Geometric flows and gravitational instantons

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Highlights

Motivations and summary

Gravitational instantons: homogeneity and self-duality

The view from the leaf: geometric flows

Extensions

Outlook

Framework

The Ricci flow describes the parametric evolution of a geometry as

$$\frac{\partial g_{ij}}{\partial t} = -R_{ij}$$

- Introduced by R. Hamilton in 1982 as a tool for proving Poincaré's (1904) and Thurston's (late 70s) 3D conjectures
- In non-critical string theory Ricci flow is an RG flow [Friedan, 1985] – can mimic time evolution as UV $\rightarrow$ IR

$$t = \log {}^{1}/_{\mu}$$

Basic features: a reminder

- Volume is not preserved along the flow

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{2}\int \mathrm{d}^D x \sqrt{\det g}\, g^{ij} \frac{\partial g_{ij}}{\partial t} = -\frac{1}{2}\int \mathrm{d}^D x \sqrt{\det g}\, R$$

  Consequence:
  - positive curvature $\longrightarrow$ space contracts
  - negative curvature $\longrightarrow$ space expands
- Killing vectors are preserved in time: the isometry group remains unaltered – or grows in limiting situations

Example

- At initial time: $R_{ij}^{(0)} = a g_{ij}^{(0)}$ with a constant
- Subsequent evolution: linear rescaling

$$g_{ij}(t) = (1 - at) g_{ij}^{(0)}$$
$$R_{ij}(t) = R_{ij}^{(0)}$$

- Properties
  - $a > 0 \Rightarrow$ uniform contraction $\rightarrow$ singularity at $t = 1/a$
  - $a < 0 \Rightarrow$ indefinite expansion

Gravitational instantons

- Useful for non-perturbative transitions in quantum gravity
- Appear in string compactifications e.g. in heterotic: $\mathbb{C}_2/\Gamma \to$ ALE spaces $\to$ Gibbons–Hawking multi-instantons as Eguchi–Hanson (blow-up of the $\mathbb{C}_2/\mathbb{Z}_2$ A_1 singularity)
- Describe hyper moduli spaces e.g. in IIA:
  - Taub–NUT ($SU(2) \times U(1)$, $\Lambda = 0$): tree-level
  - Pedersen/Fubini–Study ($SU(2) \times U(1)$, $\Lambda \neq 0$): supergravity
  - Calderbank–Pedersen (Heisenberg $\times U(1)$, $\Lambda \neq 0$): string pert
  - Calderbank–Pedersen ($U(1) \times U(1)$, $\Lambda \neq 0$): string non-pert

  or in heterotic: Atiyah–Hitchin ($SU(2)$, $\Lambda = 0$)

Geometric flows arise in gravitational instantons with time foliation

- In 4D self-dual gravitational instantons with homogeneous Bianchi spatial sections: time evolution is a Ricci flow of the 3D homogeneous space
- In non-relativistic gravity with invariance explicitly broken to foliation-preserving diffeomorphisms and with detailed-balance dynamics: time evolution is a geometric flow of the 3D space (valid actually in $D+1 \longrightarrow D$)

Geometric flows might carry information on holographic evolution in some gravitational set ups – yet to be unravelled

Highlights

Cartan's formalism

Metric and torsionless connection one-form $\omega^a{}_b$ and curvature two-form $\mathcal{R}^a{}_b$ in an orthonormal frame:

$$\mathrm{d}s^2 = \delta_{ab}\theta^a\theta^b$$

- Riemann tensor: $\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd}\theta^c \wedge \theta^d$
- Torsion tensor: $\mathcal{T}^a = \mathrm{d}\theta^a + \omega^a{}_b \wedge \theta^b = \frac{1}{2} T^a{}_{bc}\theta^b \wedge \theta^c$
- Cartan structure equations: $\omega_{ab} = -\omega_{ba}$, $\mathcal{T}^a = 0$
- Bianchi identity: $\mathrm{d}\mathcal{R}^a{}_b + \omega^a{}_c \wedge \mathcal{R}^c{}_b - \mathcal{R}^a{}_c \wedge \omega^c{}_b = 0$
- Cyclic identity: $\mathrm{d}\mathcal{T}^a + \omega^a{}_b \wedge \mathcal{T}^b = \mathcal{R}^a{}_b \wedge \theta^b = 0$

Holonomy

- $\mathrm{d}s^2 = \delta_{ab}\theta^a\theta^b$ invariant under local $SO(D)$ transformations
$$\theta^{a\prime} = \Lambda^{-1\,a}{}_b\theta^b$$
- Connection and curvature transform
  - $\omega^{a\prime}{}_b = \Lambda^{-1\,a}{}_c\omega^c{}_d\Lambda^d{}_b + \Lambda^{-1\,a}{}_c\mathrm{d}\Lambda^c{}_b$
  - $\mathcal{R}^{a\prime}{}_b = \Lambda^{-1\,a}{}_c\mathcal{R}^c{}_d\Lambda^d{}_b$

Connection and curvature are both antisymmetric-matrix-valued two-forms $\in$ $\mathbf{D(D-1)/2}$ *representation of* $SO(D)$

Self-dual/anti-self-dual decomposition in 4D

Duality supported by the fully antisymmetric symbol ϵ_{abcd}

- Dual connection:

$$\tilde{\omega}^a{}_b = \frac{1}{2}\epsilon^a{}_{bc}{}^d\omega^c{}_d$$

- Dual curvature:

$$\tilde{\mathcal{R}}^a{}_b = \frac{1}{2}\epsilon^a{}_{bc}{}^d\mathcal{R}^c{}_d$$

Curvature and connection $\in \mathbf{6}$ *(antisymmetric) of* $SO(4)$ *– reducible as* $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ *under* $SU(2)_{\mathrm{sd}} \otimes SU(2)_{\mathrm{asd}} \cong SO(4)$

Adapting the frame $\{\theta^0, \theta^i\}$ *to the action of* $SU(2)_{\mathrm{sd}} \otimes SU(2)_{\mathrm{asd}}$

- Connection one-form

$$(\mathbf{3},\mathbf{1}) \quad \Sigma_i = {}^1/_2 \left(\omega_{0i} + {}^1/_2 \epsilon_{ijk}\omega^{jk}\right)$$
$$(\mathbf{1},\mathbf{3}) \quad A_i = {}^1/_2 \left(\omega_{0i} - {}^1/_2 \epsilon_{ijk}\omega^{jk}\right)$$

- Curvature two-form

$$(\mathbf{3},\mathbf{1}) \quad \mathcal{S}_i = {}^1/_2 \left(\mathcal{R}_{0i} + {}^1/_2 \epsilon_{ijk}\mathcal{R}^{jk}\right)$$
$$(\mathbf{1},\mathbf{3}) \quad \mathcal{A}_i = {}^1/_2 \left(\mathcal{R}_{0i} - {}^1/_2 \epsilon_{ijk}\mathcal{R}^{jk}\right)$$

- $\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ decomposes
  - $\mathcal{S}_i = \mathrm{d}\Sigma_i - \epsilon_{ijk}\Sigma^j \wedge \Sigma^k$
  - $\mathcal{A}_i = \mathrm{d}A_i + \epsilon_{ijk}A^j \wedge A^k$

$\{\Sigma_i, \mathcal{S}_i\}$ *vectors of* $SU(2)_{\mathrm{sd}}$ *and singlets of* $SU(2)_{\mathrm{asd}}$ *and vice-versa for* $\{A_i, \mathcal{A}_i\}$

Dynamics in 4D

Einstein–Hilbert action in Palatini formalisms

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}_4} \tilde{\mathcal{R}}_{cd} \wedge \theta^c \wedge \theta^d$$

- Vacuum equations: $\tilde{\mathcal{R}}^c{}_d \wedge \theta^d = 0$
- Cyclic identity for torsionless connection: $\mathcal{R}^c{}_d \wedge \theta^d = 0$

