Scattering amplitudes in AdS/CFT integrability

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based on work with

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The setting

AdS/CFT correspondence: Fascinating link between conformal quantum field theories without gravity and string theory, a theory with gravity (both classical and quantized).

Two major (recent) developments in the maximal susy AdS$_5$/CFT$_4$ system:

4d max. susy Yang-Mills theory ⇔ Superstring theory on AdS$_5$ × S$^5$

1. Integrability in AdS/CFT:
   - Scaling dimensions alias string spectrum from Bethe equations ⇒ (close) to solution of the spectral problem

2. Scattering amplitudes in maximally susy Yang-Mills
   - Generalized unitarity methods and recursion relations ⇒ all tree-level amplitudes and many high-loop/high-multiplicity results available
   - Relation to light-like Wilson loops/strongly coupled string description ⇒ emergence of dual superconformal or Yangian symmetry

This talk: Review some of the progress and show how to connect the two
1. Introduction

2. Trees: Complete analytic result and relation to massless QCD

3. Symmetries: Superconformal, dual conformal and Yangian invariance
   [Drummond, Henn, JP; JHEP 0905, arXiv:0902]

4. Loops: Overview and novel Higgs regulator
$N = 4$ super Yang Mills: The simplest interacting 4d QFT

- **Field content:** All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: $A_\mu$, $\mu = 0, 1, 2, 3$, $\Delta = 1$
  - 6 real scalars: $\Phi_I$, $I = 1, \ldots, 6$, $\Delta = 1$
  - $4 \times 4$ real fermions: $\Psi_{\alpha A}$, $\bar{\Psi}_{\dot{\alpha}}$, $\alpha, \dot{\alpha} = 1, 2$, $A = 1, 2, 3, 4$, $\Delta = 3/2$
  - Covariant derivative: $D_\mu = \partial_\mu - i [A_\mu, \ast]$, $\Delta = 1$

- **Action:** Unique model completely fixed by SUSY
  \[
  S = \frac{1}{g_{YM}^2} \int d^4 x \text{Tr} \left[ \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \Phi_I)^2 - \frac{1}{4} [\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \bar{\Psi}^A \sigma^\alpha_\mu D_\mu \Psi^A_{\dot{\alpha}} - \frac{i}{2} \bar{\Psi}^A_{\dot{\alpha}} \sigma^A_{\mu} \epsilon^{\alpha\beta} [\Phi_I, \Psi^B_{\beta}] - \frac{i}{2} \Psi^A_{\alpha} \sigma^A_{\mu} \epsilon^{\dot{\alpha}\dot{\beta}} [\Phi_I, \bar{\Psi}^B_{\dot{\beta}}] \right]
  \]

- $\beta g_{YM} = 0$: Quantum Conformal Field Theory, 2 parameters: $N$ & $\lambda = g_{YM}^2 N$

- Shall consider ’t Hooft planar limit: $N \to \infty$ with $\lambda$ fixed.

- Is the 4d interacting QFT with highest degree of symmetry!

  $\Rightarrow$ **“H-atom of gauge theories”**
Superconformal symmetry

- Symmetry: $\mathfrak{so}(2, 4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2, 2|4)$

- Poincaré: $p^{\alpha \dot{\alpha}} = p^\mu (\sigma^\mu)^{\dot{\alpha} \beta}$, $m_{\alpha \beta}$, $\bar{m}_{\dot{\alpha} \dot{\beta}}$

- Conformal: $k_{\alpha \dot{\alpha}}$, $d$ ($c$ : central charge)

- R-symmetry: $r_{AB}$

- Poncaré Susy: $q^{\alpha A}$, $\bar{q}^{\dot{\alpha}}_A$  Conformal Susy: $s^{\alpha}_A$, $\bar{s}^{\dot{\alpha}}_A$

- $4 + 4$ Supermatrix notation $\bar{A} = (\alpha, \dot{\alpha}|A)$

\[
J^{\bar{A}}_{\bar{B}} = \begin{pmatrix}
  m^{\alpha}_{\beta} - \frac{1}{2} \delta^{\alpha}_{\beta} (d + \frac{1}{2} c) & k^{\alpha}_{\dot{\beta}} & s^{\alpha}_{\dot{B}} \\
p^{\dot{\alpha} \beta} & m^{\dot{\alpha}}_{\dot{\beta}} + \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} (d - \frac{1}{2} c) & \bar{q}^{\dot{\alpha}}_B \\
q^{A}_{\beta} & \bar{s}^{A}_{\dot{\beta}} & -r^{A}_{\dot{B}} - \frac{1}{4} \delta^{A}_{\dot{B}} c
\end{pmatrix}
\]

- Algebra:

\[
[J^{\bar{A}}_{\bar{B}}, J^{\bar{C}}_{\bar{D}}] = \delta^{\bar{C}}_{\bar{B}} J^{\bar{A}}_{\bar{D}} - (-1)^{(|\bar{A}|+|\bar{B}|)(|\bar{C}|+|\bar{D}|)} \delta^{\bar{A}}_{\bar{D}} J^{\bar{C}}_{\bar{B}}
\]
Scaling dimensions:
Local operators $O_n(x) = \text{Tr}[\mathcal{W}_1 \mathcal{W}_2 \ldots \mathcal{W}_n]$ with $\mathcal{W}_i \in \{D^k \Phi, D^k \Psi, D^k F\}$

$$\langle O_a(x_1) O_b(x_2) \rangle = \frac{\delta_{ab}}{(x_1 - x_2)^2 \Delta_a(\lambda)}$$

$$\Delta_a(\lambda) = \sum_{l=0}^{\infty} \lambda^l \Delta_{a,l}$$

Wilson loops:

$$\mathcal{W}_C = \left\langle \text{Tr} P \exp i \oint_C ds \left( \dot{x}^\mu A_\mu + i |\dot{x}| \theta^I \Phi_I \right) \right\rangle$$

Scattering amplitudes:

$$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda) = \begin{cases} \text{UV-finite} \\ \text{IR-divergent} \end{cases}$$

helicities: $h_i \in \{0, \pm \frac{1}{2}, \pm 1\}$
Superstring in $\text{AdS}_5 \times S^5$

\[ I = \sqrt{\lambda} \int d\tau d\sigma \left[ G_{mn}^{(\text{AdS}_5)} \partial_a X^m \partial^a X^n + G_{mn}^{(S^5)} \partial_a Y^m \partial^a Y^n + \text{fermions} \right] \]

- $ds^2_{\text{AdS}} = R^2 \frac{dx^2 + dz^2}{z^2}$ has boundary at $z = 0$
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, classical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $O(1/\sqrt{\lambda})$
- $\text{AdS}_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- Quantization unsolved!
- String coupling constant $g_s = \frac{\lambda}{4\pi N} \to 0$ in 't Hooft limit
- Isometries: $\mathfrak{so}(2, 4) \times \mathfrak{so}(6) \subset \mathfrak{psu}(2, 2|4)$
- Include fermions: Formulate as $\frac{\text{PSU}(2,2|4)}{\text{SO}(1,4) \times \text{SO}(5)}$ supercoset model [Metsaev, Tseytlin]
$\Delta a(\lambda)$ spectrum of scaling dimensions $\iff E(\lambda)$ string excitation spectrum solved (?)

Wilson loop $\mathcal{W}_C$ $\iff$ minimal surface

$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda)$ $\iff$ open string amps

Figure 1: (a) String theory description for the scattering of $M$ gluons in the large $N$ limit. Putting the $M$D3-branes at different positions $z_i \neq 0$ serves as a regulator and also allows us to exhibit dual conformal symmetry. (b) Gauge theory analogue of (a): a generic scattering amplitude at large $N$ (here: a sample two-loop diagram).

between the $M$ separated D3-branes, which are our external scattering states. Then there are the "heavy" gauge fields corresponding to the strings stretching between the coincident $N$D3-branes and one of the $M$ branes. These are the massive particles running on the outer line of the diagrams, see figure 1. In doing so, we argue that dual conformal symmetry, suitably extended to act on the Higgs masses as well, is an exact, i.e. unbroken, symmetry of the scattering amplitudes.

This unbroken symmetry has very profound consequences. It was already noticed in [17] that the integrals contributing to loop amplitudes in $N = 4$ SYM have very special properties under dual conformal transformations, but this observation was somewhat obscured by the infrared regulator. With our infrared regularisation, the dual conformal symmetry is exact and hence so is the symmetry of the integrals. Therefore, the loop integrals appearing in our regularisation will have an exact dual conformal symmetry. This observation severely restricts the class of integrals allowed to appear in an amplitude. As a simple application, triangle sub-graphs are immediately excluded.

The alert reader might wonder whether computing a scattering amplitude with several, distinct Higgs masses might not be hopelessly complicated. In fact, this is not the case. The different masses are crucial for the unbroken dual conformal symmetry to work. However, once we have used this symmetry in order to restrict the number of basis loop integrals, we can set all Higgs masses equal and think about the common mass as a regulator. As we will show in several examples, computing the small mass expansion in this regulator is particularly simple. In fact, to two loops, only very simple (two-) and (one-)fold Mellin-Barnes integrals were needed.
Scattering amplitudes in $\mathcal{N} = 4$ SYM

- Consider $n$-particle scattering amplitude

- Planar amplitudes most conveniently expressed in color ordered formalism:

$$A_n(\{p_i, h_i, a_i\}) = (2\pi)^4 \delta^{(4)}(\sum_{i=1}^{n} p_i) \sum_{\sigma \in S_n/\mathbb{Z}_n} g^{n-2} \text{tr}[t^{a_{\sigma_1}} \ldots t^{a_{\sigma_n}}]$$

\[ \times A_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\}; \lambda = g^2 N) \]

$A_n$: Color ordered amplitude. Color structure is stripped off.

Helicity of $i$th particle: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm \frac{1}{2}$ gluino
Spinor helicity formalism

Express momentum and polarizations via commuting spinors $\lambda^\alpha$, $\tilde{\lambda}^{\dot{\alpha}}$:

$$p^{\alpha\dot{\alpha}} = (\sigma^\mu)^{\alpha\dot{\alpha}} p_\mu = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \quad \Leftrightarrow \quad p_\mu p^\mu = \det p^{\alpha\dot{\alpha}} = 0$$

Choice of helicity determines polarization vector $\varepsilon^\mu$ of external gluon

$$h = +1 \quad \varepsilon^{\alpha\dot{\alpha}} = \frac{\lambda^\alpha \tilde{\mu}^{\dot{\alpha}}}{[\tilde{\lambda} \tilde{\mu}]} \quad [\tilde{\lambda} \tilde{\mu}] := \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{\dot{\alpha}} \tilde{\mu}^{\dot{\beta}}$$

$$h = -1 \quad \tilde{\varepsilon}^{\dot{\alpha}\alpha} = \frac{\mu^{\alpha} \tilde{\lambda}^{\dot{\alpha}}}{\langle \lambda \mu \rangle} \quad \langle \lambda \mu \rangle := \epsilon_{\dot{\alpha}\dot{\beta}} \lambda^{\dot{\alpha}} \mu^{\dot{\beta}}$$

$\mu$, $\bar{\mu}$ arbitrary reference spinors.

E.g. scalar products: $2p_1 \cdot p_2 = \langle \lambda_1, \lambda_2 \rangle [\tilde{\lambda}_2, \tilde{\lambda}_1] = \langle 1, 2 \rangle [2, 1]$
Trees
Classify gluon amplitudes by number of helicity flips.

- By SUSY Ward identities: \( \mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+) \)
  true to all loops.

- Maximally helicity violating (MHV) amplitudes

\[
\mathcal{A}_n(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \delta^4\left(\sum_i p_i\right) \frac{\langle i, j \rangle^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle} \quad \text{[Parke, Taylor]}
\]

- Next-to-maximally helicity amplitudes (\( N^k \)MHV) have more involved structure!

\[
A_{n,m} : g_+^{n-m} g_-^m
\]

[Picture from T. McLoughlin]
On-shell superspace

- Augment $\lambda_i^\alpha$ and $\tilde{\lambda}_i^{\dot{\alpha}}$ by Grassmann variables $\eta_i^A$, $A = 1, 2, 3, 4$
- On-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$ with on-shell superfield:

$$
\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2} \eta^A \eta^B S_{AB}(p) + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \tilde{\Gamma}^D(p) + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p)
$$

- Superamplitudes: $\langle \Phi(\lambda_1, \tilde{\lambda}_1, \eta_1) \Phi(\lambda_2, \tilde{\lambda}_2, \eta_2) \ldots \Phi(\lambda_n, \tilde{\lambda}_n, \eta_n) \rangle$

Packages all $n$-parton gluon$^\pm$-gluino$^{\pm1/2}$-scalar amplitudes

- General form of tree superamplitudes:

$$
A_n = \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})
$$

Conservation of super-momentum: $\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^A) = (\sum_i \lambda^\alpha \eta_i^A)^8$

- $\eta$-expansion of $\mathcal{P}_n$ yields $N^k$ MHV-classification of superamps as $h(\eta) = -1/2$

$$
\mathcal{P}_n = \mathcal{P}_n^{\text{MHV}} + \eta^4 \mathcal{P}_n^{\text{NMHV}} + \eta^8 \mathcal{P}_n^{\text{NNMHV}} + \ldots + \eta^{4n-8} \mathcal{P}_n^{\text{MHV}}
$$
Augment $\lambda^\alpha_i$ and $\tilde{\lambda}^\dot{\alpha}_i$ by Grassmann variables $\eta^A_i, A = 1, 2, 3, 4$

On-shell superspace $(\lambda^\alpha_i, \tilde{\lambda}^\dot{\alpha}_i, \eta^A_i)$ with on-shell superfield:

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$$

Superamplitudes: $\langle \Phi(\lambda_1, \tilde{\lambda}_1, \eta_1) \Phi(\lambda_2, \tilde{\lambda}_2, \eta_2) \ldots \Phi(\lambda_n, \tilde{\lambda}_n, \eta_n) \rangle$

Packages all $n$-parton gluon$^\pm$-gluino$^{\pm1/2}$-scalar amplitudes

General form of tree superamplitudes:

$$
\mathcal{A}_n = \frac{\delta^{(4)}(\sum_i \lambda^\alpha_i \tilde{\lambda}^\dot{\alpha}_i) \delta^{(8)}(\sum_i \lambda^\alpha_i \eta^A_i)}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})
$$

Conservation of super-momentum: $\delta^{(8)}(\sum_i \lambda^\alpha \eta^A_i) = (\sum_i \lambda^\alpha \eta^A_i)^8$

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\mathcal{P}_n = \mathcal{P}_n^{\text{MHV}} + \eta^4 \mathcal{P}_n^{\text{NMHV}} + \eta^8 \mathcal{P}_n^{\text{NNMHV}} + \ldots + \eta^{4n-8} \mathcal{P}_n^{\text{MHV}}
$$
Superamplitudes and BCFW recursion

- Efficient way of constructing tree-level amplitudes via BCFW recursion
  [Britto, Cachazo, Feng + Witten '04, 05]

\[
A_n = \sum_i A^{h}_{i+1} \frac{1}{P_i^2} A^{-h}_{n-i+1}
\]

- \(N\)-point amplitudes are obtained recursively from lower-point amplitudes
- All amplitudes are on-shell
- Special cases can be solved analytically, e.g. split-helicity amplitudes
  \(\mathcal{A}(-, \ldots, -, +, \ldots, +)\)  [Roiban, Spradlin, Volovich]

- Reformulation of recursion relations in on-shell superspace via shift in \((\lambda_i, \tilde{\lambda})\) and \(\eta_i\)  [Elvang et al, Arkani-Hamed et al, Brandhuber et al]

- **Super BCFW recursion** is much simpler and can be solved analytically!

\[\mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})\] known in closed analytical form at tree-level  [Drummond, Henn]

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The Drummond-Henn solution

\( \mathcal{P}_n \) expressed as sums over \( R \)-invariants determined by paths on rooted tree

\[
\mathcal{P}_n^{N^k \text{MHV}} = \sum_{\text{all paths of length } k} 1 \cdot R_{n,a_1b_1} \cdot R_{n,\{I_2\},a_2b_2} \cdots R_{n,\{I_p\},a_pb_p}
\]

E.g.

\[
\mathcal{P}^{\text{NMHV}} = \sum_{1 < a_1, b_1 < n} R_{n,a_1b_1}
\]

\[
\mathcal{P}_n^{N^2 \text{MHV}} = \sum_{1 < a_1, b_1 < n} R_{n,a_1b_1} \times \left[ \sum_{a_1 < a_2, b_2 \leq b_1} R_{n,\{I_2\},a_1b_1}^0 \cdot a_2b_2 \right] + \sum_{b_1 \leq a_2, b_2 < n} R_{n,a_2b_2}^{a_1b_1}
\]

Goal: Project onto component field amplitudes  

[Dixon, Henn, Plefka, Schuster]
Region momenta or dual coordinates

\[ x_i - x_{i+1} = p_i \quad x_{ij} := x_i - x_j = p_i + p_{i+1} + \cdots + p_{j-1} \]

- All amplitudes expressed via momentum invariants \( x_{ij}^2 \) and the scalar quantities:

\[
\langle n_{a_1 a_2 \ldots a_k} | a \rangle := \langle n | x_{n a_1} x_{a_1 a_2} \ldots x_{a_{k-1} a_k} | a \rangle = \lambda_n^\alpha (x_{n a_1})^\alpha \beta (x_{a_1 a_2})^\beta \gamma \ldots (x_{a_{k-1} a_k})^\delta \rho \lambda_a \rho
\]

- