Entanglement entropy: hints from the two intervals case

Erik Tonni

MIT

based on

P. Calabrese, J. Cardy and E.T.; [0905.2069] (JSTAT)

P. Calabrese, J. Cardy and E.T.; [1011.5482]

E.T.; [1011.0166]

GGI, Firenze, October 2010

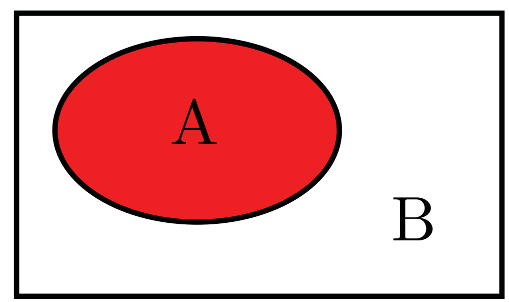
Plan of the talk

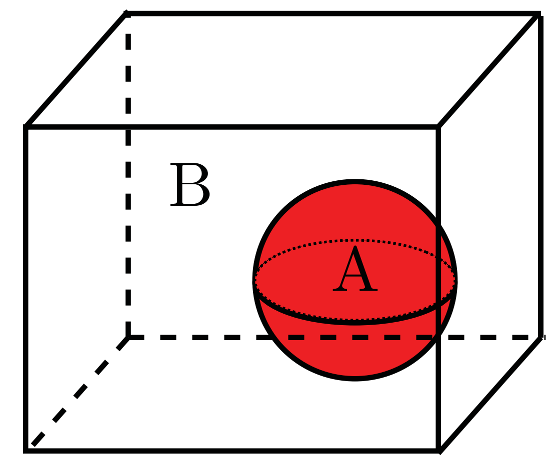
- Introduction and definition of the entanglement entropy
- Replica trick
- Twist fields
- Entanglement entropy of one interval in CFT
- Entanglement entropy of two disjoint intervals
  for the free compactified boson (Luttinger liquid)
  - Special cases and special regimes
  - Analytic continuation
  - Comparison with numerical data from XXZ spin chain
- Ising model
- Holographic entanglement entropy
- Conclusions and open problems

Entanglement entropy: definition

- Quantum system ($\mathcal{H}$) in the ground state $|\Psi\rangle$
  Density matrix $\rho = |\Psi\rangle\langle\Psi| \quad \Longrightarrow \quad \mathrm{Tr}\rho^n = 1$

- Two observers: each one measures only a subset of a complete set of cummuting observables $\qquad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$





- A's reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$

- Entanglement entropy $\equiv$ Von Neumann entropy of ρ_A

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

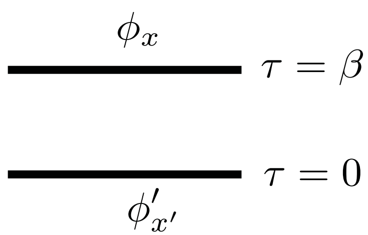
It measures the amount of information shared by A and B

Replica trick

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\, \rho_A^n$$

[Holzhey, Larsen, Wilczek, NPB (1994)]
[Calabrese, Cardy, JSTAT (2004)]

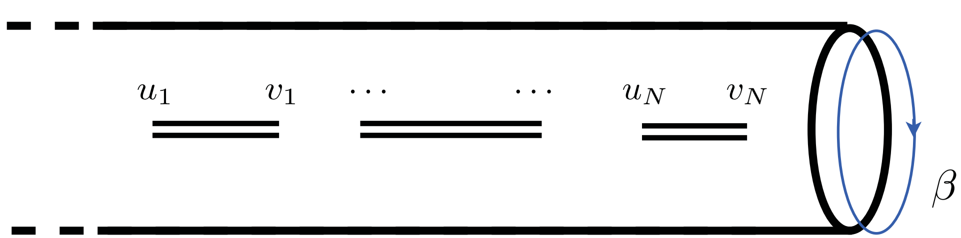
- QFT with Hamiltonian H.
  Density matrix ρ in a thermal state at temperature $T = 1/\beta$



$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) = Z^{-1} \int [d\phi(y,\tau)] \prod_{x'} \delta(\phi(y,0) - \phi'_{x'}) \prod_{x} \delta(\phi(y,\beta) - \phi_x)\, e^{-S_E}$$

$Z = \mathrm{Tr}\, e^{-\beta H}$. The trace sews together the edges at $\tau = 0$ and $\tau = \beta$ providing a cylinder with circumference of length β.

- $\rho_A = \mathrm{Tr}_B \rho$



$A = (u_1, v_1) \cup \cdots \cup (u_N, v_N)$

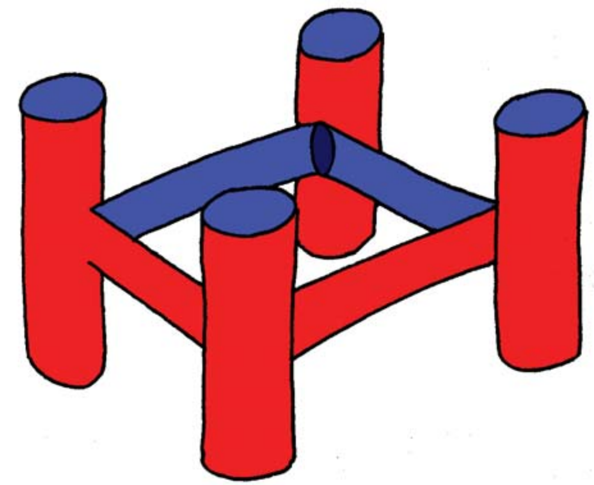
The trace Tr_B sews together only the points $\notin A$.
Open cuts are left along the disjoint intervals (u_j, v_j).

Replica trick and Riemann surfaces

- n copies of the cylinder above
  sewed together cyclically along the cuts

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$$

$$\text{"} \rho_A^{ij}\, \rho_A^{jk}\, \rho_A^{kl}\, \rho_A^{li} \text{"} =$$



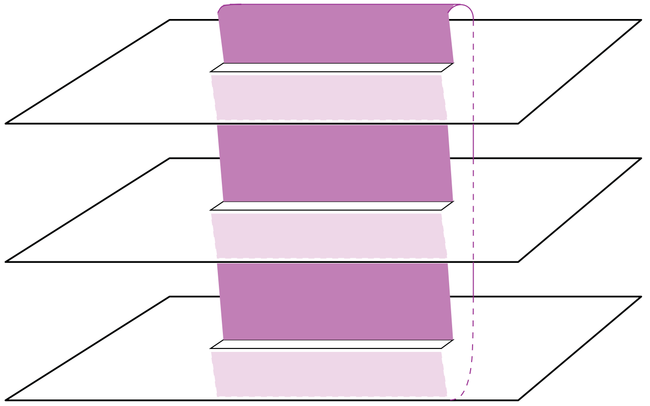
- $\mathrm{Tr}\, \rho_A^n$ as a partition function
  on the n sheeted Riemann surface $\mathcal{R}_{n,N}$

$$Z_{\mathcal{R}_{n,N}} = \int_{\mathcal{C}_{u_j,v_j}} [d\varphi_1 \cdots d\varphi_n]_{\mathbf{C}} \exp\left[-\int_{\mathbf{C}} dz d\bar{z} \left(\mathcal{L}[\varphi_1](z,\bar{z}) + \ldots + \mathcal{L}[\varphi_n](z,\bar{z})\right)\right]$$

$$\mathcal{C}_{u_j,v_j} : \quad \varphi_i(x, 0^+) = \varphi_{i+1}(x, 0^-)$$

$$x \in \bigcup_{j=1}^{N} [u_j, v_j] \qquad i = 1, \ldots, n$$

$\mathcal{R}_{3,1}$



[Cardy, Castro-Alvaredo, Doyon, JSP (2007)]

Twist fields

- Global symmetry

$$\sigma \ :\ \ i \mapsto i+1 \ \mathrm{mod}\, n \qquad \sigma^{-1} \ :\ \ i+1 \mapsto i \ \mathrm{mod}\, n$$

$$\int dxdy\, \mathcal{L}[\sigma\varphi](x,y) = \int dxdy\, \mathcal{L}[\varphi](x,y)$$

- The twist fields implement this global symmetry

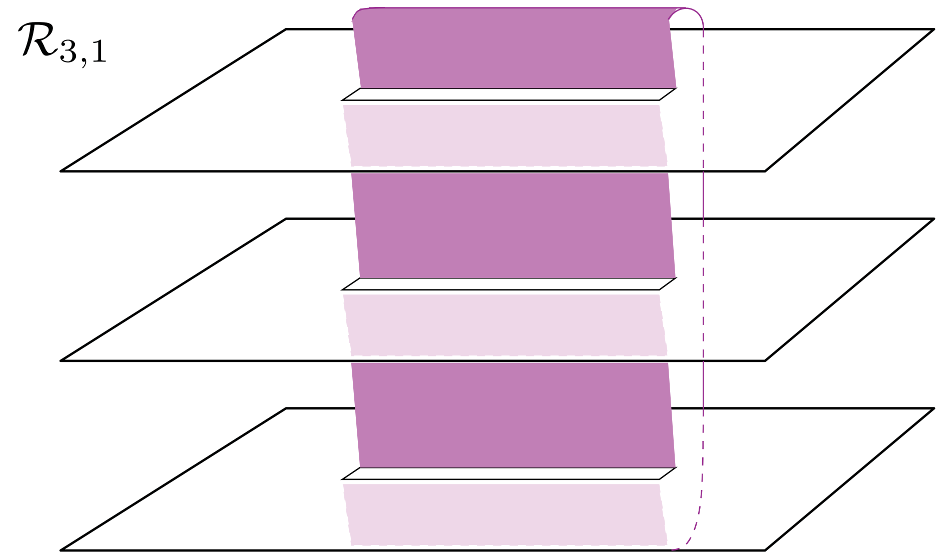
$$\mathcal{T}_n \equiv \mathcal{T}_\sigma \qquad \tilde{\mathcal{T}}_n \equiv \mathcal{T}_{\sigma^{-1}}$$

$$Z_{\mathcal{R}_{n,N}} = \langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\cdots\mathcal{T}_n(u_N,0)\tilde{\mathcal{T}}_n(v_N,0)\rangle_{\mathcal{L}^{(n)},\mathbf{C}}$$

$$\mathcal{T}_n = \prod_{k=0}^{n-1} \mathcal{T}_{n,k} \qquad \tilde{\mathcal{T}}_n = \prod_{k=0}^{n-1} \tilde{\mathcal{T}}_{n,k}$$

$$Z_{\mathcal{R}_{n,N}} = \prod_{k=0}^{n-1} \langle \mathcal{T}_{n,k}(u_1,0)\tilde{\mathcal{T}}_{n,k}(v_1,0)\cdots\mathcal{T}_{n,k}(u_N,0)\tilde{\mathcal{T}}_{n,k}(v_N,0)\rangle_{\mathcal{L}^{(n)},\mathbf{C}}$$

Boundary conditions and twist fields



- Boundary conditions:

$$\varphi_j(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \varphi_{j-1}(z, \bar{z})$$

- Linear combinations of basic fields which diagonalize the twist

$$\tilde{\varphi}_k \equiv \sum_{j=1}^{n} e^{2\pi i \frac{k}{n} j} \varphi_j$$

$$k = 0, 1, \ldots, n-1$$

$$\tilde{\varphi}_k(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = e^{2\pi i \frac{k}{n}} \tilde{\varphi}_k(z, \bar{z}) = \theta_k \tilde{\varphi}_k(z, \bar{z}) \qquad \theta_k \equiv e^{2\pi i \frac{k}{n}}$$

- Branch-point twist field $\mathcal{T}_{n,k}$ in the origin

[Dixon, Friedan, Martinec, Shenker, NPB (1987)] [Zamolodchikov, NPB (1987)]

Entanglement of a single interval

- Two-point function of twist fields for a free complex boson φ

[Dixon, Friedan, Martinec, Shenker, NPB (1987)]

$$\langle \mathcal{T}_{k,n}(u)\tilde{\mathcal{T}}_{k,n}(v)\rangle \propto \frac{1}{|u-v|^{4\Delta_{k/n}}}$$

$$\Delta_{\frac{k}{n}} = \bar{\Delta}_{\frac{k}{n}} = \frac{1}{2}\frac{k}{n}\left(1-\frac{k}{n}\right)$$

- Partition function on $\mathcal{R}_{n,1}$

$$Z_{\mathcal{R}_{n,1}} = \prod_{k=0}^{n-1} Z_{k,n} = \prod_{k=0}^{n-1}\langle \mathcal{T}_{k,n}(u)\tilde{\mathcal{T}}_{k,n}(v)\rangle = \frac{c_n}{|u-v|^{\frac{1}{3}\left(n-\frac{1}{n}\right)}}$$

Entanglement entropy of a single interval for the free real boson

$c = 1$

$$S_A = -\,\partial_n \mathrm{Tr}\rho_A^n\big|_{n=1} = \frac{1}{3}\log\frac{\ell}{a} + c_1'$$

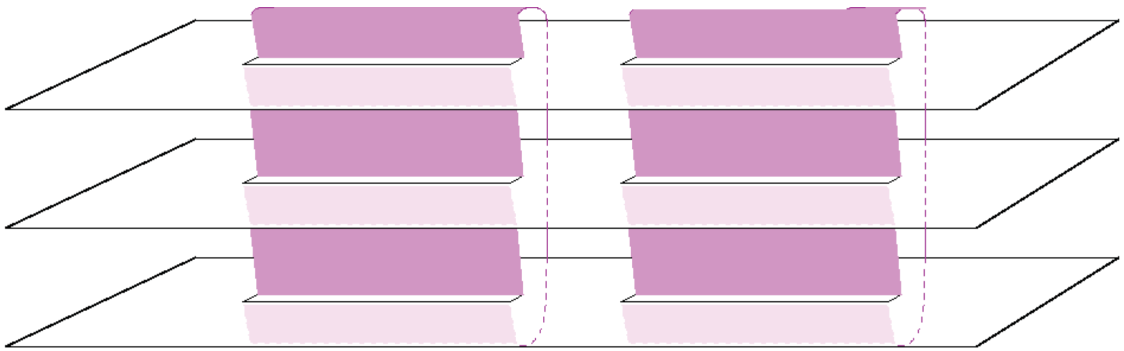
[Holzhey, Larsen, Wilczek, NPB (1994)]

Entanglement of two disjoint intervals

[Calabrese, Cardy and E.T.; JSTAT (2009)]

- $A = A_1 \cup A_2 = [u_1, v_1] \cup [u_2, v_2] \quad \Longrightarrow \quad \mathcal{R}_{n,2}$

e.g.: $\mathcal{R}_{3,2}$



- Four-point function of twist fields for a free, real, compactified boson φ

$$\mathrm{Tr}\rho_A^n \equiv Z_{\mathcal{R}_{n,2}} = c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)} \mathcal{F}_n(x)$$

$$x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)}$$

$Z^W_{\mathcal{R}_{n,2}}$

[Calabrese, Cardy, JSTAT (2004)]

Computation of $\mathcal{F}_n(x)$ (I)

- Compactification condition

$$\varphi_j(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \varphi_{j-1}(z, \bar{z}) + R(m_{j,1} + i m_{j,2}) \qquad m_j \in \mathbf{Z} + i\mathbf{Z}$$

$$\tilde{\varphi}_k(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \theta_k \tilde{\varphi}_k(z, \bar{z}) + \boxed{R \sum_{j=1}^{n} \theta_k^j m_j} \rightarrow \xi \in R\Lambda_{\frac{k}{n}} \qquad \theta_k \equiv e^{2\pi i \frac{k}{n}}$$

- Partition function on $\mathcal{R}_{n,2}$ from the four-point function of twist fields

$$Z_{\mathcal{R}_{n,2}} = \sum_{m \in \mathbf{Z}^{2n}} \prod_{k=0}^{n-1} Z_{k,n}^{\mathrm{qu}} Z_{k,n}^{\mathrm{cl}} \rightarrow$$

[Dixon, Friedan, Martinec, Shenker, NPB (1987)]

$$\mathcal{F}_n(x) = \sum_{m \in \mathbf{Z}^{2n}} \prod_{k=0}^{n-1} \frac{\mathrm{const}}{\beta_{k/n} \left[F_{k/n}(x)\right]^2} \exp\left\{ -\frac{2g\pi \sin\left(\pi \frac{k}{n}\right)}{n} \left[|\xi_1|^2 \beta_{k/n} + \frac{|\xi_2|^2}{\beta_{k/n}} \right] \right\}$$

$$\beta_y \equiv \frac{F_y(1-x)}{F_y(x)} \qquad F_y(x) \equiv {}_2F_1(y, 1-y; 1; x)$$

- Z^{cl} does not contribute in the decompactification limit

Computation of $\mathcal{F}_n(x)$ (II)

- $$\mathcal{F}_n(x) = \frac{\text{const}}{\prod_{k=0}^{n-1} \beta_{k/n} \left[F_{k/n}(x)\right]^2} \left[\sum_{m\in \mathbf{Z}^n} \exp\left\{ i\,\pi \left[m^{\mathrm{t}} \cdot \Omega \cdot m + m^{\mathrm{t}} \cdot \widetilde{\Omega} \cdot m \right] \right\}\right]^2$$

$$\Omega_{rs} \equiv 2gR^2\, \frac{i}{n} \sum_{k=0}^{n-1} \sin\left(\pi \frac{k}{n}\right) \beta_{\frac{k}{n}} \cos\left[2\pi \frac{k}{n}(r-s)\right] \qquad \widetilde{\Omega}_{rs} \equiv 2gR^2\, \frac{i}{n} \sum_{k=0}^{n-1} \sin\left(\pi \frac{k}{n}\right) \frac{1}{\beta_{\frac{k}{n}}} \cos\left[2\pi \frac{k}{n}(r-s)\right]$$

$$r, s = 1, \ldots, n$$

- Regularize the sum by eliminating the contribution of the eigenvalue generating the kernel of both Ω and $\widetilde{\Omega}$ (non trivial step!)

- Riemann-Siegel theta function

$$\Theta(z|\Gamma) \equiv \sum_{m\in \mathbf{Z}^G} \exp\left[i\pi\, m^{\mathrm{t}} \cdot \Gamma \cdot m + 2\pi i\, m^{\mathrm{t}} \cdot z \right] \qquad z \in \mathbb{C}^G$$

  - Γ is a symmetric, $G \times G$ matrix with positive imaginary part

Computation of $\mathcal{F}_n(x)$: main result

$$\Gamma_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \beta_{k/n} \cos\left[2\pi \frac{k}{n}(r-s)\right] \qquad \widetilde{\Gamma}_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \frac{1}{\beta_{k/n}} \cos\left[2\pi \frac{k}{n}(r-s)\right]$$

$r, s = 1, \dots, n-1$

$$\eta \equiv gR^2$$

$$\mathcal{F}_n(x) = \text{const} \frac{[\Theta(0|\eta\Gamma)\,\Theta(0|\eta\widetilde{\Gamma})]^2}{\prod_{k=1}^{n-1} F_{k/n}(x) F_{k/n}(1-x)}$$

nasty
n dependence

→ $\mathcal{F}_n(x)$ is invariant under $x \leftrightarrow 1-x$

- Fix the constant s.t. $\mathcal{F}_n(0) = 1$
- Riemann-Siegel theta function manipulations
- Final result

$$\mathcal{F}_n(x) = \left[\frac{\Theta(0|\eta\Gamma)\,\Theta(0|\Gamma/\eta)}{\Theta(0|\Gamma)^2}\right]^2$$

→ $\mathcal{F}_n(x)$ is invariant under $\eta \leftrightarrow 1/\eta$

Special cases

- $n = 2$ [Furukawa, Pasquier, Shiraishi, PRL (2009)] [Zamolodchikov, NPB (1987)]

$$\mathcal{F}_2(x) = \left[\frac{\theta_3(\tau_{1/2}\eta)\theta_3(\tau_{1/2}/\eta)}{\theta_3^2(\tau_{1/2})}\right]^2 \qquad \tau_{1/2} = i\beta_{1/2}$$

