Holographic Quantum Criticality via Magnetic Fields

Per Kraus (UCLA)

Based on work with Eric D'Hoker

Introduction

- Study gravity solutions dual to D=3+1 gauge theories at finite charge density and in background magnetic field

Motivations:

- Because it's there
- Applications
  - condensed matter
  - QCD
- Status of extremal black hole entropy (Nernst theorem?)

Executive Summary

- finite density of fermions in a well understood gauge theory
- in bulk, fermionic charge is carried entirely by flux
- vanishing ground state entropy
- B-field tuned quantum critical point:

$$S \sim \begin{cases} T & (B > B_c) \\ T^{1/3} & (B = B_c) \end{cases}$$

- critical point has near horizon warped AdS_3
- solutions provide microscopic realization of, and holographic dictionary for, IR WAdS_3

Einstein-Maxwell theory

- AdS duals to susy gauge theories can be described by Einstein-Maxwell (+CS) theory

$$S = \int d^D x \left(R + F^2 - 2\Lambda \right) + S_{CS}$$

consistent truncation

bulk gauge field dual to boundary R-current

Charged black brane

- Simple solution: charged black brane (Reissner-Nordstrom)

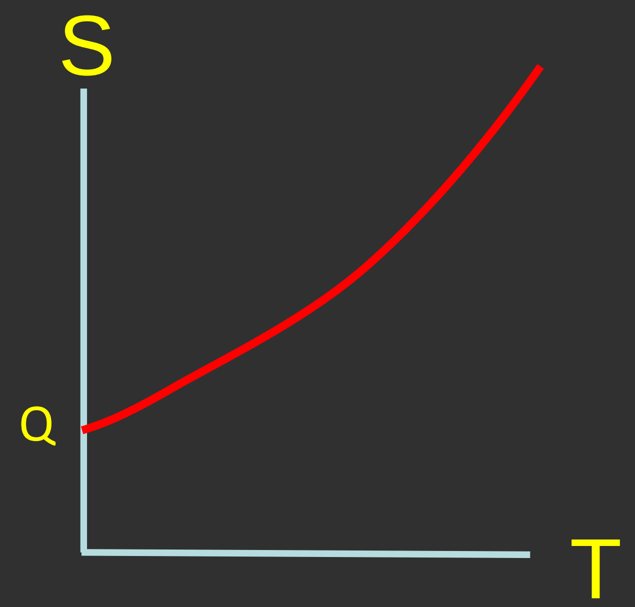
$$ds^2 = \frac{1}{U(r)}dr^2 - U(r)dt^2 + r^2 d\vec{x}^2$$

$$F_{rt} = \frac{Q}{r^{D-2}} \qquad Q \sim \text{charge density in CFT}$$

$$U(r) = r^2 - \frac{M}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}}$$

Asymptotically AdS

- Entropy density of these solutions behaves as:



- Smooth extremal limit with $S \sim Q$ (~~susy~~)

near horizon $AdS_2 \times R^{D-2}$

Fermi surface (Lee, Cubrovic et. al., Liu et, al., …)

- Extremal entropy is puzzling from CFT standpoint

In gauge theory expect Bose condensation: S=0

Magnetic fields

- Look for solutions with boundary magnetic field

$F_{ij} \to \mathrm{const}$ approaching AdS boundary

D=4

- AdS_4 solution easily obtained by duality rotation:

$$ds^2 = \frac{1}{U(r)}dr^2 - U(r)dt^2 + r^2 d\vec{x}^2$$

$$F = \frac{Q}{r^2}dr \wedge dt + B dx \wedge dy$$

$$U(r) = r^2 - \frac{M}{r} + \frac{Q^2+B^2}{r^2}$$

dyonic black brane

ground state entropy: $S \sim \sqrt{Q^2 + B^2}$

Entropy from free fields

- compare with entropy of D=2+1 charged bosons and fermions in B-field:

  - relativistic Landau levels

$$E_n = \begin{cases} \sqrt{(2n+1)B} & \text{bosons} \\ \sqrt{2nB} & \text{fermions} \end{cases} \qquad n = 0, 1, 2, \ldots \qquad \text{degeneracy} \sim BA$$

ground state degeneracy from filling up fermion zero modes: $S_{free}(Q=0,B) \sim B \sim S_{grav}(Q=0,B)$

  - For nonzero Q agreement gets worse, and eventually bosons condense when $\mu = \sqrt{B}$

- Extremal entropy is associated with charge hidden behind the horizon

- To reach unique ground state the black hole needs to expel the charge:

  e.g. by forming a charged bose/fermi condensate

- Another variation involves Chern-Simons terms for the gauge fields, since these allow the gauge field itself to carry charge

$$d \star F + F \wedge F = 0$$

D=5

- AdS_5 story is much richer
- Einstein-Maxwell-Chern-Simons action:

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R + F^2 - \frac{12}{L^2}\right) + \frac{k}{12\pi G_5}\int A \wedge F \wedge F$$

k gives anomaly of boundary R-current: $\partial_\mu j^\mu \sim k\vec{E}\cdot\vec{B}$

susy requires $k = \sqrt{\frac{4}{3}}$

- All susy IIB/M-theory AdS_5 backgrounds admit a consistent truncation to EMCS action

(Buchel/Liu; Gauntlett et. al.)

