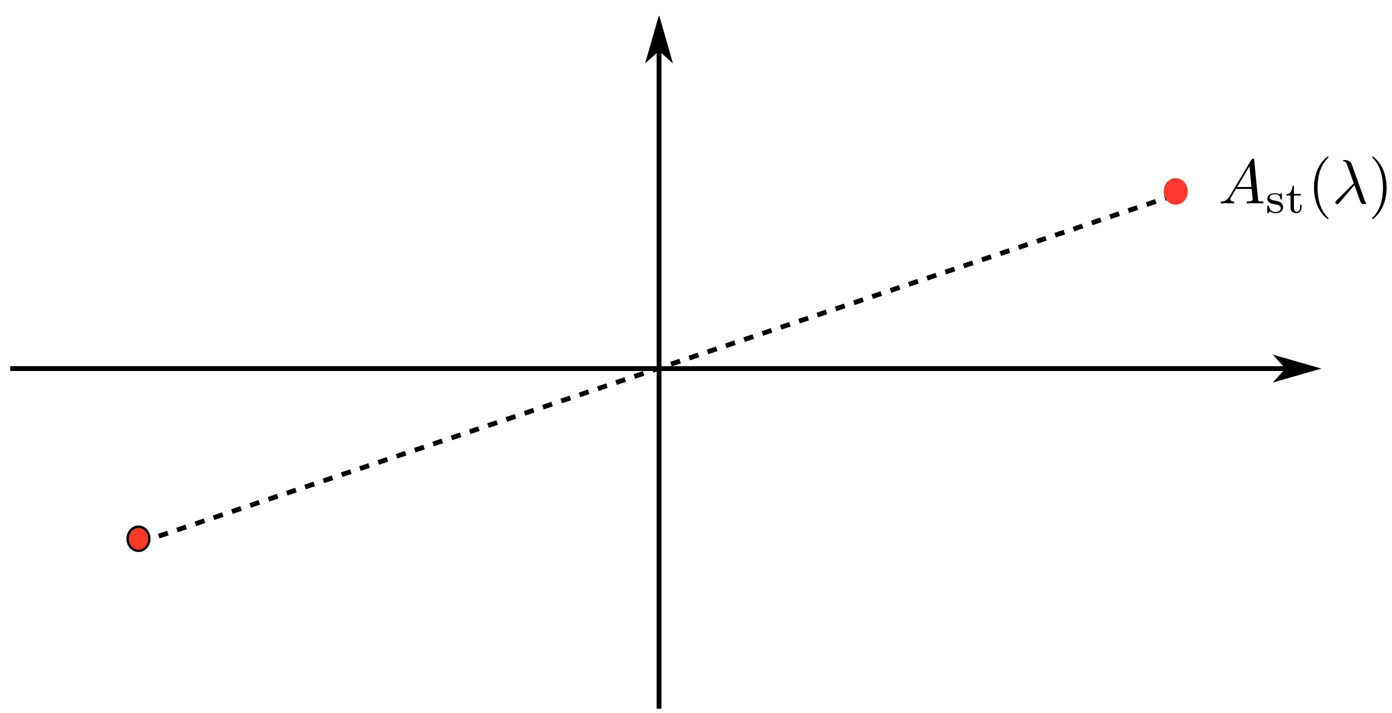
one-cut: [Shenker, David, Seiberg-Shih, M.M.-Schiappa-Weiss]



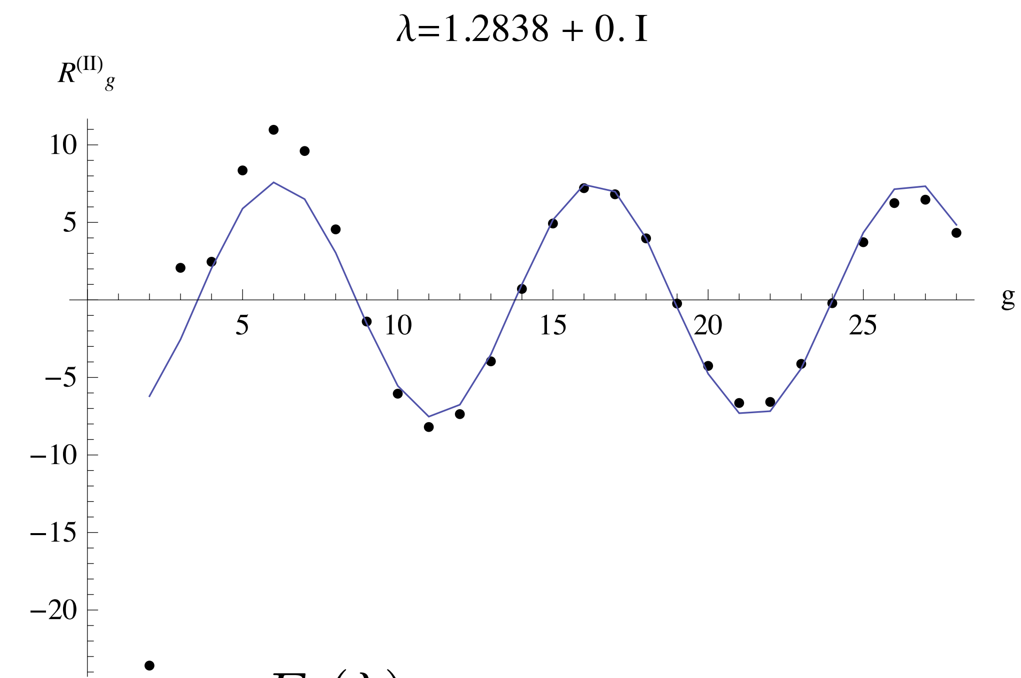
two-cuts: [Klemm-M.M.-Rauch]



Complex instantons: superstring perturbation theory on AdS4xCP3 is Borel summable for all nonzero ‘t Hooft coupling/radius!



Borel plane of the string coupling constant



$$\frac{F_g(\lambda)}{(2g)!|A_{\mathrm{st}}(\lambda)|^{-2g}} \sim \cos(2g\theta_A(\lambda)+\delta)$$

At strong coupling we find: $$A_{\mathrm{st}} \approx \frac{R^3}{4}\left(1+\frac{2\pi \mathrm{i}}{R^2}\right)$$

What is the ZZ brane in this theory?

Euclidean brane wrapping $\mathbb{RP}^3$?

Borel summability *invisible* in SUGRA -a stringy effect!

Conifold singularity and analytic continuation

In the ABJ theory, expanding around the conifold locus means expanding around pure CS theory (one A cycle/node collapses)

In the ABJM slice, the conifold locus takes place at *imaginary* ‘t Hooft coupling and there is a double-scaling limit giving the *c=1* string:

$$F_g \sim \frac{B_{2g}}{2g(2g-2)} \left(\frac{\lambda - \lambda_c}{\log(\lambda - \lambda_c)} \right)^{2-2g} \qquad \lambda_c = -\frac{2\mathrm{iK}}{\pi^2}$$

In this regime (with imaginary CS coupling) the genus expansion is *no longer* Borel summable (real instantons)

All this seems to give a concrete realization of the scenario advocated for Polyakov to go to de Sitter space

Conclusions and open problems

- We have used matrix models/topological strings to derive important aspects of ABJM theory at strong coupling. It is of course possible to analyze related 3d SCFTs with the same tools
  [in progress]

- Concrete predictions for worldsheet instanton corrections, which should be better understood. Direct calculation? Localization in the superstring?

- Is there an *a priori* reason for the connection with topological strings?

- Nonperturbative effects in the string coupling constant: identify them in both the gauge theory (large *N* instantons?) and in the superstring theory (wrapped D-branes?)

Appendix: Supermatrix models

Hermitian supermatrix

$$\Phi = \begin{pmatrix} A & \Psi \\ \Psi^\dagger & C \end{pmatrix}$$

A, C Hermitian, Grassmann even

Ψ complex, Grassmann odd

$$Z_{\mathrm{s}}(N_1|N_2) = \int \mathcal{D}\Phi \, e^{-\frac{1}{g_s}\mathrm{Str}V(\Phi)}$$

[Yost, Alvarez-Gaume-Mañes, Dijkgraaf-Vafa, ...]

Assume the eigenvalues are *real* (physical supermatrix model):

$$Z_{\mathrm{s}}(N_1|N_2) = \int \prod_{i=1}^{N_1} d\mu_i \prod_{j=1}^{N_2} d\nu_j \frac{\prod_{i<j}(\mu_i-\mu_j)^2(\nu_i-\nu_j)^2}{\prod_{i,j}(\mu_i-\nu_j)^2} e^{-\frac{1}{g_s}\left(\sum_i V(\mu_i)-\sum_j V(\nu_j)\right)}$$

compare to

$$Z_{\mathrm{b}}(N_1,N_2) = \int \prod_{i=1}^{N_1} d\mu_i \prod_{j=1}^{N_2} d\nu_j \prod_{i<j}(\mu_i-\mu_j)^2(\nu_i-\nu_j)^2 \prod_{i,j}(\mu_i-\nu_j)^2 \, e^{-\frac{1}{g}\left(\sum_i V(\mu_i)+\sum_j V(\nu_j)\right)}$$