- $n = 3$

$$\Gamma = \frac{\tau_{1/3}}{\sqrt{3}}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \gamma = \sqrt{3}\,\tau_{1/3}$$

$$\mathcal{F}_3(x) = \frac{1}{4[F_{1/3}(x)]^4}\left[\theta_2(\eta\gamma)^2\theta_2\left(\frac{\eta\gamma}{3}\right)^2 + \theta_3(\eta\gamma)^2\theta_3\left(\frac{\eta\gamma}{3}\right)^2 + \theta_4(\eta\gamma)^2\theta_4\left(\frac{\eta\gamma}{3}\right)^2\right]$$
$$\times\left[\theta_2\left(\frac{\gamma}{\eta}\right)^2\theta_2\left(\frac{\gamma}{3\eta}\right)^2 + \theta_3\left(\frac{\gamma}{\eta}\right)^2\theta_3\left(\frac{\gamma}{3\eta}\right)^2 + \theta_4\left(\frac{\gamma}{\eta}\right)^2\theta_4\left(\frac{\gamma}{3\eta}\right)^2\right]$$

- $\ldots$

Special regimes and a generalization

- decompactification regime: large η (recall the symmetry $\eta \leftrightarrow 1/\eta$)

$$\mathcal{F}_n(x) = \frac{\eta^{n-1}}{\prod_{k=1}^{n-1} F_{k/n}(x) F_{k/n}(1-x)}$$

In this regime we can perform the analytic continuation $n \to 1$.

- $x \to 0$ regime

$$\mathcal{F}_n(x) = 1 + x^{\min(\eta,1/\eta)} \sum_{l=1}^{n-1} \frac{2(n-l)}{\left[2n \sin\left(\pi \frac{l}{n}\right)\right]^{2\min(\eta,1/\eta)}} + \dots$$

- different compactification radii

$$\xi_p = \sum_{l=0}^{n-1} \theta_k^l \left(R_1 m_{l,1}^{(p)} + i\, R_2 m_{l,2}^{(p)} \right)$$

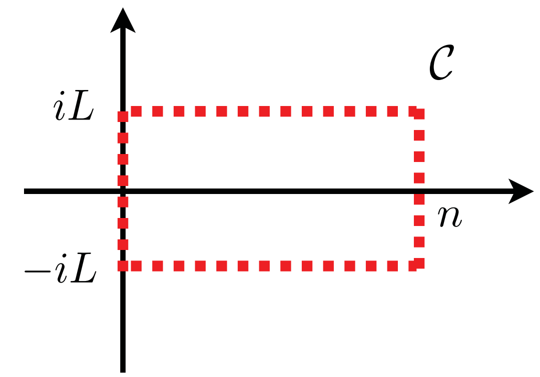
$$\mathcal{F}_n(x) = \left[\frac{\Theta(0|\eta_1\Gamma)\,\Theta(0|\Gamma/\eta_1)}{\Theta(0|\Gamma)^2}\right] \left[\frac{\Theta(0|\eta_2\Gamma)\,\Theta(0|\Gamma/\eta_2)}{\Theta(0|\Gamma)^2}\right]$$

Analytic continuation

- decompactification regime: large η (recall the symmetry $\eta \leftrightarrow 1/\eta$)

$$\mathcal{F}_n(x) = \frac{\eta^{n-1}}{\prod_{k=1}^{n-1} F_{k/n}(x) F_{k/n}(1-x)}$$

- Useful representation:



$$D_n(x) = \sum_{k=1}^{n-1} \log F_{k/n}(x) = \int_{\mathcal{C}} \frac{dz}{2\pi i} \pi \cot(\pi z) \log F_{z/n}(x)$$

$$D_1'(x) \equiv - \left.\frac{\partial D_n(x)}{\partial n}\right|_{n=1} = \int_{-i\infty}^{i\infty} \frac{dz}{i} \frac{\pi z}{\sin^2 \pi z} \log F_z(x)$$

- **Mutual information**

$$I_{A_1:A_2} \equiv S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$$

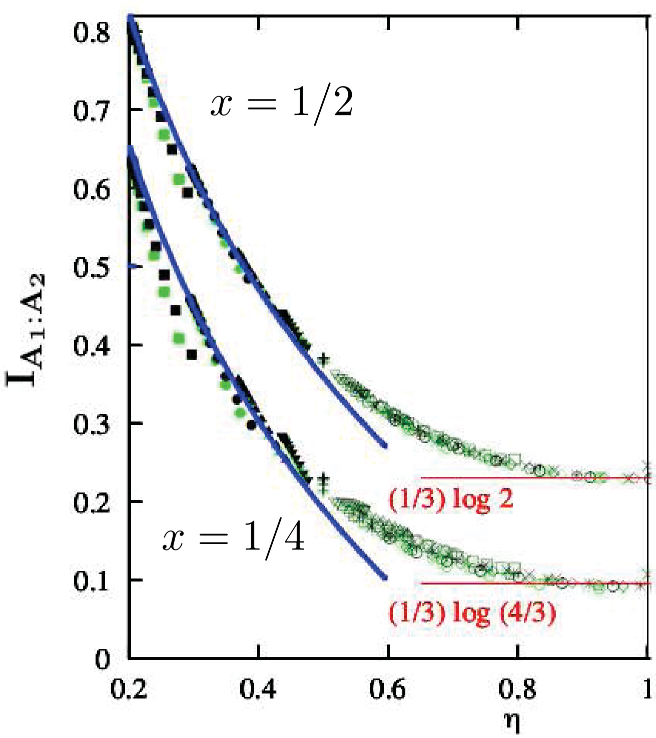
$$I_{A_1:A_2}(\eta \ll 1) - I^W_{A_1:A_2} \simeq -\frac{1}{2} \ln \eta + \frac{D_1'(x) + D_1'(1-x)}{2}$$