No AdS_2

- Easy to check that finite magnetic field is:
  - Incompatible with existence of AdS_2 factor
  - Incompatible with smooth, finite entropy, extremal horizon
- What is nature of zero temperature solution?

Uncharged solutions

- Look for solution corresponding to gauge theory on plane with constant magnetic field

$$F = B dx_1 \wedge dx_2$$

- Challenging to find fully analytical asymptotically AdS_5 solutions ~~susy~~
- But a simple near horizon solution is:

$$\begin{aligned} ds^2 &= \tfrac{dr^2}{3r^2} - 3r^2 dt^2 + r^2 dx_3^2 + \tfrac{B}{\sqrt{3}}(dx_1^2 + dx_2^2) \\ &= AdS_3 \times R^2 \end{aligned}$$

- Generalization: $AdS_3 \rightarrow BTZ$

central charge

- Brown-Henneaux: $c_{AdS} = \frac{3l_3}{2G_3} = \frac{N^2 BV_2}{\pi}$

- Compare with free N=4 SYM in B-field.

  Landau levels again, but now with continuous momentum parallel to $\vec{B}$

- At low energies fermion zero modes dominate, and theory flows to D=1+1 CFT

$$\frac{q_\psi \mathcal{B} V_2}{2\pi} \quad \text{zero modes per fermion}$$

$$\Rightarrow \quad c_{N=4} = \sum_\psi \frac{1}{2}\frac{q_\psi \mathcal{B} V_2}{2\pi} = \sqrt{3}\,\frac{BV_2 N^2}{2\pi}$$

$$\boxed{= \sqrt{\frac{3}{4}}\,c_{AdS}}$$

note: $c_{N=4} < c_{AdS}$

interpolating solution

- Look for solution interpolating between $\mathrm{AdS}_3 \times R^2$ and AdS_5

$$ds^2 = \frac{dr^2}{L(r)^2} + L(r)(-dt^2 + dx_3^2) + e^{2V}(dx_1^2 + dx_2^2)$$

zero temperature ➡ boost invariant

  - Solve for L(r) in terms of V(r) analytically
  - find unique V(r) solution numerically

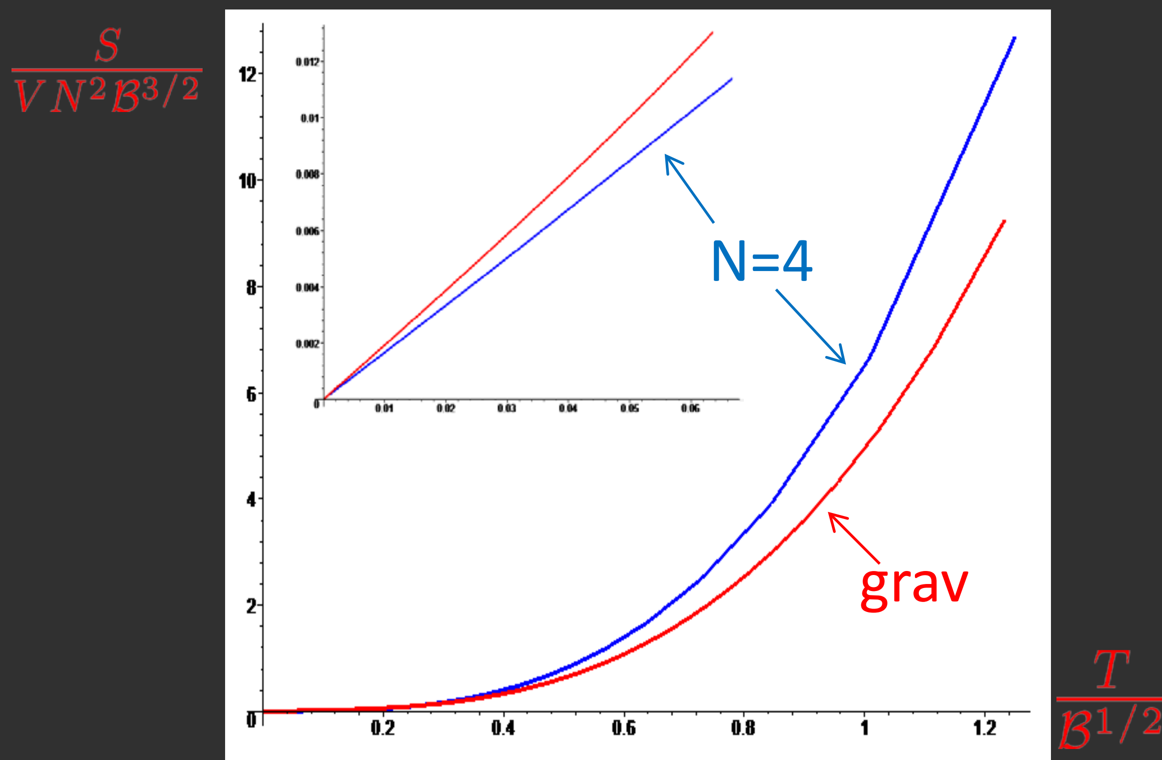
- solution describes RG flow between UV D=3+1 CFT (N=4 SYM) and IR D=1+1 CFT (fermion zero modes).

