(A Holographic fractional) topological insulator

Carlos Hoyos
University of Washington

Galileo Galilei Institute,
October 14, 2010

C.H., K. Jensen & A. Karch, arXiv:1007.3253

Outline

- Small review of topological insulators
- Models for fractional topological insulators
- Holographic model of FTI

Why are topological insulators interesting?

Experiment:

- New class of materials
- Effective 2+1 massless Dirac fermions at the surface
- Realized in the lab: $Bi_{1-x}Sb_x$, Bi_2Te_3, Bi_2Se_3 , Sb_2Tb_3
- Unusual Quantum Hall Effect (half-integer)
- Robust against disorder

Theory:

- Topological classification of theories with fermions (of which 3+1 topological insulators are an example)
- Some theories related by dimensional reduction
- Map to stable D-brane configurations (K-theory)

Review: [Hasan & Kane, 1002.3895]

Insulators

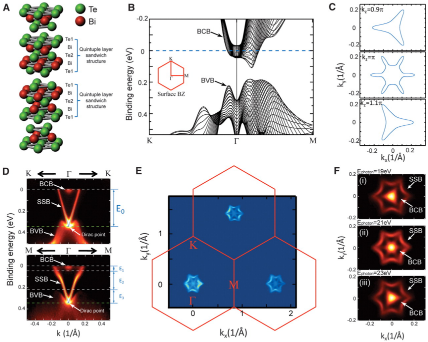


Figure: An insulator for a condensed matter physicist (experimentalist)

$$\mathcal{L}_M = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\gamma^\mu D_\mu - M \right) \psi$$

Equation: Simplified version for theoretists

Topological phases of insulators

- Consider 2+1 massless fermions in a magnetic field ($\mathcal{T}$,$\mathcal{C}$ broken)
- Gapped Landau levels (insulator)
- Gapless states at the boundary
- Integer Quantum Hall Effect $\sigma_{xy} = n\frac{e^2}{h}$

  $\Rightarrow$ topologically inequivalent to normal insulators (where $\sigma_{xy} = 0$)

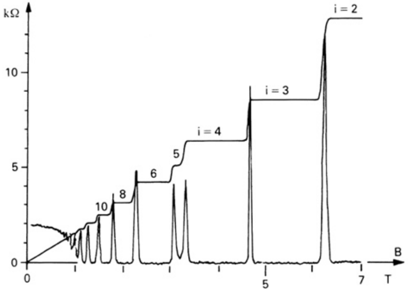


Figure: Hall resistivity as a function of the magnetic field

3+1 Topological Insulators

- Consider an insulator in 3+1 dimensions:
- $\mathcal{L}_M = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\left(i\gamma^\mu D_\mu - M\right)\psi$
- We can make M complex
- $M\overline{\psi}\psi \to \overline{\psi}|M|e^{i\gamma_5\phi}\psi = |M|\cos\phi\,\overline{\psi}\psi + i|M|\sin\phi\,\overline{\psi}\gamma_5\psi$
- Under $\mathcal{T}$: $\overline{\psi}\psi$ even, $\overline{\psi}\gamma_5\psi$ odd
- Time reversal invariance requires $\phi = 0, \pi$ (real mass)
- Normal insulator $\phi = 0$
- Topological insulator $\phi = \pi$
- Topologically protected by $\mathcal{T}$:
  $|M| \to 0$ in order to change the phase of the mass

Topological Effective Theories

- An axial rotation $\psi \to e^{-i\gamma_5\phi/2}\psi$ shifts $M \to e^{-i\gamma_5\phi}M$
- The mass is now real and positive, but the ABJ anomaly gives a theta term to the action $\theta = \phi$.
- After integrating out the fermions the effective theory is

$$\mathcal{L}_{TET} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2\theta}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

- The θ-term $\sim \theta\, \mathbf{E}\cdot\mathbf{B}$ is $\mathcal{T}$-odd.
- BUT: the quantum theory is invariant under $\theta \to \theta + 2\pi$
- Two $\mathcal{T}$-symmetric values: $\theta = 0, \pi$

- Normal insulator: $\theta = 0$
- Topological insulator: $\theta = \pi$

Normal/Topological insulator interface

Massive Dirac fermions with space-dependent mass $M(x)$

$$M(x) = \begin{cases} \text{positive}, & x > 0 \\ \text{negative} & x < 0 \end{cases}$$

$\mathcal{T}$-symmetry implies M is real, so $M(x = 0) = 0$
Massless fermions at the interface:

- 2+1 dimensions: Operator $\overline{\psi}\psi$: $\mathcal{T}$-odd
- $\mathcal{T}$-symmetric mass needs two flavors: $m(\overline{\psi}_1\psi_1 - \overline{\psi}_2\psi_2)$
- Normal/Topological insulator: odd number of flavors at the interface
- Robustnees, $\mathcal{T}$-symetric deformations do not lift massless excitations

Normal/Topological insulator interface

Gauge theory with space-dependent theta term (axion) $\theta(x)$

$$\theta(x) = \begin{cases} 0 & , \quad x > 0 \\ \Delta\theta & , \quad x < 0 \end{cases}$$

In the $\mathcal{T}$-symmetric case $\Delta\theta = 0, \pi$. The 3+1 eom's are

$$\partial_\mu F^{\mu\nu} + \frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu\theta\partial_\alpha A_\beta = 0$$

Since $\partial_\nu\theta = \Delta\theta\delta(x)\delta_\nu^x$, the effective 2+1 theory at the interface is

$$\mathcal{L}_k = \frac{1}{4}F_{ij}F^{ij} - \frac{e^2 k}{4\pi}\epsilon^{ijk}A_i F_{jk}$$

With a Chern-Simons level $\quad k = \frac{\Delta\theta}{2\pi} = \frac{1}{2} \quad \Rightarrow \quad \sigma_{xy} = ke^2/h.$

$\Rightarrow$ half-integer QHE

Fractional Topological Insulator

Effective theory:

- Gauge theory: $U(1)_{\rm em} \times SU(N_c)$, N_c odd (baryon is a fermion)
- $\mathbf{F}_{\mu\nu} = \frac{e}{N_c} F_{\mu\nu} \mathbf{1} + g f^a_{\mu\nu} \mathbf{T}^a$
- Quarks: fundamental representation, charge $q = e/N_c$
- Baryon: charge e, composite electron

ABJ axial anomaly: $\mathcal{L}_\theta = \frac{\theta}{32\pi^2} \,\mathrm{Tr}\, \left(\epsilon^{\mu\nu\alpha\beta} \mathbf{F}_{\mu\nu} \mathbf{F}_{\alpha\beta}\right)$

- Symmetry $\theta \to \theta + 2\pi$
- $U(1)$ part: $\mathcal{L}_\theta = N_c \frac{e^2}{N_c^2} \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$
- $\theta_{\rm eff} = \frac{\theta}{N_c}$: $\mathcal{T}$-symmetric values $\theta_{\rm eff} = 0, \pi/N_c, \cdots$
- Interface: Chern-Simons level $k = 1/(2N_c), \cdots$
- Hall conductivity $\sigma_{xy} = \frac{1}{2N_c} \frac{e^2}{h}$

[Maciejko, Qi, Karch & Zhang, 1004.3628]

Why an AdS/CFT model?

A lot is known about weakly coupled TIs: Complete classification using free fermions, there are no fractional TIs

[Schnyder, Ryu, Furusaki & Ludwig 2008-10; Kitaev 2009]

- Non-Abelian gauge theories could be used as models
- Problem # 1: Charge carriers with fractional charge should be deconfined [Swingle,Barkeshli,McGreevy,Senthil,1005.1076]
- Solution # 1: Use a CFT, like $\mathcal{N} = 4$ SYM
- We could use a weakly coupled description, like field theories on D-branes
- Problem # 2: A model describing an interface breaks conformal invariance (and supersymmetry)
- With D-branes one needs to bend or rotate the D-brane: generically unstable, non-perturbative problems in the field theory?
- Solution # 2: Use AdS/CFT with probe branes, the configuration is fixed by the boundary conditions

Holographic model

- $\mathcal{N} = 4$ $SU(N_c)$ SYM: gauge theory $\rightarrow$ $AdS_5 \times S^5$ dual
- Hypermultiplet in the fundamental representation:
  quasiparticles/quarks $\rightarrow$ probe D7 brane
- $U(1)$ global flavor symmetry:
  $U(1)_{\rm em}$ weakly coupled $\rightarrow$ $U(1)$ gauge field on the D7
- Charge quasiparticles $q_p = e \Rightarrow$ charge electron $q_e = N_c e$
- Chern-Simons level $k = N_c/2$ (equivalent)

Holographic model

Interface between normal and topological insulator

$$ds^2 = (r^2 + \rho^2)\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dr^2 + r^2 d\Omega_3^2 + d\rho^2 + \rho^2 d\phi^2}{r^2 + \rho^2}$$

D7 embedding: wraps $\{x^\mu, \Omega_3, r\}$ directions. Profile $\rho(r, x)$,

- $\lim_{r\to\infty} \rho(r, x) \simeq M(x) + c(x)/r^2 + \cdots$
- $M(x) \sim$ quark mass
- $c(x) \sim$ quark condensate
- Real mass: $\phi = 0$ or $\phi = \pi$
- $d\rho^2 + \rho^2 d\phi^2 \to dX^2 + dY^2$, $X = 0$, $Y = \rho$
- $\lim_{x\to\infty} M(x) = M_0$ positive mass ($\phi = 0$)
- $\lim_{x\to-\infty} M(x) = -M_0$ negative mass ($\phi = \pi$)

Topological term at the interface

- Background five-form flux: $\int_{S^5} F_5 = 2\kappa_{10}^2 T_3 N_c$
- Induced Chern-Simons term in D7 action

$$S_{WZ} = -\frac{1}{2}(2\pi\alpha')^2 T_7 \int F_5 \wedge A \wedge F = -\frac{1}{2}(2\pi\alpha')^2 T_7 \int F_5 \int_{tyz} A \wedge F$$

- Where $F_5 \sim d\Omega_3 \wedge d\theta \wedge d\phi$
- We are now considering the embedding as $\theta = \theta(r, x)$, $\phi = \Delta\phi\Theta(-x)$
- The embedding covers $\theta \in [0, \pi)$. Real mass implies $\Delta\phi = \pi$.
- Then, $\int F_5 = \frac{\Delta\phi}{2\pi} \int_{S^5} F_5 = \frac{1}{2} \int_{S^5} F_5$

Chern-Simons term

$S_{WZ} = -\frac{k}{4\pi} \int_{tyz} A \wedge F, \qquad k = \frac{N_c}{2}$

Embedding: linearized solution

DBI action for the ansatz

$$\mathcal{L}_{DBI} = r^3 \sqrt{1 + (\partial_r Y)^2 + \frac{(\partial_x Y)^2}{(r^2 + Y^2)^2}}$$

Scaling symmetry $Y, r \to \xi Y, \xi r$, $x \to x/\xi$
Expanding to second order

$$\mathcal{L}^{(2)} = \frac{r^3}{2}(\partial_r Y)^2 + \frac{1}{2r}(\partial_x Y)^2$$

Solution:

$$Y_0(r,x) = \frac{M_0 x r}{\sqrt{1 + (xr)^2}}$$

$$\lim_{r\to\infty} Y_0(x,r) = \operatorname{sign}(x) M_0 = M(x)$$

Embedding: beyond linear order

Series solution:

$$Y(r,x) = M_0 \sum_{n=0}^{\infty} (M_0 x)^{2n} f_n(xr) = Y_0(r,x) + M_0^3 x^2 f_1(xr) + \cdots$$

- Regularity conditions fix $f_n(xr)$
- Accurate for $r \gg M_0$, $M_0 x \ll 1$
- Mass: $M(x) = \operatorname{sign}(x) M_0 + O(M_0^3 x^2)$
- Condensate follows from dimensional analysis

$$c(x) = \operatorname{sign}(x) \frac{M_0}{x^2} \sum_{n=0}^{\infty} c_n (M_0 x)^{2n}$$

Issues: wrong behavior as $r \to 0$, $\partial_r Y(0,x) \neq 0 \to$ conical defect