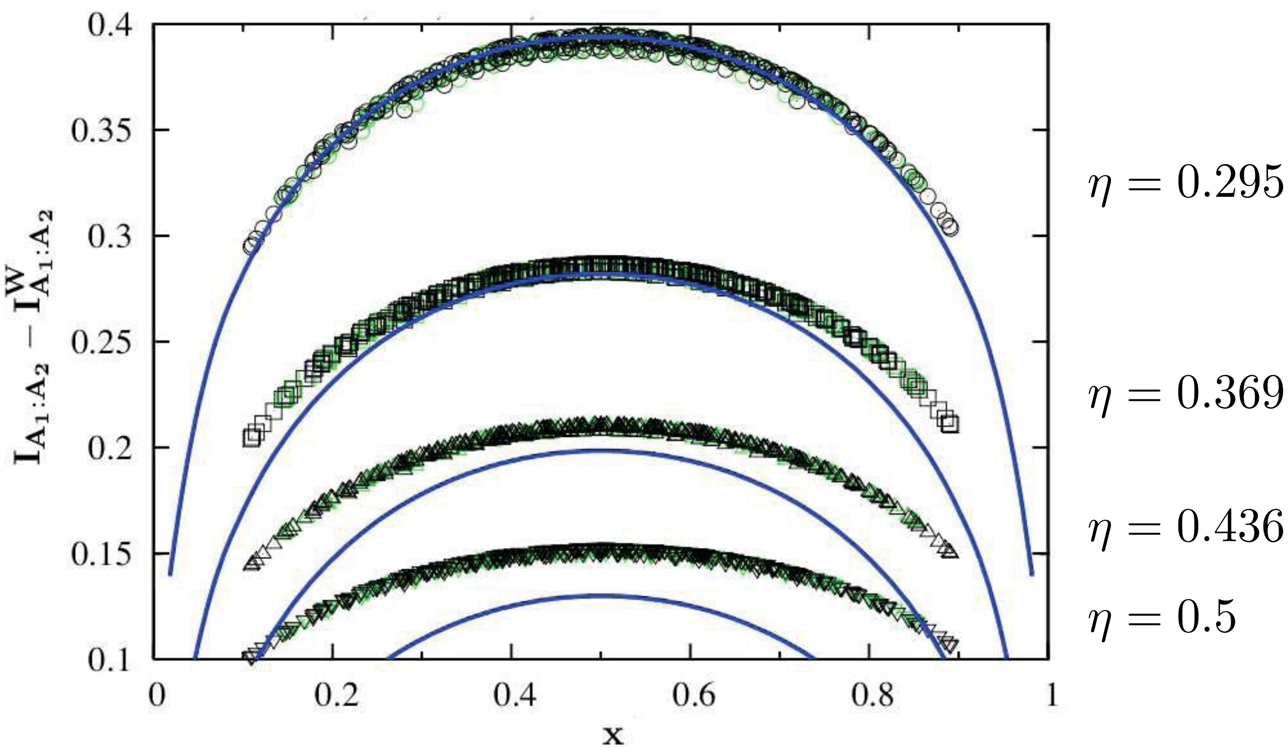
Comparison with the numerical data

- Exact diagonalization of the XXZ spin chain in a magnetic field (up to $L = 30$) [Furukawa, Pasquier, Shiraishi, PRL (2009)]

$$H \equiv \sum_{j=1}^{L} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta\, S_j^z S_{j+1}^z - h\, S_j^z \right)$$

$$\left.\begin{array}{l} \Delta \in (-1, 1] \\ h = 0 \end{array}\right\} \eta = 1 - \frac{1}{\pi} \arccos \Delta$$





confirms the formula

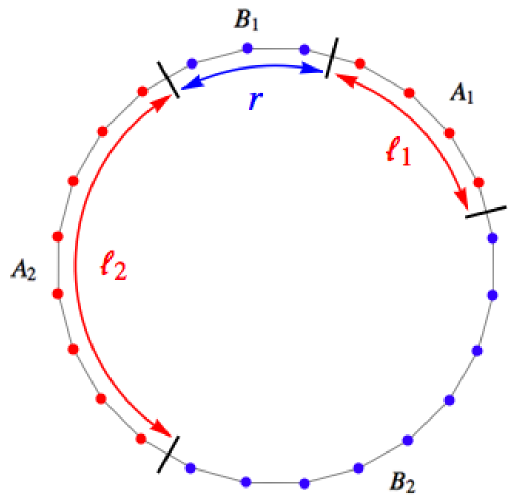
$$I_{A_1:A_2}(\eta \ll 1) - I^W_{A_1:A_2} \simeq -\frac{1}{2} \ln \eta + \frac{D_1'(x) + D_1'(1-x)}{2}$$

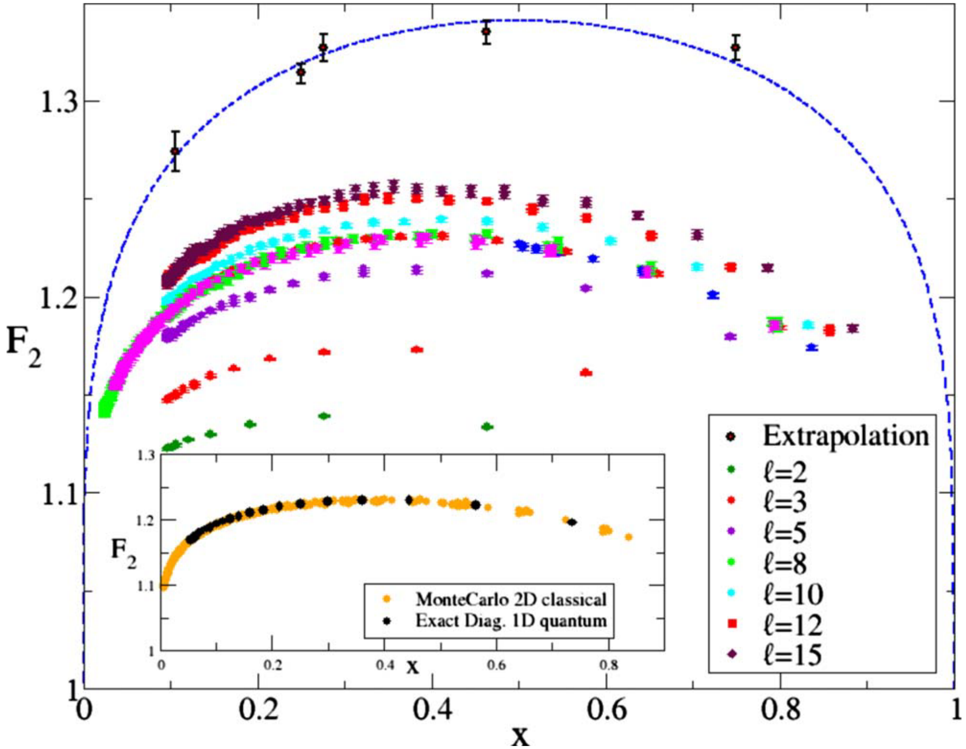
Ising model: 2 sheets

$$H_{XY} \equiv -\sum_{j=1}^{L} \left(\frac{1+\gamma}{4}\, \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{4}\, \sigma_j^y \sigma_{j+1}^y + \frac{h}{2}\, \sigma_j^z \right)$$

$\gamma =$ anisotropy $\begin{cases} 1 & \text{Ising model} \\ 0 & \text{XX model} \end{cases}$

$h =$ magnetic field





[Alba, Tagliacozzo and Calabrese; PRB (2010)]

$$\mathcal{F}_2(x) = \frac{1}{\sqrt{2}} \left\{ \left[\frac{(1+\sqrt{x})(1+\sqrt{1-x})}{2} \right]^{1/2} + x^{1/4} + [x(1-x)]^{1/4} + (1-x)^{1/4} \right\}$$