Finite temperature

- Now interpolate between $\mathrm{BTZ} \times R^2$ and AdS_5
- Two parameters: temperature and B-field
  One dimensionless combination: $T/\sqrt{B}$
- Using gauge freedom, solutions can be parameterized by B-field at horizon. Choose value and integrate out.
- Find smooth interpolating solutions for all values of $T/\sqrt{B}$

Thermodynamics

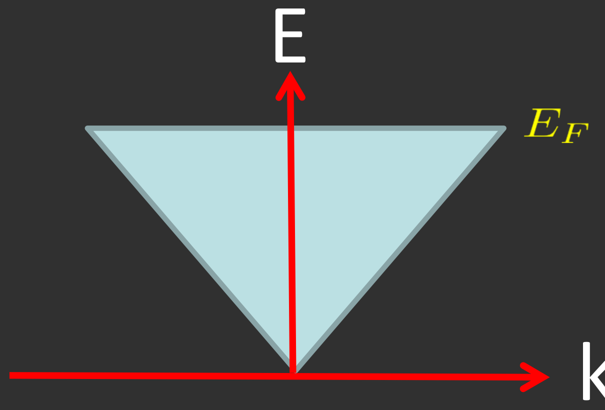
- Numerically compute S vs. T and compare with free N=4 SYM in B-field



- high T: $S_{grav} = \frac{3}{4} S_{N=4} \sim T^3$
- low T: $S = \frac{\pi}{3} cT \ , \quad S_{grav} = \sqrt{\frac{4}{3}} S_{N=4}$

Adding charge

- In CFT, adding charge builds up a Fermi sea



- New behavior can set it when $\rho \sim B^{3/2}$

Energetically favorable to start filling up higher fermionic, and bosonic, Landau levels

Charged solutions

- Construct solutions with nonzero T, B, and Q
- General ansatz:

$$ds^2 = \frac{dr^2}{L^2 - MN} + M dt^2 + 2L dt dx_3 + N dx_3^2 + e^{2V}(dx_1^2 + dx_2^2)$$

$$F = E(r) dr \wedge dt + B dx_1 \wedge dx_2 + P(r) dx_3 \wedge dr$$

horizon: $L^2 - MN = 0$

- Solutions stationary but not static, due to combined effect of charge, B-field and CS term

Near horizon geometry

- Look for factorized near horizon solutions

$$M_3 \times R^2$$ free parameter

- Can find the general such solution assuming translation invariance along the boundary

$$ds^2 = \frac{dr^2}{4B^2r^2} - \left(\tilde{\alpha}r + \frac{q^2}{k(k-\frac{1}{2})}r^{2k}\right)dt^2 + 4Brdtdx_3 + \frac{B}{\sqrt{3}}(dx_1^2 + dx_2^2)$$

$$F = Bdx_1 \wedge dx_2 + qr^{k-1}dr \wedge dt$$

- 3D part: "null warped", "Schrodinger", "pp-wave"

3D geometry studied in context of TMG

e.g. (Anninos et. al)

Scaling

- $\tilde{\alpha} = 0$ solution is scale invariant under

$$r \to \lambda r, \quad t \to \frac{1}{\lambda^k} t, \quad x_3 \to \frac{1}{\lambda^{1-k}} x_3$$

$$\Rightarrow \quad t \sim (x_3)^{\frac{k}{1-k}} \quad \Rightarrow \quad \boxed{z = \frac{k}{1-k}}$$

z = dynamical critical exponent?

- Naively, scale invariance fixes entropy density:

$$s \sim T^{1/z} \sim T^{\frac{1-k}{k}}$$

but z is negative when k>1 !?

