(A Holographic fractional) topological insulator

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Outline

- Small review of topological insulators
- Models for fractional topological insulators
- Holographic model of FTI
Why are topological insulators interesting?

**Experiment:**
- New class of materials
- Effective 2+1 massless Dirac fermions at the surface
- Realized in the lab: $\text{Bi}_{1-x}\text{Sb}_x$, $\text{Bi}_2\text{Te}_3$, $\text{Bi}_2\text{Se}_3$, $\text{Sb}_2\text{Tb}_3$
- Unusual Quantum Hall Effect (half-integer)
- Robust against disorder

**Theory:**
- Topological classification of theories with fermions (of which 3+1 topological insulators are an example)
- Some theories related by dimensional reduction
- Map to stable D-brane configurations (K-theory)

Review: [Hasan & Kane, 1002.3895]
Figure: An insulator for a condensed matter physicist (experimentalist)

\[
\mathcal{L}_M = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \overline{\psi} (i \gamma^\mu D_\mu - M) \psi
\]

Equation: Simplified version for theorists
Topological phases of insulators

- Consider $2+1$ massless fermions in a magnetic field ($\mathcal{T}, \mathcal{C}$ broken)
- Gapped Landau levels (insulator)
- Gapless states at the boundary
- Integer Quantum Hall Effect $\sigma_{xy} = n\frac{e^2}{h}$
  ⇒ topologically inequivalent to normal insulators (where $\sigma_{xy} = 0$)

Figure: Hall resistivity as a function of the magnetic field
Consider an insulator in 3+1 dimensions:

\[ \mathcal{L}_M = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \overline{\psi} (i \gamma^\mu D_\mu - M) \psi \]

We can make \( M \) complex

\[ M \overline{\psi} \psi \rightarrow \overline{\psi} |M| e^{i \gamma^5 \phi} \psi = |M| \cos \phi \overline{\psi} \psi + i |M| \sin \phi \overline{\psi} \gamma^5 \psi \]

Under \( \mathcal{T} \): \( \overline{\psi} \psi \) even, \( \overline{\psi} \gamma^5 \psi \) odd

Time reversal invariance requires \( \phi = 0, \pi \) (real mass)

Normal insulator \( \phi = 0 \)

Topological insulator \( \phi = \pi \)

Topologically protected by \( \mathcal{T} \):

\( |M| \rightarrow 0 \) in order to change the phase of the mass
Topological Effective Theories

- An axial rotation $\psi \rightarrow e^{-i\gamma^5\phi/2} \psi$ shifts $M \rightarrow e^{-i\gamma^5\phi} M$
- The mass is now real and positive, but the ABJ anomaly gives a theta term to the action $\theta = \phi$.
- After integrating out the fermions the effective theory is

$$\mathcal{L}_{TET} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 \theta}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

- The $\theta$-term $\sim \theta \mathbf{E} \cdot \mathbf{B}$ is $\mathcal{T}$-odd.
- BUT: the quantum theory is invariant under $\theta \rightarrow \theta + 2\pi$
- Two $\mathcal{T}$-symmetric values: $\theta = 0, \pi$

- Normal insulator: $\theta = 0$
- Topological insulator: $\theta = \pi$
Massive Dirac fermions with space-dependent mass $M(x)$

$$M(x) = \begin{cases} \text{positive, } & x > 0 \\ \text{negative, } & x < 0 \end{cases}$$

$\mathcal{T}$-symmetry implies $M$ is real, so $M(x = 0) = 0$

Massless fermions at the interface:
- 2+1 dimensions: Operator $\bar{\psi}\psi$: $\mathcal{T}$-odd
- $\mathcal{T}$-symmetric mass needs two flavors: $m(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2)$
- Normal/Topological insulator: odd number of flavors at the interface
- Robustness, $\mathcal{T}$-symetric deformations do not lift massless excitations
Gauge theory with space-dependent theta term (axion) $\theta(x)$

$$\theta(x) = \begin{cases} 
0 , & x > 0 \\
\Delta \theta , & x < 0 
\end{cases}$$

In the $T$-symmetric case $\Delta \theta = 0, \pi$. The 3+1 eom’s are

$$\partial_\mu F^{\mu\nu} + \frac{e^2}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} \partial_\mu \theta \partial_\alpha A_\beta = 0$$

Since $\partial_\nu \theta = \Delta \theta \delta(x) \delta^x_\nu$, the effective 2+1 theory at the interface is

$$\mathcal{L}_k = \frac{1}{4} F_{ij} F^{ij} - \frac{e^2 k}{4\pi} \epsilon^{ijk} A_i F_{jk}$$

With a Chern-Simons level $k = \frac{\Delta \theta}{2\pi} = \frac{1}{2} \Rightarrow \sigma_{xy} = ke^2/h$.

$\Rightarrow$ half-integer QHE
Effective theory:

- Gauge theory: $U(1)_{\text{em}} \times SU(N_c)$, $N_c$ odd (baryon is a fermion)
- $F_{\mu\nu} = \frac{e}{N_c} F_{\mu\nu} 1 + gf_{\mu\nu} T^a$
- Quarks: fundamental representation, charge $q = e/N_c$
- Baryon: charge $e$, composite electron

ABJ axial anomaly: $\mathcal{L}_\theta = \frac{\theta}{32\pi^2} \text{Tr} \left( \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right)$

- Symmetry $\theta \rightarrow \theta + 2\pi$
- $U(1)$ part: $\mathcal{L}_\theta = N_c \frac{e^2}{N_c^2} \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$
- $\theta_{\text{eff}} = \frac{\theta}{N_c}$: $\mathcal{T}$-symmetric values $\theta_{\text{eff}} = 0, \pi/N_c, \cdots$
- Interface: Chern-Simons level $k = 1/(2N_c), \cdots$
- Hall conductivity $\sigma_{xy} = \frac{1}{2N_c} \frac{e^2}{h}$

[Maciejko, Qi, Karch & Zhang, 1004.3628]
Why an AdS/CFT model?

A lot is known about weakly coupled TIs: Complete classification using free fermions, there are no fractional TIs

[Schnyder, Ryu, Furusaki & Ludwig 2008-10; Kitaev 2009]

- Non-Abelian gauge theories could be used as models
- Problem # 1: Charge carriers with fractional charge should be deconfined  [Swingle,Barkeshli,McGreevy,Senthil,1005.1076]
- Solution # 1: Use a CFT, like $\mathcal{N} = 4$ SYM
- We could use a weakly coupled description, like field theories on D-branes
- Problem # 2: A model describing an interface breaks conformal invariance (and supersymmetry)
- With D-branes one needs to bend or rotate the D-brane: generically unstable, non-perturbative problems in the field theory?
- Solution # 2: Use AdS/CFT with probe branes, the configuration is fixed by the boundary conditions
**Holographic model**

- $\mathcal{N} = 4 \ SU(N_c) \ SYM$: gauge theory $\rightarrow$ $AdS_5 \times S^5$ dual

- Hypermultiplet in the fundamental representation: quasiparticles/quarks $\rightarrow$ probe D7 brane

- $U(1)$ global flavor symmetry:
  $U(1)_{em}$ weakly coupled $\rightarrow$ $U(1)$ gauge field on the D7

- Charge quasiparticles $q_p = e \Rightarrow$ charge electron $q_e = N_c e$

- Chern-Simons level $k = N_c/2$ (equivalent)
Holographic model

Interface between normal and topological insulator

\[ ds^2 = (r^2 + \rho^2) \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2 + r^2 d\Omega_3^2 + d\rho^2 + \rho^2 d\phi^2}{r^2 + \rho^2} \]

D7 embedding: wraps \( \{x^\mu, \Omega_3, r\} \) directions. Profile \( \rho(r, x) \),

- \( \lim_{r \to \infty} \rho(r, x) \simeq M(x) + c(x)/r^2 + \cdots \)
- \( M(x) \sim \) quark mass
- \( c(x) \sim \) quark condensate
- Real mass: \( \phi = 0 \) or \( \phi = \pi \)
- \( d\rho^2 + \rho^2 d\phi^2 \to dX^2 + dY^2, \ X = 0, \ Y = \rho \)
- \( \lim_{x \to \infty} M(x) = M_0 \) positive mass (\( \phi = 0 \))
- \( \lim_{x \to -\infty} M(x) = -M_0 \) negative mass (\( \phi = \pi \))
Topological term at the interface

- Background five-form flux: \( \int_{S^5} F_5 = 2\kappa_{10}^2 T_3 N_c \)
- Induced Chern-Simons term in D7 action

\[
S_{WZ} = -\frac{1}{2}(2\pi \alpha')^2 T_7 \int F_5 \wedge A \wedge F = -\frac{1}{2}(2\pi \alpha')^2 T_7 \int F_5 \int_{tyz} A \wedge F
\]

Where \( F_5 \sim d\Omega_3 \wedge d\theta \wedge d\phi \)
- We are now considering the embedding as \( \theta = \theta(r, x), \phi = \Delta \phi \Theta(-x) \)
- The embedding covers \( \theta \in [0, \pi) \). Real mass implies \( \Delta \phi = \pi \).
- Then, \( \int F_5 = \frac{\Delta \phi}{2\pi} \int_{S^5} F_5 = \frac{1}{2} \int_{S^5} F_5 \)