Ising model

[Calabrese, Cardy and E.T.; [1011.5482]]

$$\mathcal{F}_n(x) = \frac{1}{2^{n-1}\Theta(0|\Gamma)} \sum_{\varepsilon,\delta} \left| \Theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (0|\Gamma) \right|$$

$$\Gamma_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \beta_{k/n} \cos\left[2\pi \frac{k}{n}(r-s)\right] \qquad \beta_y \equiv \frac{F_y(1-x)}{F_y(x)} \qquad F_y(x) \equiv {}_2F_1(y, 1-y; 1; x)$$

$$r, s = 1, \dots, n-1$$

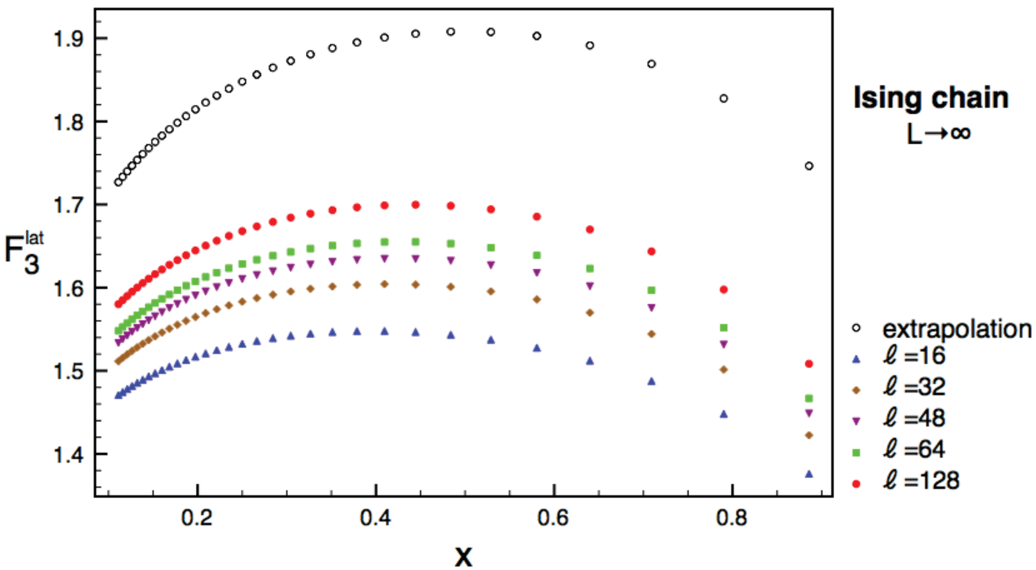
- Riemann-Siegel theta function with characteristic

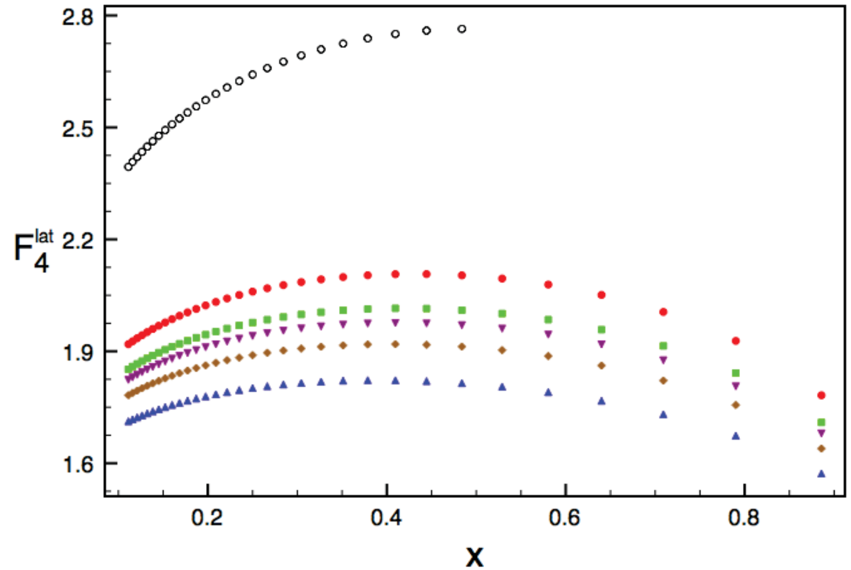
$$\Theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (z|\Gamma) \equiv \sum_{m \in \mathbf{Z}^G} \exp\left[\, i\pi\, (m+\varepsilon)^{\mathrm{t}} \cdot \Gamma \cdot (m+\varepsilon) + 2\pi i\, (m+\varepsilon)^{\mathrm{t}} \cdot (z+\delta) \,\right]$$

- ε and δ are vectors with $n-1$ elements which are either 0 or $1/2$
- $\mathcal{F}_n(x)$ is invariant under $x \leftrightarrow 1-x$

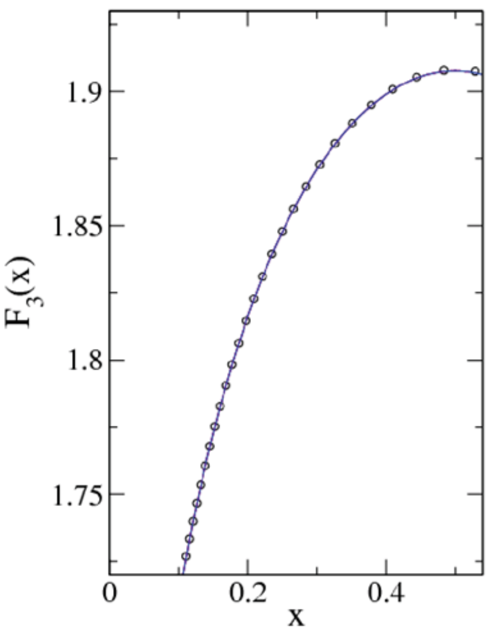
Ising model: 3,4, ... sheets

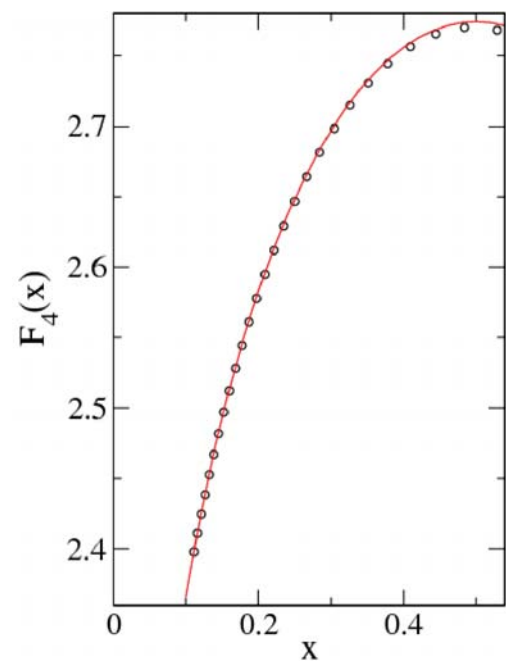
[Fagotti, Calabrese; JSTAT (2010)]





[Calabrese, Cardy and E.T.; [1011.5482]]