Embedding: numerical solution

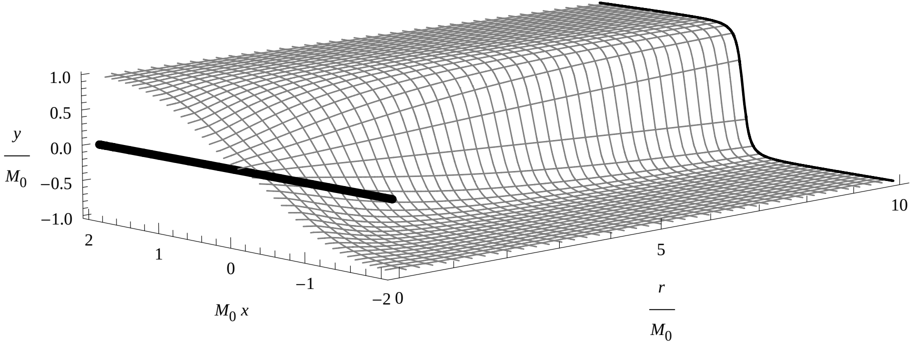
Numerical problem: relaxation to the ground state

$$\partial_\tau Y(r,x,\tau) = -|\delta\mathcal{L}_{DBI}\left[Y,\partial_r Y,\partial_x Y\right]|$$

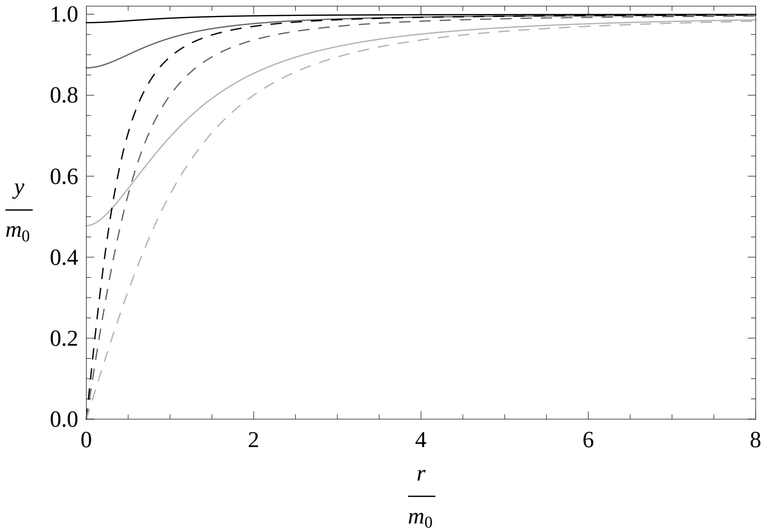
As $\tau \to \infty$, $|\delta\mathcal{L}_{DBI}| \to 0$ so $Y(r,x,\infty)$ satisfies the eom's

- Initial condition $Y(r,x,0) = Y_0(r,x)$
- Boundary conditions $(r=0)$ $\partial_r Y(0,x,\tau) = 0$
- Boundary conditions $(r_c = 20M_0)$ $Y(r_c,x,\tau) = Y_0(r_c,x)$
- Boundary conditions $(x_c = 4/M_0)$ $Y(r,\pm x_c,\tau) = Y_0(r_c,\pm x_c)$

Embedding: numerical solution



Embedding: numerical solution



Other string theory constructions

- Stable D-brane intersections (K-theory classification)

  [Ryu & Takayanagi, 1007.4234]

- Holographic FTI in 2+1 and 1+1 dimensions

  [Karch, Maciejko & Takayanagi, 1009.2991]

  - D3/D5 2+1 intersection/probe D5 in $AdS_4 \times S^2$
    $\mathcal{T}$-symmetric, Spin Hall Effect $U(1)_R \sim$ spin symmetry
    $S_{\rm bulk} = \frac{N_c}{4\pi} \int (A_R \wedge F + A \wedge F_R)$
  - D3/D5 1+1 intersection/probe D5 in $AdS_3 \times S^3$
    $\mathcal{C}$-symmetric, boundary charge density (holographic model unstable)
    $S_{\rm bulk} = \frac{N_c}{2} \int F \;\Rightarrow\; S_{\rm interface} = \frac{N_c}{2} \int A_t$
  - D3/D3 1+1 intersection/probe D3 in $AdS_3 \times S^1$
    $\mathcal{T}$-symmetric, boundary spin density
    $S_{\rm bulk} = N_c \int F^R \;\Rightarrow\; S_{\rm interface} = N_c \int A_t^R$

Conclusions and future directions

- There is a complete classification of TI at weak coupling
- At strong coupling there could be new classes, as the fractional TI
- AdS/CFT models can be used to explore these new possibilities
- We have constructed an explicit example of 3+1 TI with fractional Hall conductivity at the interface $\sigma_{xy} = 1/(2N_c)e^2/h$
- Other examples in 2+1 and 1+1 have been found using holography
- Are there fractional versions of all weakly coupled TIs? What are the possible cases?