Entanglement entropy of disconnected regions in Conformal Field Theories

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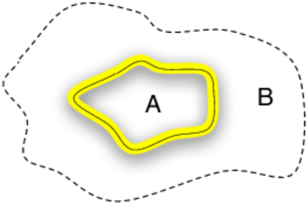
Mainly joint work with John Cardy
also V Alba, M Campostrini, F Essler, M. Fagotti, A Lefevre, J Moore, B Nienhuis, I Peschel, L Tagliacozzo, E Tonni

Review: PC & JC JPA 42, 504005 (2009)

Entanglement: what is it?

Quantum system in a pure state $|\Psi\rangle$
The density matrix is $\rho = |\Psi\rangle\langle\Psi|$
$(\mathrm{Tr}\rho^n = 1)$

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$



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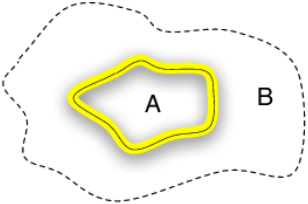

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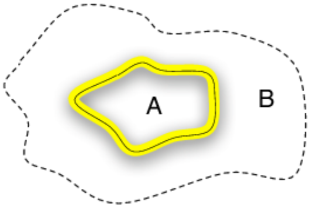

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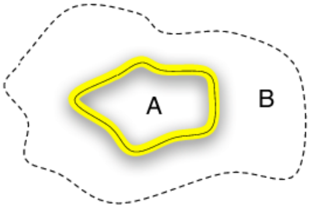

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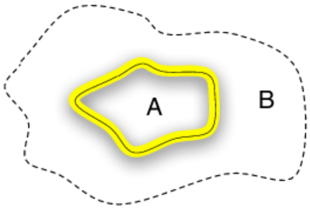
A natural measure is the *entanglement entropy* $(\rho_A = \mathrm{Tr}_B \rho)$

$$S_A \equiv -\mathrm{Tr}\rho_A \log \rho_A = -\sum_n c_n^2 \log c_n^2 = S_B$$

$|\Psi\rangle$ is the ground state of a local Hamiltonian H
Is entanglement special?

Yes, if A corresponds to a spatial subset

(Area law)



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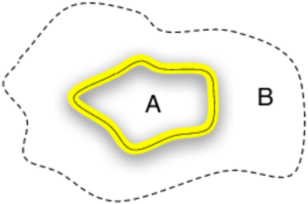

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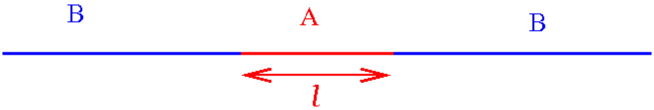


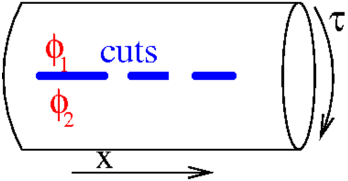
In a 1+1D CFT Holzhey, Larsen, Wilczek '94

$$S_A = \frac{c}{3} \ln \ell$$

This is the most effective way to measure the central charge c

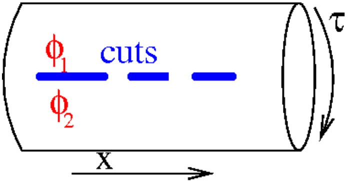
Path integral and Riemann surfaces PC and J Cardy '04

$$\langle \Phi_1(x) | \rho_A | \Phi_2(x) \rangle =$$



Path integral and Riemann surfaces PC and J Cardy '04

$$\langle \Phi_1(x)|\rho_A|\Phi_2(x)\rangle =$$



$$\mathrm{Tr}\rho_A^n =$$

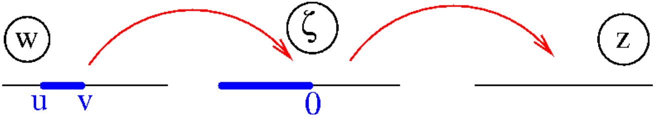
$\mathrm{Tr}\rho_A^n =$ for n integer is the partition function on a n-sheeted Riemann surface $\mathcal{R}_{n,1}$

Replica trick: $S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n$

Riemann surfaces and CFT PC and J Cardy '04

The Riemann surface $\mathcal{R}_{n,1}$ is topological equivalent to the complex plane on which is mapped by

$$w \to \zeta = \frac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow\ w \to z = \left(\frac{w-u}{w-v}\right)^{1/n}$$



$$\mathrm{Tr}\rho_A^n = $$

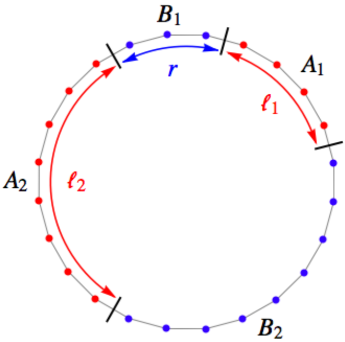
$$= c_n |u-v|^{-\frac{c}{6}(n-1/n)}$$

$$|u-v| = \ell$$

$$\Rightarrow S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n = \frac{c}{3}\log \ell$$

More difficult problem

A = Disconnected regions:



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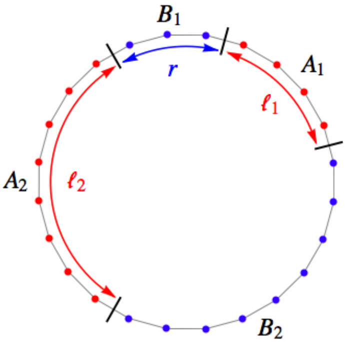
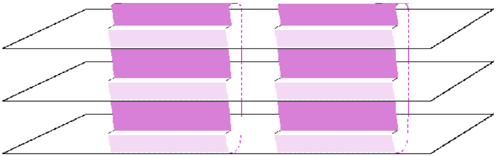


More complex Riemann surface:



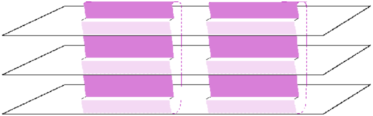
$\mathcal{R}_{n,2}$ of genus $(n-1)$

[$\mathcal{R}_{n,N}$ has genus $(n-1)(N-1)$]

$\mathrm{Tr}\rho_A^n$, S_A ?

Disjoint intervals: History

$$A = [u_1, v_1] \cup [u_2, v_2]$$



In 2004 we obtained

$$\mathrm{Tr}\rho_A^n = c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)}$$

Tested for free fermions in different ways Casini-Huerta, Florio et al.

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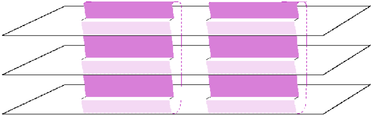

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For more complicated theories in 2008 Furukawa-Pasquier-Shiraishi and Caraglio-Gliozzi showed that it is incorrect!

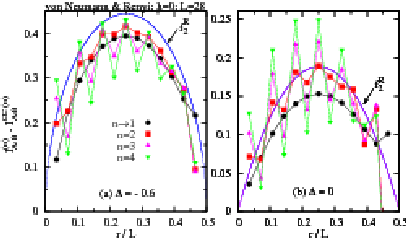
$$\mathrm{Tr}\rho_A^n = c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)} F_n(x)$$

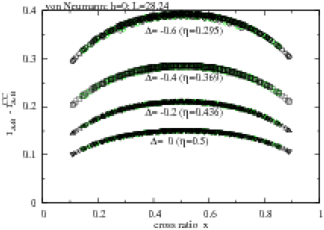
$$x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)} = 4 - \text{point ratio}$$

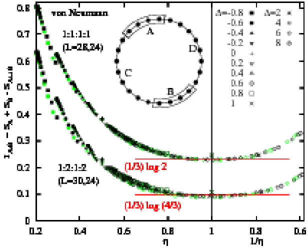
Compactified boson (Luttinger) Furukawa Pasquier Shiraishi

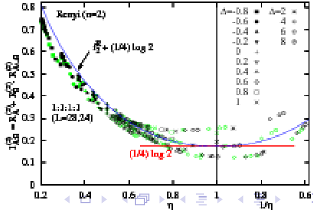
$$F_2(x) = \frac{\theta_3(\eta\tau)\theta_3(\tau/\eta)}{[\theta_3(\tau)]^2}, \qquad x = \left[\frac{\theta_2(\tau)}{\theta_3(\tau)}\right]^4 \qquad \eta \propto R^2$$

Compared against exact diagonalization in XXZ chain









Compactified boson PC Cardy Tonni '09

Using old results of CFT
on orbifolds Dixon et al 86

$$F_n(x) = \frac{\Theta(0|\eta\Gamma)\,\Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2}$$

Γ is an $(n-1)\times(n-1)$ matrix

$$\Gamma_{rs} = \frac{2i}{n}\sum_{k=1}^{n-1}\sin\left(\pi\frac{k}{n}\right)\beta_{\frac{k}{n}}\cos\left[2\pi\frac{k}{n}(r-s)\right]$$

$$\text{with} \qquad \beta_y = \frac{H_y(1-x)}{H_y(x)}, \qquad H_y(x) = {}_2F_1(y, 1-y; 1; x)$$

Riemann theta function $\Theta(z|\Gamma) \equiv \sum_{m\in\mathbf{Z}^{n-1}} \exp\left[i\pi\, m\cdot\Gamma\cdot m + 2\pi i m\cdot z\right]$

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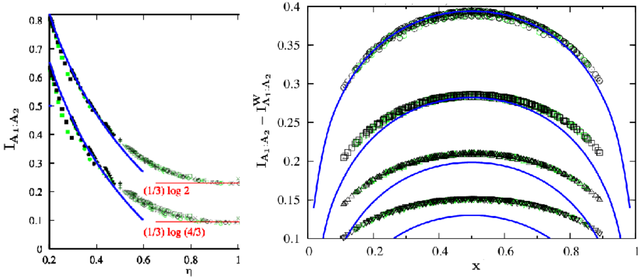
- $F_n(x)$ invariant under $x \to 1-x$ and $\eta \to 1/\eta$
- We are unable to analytic continue to real n for general x and η
- Only for $\eta \ll 1$ and for $x \ll 1$

$\eta \ll 1$

$$-F_1'(x) = \frac{1}{2}\ln\eta - \frac{D_1'(x) + D_1'(1-x)}{2}$$

with

$$D_1'(x) = -\int_{-i\infty}^{i\infty} \frac{dz}{i}\frac{\pi z}{\sin^2 \pi z}\log H_z(x)$$



Compactified boson III PC Cardy Tonni

$x \ll 1$

$$F_n(x) = 1+2n\left(\frac{x}{4n^2}\right)^{\alpha} P_n+2n\left(\frac{x}{4n^2}\right)^{2\alpha} P_n^{(2)}+\cdots \qquad \alpha = \min(\eta, 1/\eta)$$

$$P_n = \sum_{l=1}^{n-1} \frac{l/n}{[\sin(\pi l/n)]^{2\alpha}} = \frac{1}{2}\sum_{l=1}^{n-1} \frac{1}{[\sin(\pi l/n)]^{2\alpha}}$$

$$-F_1'(x) = 2^{1-2\alpha} x^{\alpha} P_1' + \ldots$$

Compactified boson III PC Cardy Tonni

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NEW

$$\boxed{P_1' = \frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{4\Gamma\left(\alpha+\frac{3}{2}\right)}}$$

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NEW^2

$$P_n^{(2)} = \frac{n}{2} \sum_{L=3}^{n-1} \sum_{\ell_1=1}^{L-2} \sum_{\ell_2=1}^{L-\ell_1-1} \left[Q_1^{2\alpha} + 2Q_2^{2\alpha}\right]$$

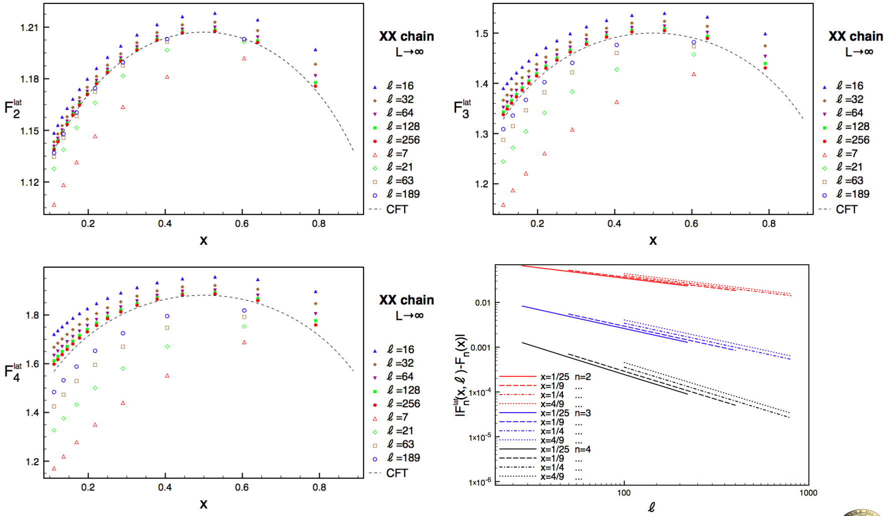
$$Q_1 \equiv \frac{\sin(\pi(L-\ell_1)/n)\sin(\pi(L-\ell_2)/n)}{\sin(\pi\ell_1/n)\sin(\pi\ell_2/n)\sin(\pi L/n)\sin(\pi(L-\ell_1-\ell_2)/n)}$$

$$Q_2 \equiv \frac{\sin(\pi\ell_1/n)\sin(\pi\ell_2/n)}{\sin(\pi(L-\ell_1)/n)\sin(\pi(L-\ell_2)/n)\sin(\pi L/n)\sin(\pi(L-\ell_1-\ell_2)/n)},$$

$P_n^{(m)}$ have an OPE interpretation (for $m=1$ [Headrick '10])

The XX model Fagotti PC '10

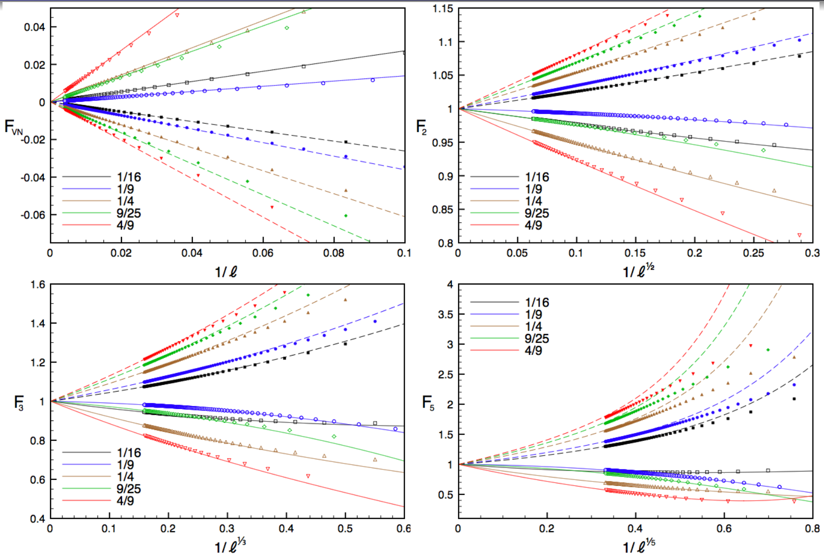
The RDM of two intervals is not trivial because of JW string Igloi-Peschel



$$F_n^{\rm lat}(x) = F_n^{CFT}(x) + (-)^\ell \ell^{-\delta_n} f_n(x) + \dots$$

CFT OK and $\delta_n = 2/n$

The XX model with Open BC Fagotti PC '10

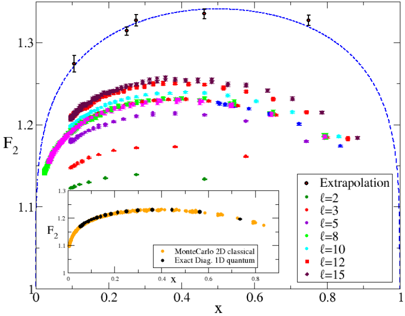


$F_n(x) = 1$ because it is free fermions!

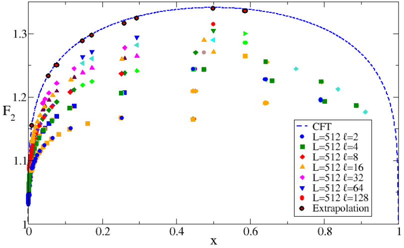
The Ising model Alba Tagliacozzo PC '09

Monte Carlo for 2D and TTN for 1D

$\mathrm{Tr}\rho_A^2$ MC

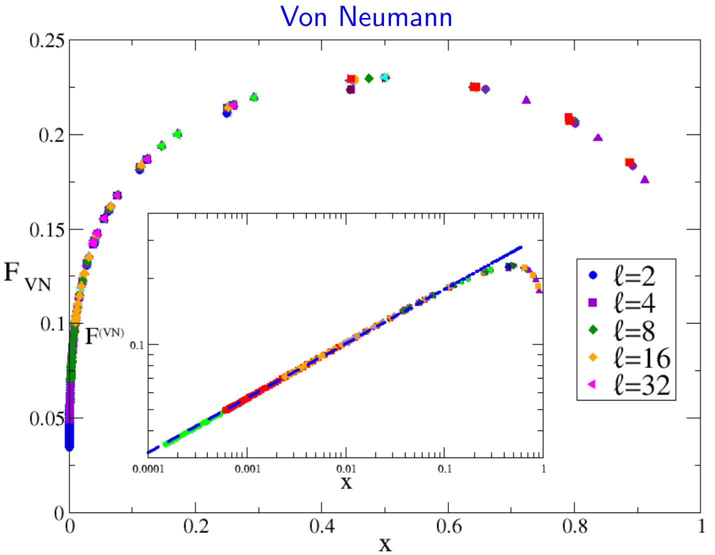


$\mathrm{Tr}\rho_A^2$ TTN



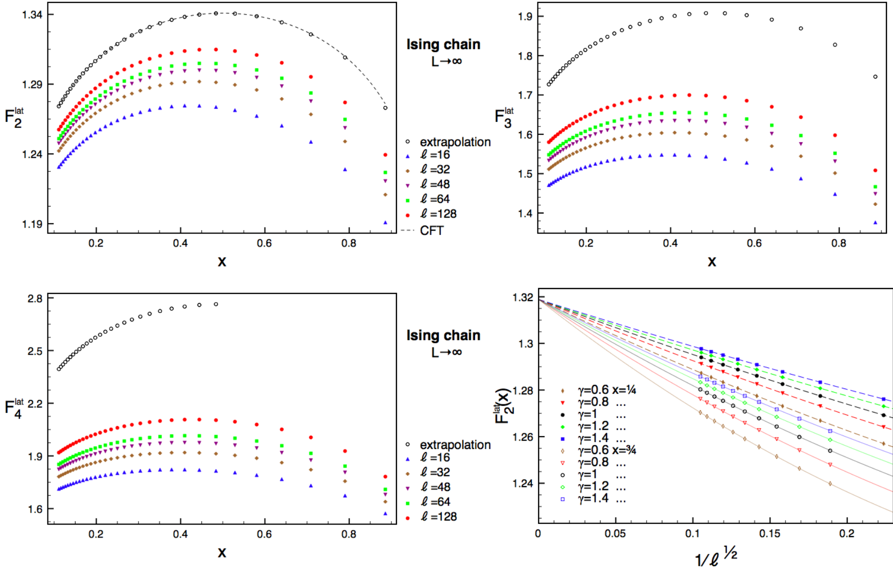
Large monotonic corrections to the scaling! FSS analysis confirms:

$$F_2(x) = \frac{1}{\sqrt{2}}\left[\left(\frac{(1+\sqrt{x})(1+\sqrt{1-x})}{2}\right)^{1/2} + x^{1/4} + ((1-x)x)^{1/4} + (1-x)^{1/4}\right]^{1/2}$$


Von Neumann
0.25
0.2
0.15
0.1
0.05
0
F_{VN}
$F^{(VN)}$
0.1
0.0001
0.001
0.01
0.1
1
x
ℓ=2
ℓ=4
ℓ=8
ℓ=16
ℓ=32
0
0.2
0.4
0.6
0.8
1
x

The Ising model Fagotti PC '10

$\delta_n = 1/n$ because of Ising fermion!



Using Dijkgraaf, Verlinde, Verlinde '88 we showed

$$F_n(x) = \frac{1}{2^{n-1}\Theta(0|\Gamma)} \sum_{\epsilon_1,\epsilon_2} \left| \Theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} (0|\Gamma) \right|$$

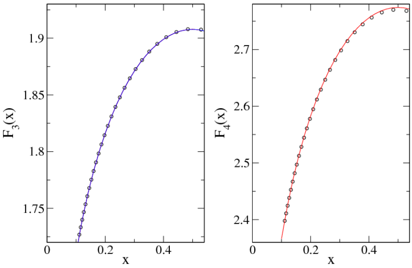
Riemann theta function with characteristics

$$\Theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\Gamma) \equiv \sum_{m \in \mathbf{Z}^{n-1}} \exp \left[i\pi\, (m+\alpha)\cdot\Gamma\cdot(m+\alpha) + 2\pi i (m+\alpha)\cdot(z+\beta) \right]$$

ϵ_1, ϵ_2 are vectors of length $n-1$ with elements $= 0, 1/2$
Γ is the same matrix as for Luttinger

The Ising model: CFT, very new II PC Cardy Tonni '10

- For $n = 2$ it reduces to the simple function above
- It reproduces perfectly numerical data for $n = 3, 4$.



- Small x expansion

$$F_n(x) = 1 + \left[\frac{x}{4n^2}\right]^{\frac{1}{4}} \sum_{\ell=1}^{n-1} \frac{n/2}{[\sin(\pi\ell/n)]^{\frac{1}{2}}} + \frac{n}{4}\left[\frac{x}{4n^2}\right]^{\frac{1}{2}} \sum_{L=3}^{n-1} \sum_{\ell_1=1}^{L-2} \sum_{\ell_2=1}^{L-\ell_1-1} Q_1^{\frac{1}{2}} + \cdots$$

Again OPE interpretation and analytic continuation to $n \to 1$

Entanglement entropy provides many universal features of quantum systems.

Not only the central charge

Open problems:

- The analytic continuation of $F_n(x)$ is unknown and so is S_A
- No results for more than two intervals
- No understanding of $F_n(x)$ in AdS/CFT
- The simplicity of $F_n(x)$ suggests a deeper connection between entanglement entropy and the Riemann Θ functions
- What about non-conformal systems?