- Also: no finite T version of above solution
- Need to recall that solution is embedded in AdS_5

Numerics for charged solutions

- write general ansatz:

$$ds^2 = \frac{dr^2}{L^2 - MN} + M dt^2 + 2L dt dx_3 + N dx_3^2 + e^{2V}(dx_1^2 + dx_2^2)$$

$$F = E(r) dr \wedge dt + B dx_1 \wedge dx_2 + P(r) dx_3 \wedge dr$$

- fix gauge near the horizon:

$$L(0) = M(0) = V(0) = 0, \quad N(0) = -M'(0) = 1$$

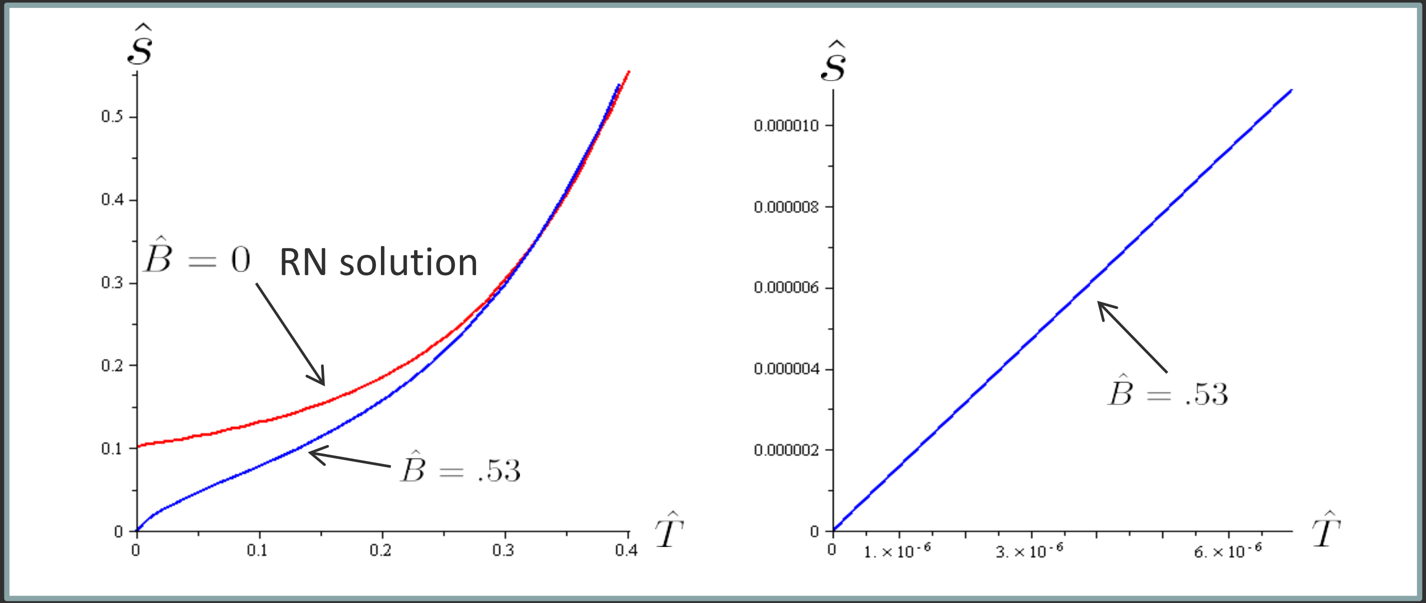
$$B = b\ , \quad E = q$$

free parameters (b,q) equivalent to two dimensionless combinations of (B,Q,T)

- Shoot out to infinity and compute physical parameters. Repeat for new (b,q)

Numerical results

- Compute $\hat{s}(\hat{T})$ at $k=\sqrt{4/3}$ and “large enough” $\hat{B}$

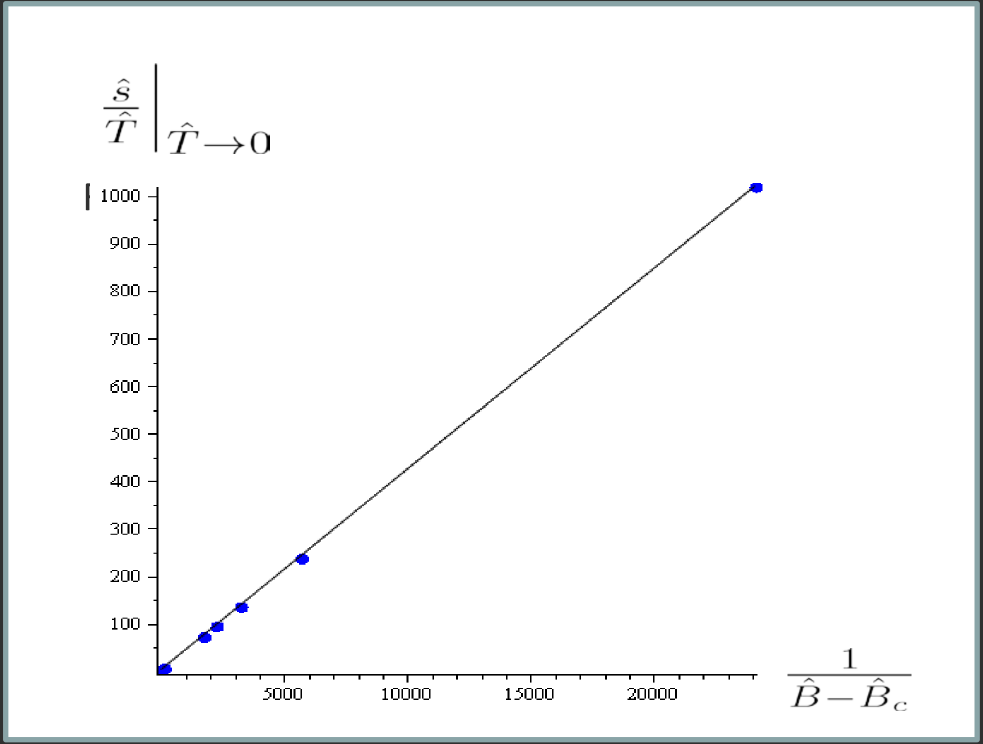


$$\hat{s}=\frac{s}{(B^3+\rho^2)^{1/2}} \qquad \hat{T}=\frac{T}{(B^3+\rho^2)^{1/6}} \qquad \hat{B}=\frac{B}{\rho^{2/3}}$$

- low temperature entropy vanishes linearly

Numerical results

- repeating for smaller $\hat{B}$ again yields linear behavior, but with diverging coefficient as $\hat{B} \to \hat{B}_c$



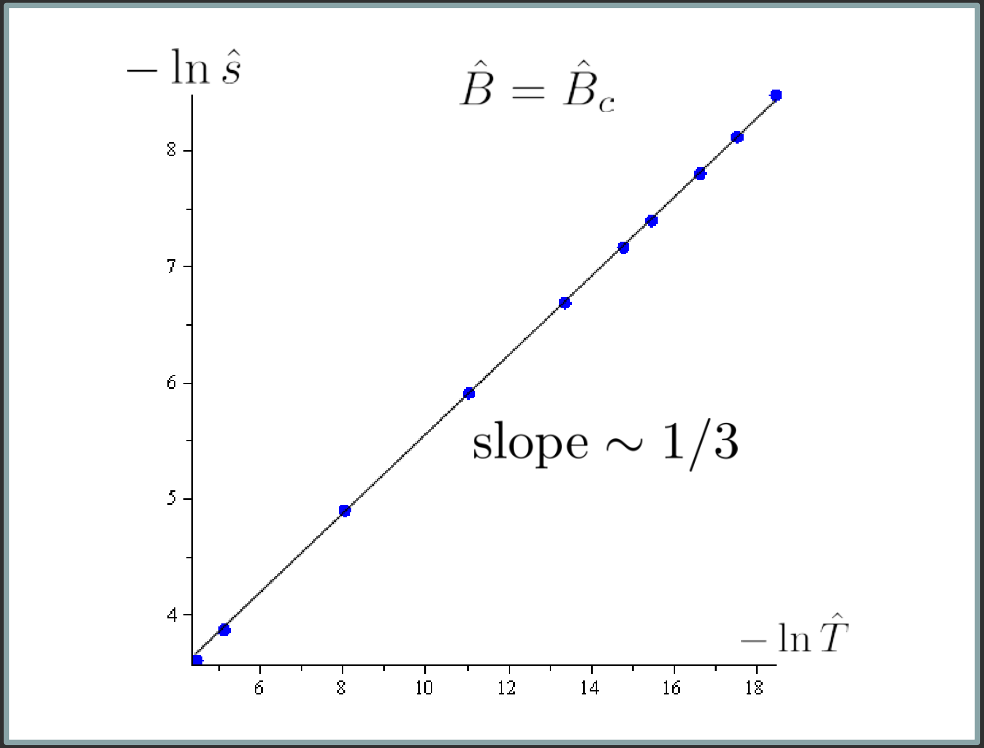
$$\hat{s} \sim \frac{\hat{T}}{\hat{B}-\hat{B}_c}$$

$$\hat{B}_c = .499424\ldots$$

Numerical results

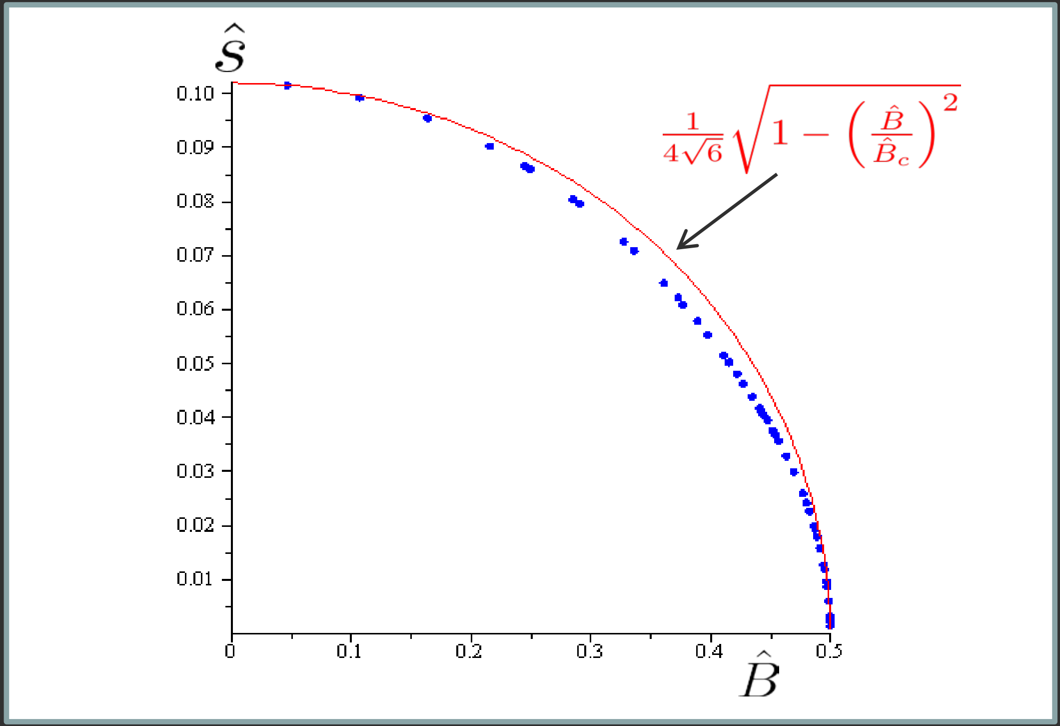
- Sitting right at $\hat{B} = \hat{B}_c$ gives new scaling:

$$\hat{s} \sim \hat{T}^{1/3}$$

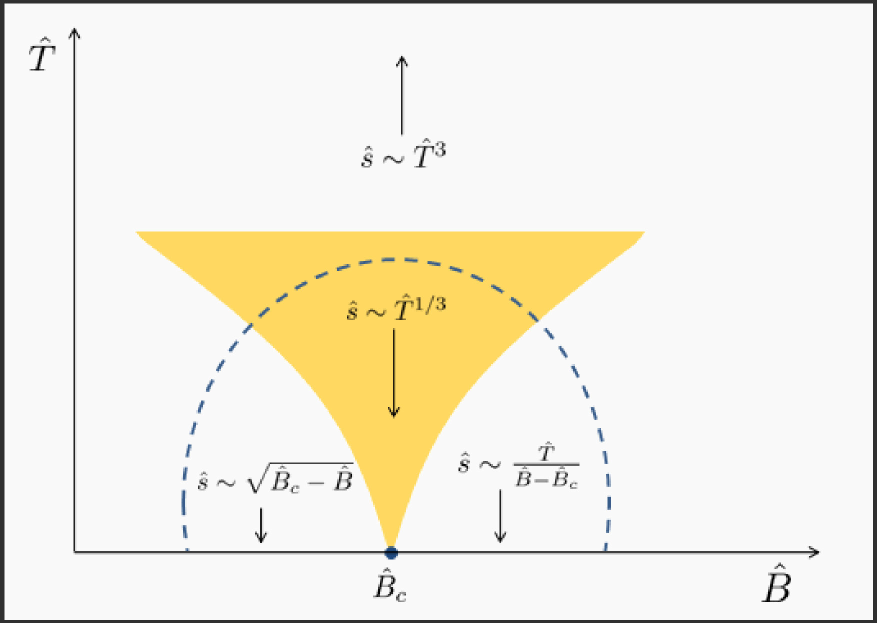


Numerical results

- Decreasing the magnetic field to $\hat{B} < \hat{B}_c$ gives nonzero extremal entropy



Summary of thermodynamics



$$\hat{s} = \hat{T}^{1/3} f\left(\frac{\hat{B}-\hat{B}_c}{\hat{T}^{2/3}}\right)$$

in scaling region

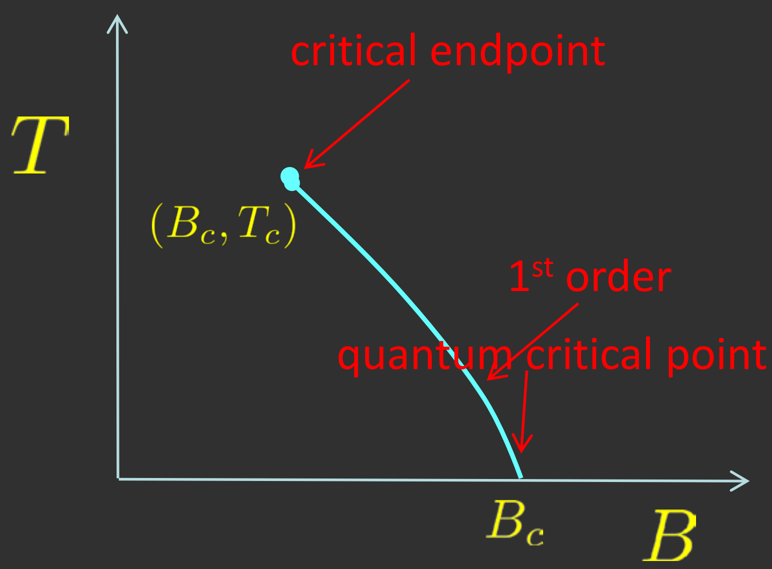
- Near d spatial-dim critical point with dynamical exponent z and relevant coupling g of dimension Δ

$$\hat{s} = \hat{T}^{d/z} f\left(\frac{g^{2/\Delta}}{\hat{T}^{2/z}}\right)$$

$$\Rightarrow \quad d = 1, \quad z = 3, \quad \Delta = 2$$

Metamagnetic quantum criticality

- *Finite temperature* metamagnetic phase transition analogous to liquid-vapor transition



• magnetization jumps, but no change in symmetry

holographic version: (Lifschytz/Lippert)

- Tune some parameter to bring $T_c \to 0$
- Scale invariant QFT with relevant operator corresponding to change of B

- Entropic landscape of $Sr_3Ru_2O_7$

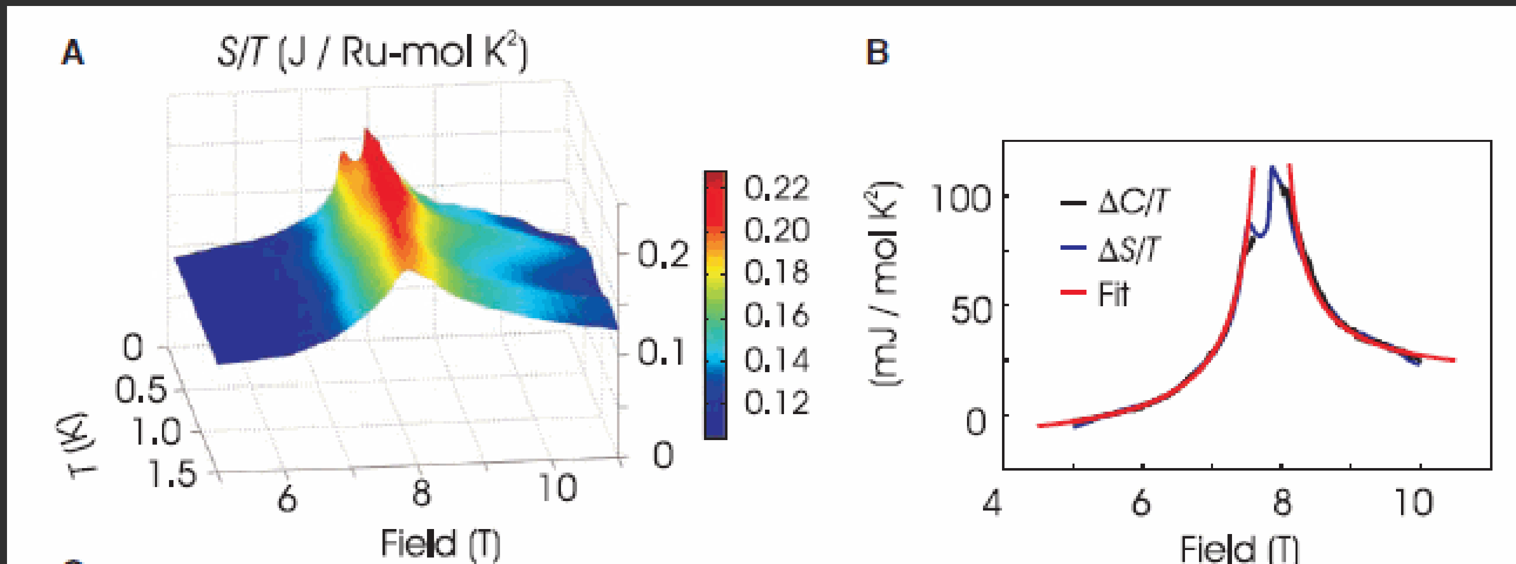


Fig. 4. (**A**) Entropy divided by temperature as a function of applied field and temperature for $Sr_3Ru_2O_7$. (**B**) Both $\Delta S/T$ (blue) and the specific heat, $\Delta C/T$ (black), diverge as $[(H - H_c)/H_c]^{-1}$ outside the cut off region (red lines). $\Delta C/T$ is fixed relative to $\Delta S/T$ at 5.5 T on the low field side and 9.5 T on the high field side. (**C**) Isoentropes dip to