Curvature (anti)self-duality guarantees vacuum solution

$$\begin{array}{ccccc} \mathcal{R}^a{}_b & = & \pm\tilde{\mathcal{R}}^a{}_b & \Rightarrow & \text{Ricci flatness} \\ & \Updownarrow & & & \\ \mathcal{A}_i = 0 & \text{or} & \mathcal{S}_i = 0 & & \end{array}$$

The $\mathcal{M}_4$ geometry

Foliation and spatial homogeneity *[textbook: Ryan and Shepley, 1975]*

- Topologically $\mathcal{M}_4 = \mathbb{R} \times \mathcal{M}_3$
- Bianchi 3D group G acts simply transitively on the leaves $\mathcal{M}_3$
  $\mathcal{M}_3$ is locally G
  - left-invariant Maurer–Cartan forms σ^i:

$$\mathrm{d}\sigma^i = \frac{1}{2} c^i{}_{jk} \sigma^j \wedge \sigma^k$$

  - 3 linearly independent Killing vectors tangent to $\mathcal{M}_3$:

$$[\xi_i, \xi_j] = c^i{}_{jk} \xi_k$$

  - Classes A (T_3, Heisenberg, $E_{1,1}$, E_2, $SL(2, \mathbb{R})$, $SU(2)$) & B

Self-dual vacuum solutions

Geometry

Foliation plus spatial homogeneity $\longrightarrow$

- Good ansatz for the metric (g_{ij}s functions of t):

$$\mathrm{d}s^2 = \mathrm{d}t^2 + g_{ij}\sigma^i\sigma^j = \delta_{ab}\theta^a\theta^b$$

- Minimalistic (diagonal) ansatz:

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \sum_i \left(\gamma_i\sigma^i\right)^2$$

(the most general in most Bianchi classes)

Second-order equations:

$$\mathcal{A}_i = \mathrm{d}A_i + \epsilon_{ijk} A^j \wedge A^k = 0$$

Solutions: anti-self-dual flat connections

$$A_i = \frac{\lambda_{ij}}{2} \sigma^j$$

$$\lambda_{i\ell} c^\ell{}_{jk} + \epsilon_{imn} \lambda^m{}_{[j} \lambda^n{}_{k]} = 0$$

$G \to SU(2)$ homomorphisms [Bourliot, Estes, Petropoulos, Spindel, 2009]

- $\lambda_{ij} = 0$ rank-0 (trivial) homomorphism: Class A, Class B
- $\lambda_{ij} \neq 0$
  - rank-1: I, II, $\mathrm{VI}_{h=-1}$, $\mathrm{VII}_{h=0}$ & III, IV, V ,$\mathrm{VI}_{h\neq -1}$, $\mathrm{VII}_{h\neq 0}$
  - rank-3: VIII, IX

Bianchi IX: $G \equiv SU(2)$ *and* $\mathcal{M}_3 \equiv S^3$

Convenient parameterization: $\Omega^i = \gamma_j \gamma_k$

$$\mathrm{d}s^2 = \Omega^1\Omega^2\Omega^3\,\mathrm{d}T^2 + \frac{\Omega^2\Omega^3}{\Omega^1}\left(\sigma^1\right)^2 + \frac{\Omega^3\Omega^1}{\Omega^2}\left(\sigma^2\right)^2 + \frac{\Omega^1\Omega^2}{\Omega^3}\left(\sigma^3\right)^2$$

General self-duality equations: $A_i = \frac{\lambda_{ij}}{2}\sigma^j$

$\lambda_{ij} = 0$ Lagrange system (Euler-top) [Jacobi]

$$\dot{\Omega}^1 = -\Omega^2\Omega^3, \quad \dot{\Omega}^2 = -\Omega^3\Omega^1, \quad \dot{\Omega}^3 = -\Omega^1\Omega^2$$

$\lambda_{ij} = \delta_{ij}$ Darboux–Halphen system [Darboux 1878; Halphen 1881]

$$\begin{cases} \dot{\Omega}^1 = \Omega^2\Omega^3 - \Omega^1\left(\Omega^2 + \Omega^3\right) \\ \dot{\Omega}^2 = \Omega^3\Omega^1 - \Omega^2\left(\Omega^3 + \Omega^1\right) \\ \dot{\Omega}^3 = \Omega^1\Omega^2 - \Omega^3\left(\Omega^1 + \Omega^2\right) \end{cases}$$

Solutions with $\gamma_1 = \gamma_2 \to SU(2) \times U(1)$ symmetry

1. Lagrange: Eguchi–Hanson [Eguchi, Hanson, April 1978]

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{1-\frac{a^4}{\rho^4}} + \rho^2 \frac{\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \left(1-\frac{a^4}{\rho^4}\right)\left(\sigma^3\right)^2}{4}$$

   with a removable bolt at $\rho = a$

2. Darboux–Halphen: Taub–NUT [Newman, Tamburino, Unti, 1963]

$$\mathrm{d}s^2 = \frac{r+m}{r-m}\frac{\mathrm{d}r^2}{4} + \left(r^2 - m^2\right)\frac{\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2}{4} + \frac{r-m}{r+m}\left(m\sigma^3\right)^2$$

   with a removable nut at $r = m$

Note: not the most general

- $\gamma_1 = \gamma_2 = \gamma_3 \to SU(2) \times SU(2)$: solution is flat space
- $\gamma_1 \neq \gamma_2 \neq \gamma_3 \to$ strict-$SU(2)$: solutions exist but have often naked singularities
  - Lagrange system: $\exists$ naked singularities [Belisnky, Gibbons, Page, Pope, June 1978]
  - Darboux–Halphen system: solvable in terms of quasi-modular forms [Halphen, 1881], $\exists$ naked singularities except for one solution with a bolt [Atiyah, Hitchin, 1985] describing the configuration space of two slowly moving BPS $SU(2)$ Yang–Mills–Higgs monopoles [Manton, 1981]

Reminder: bolts and nuts

Fixed points of isometries generated by ξ

- *characterised by the rank of $\nabla_{[\nu}\xi_{\mu]}$*
- *potential removable or non-removable singularities, depending on the precise behaviour of $g_{\mu\nu}$*
- $\chi_{\mathsf{bolt}} = 2, \chi_{\mathsf{nut}} = 1$

Around $t = 0$

- rank 4: nut – removable if $\gamma_i \simeq {}^t/_2 \; \forall i$
- rank 2: bolt – removable if $\gamma_1 \simeq \gamma_2 \simeq$ finite and $\gamma_3 \simeq {}^{nt}/_2$

Gravitational instantons of GR are classified according to bolts, nuts and asymptotic behaviours (Euclidean vs. Taubian) within the positive-action conjecture [Gibbons, Hawking, 1979]

Highlights

Curvature for 3D homogeneous spaces

$\tilde{\mathcal{M}}_3$: homogeneous 3D Bianchi IX space with metric

$$\mathrm{d}\tilde{s}^2 = \gamma_{ij}\sigma^i\sigma^j = \delta_{ij}\tilde{\theta}^i\tilde{\theta}^j$$

(Γ^{ij} inverse of γ_{ij})

Bianchi A classes: $c^k{}_{ij} = -\epsilon_{ij\ell}n^{\ell k}$

- Cartan–Killing: $\mathcal{C}_{ij} = -\frac{1}{2}\epsilon_{\ell im}\epsilon_{kjn}n^{mk}n^{n\ell}$
- Ricci:

$$\mathrm{Ric}[\gamma] = \mathcal{C} - \frac{1}{2}\frac{\mathrm{tr}\,(n\gamma)^2}{\det\gamma}\gamma + \frac{\gamma n\gamma n\gamma}{\det\gamma}$$

Back to 4D: self-duality equations

$\mathcal{M}_4$ *with* $\mathrm{d}s^2 = \mathrm{d}t^2 + g_{ij}(t)\sigma^i\sigma^j$

Self-duality over $\mathcal{M}_4$ with $g_{ij} = \gamma_{ik}\mathcal{K}^{k\ell}\gamma_{\ell j}$

$$A_i \equiv \frac{1}{2}\left(\omega_{0i} - \frac{1}{2}\epsilon_{ijk}\omega^{jk}\right) = \frac{\lambda_{ij}}{2}\sigma^j \Leftrightarrow \frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t} = -R_{ij}[\gamma] - \frac{1}{2}\mathrm{tr}\left(\alpha_i\alpha_j\right)$$

$\alpha = \alpha_i\tilde{\theta}^i$ $SU(2)$ Yang–Mills connection over $\tilde{\mathcal{M}}_3$

$$\alpha_i = \left(\mathcal{C}_{ij} - \lambda_{ij}\right)t^j$$

with $\mathrm{tr}(t^i t^j) = -2\delta^{ij}$

- t-independent: ${}^{\mathrm{d}\alpha}/_{\mathrm{d}t} = 0$
- flat: $F \equiv \mathrm{d}\alpha + [\alpha, \alpha] = 0$ $(\Leftrightarrow \lambda_{i\ell}c^{\ell}{}_{jk} + \epsilon_{imn}\lambda^m{}_{[j}\lambda^n{}_{k]} = 0)$

Output: self-duality in $\mathcal{M}_4 = \mathbb{R} \times \mathcal{M}_3 \leftrightarrow$ *Ricci flow plus pure-gauge* $SU(2)$ *Yang–Mills background over* $\tilde{\mathcal{M}}_3$

- Valid for Bianchi A class
- For Bianchi IX ($\mathcal{C}_{ij} = \delta_{ij}$) with diagonal metric $\gamma_{ij} = \gamma_i \delta_{ij}$

  $\lambda_{ij} = \delta_{ij}$ pure Ricci flow on $S^3 \leftrightarrow$ Darboux–Halphen (branch of Taub–NUT and Atiyah–Hitchin)

  $\lambda_{ij} = 0$ Ricci plus YM flow on $S^3 \leftrightarrow$ Lagrange (branch of Eguchi–Hanson and Belisnky *et al*)

Highlights

Can α flow? Can it have $F \neq 0$?

Comments on the emerging geometric flow of the 3D leaves

- α is a background $SU(2)$ gauge field inherited from the anti-self-dual part of the 4D Levi–Civita connection
- The geometric flow *is not* gauge invariant – not supposed to be
- The gauge field
  - *does not* flow ($\dot{\alpha} = 0$)
  - its strength is set to $F = 0$

Adding Λ: milder self-duality condition (Weyl) but major difference

$\mathcal{A}_i \neq 0 \rightarrow$ dynamical $SU(2)$ gauge field on the 3D leaf

- Flowing connection α
- Non-vanishing field strength F

(breakdown of genuine self-duality)

Genuine Ricci plus $SU(2)$ Yang–Mills flow

- Example in Bianchi IX: the Fubini–Study or Pedersen solutions (metric on $\mathbb{C}P_2$ and relatives)
- Example in Bianchi II: Calderbank–Pedersen solution

Can one go beyond 4D?

Self-duality in $D = 7, 8$

The octonionic structure constants $\psi_{\alpha\beta\gamma}$ $\alpha, \beta, \gamma \in \{1, \ldots, 7\}$ and the dual G_2-invariant antisymmetric symbol $\psi^{\alpha\beta\gamma\delta}$ allow to define

- Duality in 7D: $SO(7) \supset G_2$
- Duality in 8D: $SO(8) \supset \mathrm{Spin}_7$

However

- $SO(7) \not\supseteq H \otimes G_2$
- $SO(8) \not\supseteq H \otimes \mathrm{Spin}_7$

In foliations $\mathcal{M}_{D+1} = \mathbb{R} \times \mathcal{M}_D$ *with* $\mathcal{M}_D$ *a fibration over a Bianchi group:* $\mathcal{A} = 0 \Rightarrow A = 0$ – *geometric flow under investigation*

Non-relativistic gravity [Hořava, 2008–09]

Foliation $\mathcal{M}_{D+1} = \mathbb{R} \times \mathcal{M}_D$*: explicit breaking of diffeomorphisms*

$$S = \int \mathrm{d}t\, \mathrm{d}^D x \sqrt{g} \left(\frac{2}{\kappa^2} \left(K_{ij} K^{ij} - \lambda K^2 \right) + V \right)$$

$\mathrm{d}s^2 = \mathrm{d}t^2 + g_{ij} \mathrm{d}x^i \mathrm{d}x^j$, $K_{ij} = {}^1\!/\!_2 \partial_t g_{ij}$, $[x] = -1$, $[t] = -z$

- GR: $\lambda = 1$, $z = 1$ and $V = {}^2\!/\!_{\kappa^2}(2\Lambda - R_D)$
- HL: $\lambda \in \mathbb{R}$ and $V = {}^{\kappa^2}\!/\!_2 E^{ij} \mathcal{G}_{ijk\ell} E^{k\ell}$
  - $\mathcal{G}_{ijk\ell} = \frac{1}{2} \left(g_{ik} g_{j\ell} + g_{i\ell} g_{jk} \right) - \frac{\lambda}{D\lambda - 1} g_{ij} g_{k\ell}$ (zero at $\lambda = {}^1\!/\!_D$)
  - *power-counting* (super)renormalizability: $z(>) = D$
  - detailed balance: $E^{ij} = -\frac{1}{2\sqrt{g}} \frac{\delta W_D[g]}{\delta g_{ij}}$

$D = z = 3$: $W_3 = W_{\mathrm{CS}} + W_{\mathrm{EH}}$ *(topologically massive gravity)*

Ground states in the positive-definite case ($\lambda < 1/D$)

- Detailed balance $\rightarrow$ S (up to boundary term: $1/2\, |W_D|_{t_{\rm in}}^{t_{\rm fin}} \geq 0$)

$$\frac{2}{\kappa^2}\int \mathrm{d}t\,\mathrm{d}^D x\,\sqrt{g}\left(K_{ij} \pm \frac{\kappa^2}{2}\mathcal{G}_{ijmn}E^{mn}\right) G^{ijk\ell}\left(K_{k\ell} \pm \frac{\kappa^2}{2}\mathcal{G}_{k\ell rs}E^{rs}\right)$$

- Ground-state extremums $\rightarrow$ geometric flow

$$\partial_t g_{ij} = \mp\kappa^2\mathcal{G}_{ijk\ell}E^{k\ell}$$

- Static solutions – fixed-points of the flow $\rightarrow$ extremums of W_D

$$E^{ij} \equiv -\frac{1}{2\sqrt{g}}\frac{\delta W_D[g]}{\delta g_{ij}} = 0$$

Gravitational instantons

Flow lines $\leftrightarrow$ Hořava–Lifshitz classical solutions

- Static solutions $\longrightarrow$ $V = 0$ (D-dim extremums) and $S = 0$
- Generic flow lines $\longrightarrow$ infinite-action solutions with singularities at finite proper time
- Flow lines interpolating two fixed points (D-dim extremums)
  - *finite* action

$$S_{\text{ground state}} = \frac{1}{2}\,|\Delta W_D|$$

  - the end-points would be singular but are at *infinite* proper time

4D Euclidean space–time ($D = 3$)

Detailed balance with Chern–Simons and Einstein–Hilbert

$$\partial_t g_{ij} = \frac{\kappa^2}{w_{\mathrm{CS}}} C_{ij} - \frac{\kappa^2}{\kappa_W^2}\left(R_{ij} - \frac{2\lambda - 1}{2(3\lambda - 1)} R g_{ij} + \frac{\Lambda_W}{1 - 3\lambda} g_{ij}\right)$$

Cotton–Ricci flows – highly intricate mathematical problem

Can be better studied assuming e.g. Bianchi IX symmetry for the 3D leaves ($SU(2)$-homogeneous) *[Bakas, Bourliot, Lüst, Petropoulos, 2010]*

$$g_{ij} \mathrm{d}x^i \mathrm{d}x^j = \sum_i \gamma_i(t) \left(\sigma^i\right)^2$$

Rich (analytic/numerical) behaviour: fixed points (isotropic, axisymmetric, anisotropic), convergence, stability ...

$\lambda \to -\infty$: normalized Ricci plus Cotton flow

$$\partial_t g_{ij} = \frac{\kappa^2}{w_{CS}} C_{ij} - \frac{\kappa^2}{\kappa_W^2}\left(R_{ij} - \frac{1}{3} R g_{ij}\right)$$

- The volume is conserved: $V = 16\pi^2 \sqrt{\gamma_1\gamma_2\gamma_3} = 2\pi^2 L^2$
- Typical phase portrait ($x = {}^{4\gamma_1}/_{L^2}$, $y = {}^{4\gamma_2}/_{L^2}$)

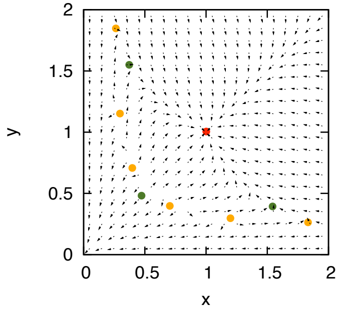

Figure: Flow lines for $\mu \equiv {}^{w_{CS}L}/_{\kappa_W^2} < -6\sqrt[3]{2}$

Note: 3D detailed balance with Einstein–Hilbert $\rightarrow$ pure Ricci

Poincaré's conjecture: unique (isotropic) fixed point

$$\Downarrow$$

No gravitational instantons: solutions have infinite (generic) or zero (static) action

Hořava–Lifshitz Bianchi IX gravitational instantons: time-dependent solutions interpolating between genuine 4D static solutions

Look like ordinary instantons of particle theory ...

Smooth evolution of the S^3 – globally $\mathbb{R} \times S^3$

- no nuts, no bolts
- zero Euler number χ and signature τ
- no $SO(3)$, no taubian infinity

... rather than GR gravitational instantons – universal behaviour

Reason: detailed-balance condition $\rightarrow$ geometric flows

Relaxing the detailed balance $\rightarrow$ richer spectrum of instantons, black holes ... closer to GR in the IR

Highlights

Motivations and summary

Gravitational instantons: homogeneity and self-duality

The view from the leaf: geometric flows

Extensions

Outlook

Geometric flows and gravitational instantons

4D Einstein dynamics versus 3D geometric flows in spaces with time foliation, homogeneous spatial sections and self-duality

- Role of 4D: $SO(4) \cong SU(2) \times SU(2) \Rightarrow$ reduction is sd $\oplus$ asd
- Role of the 3D homogeneity: $G \longrightarrow SU(2) \Rightarrow$ gauge choice
- Role of the self-duality: effectively reduces the system to 3D
  - geometric flow driven by Ricci plus $SU(2)$ gauge field
  - no degree of freedom for the gauge field ($\tilde{F} = 0$)

- *Possible generalizations in* $D + 1 = 8, 7$ *or to include* $\Lambda \neq 0$
- *Possible holographic applications: flows along the radial direction towards to boundary*

Gravitational instantons in non-relativistic gravity: general framework to embed various geometric flows

- Similar set-up: foliation $\mathcal{M}_{D+1} = \mathbb{R} \times \mathcal{M}_D$
- Major difference: explicit breaking of the diffeomorphism invariance – in Einstein this breaking is spontaneous
- Similar constraint: detailed balance and ground states instead of self-duality
- Similar effect: dynamics locked by the D-dim ancestor – instantons are flow lines interpolating between D-dim extremums (degenerate static $D+1$-dim solutions)
- Important differences: anistotropy scaling $z = D$, $\lambda < 1/D$ – "smoother" instantons

Example: 4D → 3D dynamics governed by Ricci–Cotton flows – analytic and numerical available results – more to be done wrt z, D