Electromagnetic duality in AdS/QHE:

magnetic monopoles and the quantum Hall effect

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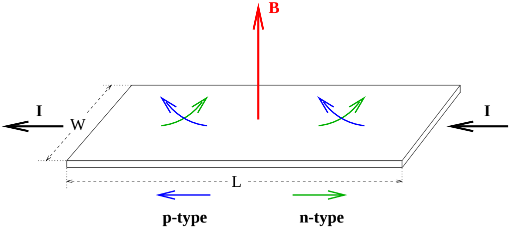
Outline

The Quantum Hall effect
- Review of QHE
- Modular symmetry
  - Quantum phase transitions and temperature flow
  - Selection rule

AdS/CFT correspondence
- Duality in bulk theory
- Schwinger-Zwanziger quantisation
- Bulk solution and scaling exponents

The Classical Hall Effect

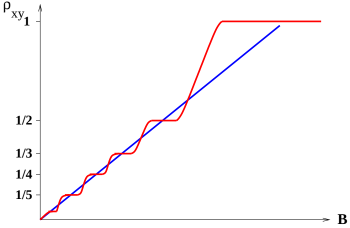
Edwin Hall (1879)



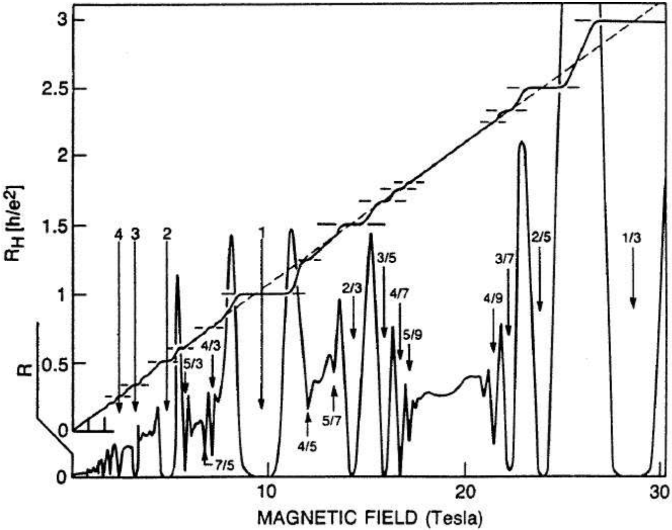
- $J_\alpha = \sigma_{\alpha\beta} E^\beta$ $\qquad (\sigma_{xx} = \sigma_{yy})$.
- $z = x + iy \;\Rightarrow\; \rho := \rho_{xy} + i\rho_{xx}, \quad \sigma = \sigma_{xy} + i\sigma_{xx} = -1/\rho.$
- Classically: $\rho^{cl}_{xy} = -\frac{B}{en}$, $\rho^{cl}_{xx} = \frac{m}{e^2 n \tau_c}$, $\tau_c =$ collision time.
- $\mathrm{Im}(\rho) \geq 0 \;\Leftrightarrow \mathrm{Im}(\sigma) \geq 0.$

The Quantum Hall Effect

von Klitzing (1980); Tsui + Störmer (1982)

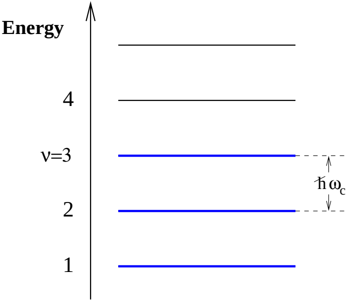


- For low T, high purity and high particle density, resistance is quantised: $R_H = h/e^2 = 25.812807449(86)\ k\Omega$.
- $\rho = \frac{1}{p}\left(\frac{h}{e^2}\right),\ p \in \mathbf{Z}; \qquad \sigma = p\left(\frac{e^2}{h}\right),$ von Klitzing (1980).
  Integer QHE
- $\sigma = \frac{p}{q}\left(\frac{e^2}{h}\right),\ p, q \in \mathbf{Z},\ q$ odd, Tsui + Störmer (1982).
  Fractional QHE



Stormer (1992)

The Quantum Hall Effect



$$\omega_c = \frac{eB}{m}$$

- Free particles in transverse $B \Rightarrow$ Harmonic Oscillator.
- Degeneracy/unit area: $g = \left|\frac{eB}{h}\right| = \left|\frac{B}{e}\right|\left(\frac{e^2}{h}\right)$.
- Filling factor, $\nu := n/g = \frac{ne}{B}\left(\frac{h}{e^2}\right) \Rightarrow |\sigma_{xy}^{cl}| = \nu\left(\frac{e^2}{h}\right)$, $(\sigma_{xx} = 0)$.
- Filled Landau Levels inert $\Rightarrow$ pseudo-particle excitations are the same for $\sigma \to \sigma + 1$, $\left(\frac{e^2}{h} = 1\right)$.
- Particle-hole symmetry (one-third full = one-third empty):

$$\sigma \to 1 - \bar{\sigma}.$$

The Law of Corresponding States

Kivelson, Lee and Zhang (1992); Lütken+Ross (1992)

