A Theoretical Realization of a Quantum Hall Nematic

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GGI Seminar

October 6, 2010

based on work with Shamit Kachru and Chetan Nayak

Introduction

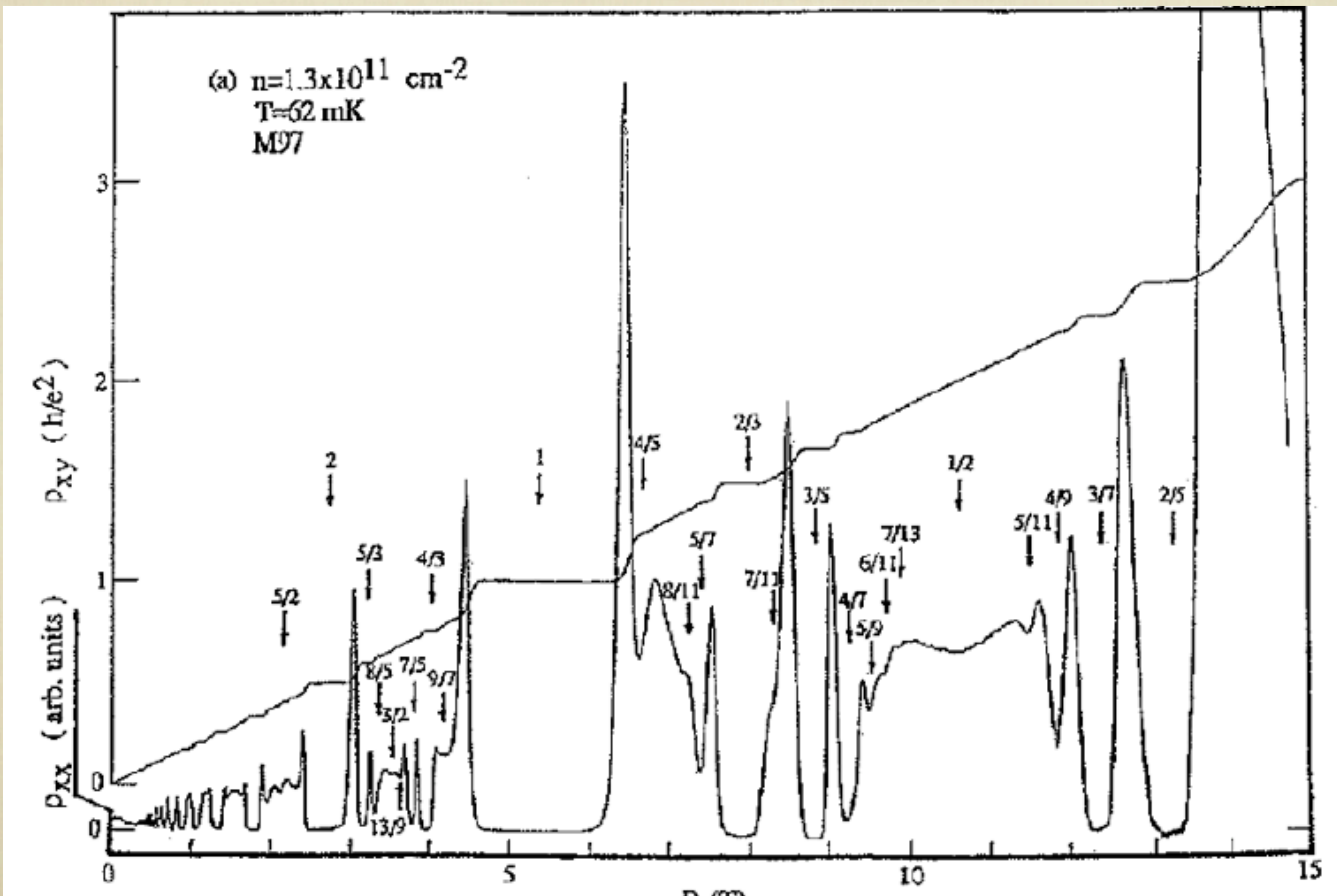
There are two motivations for this work. The first comes from an unpublished experiment on a 2 dimensional electron gas (DEG) from the Eisenstein group. Before I tell you about their results, let me first remind you about the quantum Hall effect.

The QHE comes in two varieties:
(1) the integer QHE discovered by von Klitzing, Dorda, and Pepper in 1980 and
(2) the fractional QHE found by Tsui, Stormer, and Goddard in 1982.
They found that if you study a 2 DEG (of density around $10^{11} e^{-}/cm^{2}$) made from an AlAs-GaAs-AlAs heterostructure at low temperatures (around 10 mK) and in a strong magnetic field (around 5 T), the DC longitudinal and off-diagonal conductivities have the following low temperature behavior:

$$\lim_{T\to 0}\sigma_{xx} = \lim_{T\to 0}\sigma_{yy} = 0$$

$$\lim_{T\to 0}\sigma_{xy} = \nu\frac{e^2}{h}$$

for certain discrete choices of $\nu = n/(2p+1)$.



This should be contrasted with the classical behavior where $\sigma_{xy} \sim n/B$.

If you like, you could begin with a 2D non-interacting fermi liquid at zero temperature and imagine slowly turning on a background magnetic field, however, you would encounter phase transitions once the magnetic field was sufficiently strong.
I.e., the initial starting point is not good for studying the QH regime -- denoted by parameters roughly in the range stated above.

Laughlin, Halperin, and many others explained the basic effect back in the 80s and 90s.

In addition to the above response, other properties of the QH liquid were discovered (still some only theoretically):

-- gapless chiral edge modes (Halperin, Wen),

-- ground state degeneracy when the liquid is studied on a non-trivial Riemann surface (Wen), and

-- anyonic quasi-particle excitations (Halperin, Wen, Wilczek, Zee).

For certain fillings, there are even candidate states that have quasi-particle excitations that obey nonabelian statistics (Moore-Read).

Recent Experiments

I'd like to describe two different sets of remarkable experiments by the Eisenstein group who studied properties of the 2 DEG at filling fractions $\nu > 2$.

We shall not discuss a theory that explains the first.

Rather, we mention it in order to contrast the behavior found in the second experiment and to partially motivate the construction of the theoretical model I'll describe in the bulk of the talk.

Further, certain transport properties observed in the second experiment will have overlap with the theory I wish to describe.

“Experiment I” at $\nu = 5/2$

In the late 90s and early (20)00s, Lilly et al. found nematic behavior of the 2 DEG when an in-plane magnetic field was turned on for filling fractions $\nu = (2n+1)/2$. SO(2) rotation symmetry is broken by the in-plane field and is reflected in anisotropic longitudinal conductivities, $\sigma_{xx} \neq \sigma_{yy}$.

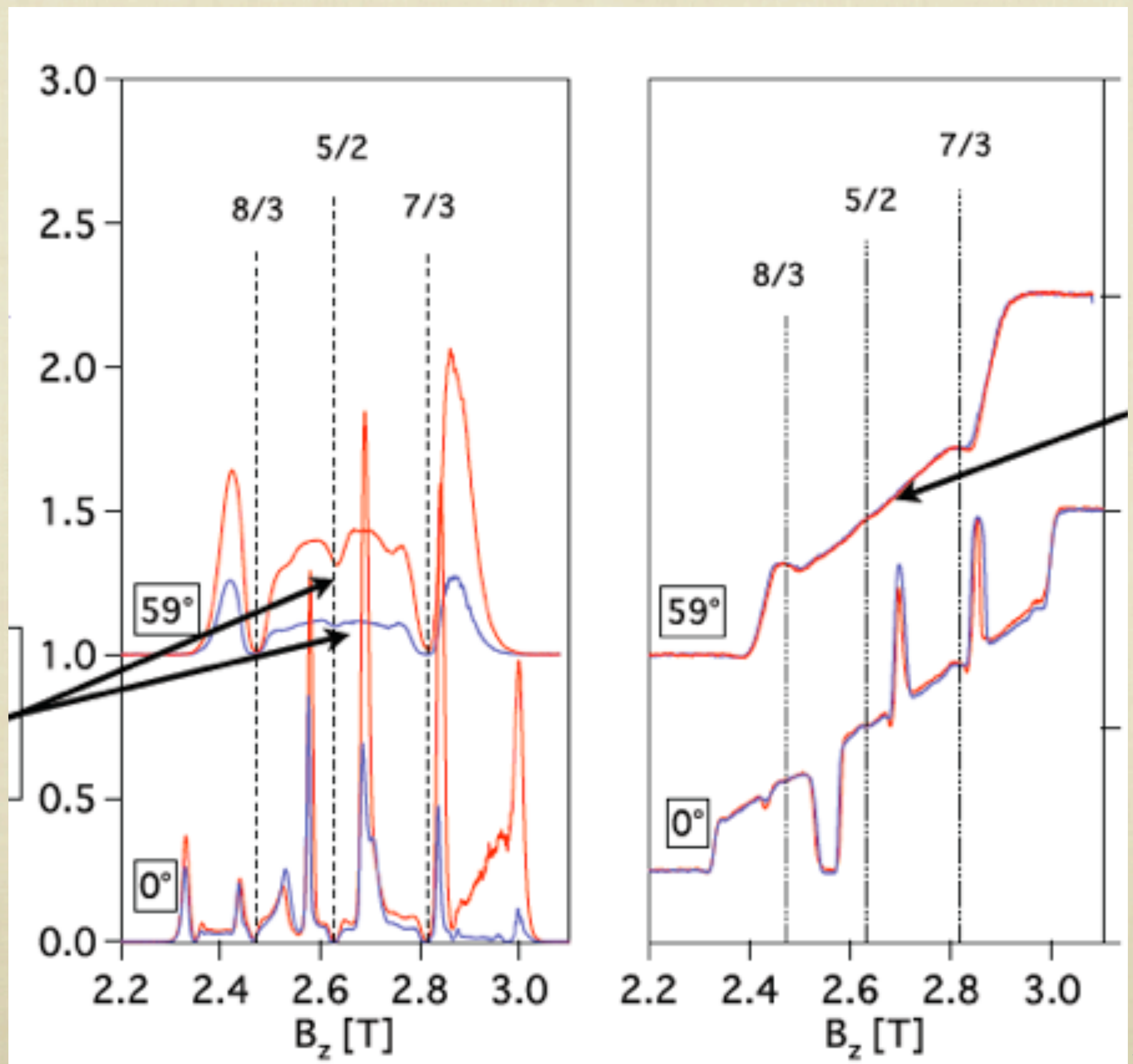
To be specific, let us discuss the experiments performed at $\nu = 5/2$.

When an in-plane $B_{||}$ field is turned on, it is found:

-- for sufficiently large $B_{||}$, $\theta = \tan^{-1}(B_{||}/B) = 60^\circ$, the plateau is destroyed, where B is the large magnetic field transverse to the two-dimensional plane on which the electrons move,

-- the longitudinal resistivities can differ by a factor on the order of 10.

$R_{xx}(k\Omega)$

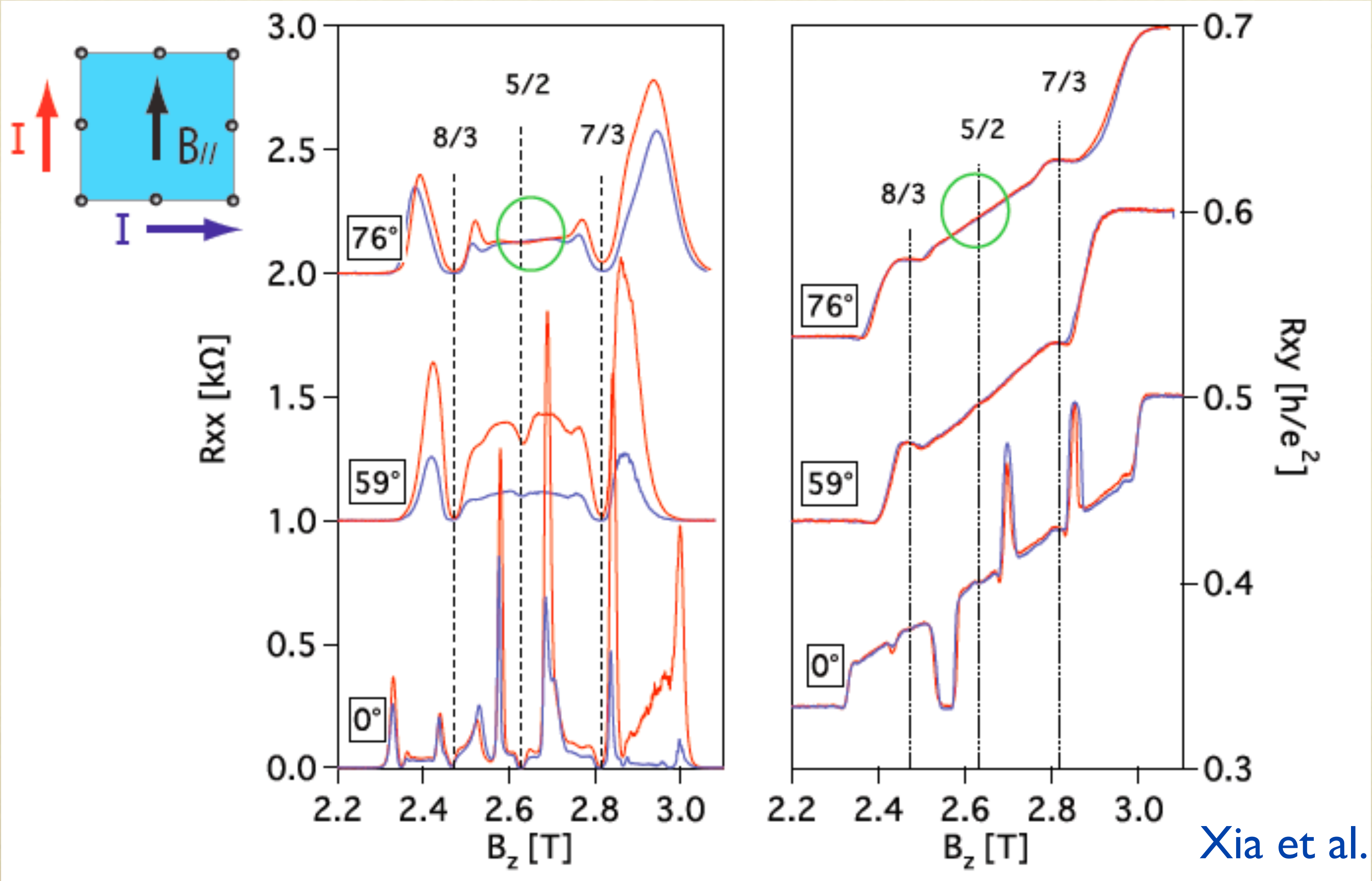


Xia et al.

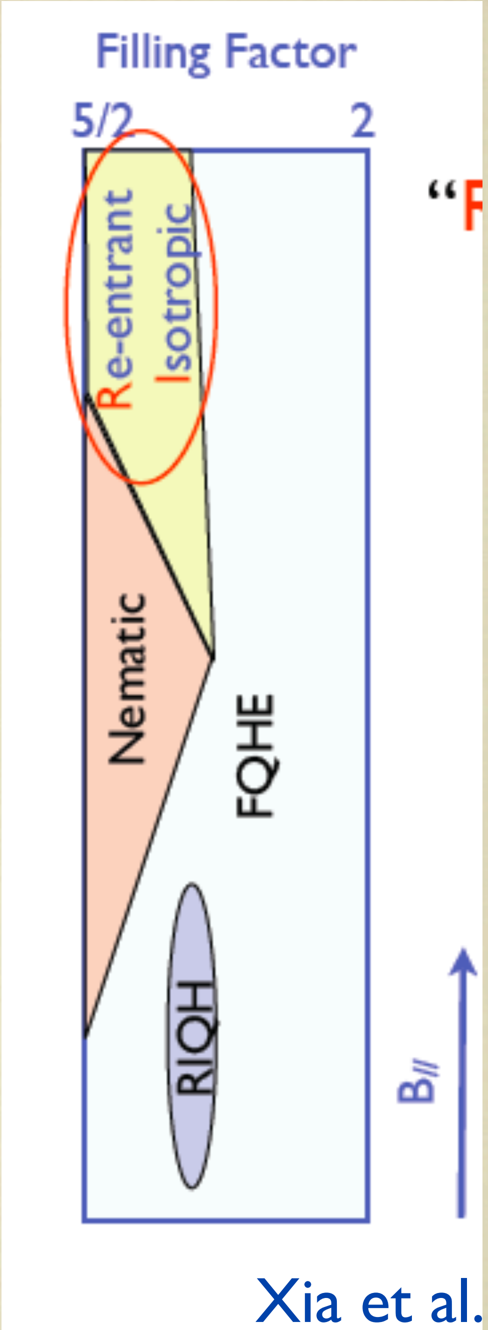
More recently (March of 2010), Xia et al. have found that when $\theta \sim 76^\circ$:

-- the plateau is still destroyed, BUT

-- SO(2) invariance of the longitudinal resistivities is restored!



The phase diagram looks like the following:



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An explanation can be given in terms of mixing of various Landau levels.

I do not want to enter into the interesting discussion of the physics responsible for the re-entrant behavior. I only want to infer the following naive conclusion:

Although the in-plane field explicitly breaks SO(2) rotation invariance from the ``lab frame," from the 2 DEG perspective, it might be more accurately acounted for by spontaneous symmetry breaking. It is this viewpoint -- that the system spontaneously breaks SO(2) after a sufficiently strong in-plane field is applied -- that we shall adopt in what follows.

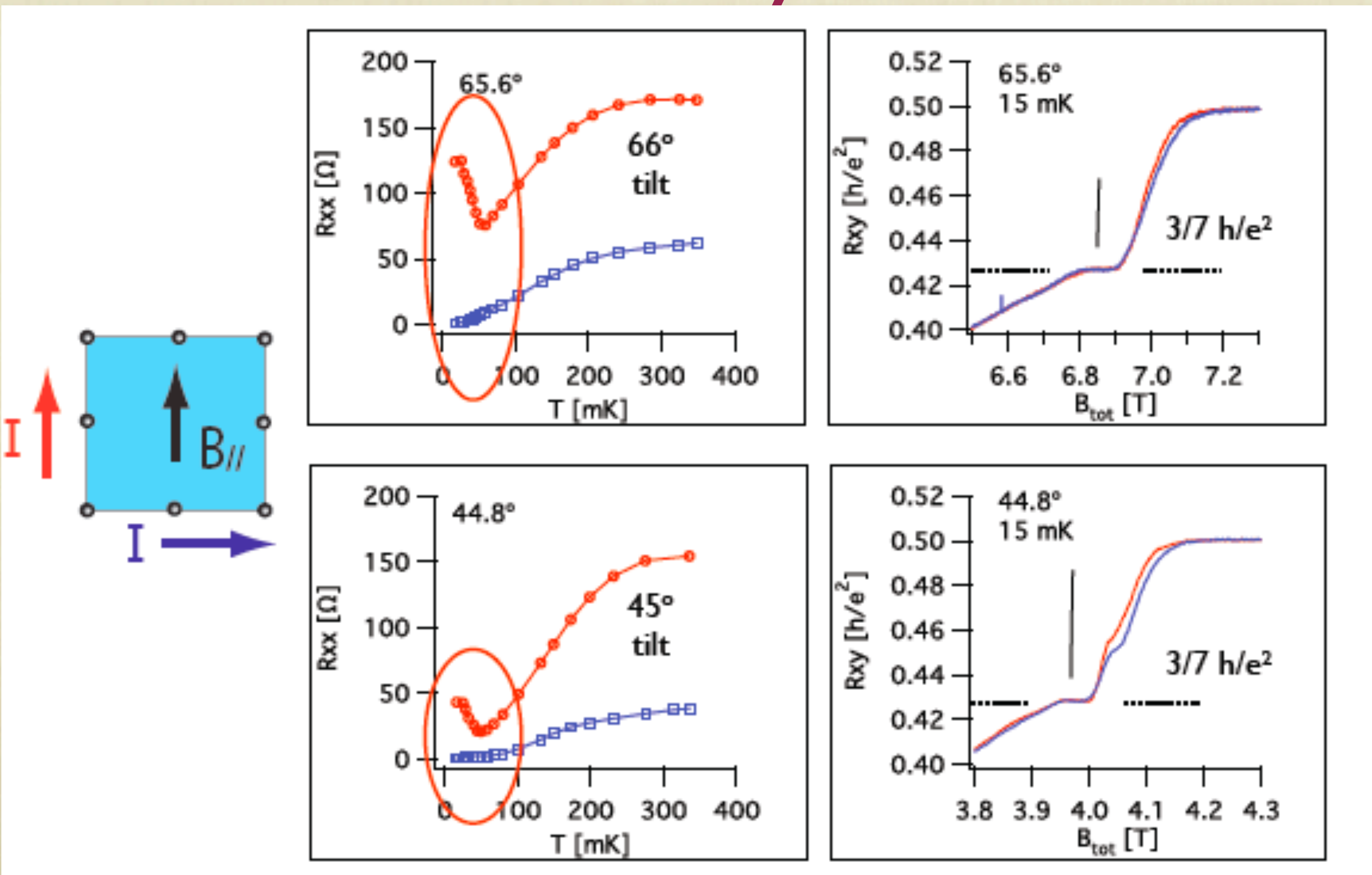
Caveat: This can't strictly be true since the magnitude of the longitudinal conductivities is tied to the direction of the in-plane field.

"Experiment 2" at $\nu = 7/3$

Near the $\nu = 5/2$ plateau lives the $\nu = 7/3$ plateau. Again, the effects of an in-plane magnetic field were studied by Xia et al. They found:

-- as $B_{||}$ is increased ($\theta \sim 45^\circ - 65^\circ$), the longitudinal conducitivities show nematic behavior, BUT

-- the Hall plateau persists, i.e., the quantization of the Hall conductance is not destroyed.



Xia et al.

Objective

It is this phenomena:

(1) quantized Hall conductance with

(2) nematic or anisotropic longitudinal conductivities

that we would like to explain.

Instead of computing the finite temperature DC conductivities, we calculate the T=0, finite frequency conductivities. Nevertheless, we will find that our theory shows both nematic behavior and quantized Hall conductance.

To do this, we first review a theory that describes the usual QHE and then deform it in such a way to obtain a theory with anisotropic longitudinal conductivities.

What's Ahead:

(1) CS Description of the FQHE

(2) The Deformation of the CS Theory

(3) General Properties of the Deformed Theory

(4) Transport Properties of the Deformed Theory

(5) A Matching of Effective and Microscopic Parameters

CS Theory Description of the FQHE

For our purposes, we shall understand the QHE in terms of an effective ``hydrodynamic" theory (Zee).

The number current of the two-dimensional electrons is conserved.

A conserved current in 2+1 d implies the existence of a gauge field:

$$\partial_\mu j^\mu = 0 \Rightarrow j_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial^\nu a^\rho.$$

We now construct an effective action for the gauge field by writing down the leading operators constructed out of a_μ .

A mass a^2 term is not allowed since this breaks the U(1) gauge invariance.

The leading operator is the Chern-Simons form,

$$L_0 = \frac{k}{4\pi}\int \epsilon_{\mu\nu\rho} a_\mu \partial_\nu a_\rho,$$

It's okay that L_0 breaks P,T because the background magnetic field in the physical system breaks these symmetries.

It is standard to take the leading irrelevant operator to be the Maxwell term,

$$L_1 = \int \frac{1}{2\Lambda}\left(e^2 - b^2\right)$$

where Λ is an UV cutoff,

$e_i = \partial_i a_t - \partial_t a_i$, and

$b = \epsilon_{ij}\partial_i a_j$.

From the effective action, $L_0 + L_1$, we can calculate the response to an external gauge field by minimally coupling a background field to the current $j_\mu A^\mu$.

Thus, our full action is:

$$L = \int \frac{k}{4\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{1}{2\Lambda}\left(e^2 - b^2\right) + j_\mu A^\mu.$$

Response functions are determined from current-current correlation functions.

Equivalently, we may determine the response of the system to the external perturbation by integrating out the internal gauge field, a_μ , to obtain an action for the background field,

$$L_{eff} = \int A_\mu K^{\mu\nu}(\omega, q) A_\nu.$$

The response matrix $K_{\mu\nu}$ should have the following properties:
Re(K) should be symmetric while Im(K) should be anti-symmetric. Further, gauge invariance, $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$, implies

$$\omega K_{0\mu} + q_i K_{i\mu} = 0.$$

The theory described by L is quadratic so it's straightforward to integrate out a_μ . Let me not write down K explicitly (but trust me, it obeys the above constraints). I will only describe a few of its long wavelength and low energy properties.

The compressibility, $C = \frac{\partial n}{\partial \mu}$, is determined by K:

$$C = \lim_{q \to 0} \lim_{\omega \to 0} K_{00}.$$

Thus, we find, C = 0. This means the fluid is incompressible.

What about charge transport? We find:

$$\sigma_{xx} = \sigma_{yy} \sim \frac{i}{2\pi k} \frac{\omega/\omega_c}{1 - \omega^2/\omega_c^2},$$

$$\sigma_{xy} = \frac{1}{2\pi k} \frac{1}{1 - \omega^2/\omega_c^2},$$

where $\omega_c = \Lambda \sim B/m_e$ is the cyclotron frequency. The pole in σ_{xx}, σ_{xy} corresponds to inter-Landau level excitations.

Lifshitz-Chern-Simons Theory

We now ask what deformations of this theory can lead to anisotropic longitudinal conductivities.

Consider the Hamiltonian of the theory where all operators are given arbitrary coefficients:

$$H = \frac{1}{2g_{e_x}^2} e_x^2 + \frac{1}{2g_{e_y}^2} e_y^2 + \frac{1}{2g_b^2} b^2.$$

Unfortunately, letting the coefficients of e_x^2 and e_y^2 be different is not a significant enough deformation to alter the real part of the longitudinal conductivities. We must do something more drastic.

To see what to do, it is helpful re-write the Lagrangian in first-order phase space form:

$$L = \int \frac{1}{g^2}\left(e_i(\partial_i a_t - \partial_t a_i) - \frac{r}{2}e^2 - \frac{1}{2}b^2\right) + \frac{k}{4\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu a_\rho.$$

In the above, the electric field e, is treated as an independent field variable.

We can recover the usual Maxwell-CS Lagrangian written before if we integrate out e using its equation of motion.

Recall that e^2 was the leading irrelevant operator to the Chern-Simons form if we assumed a z=1 dynamical critical exponent, under which

$$a_t \to \lambda^{-1} a_t, a_i \to \lambda^{-1} a_i, e_i \to \lambda^{-2} e_i.$$

and

$$t \to \lambda t, x_i \to \lambda x_i$$

However, the CS Lagrangian is topological -- so at the very least, it's consistent with any dynamical exponent z.

Let's choose z=2.

Then we may ask, in the first-order form of the Lagrangian, what are the leading operators consistent with the following z=2 scale invariance:

$$t \to \lambda^2 t, x_i \to \lambda x_i$$

and

$$a_t \to \lambda^{-2} a_t, a_i \to \lambda^{-1} a_i, e_i \to \lambda^{-1} e_i.$$

If we maintain the shift symmetry of $e \to e + \mathrm{const.}$, then there exists only the marginal deformation:

$$\delta L = -\frac{\kappa^2}{2} \int (\partial_i e_j)^2.$$

Alternatively, we may say that this theory is obtained by first fine-tuning the coefficient of the e^2 operator $r \rightarrow 0$.

We then need to add in the leading higher dimension operators, from the perspective of z=1 scaling, to the theory.

There are two possibilities. The first is δL above; it is the leading operator quadratic in **e**. In addition, e^4 can be added, but it turns out to be marginally irrelevant, assuming its coefficient is consistent with vacuum stability. Because neither we nor any real system are ever precisely at the critical point at **r=0**, we will keep the quartic e^4 operator around in our analysis.

Thus, the full theory we shall study is described by

$$L = \int \frac{1}{g^2}\Big(e_i(\partial_i a_t - \partial_t a_i) - \frac{r}{2}e^2 - \frac{\kappa^2}{2}(\partial_i e_j)^2 - \frac{1}{2}b^2\Big) - \frac{\lambda}{4}(e^2)^2 + k\epsilon_{\mu\nu\rho}a_\mu\partial_\nu a_\rho.$$

We will refer to this theory as Lifshitz-Chern-Simons theory. At r=0, it has z=2 dynamical scale invariance and has a one-dimensional space of relevant deformations parameterized by r.

Phase structure is the following:

-- r>0, the theory is equivalent to Maxwell-CS theory at long distances and so is in the QH regime,

-- r=0, the theory has z=2 scale invariance and is classically critical,

-- r<0, the theory is unstable; the e field wants to condense at a non-zero value.

Note that we have already computed the response of this theory in the r>0 phase to perturbations by an (external) background gauge field.

When r>0, the newly added operators with couplings κ^2 and λ are irrelevant in the IR. The theory reduces to the Maxwell-CS theory we described in the previous section.

Thus, we're already a third of the way done with determining the physical properties of the theory!

It remains to calculate the response at the r=0 point and in the r<0 phase.

Properties of the Critical r=0 Theory

Before we do this, however, I would like to mention a few properties of the $z=2$ critical point at $r=0$. The reason is because this is a situation where the deformation to the CS term is marginal. As we've seen, the Maxwell operator deformation is irrelevant and so decouples in the IR. We will find some new behavior when instead this marginal operator is present.

Below I will discuss three properties of the Lifshitz-CS theory: its (1) canonical quantization, (2) edge states, and (3) ground state degeneracy.

Canonical Quantization

Recall that the single fluctuation described by the gauge field of Maxwell-CS theory is massive and has spin-1 (Jackiw et al.).

The Chern-Simons term imparts a tree-level mass to the photon. (In terms of the QHE, the massive photon is related to inter-Landau level transitions.)

Because of the Chern-Simons term, the rotation generator is modified from its naive Noether form; the modification of the operator is such that the massive photon has unit eigenvalue under the rotation.

As you might expect, without any dimensionful parameters, the excitation described by the Lifshitz-CS theory is massless. Indeed, canonical quantization of the theory bears this out.

$$H = \frac{1}{2}\int \kappa^2 g^2(\partial_i \Pi_T^j)^2 + \Big(\frac{1}{g^2} + \frac{\kappa^2 k^2 g^2}{4\pi}\Big)(\partial_i a_T^j)^2,$$

where $\Pi_i = e_i/g^2 + \frac{k}{4\pi}\epsilon_{ij}a_j$ is the canonical momentum and the subscript T refers to the fact that the fluctuations are transverse -- $q_i a^i = 0$. Hamilton's equations say that a_i is gapless and has quadratic dispersion $\omega^2 \sim q^4$.

The proper definition of the rotation generator is trickier in this non-relativistic setting. In the Maxwell-CS case, the full SO(2,1) non-abelian Lorentz group fixed the appropriate operator. We do not have this symmetry. There does, however, exist a RG flow via the e^2 operator that takes us to the Maxwell-CS theory in the IR. It is possible to define the UV rotation generator by its Lorentz invariant IR limit.

Edge Modes

When CS theory is studied on a a manifold with boundary, it is well known that the action is invariant up to a boundary term (Callan and Harvey, Wen, Stone).

$$\frac{\delta L_{CS}}{\delta\alpha} = \int_{\partial} \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu},$$

where α is the gauge transformation parameter and μ, ν run over the directions of the two-dimensional boundary.

To make the full theory well defined, it is necessary to add in boundary degrees of freedom that cancel the above anomalous term. These gapless chiral modes are the so-called edge modes.

Their existence is entirely due to the presence of the Chern-Simons term. Neither the Maxwell or Lifshitz terms transform anomalously so only the CS term contributes to the anomalous transformation above.

Ground State Degeneracy

We now study the Lifshitz-CS theory on the 3-manifold $R \times \Sigma$ where Σ is some Riemann surface of genus g. Actually, it's sufficient to take $\Sigma = T^2$, the 2-torus.

Being interested in the ground state, we can set all momenta to zero, thus we reduce the problem to a quantum mechanical one. But from the above expression for the Hamiltonian, we see that it precisely vanishes at zero momentum.

We have forgotten to keep the quartic e^4 operator. It is marginally irrelevant, but on a compact space, the size of the space cuts off the running of this operator and so it must be kept in the Lagrangian.

Re-inserting it, we find

$$H = \lambda \int ((\Pi_i - \frac{k}{4\pi}\epsilon_{ij}a_j)^2)^2.$$

This is just the square of the Hamiltonian for a particle in a magnetic field of strength $k/2\pi$. Indeed the energy levels are indexed by an integer, $E \sim n^2$ for large n, and have a degeneracy equal to k. For a Riemann surface of genus g, the degeneracy is k^g.

Transport at r=0

Using the techniques described previously, it is straight forward to obtain the leading ($\lambda = 0$) transport at the r=0 critical point. We find:

$$\begin{aligned} \mathrm{Re}\,\sigma_{xx}(\omega) = \mathrm{Im}\,\sigma_{xx}(\omega) &= 0 \\ \mathrm{Re}\,\sigma_{xy}(\omega) = \mathrm{Im}\,\sigma_{xy}(\omega) &= 0 \end{aligned}$$

while the compressibility is finite,

$$C = \frac{g^2}{8\pi^2(1+\kappa^2)(k/2\pi)^2}$$

It's remarkable that the apparent critical point is more insulating that the r>0 QH regime. The intuitive reason for this is that we are interested in the long distance $q \to 0$ limit. But the Lifshitz term vanishes in this limit and so no finite frequency term can survive.

Transport in the r<0 Phase

For r<0, the Hamiltonian

$$g^2 H[e,a] = -\frac{|r|}{2} e_i^2 + \frac{\kappa^2}{2}(\partial_i e_j)^2 + \frac{\lambda}{4}(e_i^2)^2 + b^2$$

is minimized at $e_i^2 = \frac{|r|}{\lambda}, a_i = 0$.
Without loss of generality, we can choose the SO(2) symmetry-breaking vacuum to be given by $e_i = (\sqrt{|r|/\lambda}, 0)$
We expand about this point by writing $e_i = (\sqrt{|r|/\lambda} + \tilde{e}_x, e_y)$ and substitute this into H, thereby obtaining

$$H[e,a] = |r|\tilde{e}_x^2 + \frac{\kappa^2}{2}(\partial_i e_y)^2 + b^2 + \mathcal{O}(e^3) .$$

The corresponding action is

$$S = \int dt\, d^2x \Big[\frac{1}{g^2}\Big(\tilde{e}_x \partial_t a_x - |r|\tilde{e}_x^2 - (\partial_y a_x - \partial_x a_y)^2 + a_t \partial_i e_i + e_y \partial_t a_y - \frac{\kappa^2}{2}(\partial_i e_y)^2 \Big) + \frac{k}{4\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \Big].$$

The theory looks like the Maxwell theory along the x-direction and while it is Lifshitz-like along the \$y-\$direction.

If impurities in the system can be ignored, we find the following response:

$$\sigma_{xx}(\omega) = 0$$

$$\sigma_{yy}(\omega) = \frac{g^4 r}{8\pi}\delta(\omega) + \frac{i g^4 r}{8\pi^2 \omega}$$

$$\sigma_{xy}(\omega) = 0.$$

The compressibility which follows from K_{00} is finite or zero depending upon how the zero momentum limit is taken.

To determine the meaning of the delta function it is useful to include interactions. Because any real system will have some amount of ``dirt", we consider adding in the effects of impurities. This, indeed, modifies the response,

$$\begin{aligned}\sigma_{xx}(\omega) &= \frac{1}{8\pi^2}\,\frac{ig^4\omega}{g^8\tilde{k}^2 r + i(\omega\tau)^2}\\ \sigma_{yy}(\omega) &= \frac{1}{8\pi^2}\,\frac{g^4 r\,\omega\,\tau^2}{g^8\tilde{k}^2 r + i(\omega\tau)^2}\\ \sigma_{xy}(\omega) &= \frac{1}{2\pi k}\cdot\frac{1}{1 + i(\omega\tau)^2/g^8\tilde{k}^2 r},\end{aligned}$$

where τ is the scattering time between impurities. Indeed, as the impurities are removed $\tau \to \infty$, we recover the results for the clean system. Further, the compressibility vanishes as long as τ is finite.

Notice the following behavior as $\omega \to 0$:

$$\sigma_{xy} \to \frac{1}{2\pi k},$$
$$\mathrm{Re}(\sigma_{xx}) \sim \omega,$$
$$\mathrm{Re}(\sigma_{yy}) \sim \omega^3.$$

Thus, we have a system that might be called a quantum Hall nematic: it has quantized Hall conductivity and longitudinal conductivities that vary differently at finite frequency!

Note, however, that the inclusion of disorder was crucial. This should be contrasted with the quantized Hall coefficient in the r>0 phase. Somehow, both disorder and interactions are crucial for this effect.

Relation to a Microscopic Theory

So far, our discussion has been purely in terms of an effective field theory. The parameters in our model are necessarily phenomenological. Perhaps a more microscopic realization would be useful?

Indeed there exists a more microscopic description of the quantum Hall effect. This is provided by the so-called Chern-Simons/Landau Ginzburg description of the Hall effect. It was developed by Zhang, Hansson, and Kivelson in the early 90s after motivating work by Girvin and MacDonald (a related fermionic theory was developed by Fradkin and Lopez).

Unfortunately, there is not enough time today to explain the precise relation.

However, one makes use of the CS/LG description of the Hall effect and relates this theory, via particle-vortex duality (Fisher, Fisher and Lee), to the "hydrodynamic theory we have been studying.

The relation between the parameters is the following:

$$\kappa^2 = 4c\rho' eV_0,$$
$$g^2 = 4\pi^2/eV_0,$$
$$|r| = 8|b|\rho' eV_0,$$

where ρ' is the density, V_0 is the pair potential, e is the unit of electric charge and b, c are two constants specifying certain terms in the QM kinetic energy.

Conclusion

- First, I reviewed various experiments by the Eisenstein group on a 2 DEG at filling fraction $\nu > 2$.
- I noted the possible existence of a new state with quantized Hall conductivity, but anisotropic DC longitudinal conductivity at finite temperature.
- Then I introduced a model, Lifshitz-Chern-Simons theory, whose transport at finite frequency and zero temperature shows, in the disordered regime, quantized Hall conductivity and anisotropic longitudinal conductivity.
- We are currently studying a more microscopic realization of the model that shares the same transport properties.
- We hope that we will be able to have a better understanding of the theory in order to know whether or not it is related to the above experiment, or is simply an interesting toy theory (waiting to be discovered!)