Chern-Simons term

\[
S_{WZ} = -\frac{k}{4\pi} \int_{tyz} A \wedge F, \quad k = \frac{N_c}{2}
\]
Embedding: linearized solution

DBI action for the ansatz

\[ \mathcal{L}_{\text{DBI}} = r^3 \sqrt{1 + \left( \partial_r Y \right)^2 + \frac{(\partial_x Y)^2}{(r^2 + Y^2)^2}} \]

Scaling symmetry \( Y, r \rightarrow \xi Y, \xi r, x \rightarrow x/\xi \)

Expanding to second order

\[ \mathcal{L}^{(2)} = \frac{r^3}{2} (\partial_r Y)^2 + \frac{1}{2r} (\partial_x Y)^2 \]

Solution:

\[ Y_0(r, x) = \frac{M_0 xr}{\sqrt{1 + (xr)^2}} \]

\[ \lim_{r \rightarrow \infty} Y_0(x, r) = \text{sign} (x) M_0 = M(x) \]
Embedding: beyond linear order

Series solution:

\[ Y(r, x) = M_0 \sum_{n=0}^{\infty} (M_0 x)^{2n} f_n(xr) = Y_0(r, x) + M_0^3 x^2 f_1(xr) + \cdots \]

- Regularity conditions fix \( f_n(xr) \)
- Accurate for \( r \gg M_0, M_0 x \ll 1 \)
- Mass: \( M(x) = \text{sign}(x) M_0 + O(M_0^3 x^2) \)
- Condensate follows from dimensional analysis

\[ c(x) = \text{sign}(x) \frac{M_0}{x^2} \sum_{n=0}^{\infty} c_n (M_0 x)^{2n} \]

Issues: wrong behavior as \( r \to 0 \), \( \partial_r Y(0, x) \neq 0 \) \( \to \) conical defect
Embedding: numerical solution

Numerical problem: relaxation to the ground state

\[ \partial_{\tau} Y(r, x, \tau) = -|\delta \mathcal{L}_{DBI}[Y, \partial_r Y, \partial_x Y]| \]

As \( \tau \to \infty, |\delta \mathcal{L}_{DBI}| \to 0 \) so \( Y(r, x, \infty) \) satisfies the eom’s

- Initial condition \( Y(r, x, 0) = Y_0(r, x) \)
- Boundary conditions \((r = 0)\) \( \partial_r Y(0, x, \tau) = 0 \)
- Boundary conditions \((r_c = 20M_0)\) \( Y(r_c, x, \tau) = Y_0(r_c, x) \)
- Boundary conditions \((x_c = 4/M_0)\) \( Y(r, \pm x_c, \tau) = Y_0(r_c, \pm x_c) \)
Embedding: numerical solution
Embedding: numerical solution
Other string theory constructions

- Stable D-brane intersections (K-theory classification)
  [Ryu & Takayanagi, 1007.4234]

- Holographic FTI in 2+1 and 1+1 dimensions
  [Karch, Maciejko & Takayanagi, 1009.2991]
  - D3/D5 2+1 intersection/probe D5 in $AdS_4 \times S^2$
    $\mathcal{T}$-symmetric, Spin Hall Effect $U(1)_R \sim \text{spin symmetry}$
    \[ S_{\text{bulk}} = \frac{N_c}{4\pi} \int (A_R \wedge F + A \wedge F_R) \]
  - D3/D5 1+1 intersection/probe D5 in $AdS_3 \times S^3$
    $\mathcal{C}$-symmetric, boundary charge density (holographic model unstable)
    \[ S_{\text{bulk}} = \frac{N_c}{2} \int F \Rightarrow S_{\text{interface}} = \frac{N_c}{2} \int A_t \]
  - D3/D3 1+1 intersection/probe D3 in $AdS_3 \times S^1$
    $\mathcal{T}$-symmetric, boundary spin density
    \[ S_{\text{bulk}} = N_c \int F^R \Rightarrow S_{\text{interface}} = N_c \int A_t^R \]
Conclusions and future directions

- There is a complete classification of TI at weak coupling
- At strong coupling there could be new classes, as the fractional TI
- AdS/CFT models can be used to explore these new possibilities
- We have constructed an explicit example of 3+1 TI with fractional Hall conductivity at the interface $\sigma_{xy} = 1/(2N_c)e^2/h$
- Other examples in 2+1 and 1+1 have been found using holography
- Are there fractional versions of all weakly coupled TIs? What are the possible cases?