- Physics of pseudo-particle excitations is symmetric under

$$\begin{array}{rccc} \text{Landau level addition:} & \sigma & \to & \sigma + 1 \\ \text{Flux attachment:} & -\dfrac{1}{\sigma} & \to & -\dfrac{1}{\sigma} + 2 \\ \text{Particle-Hole Interchange:} & \sigma & \to & 1 - \overline{\sigma} \end{array}$$

Modular Group:

$$\Gamma_0(2) \subset \Gamma(1): \qquad \sigma \to \frac{a\sigma+b}{c\sigma+d}$$

$$a,\, b,\, c,\, d \in \mathbf{Z},\ ad - bc = 1 \text{ with } c \text{ even.}$$

$\Gamma_0(2)$: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbf{Z})$, $\det\gamma = 1$; c even.

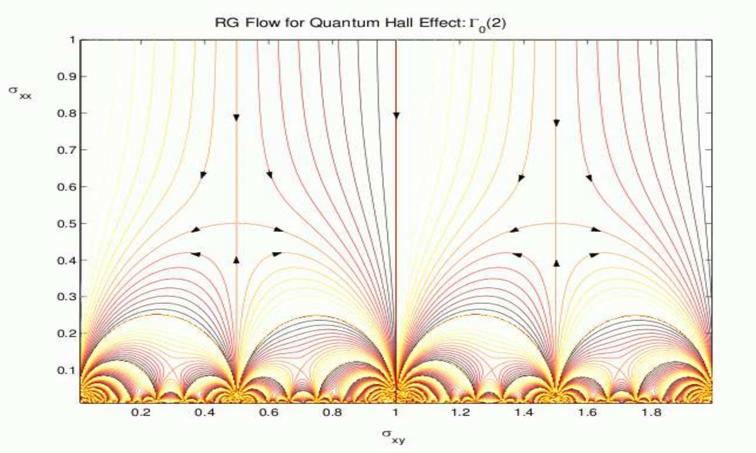
$\Gamma(1)$, Fradkin+Kivelson (1996); $\Gamma(2)$, Georgelin et al (1996)
Witten [hep-th/0307041]; Leigh+Petkou [hep-th/0309171].

Quantum Phases

- Hall Plateaux $\Leftrightarrow$ Phases of 2-D "Electron" Gas.
- Law of Corresponding States: maps between phases.
- $\sigma_{xy} : p/q \to p'/q'$ is a Quantum Phase Transition, Fisher (1990).
- For $\sigma_{xy} = 1/q$, quasi-particles have electric charge e/q, Laughlin (1983).
- Second order phase transition between phases: $\xi \approx |\Delta B|^{-\nu_\xi}$, $\Delta B = B - B_c$.
- Simple scaling $\Rightarrow \sigma(T, \Delta B, n, \dots) = \sigma(\Delta B/T^\kappa, n/T^{\kappa'}, \dots)$
  - Superuniversality: κ and κ' are the same for all transitions.
  - σ flows as T is varied.
  - Experimentally:

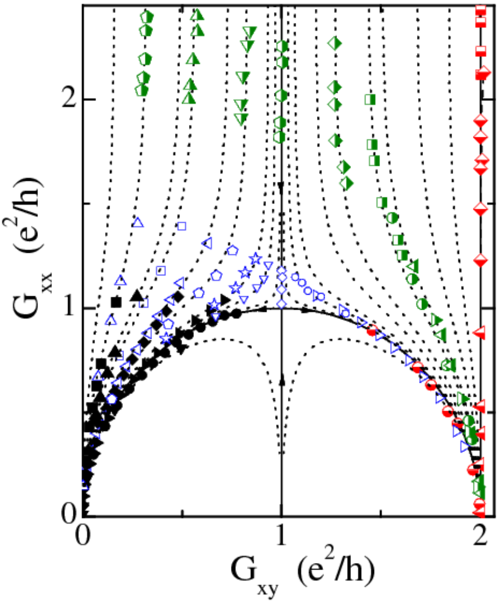
$$\kappa = 0.42 \pm 0.01 \text{ (Wanli et al (2009))}$$

Temperature flow



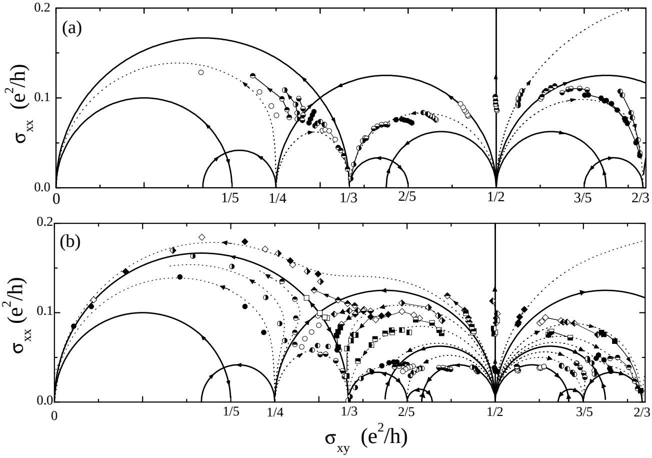
Burgess+Lütken (1997), BD (1999), Lütken+Ross (2009).