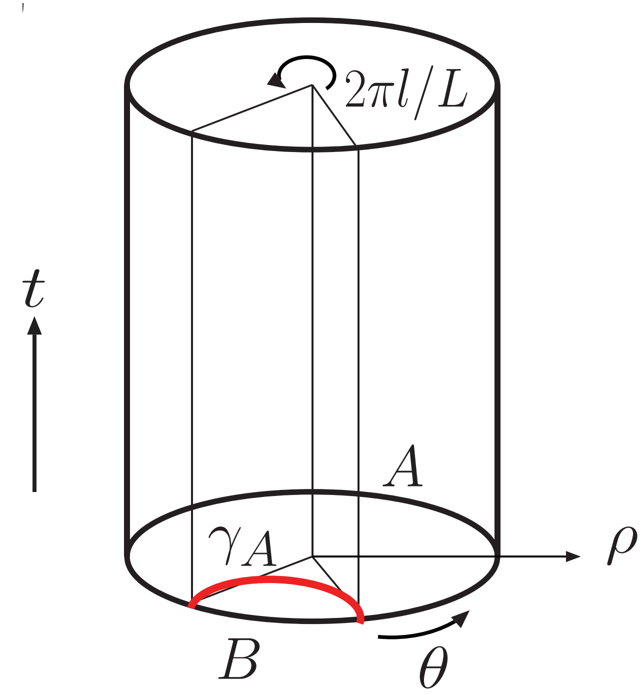


Holographic entanglement entropy

AdS_{d+2}/CFT_{d+1} correspondence

[Ryu, Takayanagi, PRL (2006), JHEP (2006)]

- Prescription: in regularized AdS_{d+2}
  - Find the *minimal area* surface γ_A s.t. $\partial\gamma_A = \partial A$
  - $$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}$$



- $d = 1$ formula $S_A = (c/3)\log(l/a)$ and the area law

[Bombelli, Koul, Lee, Sorkin, PRD (1986)]
[Srednicki, PRL (1993)]

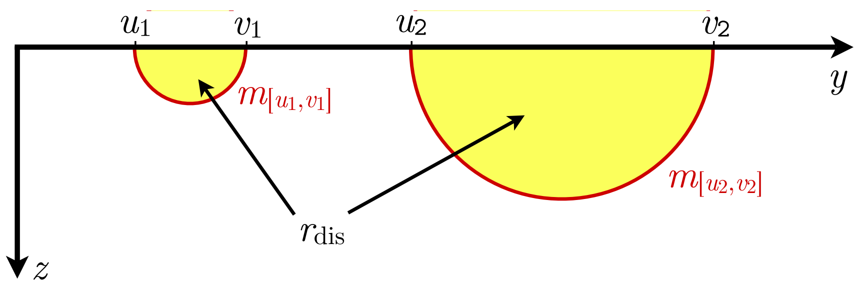
$$S_A \propto \frac{\text{Area}(\partial A)}{a^{d-1}}$$

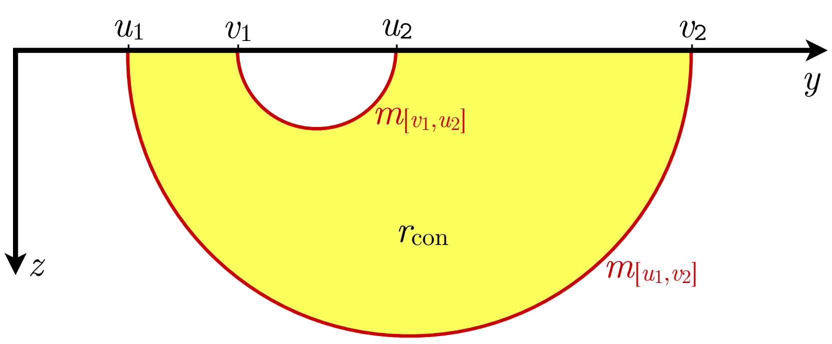
are recovered.

Transition in the holographic mutual information

- The holographic prescription predicts a transition for the mutual information

[Headrick; 1006.0047]

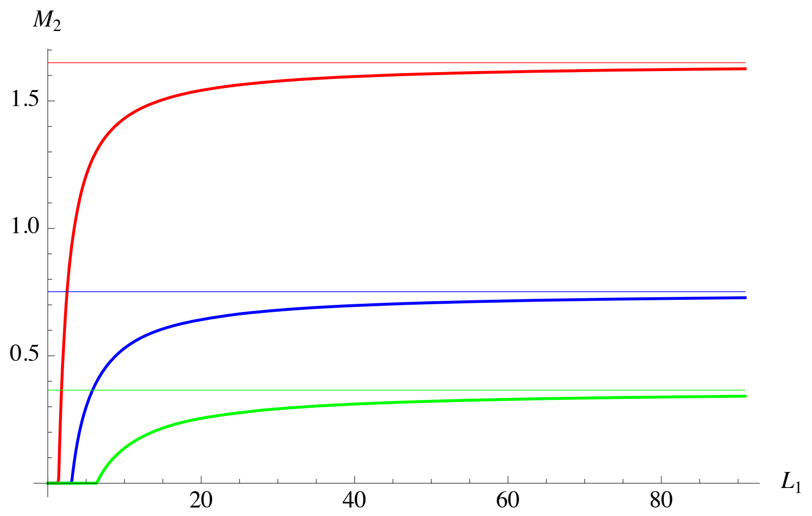


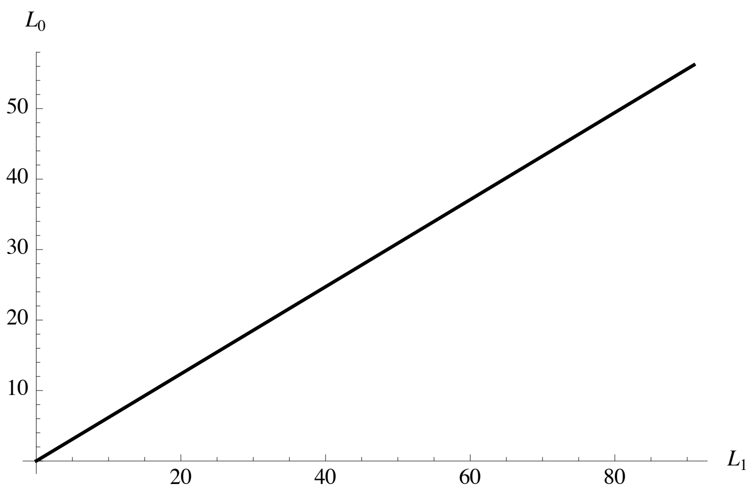


- AdS_{d+2}

$$S_d(L_1, L_2; L_0) \equiv \min\Big[\underbrace{\tilde{A}_d(L_1) + \tilde{A}_d(L_2)}_{\text{disconnected surface}}\,;\,\underbrace{\tilde{A}_d(L_0) + \tilde{A}_d(L_1 + L_0 + L_2)}_{\text{connected surface}}\Big]$$

[E.T.; [1011.0166]]



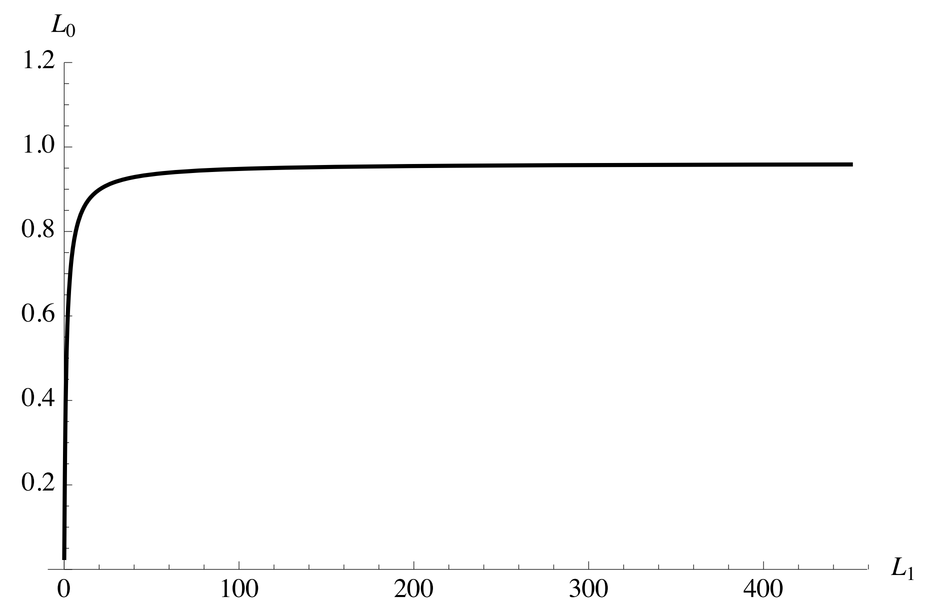


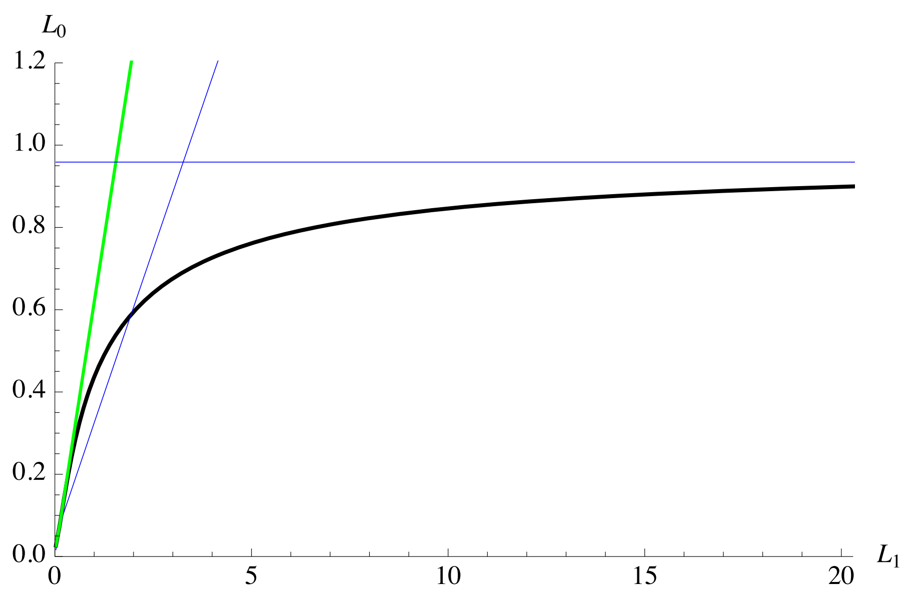
Holographic mutual information: charged black hole

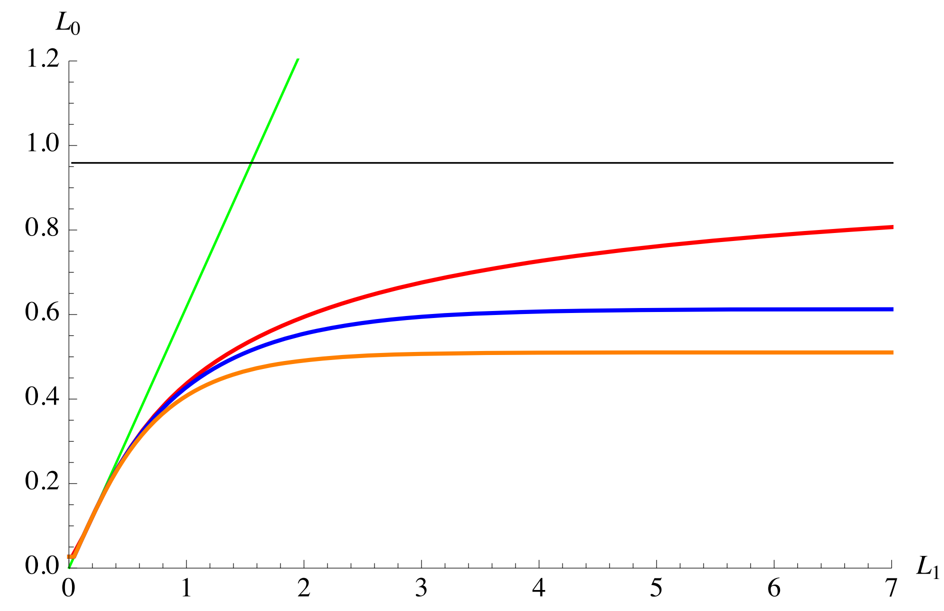
- $$\frac{ds^2}{R^2} = \frac{-f\,dt^2 + d\vec{x}^2}{z^2} + \frac{dz^2}{f\,z^2}$$

  $$f = 1 + Q^2\left(\frac{z}{R^2}\right)^{2d} - M\left(\frac{z}{R^2}\right)^{d+1}$$

- Transition curve for the mutual information when $L_1 = L_2$







About the N interval case

- The following form is expected:

$$\mathrm{Tr}\,\rho_A^n = \underbrace{c_n^N \left(\frac{\prod_{j<k}(u_k - u_j)(v_k - v_j)}{\prod_{j,k}(v_k - u_j)} \right)^{(c/6)(n-1/n)}}_{Z^W_{\mathcal{R}_{n,N}}} \mathcal{F}_{n,N}(\{x\})$$

[Calabrese, Cardy, JSTAT (2004)]

?

$\{x\}$ is the set of $2N-3$ independent ratios.

- Holographic prescription provides $Z^W_{\mathcal{R}_{n,N}}$.

Conclusions and open issues

Free compactified boson

- Two disjoint intervals: formula for $\mathrm{Tr}\rho_A^n$ found for all the parameters of the model (n, η and x).
- Mutual information in the decompactification regime (checked against numerical data from the XXZ spin chain)

Ising model

- Two disjoint intervals: formula for $\mathrm{Tr}\rho_A^n$

- Analytical continuation for $n \to 1$ of $\mathcal{F}_n(x)$
  $\Longrightarrow$ Mutual information for any value of the parameters
- Presence of boundaries
- Generalization to $N > 2$ intervals
- Holographic computation of $I_{A_1:A_2}$

Thank you!