(Rost et. al.
Science, Sept. 2009)

- Approaching the critical point from the Fermi liquid region the entropy diverges like what we had:

$$S \sim \frac{T}{B - B_c}$$

Hertz-Millis

- standard approach based on Hertz-Millis theory

- integrate out gapless fermions to get effective action for bosonic collective mode:

$$S = \int d^d k d\omega \left(\frac{\omega}{|k|} + k^2 + |B - B_c| \right) |\psi(\omega, k)|^2 + \ldots$$

$$\Longrightarrow \quad z = 3 \qquad \Delta_B = 2$$

$\boxed{\text{same as before}}$

Other values of k

- repeating numerics for other k shows:
  - k > 3/4: $\hat{s} \sim \hat{T}^{1/3}$ near critical point
  - 1/2 < k < 3/4: $\hat{s} \sim \hat{T}^{\frac{1-k}{k}}$ near critical point

    agrees with scaling predicted from WAdS_3 !
  - k < 1/2: no critical point $(\hat{B}_c \to \infty)$

Analytical treatment

- Proceed by looking for a T=0 solution that interpolates between null warped near horizon

$$ds^2 = \frac{dr^2}{4B^2r^2} - \left(\tilde{\alpha} r + \frac{q^2}{k(k-\frac{1}{2})} r^{2k}\right) dt^2 + 4Br dt dx_3 + \frac{B}{\sqrt{3}}(dx_1^2 + dx_2^2)$$

$$F = B dx_1 \wedge dx_2 + q r^{k-1} dr \wedge dt$$

and asymptotic AdS_5

- Can solve problem in terms of one “universal” function

- All charge is carried by flux outside the horizon

- implies that $\mathcal{N} = 4$ SYM at nonzero (Q, B) flows to null-warped CFT at low energies

Critical B-field

- Near horizon null-warped geometry

$$ds^2 = \frac{dr^2}{4B^2r^2} - \left(\tilde{\alpha} r + \frac{q^2}{k(k-\frac{1}{2})} r^{2k}\right) dt^2 + 4Brdtdx_3 + \frac{B}{\sqrt{3}}(dx_1^2 + dx_2^2)$$

controls value of $\hat{B}$

- require $\tilde{\alpha} \geq 0$ in order for this geometry to arise as T=0 limit of smooth finite T black hole

$$\Longrightarrow \quad \hat{B} \geq \hat{B}_c$$

- Formula for $\hat{B}_c$ agrees with numerical results

Low T Thermodynamics

- Need to carry out a matched asymptotic expansion analysis
  - near region: deformed BTZ
  - far region: T=0 charged solution discussed previously

- Although BTZ has $s \sim T$, this does not carry over to full solution, due to the nontrivial relation between near and far time and space coordinates

Low T Thermodynamics

- Full calculation gives low temperature entropy:

$$\hat{s} = \frac{\pi}{6}\left(\frac{\hat{B}^3}{\hat{B}^3 - \hat{B}_c^3}\right)\hat{T} \qquad (\hat{B} > \hat{B}_c)$$

$$\hat{s} = \left(\frac{\pi}{576k\hat{B}_c^3}\right)^{1/3}\hat{T}^{1/3} \qquad (\hat{B} = \hat{B}_c)$$

- Also get explicit result for scaling function:

$$\hat{s} = \hat{T}^{1/3} f\left(\frac{\hat{B} - \hat{B}_c}{\hat{T}^{2/3}}\right)$$

- For $\hat{B} < \hat{B}_c$ a finite extremal entropy branch arises, which is yet to be understood

1/2 < k < 3/4

- In this window, there exist hairy ANW black hole solutions

  i.e. V(r) varies nontrivially

- These solutions control low T thermodynamics, and one indeed finds

$$\hat{s} \sim \hat{T}^{1/z} = \hat{T}^{\frac{1-k}{k}}$$

in agreement with numerics

Correlators

- Low energy physics can be probed by computing correlation functions
- correlators can be computed analytically at low momentum via matched asymptotic expansion
- Results reveal emergent IR Virasoro and current algebras, connection to Luttinger liquids, etc.

Summary and future directions

- Obtained solutions corresponding to D=3+1 susy gauge theories at finite temperature, charge, and B-field
- Solutions exhibit interesting T=0 critical point
- Low T thermodynamics understood analytically from gravity side
- Correlators can be found analytically
- Goal for the future: understand what is driving the phase transition in the gauge theory