- Attractive fixed points at $\sigma_{xy} = p/q$, q odd; repulsive points for q even.
- Fractal structure near real axis (no true fractals in Nature). (Wigner crystal for $\sigma_{xy} < \frac{1}{7}$; $\hbar\omega_c < k_B T$ $(\sigma_{xy} >> 1)$.)

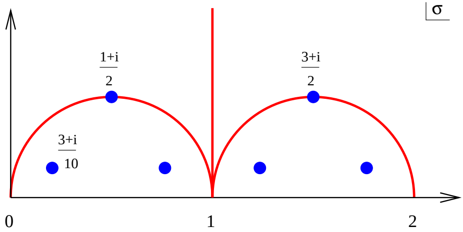


$\sigma(\Delta B/T^{\kappa}, n/T^{\kappa'})$

S.S. Murzin et al (2002)



S.S. Murzin et al., (2005)



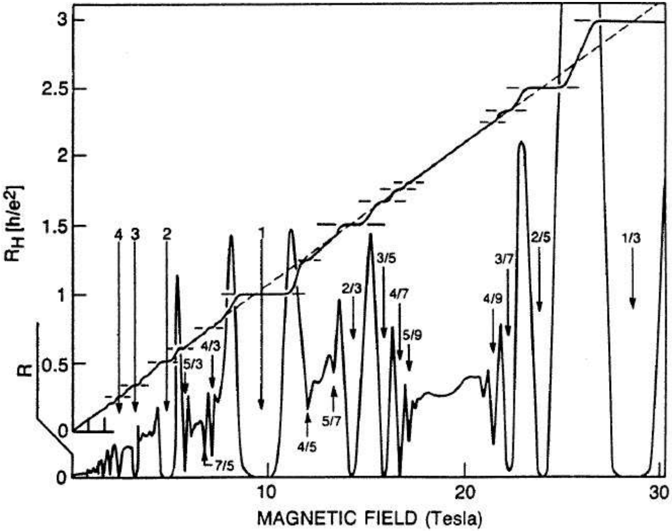
- Any $\sigma_{xy} : \frac{p}{q} \to \frac{p'}{q'}$ can be obtained from $\sigma : 0 \to 1$ by some $\gamma \in \Gamma_0(2)$,

  $\gamma(0) = \frac{p}{q}, \gamma(1) = \frac{p'}{q'} \Rightarrow \gamma = \begin{pmatrix} p' - p & p \\ q' - q & q \end{pmatrix}$, $\det\gamma = 1 \Rightarrow$

- Selection Rule:

$$p'q - pq' = 1$$

BPD (1998).



Stormer (1992)

AdS/CFT Correspondence

- AdS/CFT: $(2+1)$-d sample is boundary of $(3+1)$-d gravity coupled to matter.
- QHE: strongly interacting electrons in $2+1$ dimensions.
  - Conductivity is dimensionless $\Rightarrow$ CFT in $(2+1)$-d.
  - Use classical gravity + matter in $(3+1)$-d bulk.
- Bulk theory: AdS_4-black-hole-dyon (AdS_4-Reissner-Nordström) coupled to $U(1)$ gauge theory with charged matter.
  Hartnoll+Kovtun [0704.1160]; Keski-Vakkuri+Per Kraus [0905.4538].

Electromagnetic duality in bulk

- Include dilaton ϕ and axion χ in bulk:

$$\mathcal{S} = \int \left\{ \frac{1}{2\kappa^2}\left(R - 2\Lambda - \frac{1}{2}\left(\partial\phi.\partial\phi + e^{2\phi}\partial\chi.\partial\chi\right)\right) - \frac{1}{2}e^{-\phi}F^2 - \frac{\chi}{2}F\widetilde{F} \right\} \sqrt{-g}\,d^4x \qquad \left(\widetilde{F}^{\mu\nu} = \frac{1}{2}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}F_{\rho\sigma}\right)$$

- Constitutive relations: $D_i = G_{i0}$, $H^i = \frac{1}{2}\epsilon^{ijk}G_{jk}$

$$G^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} \mathbf{D} = e^{-\phi}\mathbf{E} + \chi\mathbf{B} \\ \mathbf{H} = e^{-\phi}\mathbf{B} - \chi\mathbf{E} \end{cases}$$

- Define $\tau := \chi + ie^{-\phi}$; $\mathcal{F} = F - i\widetilde{F}$ and $\mathcal{G} = -\widetilde{G} - iG$
- Equations of motion invariant under $Sl(2,\mathbf{R})$, $(ad - bc - 1)$.

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix}\mathcal{G}\\ \mathcal{F}\end{pmatrix} \to \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\mathcal{G}\\ \mathcal{F}\end{pmatrix}$$

Gibbons+ Rasheed (1995)

- Generalises EM duality: $\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} \to \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}$.

Modular symmetry

- Dyons $\{Q, M\}$, $\{Q', M'\}$ $\Rightarrow$

$$\begin{aligned} Q'M - M'Q &= 2\pi N\hbar \\ Q = n_e e,\; M = n_m(h/e) &\Rightarrow n_m n'_e - n'_m n_e = N \end{aligned}$$

- Dirac-Schwinger-Zwanziger quantisation condition: semi-classically, $Sl(2, \mathbf{R}) \to Sl(2, \mathbf{Z})$.
- Generalises Dirac quantisation condition: for $\{Q', 0\}$ and $\{0, M\}$, $Q'M = 2\pi N\hbar$, $SO(2) \to \mathbf{Z}_2$.
- In full quantum theory expect a sub-modular group, *e.g.* $\Gamma(2)$ for $\mathcal{N} = 2$ SUSY Yang-Mills, Seiberg+Witten (1994).

Bulk theory

- Bulk metric: $(\Lambda = -\frac{3}{L^2},\ v = \frac{L}{r})$

$$ds^2 = L^2\lambda^2 \left\{ -f(v)\frac{dt^2}{v^{2z}} + \frac{dr^2}{f(v)v^2} + \frac{dx^2 + dy^2}{v^2} \right\}$$

- $v \to 0$ $(r \to \infty)$ is UV-limit of $(2+1)$-d theory.
- z: Lifshitz scaling exponent $(x \to \ell x,\ y \to \ell y,\ t \to \ell^z t)$.
- $f(v_h) = 0 \ \Rightarrow$ finite temperature, $T = \frac{|f'(v_h)|}{4\pi v_h^{z-1} L}$.
- Matter: classical $Sl(2, \mathbf{R})$ symmetry

Einstein-dilaton-axion- $\begin{cases} \textit{Maxwell} \\ \textit{DBI} \end{cases}$ Gibbons+ Rasheed (1995)

-

$$S_{U(1)} = -\mathcal{T}\int d^4x \left[\sqrt{-det\left(g_{\mu\nu} + \ell^2 e^{-\phi/2} F_{\mu\nu}\right)} - \sqrt{-g} \right]$$
$$-\frac{1}{4}\int d^4x \sqrt{-g}\,\chi F_{\mu\nu}\widetilde{F}^{\mu\nu}.$$

Scaling exponents from AdS/CFT

- Calculate CFT conductivity using probe brane ⇒

$$\sigma\left(\frac{B}{T^{\frac{2}{z}}}, \frac{n}{T^{\frac{2}{z}}}\right)$$

  Karch+O'Bannon [0705.3870]; O'Bannon [0708.1994]; Hartnoll et al [0912.1061]; Goldstein et al [0911.3589; 1007.2490].

- Bulk solution (Taylor [0812.0530]): $\chi = 0$,

$$f(v) = 1 - \left(\frac{v}{v_h}\right)^{z+2}, \quad e^{-\phi} = v^4, \quad F^{vt} = \frac{Q\,v^2}{L^2}$$

$$Sl(2, \mathbf{R}) \quad \Rightarrow \quad z = 5.$$

Scaling dimension (Bayntum, Burgess, Lee+BPD, arXiv:[1007.1917])

$$z = 5 \quad \Rightarrow \quad \kappa = \frac{2}{z} = 0.4$$

Summary

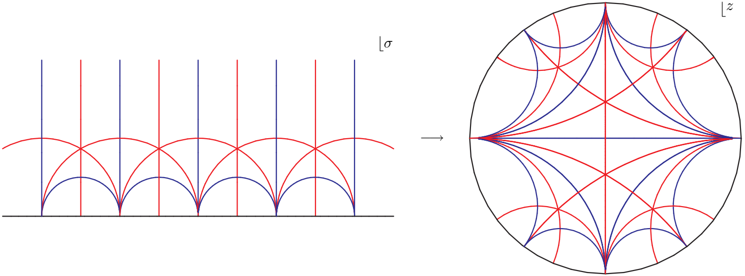
- Modular transformations, $\sigma \to \frac{a\sigma+b}{c\sigma+d}$, map between phases of the QHE $(\sigma = \sigma_{xy} + i\sigma_{xx})$.
  - The map is a symmetry of QHE vacua.
- Fractional charges in the quantum Hall effect are analogous to the Witten effect in 4-dimensions.
- 4-d bulk theory with electromagnetic duality,

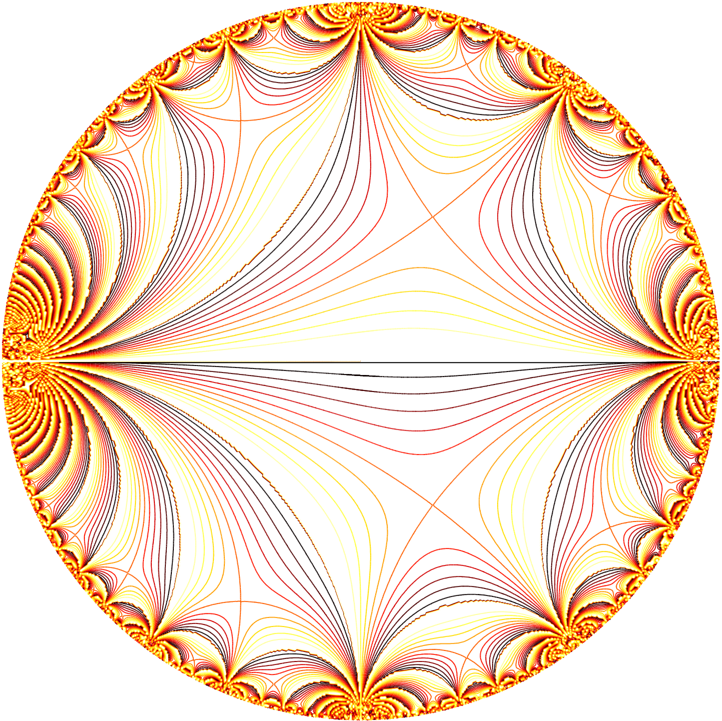
$$Sl(2, \mathbf{R}) \to \Gamma \subset Sl(2, \mathbf{Z})/\mathbf{Z}_2$$

can give AdS/CFT with QHE in $2+1$-d.

  - Probe brane in bulk gives information about conductivity in CFT.
  - Parameters in bulk solution $\Leftrightarrow$ exponents in CFT, $\kappa = 2/5$.

The symmetries of the modular group are beautifully exhibited by transforming to $z = \frac{1+i\sigma}{1-i\sigma}$, (Poincaré map):





Maxwell - Chern - Simons Theory

- Classical relation

$$B = -en\rho^{cl}_{xy} \Rightarrow \sigma^{cl}_{xy} B = J^0 \qquad (J^0 = en \text{ and } \sigma_{xx} = 0)$$

from

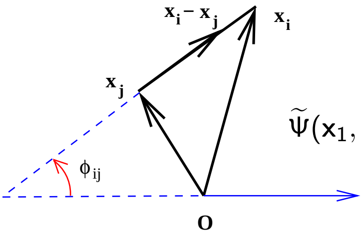
$$\begin{aligned} \mathcal{L}_{eff}[A_0] &= -\sigma_{xy} A_0 B + A_0 J^0 + \cdots \Rightarrow \\ \mathcal{L}_{eff}[A] &= -\frac{\sigma_{xy}}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu + \cdots . \end{aligned}$$

- Include Ohmic conductivity, $\sigma_{xx} = i \lim_{\omega \to 0}(\omega\epsilon(\omega))$,

$$\mathcal{L}_{eff}[A] = -\frac{\epsilon}{4}F^2 - \frac{\sigma_{xy}}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu + \cdots ,$$

$$\mathcal{L}_{eff}[A] \approx \frac{i\sigma_{xx}}{4\omega}F^2 - \frac{\sigma_{xy}}{4}\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + A_\mu J^\mu + \cdots .$$

Statistical Gauge Field



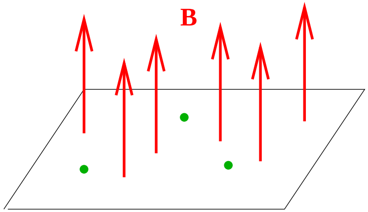
$$\widetilde{\Psi}(\mathrm{x}_1, \ldots, \mathrm{x}_N) = e^{\frac{i\vartheta}{\pi}(\Sigma_{i<j}\phi_{ij})}\Psi(\mathrm{x}_1, \ldots, \mathrm{x}_N)$$

- Interchange $i \leftrightarrow j$, $\phi_{ij} \to \phi_{ij} + \pi \Rightarrow$ phase changes by ϑ.
- $\vartheta = 2\pi k$, identity; $\vartheta = \pi(2k+1)$, Fermions $\leftrightarrow$ Bosons.
- In Hamiltonian, $-i\hbar\nabla - e\mathbf{A} \to -i\hbar\nabla - e(\mathbf{A} + \mathbf{a})$:

$$a_\alpha(\mathrm{x}_i) = \frac{\hbar\vartheta}{e\pi}\sum_{j\neq i}\nabla^{(i)}_\alpha \phi_{ij} \Rightarrow \epsilon^{\beta\alpha}\nabla^{(i)}_\beta a_\alpha(\mathrm{x}_i) = \frac{2\hbar\vartheta}{e}\sum_{j\neq i}\delta(\mathrm{x}_i - \mathrm{x}_j).$$

- $$b(\mathrm{x}) := \epsilon^{\beta\alpha}\nabla_\beta a_\alpha(\mathrm{x}) = \frac{2\hbar\vartheta}{e}n(\mathrm{x}).$$

Composite fermions and flux attachment



$\vartheta = -2\pi$

- $b := \epsilon^{\beta\alpha}\nabla_\beta a_\alpha \Rightarrow \frac{b}{n} = \frac{\vartheta}{\pi}\left(\frac{h}{e}\right) \overset{(\vartheta=-2\pi)}{=} -2\left(\frac{h}{e}\right)$.
- $A_\mu \to A'_\mu = A_\mu + a_\mu$.
- $\nu = 1/3 \Leftrightarrow \nu_{CF} = 1$, $\left(\frac{1}{\nu} \Leftrightarrow \frac{1}{\nu} + 2\right)$.

  Fractional QHE = Integer QHE for composite Fermions, Jain (1990).

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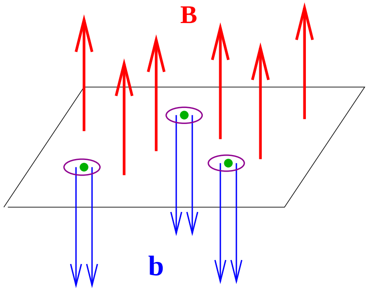

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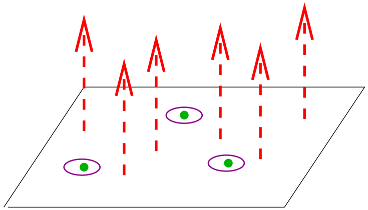

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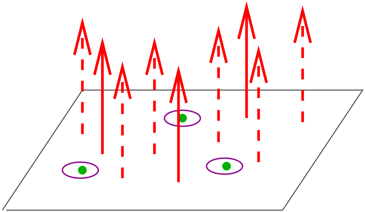

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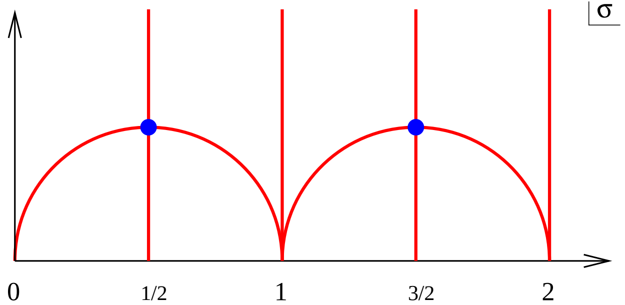
Scaling flow and modular symmetry

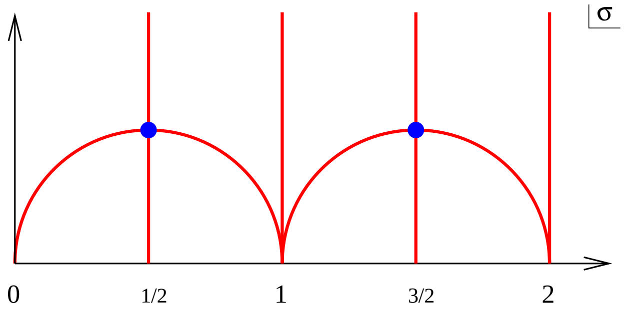


- Action of $\Gamma_0(2)$ commutes with flow $\Rightarrow$ fixed points of $\Gamma_0(2)$ are fixed points of flow $(\exists \gamma \in \Gamma_0(2)$ s.t. $\gamma(\sigma_*) = \sigma_*)$.

Assume:

- Integers are attractive.
- $\sigma_{xx} \downarrow$ as $T \downarrow$, (semi-conductor behaviour)
- Modular symmetry $\Rightarrow$ even denominators are repulsive.

Scaling flow and modular symmetry



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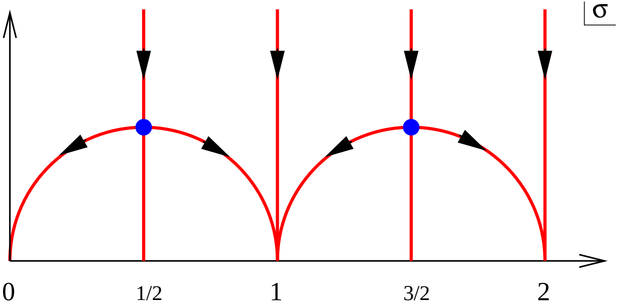

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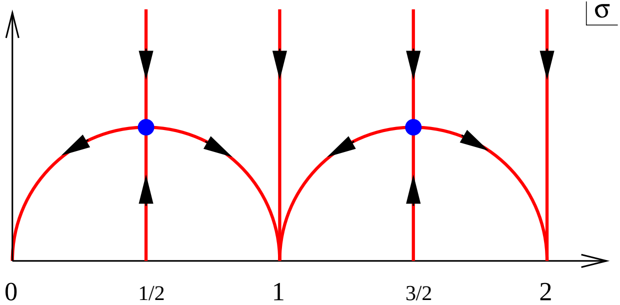

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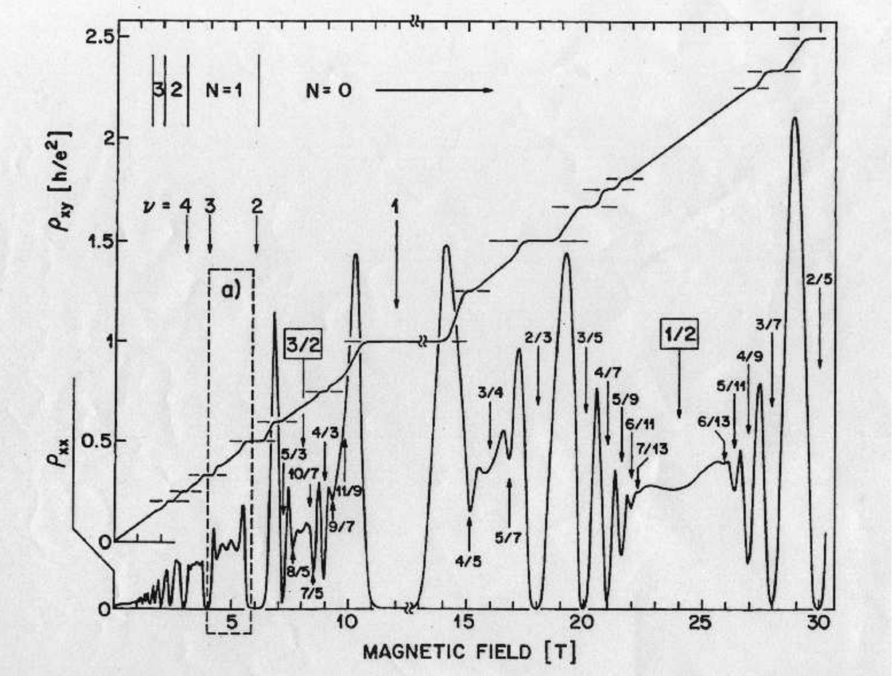
Phase Transitions and Scaling Flow

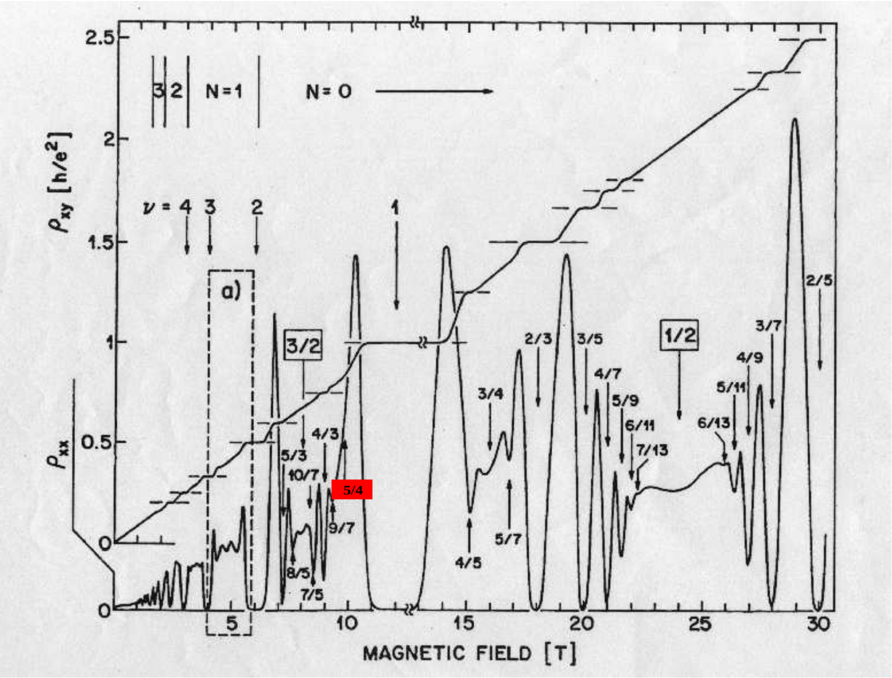
- Second order phase transition between phases: $\xi \approx |\Delta B|^{-\nu_\xi}$, $\Delta B = B - B_c$.
- Simple scaling $\Rightarrow \sigma(T, \Delta B, n) = \sigma(\Delta B/T^{\kappa}, n/T^{\kappa'})$.
- l_T =scattering length, let $s(l_T)$ be monotonic in l_T (and T). Define $\beta(\sigma,\bar{\sigma}) = \frac{d\sigma}{ds}$, then
$$\beta(\gamma(\sigma), \overline{\gamma(\sigma)}\,) = \frac{1}{(c\sigma+d)^2}\beta(\sigma,\bar{\sigma}).$$
- Action of $\Gamma_0(2)$ commutes with flow $\Rightarrow$ fixed points of $\Gamma_0(2)$ are fixed points of flow ($\exists \gamma \in \Gamma_0(2)$ s.t. $\gamma(\sigma_*) = \sigma_*$). (Fixed points of the flow need not be fixed points of the modular group.)

Scaling Flow and Modular Symmetry

- Change variables from σ to $f(\sigma) := \frac{\Theta_3^4\Theta_4^4}{\Theta_4^4 - \Theta_3^4}$ where $\Theta_3(\sigma) := \sum_{n=0}^{\infty} e^{i\pi n^2\sigma}$, $\Theta_4(\sigma) := \sum_{n=0}^{\infty}(-1)^n e^{i\pi n^2\sigma}$.
- $f(\gamma(\sigma)) = f(\sigma)$ is invariant under $\Gamma_0(2)$.
- Define $\beta_f(f, \overline{f}) := \frac{df}{ds}$.
- Let $q := e^{i\pi\sigma}$, then $f(\bar{q}) = \overline{f(q)}$ and $\sigma_{xy} \to -\sigma_{xy}$ is $q \to \bar{q}$.
- Particle-hole symmetry, $f \leftrightarrow \bar{f} \Rightarrow \frac{d\bar{f}}{ds} = \beta_f(\bar{f}, f)$.
- $\frac{d\bar{f}}{ds} = \overline{\beta_f(f, \bar{f})} = \beta_f(\bar{f}, f) \Rightarrow \beta_f$ is real if f is real $\Rightarrow$
  Any curve on which f is real is an integral curve of the flow,
  C. Burgess + BPD (2000).



ρ_{xy} [h/e²]
ρ_{xx}
2.5
2
1.5
1
0.5
0
0
3 2 N=1
N= 0
ν = 4 3 2
1
a)
3/2
1/2
2/3
3/5
4/7
5/9
6/11
7/13
3/4
4/5
5/7
5/3
4/3
10/7
11/9
9/7
8/5
7/5
6/13
5/11
4/9
3/7
2/5
5
10
15
20
25
30
MAGNETIC FIELD [T]



2.5
2
1.5
1
0.5
0
0
ρ_{xy} [h/e^2]
ρ_{xx}
3 2 N=1
N= 0
ν = 4 3 2
1
a)
3/2
1/2
5/3
4/3
10/7
5/4
9/7
8/5
7/5
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MAGNETIC FIELD [T]

Law of Corresponding States for Bosons

- $\Gamma(1) = Sl(2, \mathbf{Z})/\mathbf{Z}_2$ is generated by $\mathbf{S} : \sigma \to -1/\sigma$ and $\mathbf{T} : \sigma \to \sigma + 1$.
- For Fermionic pseudo-particles $\Gamma_0(2)$ is generated by $\mathbf{L} = \mathbf{T}$ and $\mathbf{F}^2 = \mathbf{S}^{-1}\mathbf{T}^{-2}\mathbf{S}$.
- For Bosonic pseudo-particles, start with $\Gamma_0(2)$ and turn Fermions into Bosons by adding a single unit of flux, $\mathbf{F} = \mathbf{S}^{-1}\mathbf{T}^{-1}\mathbf{S}$. This conjugates $\Gamma_0(2)$ by $\mathbf{F}$.
- Define $\Gamma_\theta := \mathbf{F}^{-1}\Gamma_0(2)\mathbf{F}$, generated by $\mathbf{S}$ and $\mathbf{T}^2$, Shapere+Wilczek (1989).

Bosonic Charge Carriers

Law of Corresponding States

$$\Gamma_\theta \subset \Gamma(1): \qquad \sigma \to \frac{a\sigma+b}{c\sigma+d}$$

$$a,\ b,\ c,\ d \in \mathbf{Z},\ ad - bc = 1$$

either a, d both odd and c, d both even or *vice versa*.

- Fixed point at $\sigma = i$ (superconductor – insulator transition), Fisher (1990).
- Realisable in 2-d bosonic systems: e.g. high mobility thin film superconductors, C. Burgess and +BPD (2001).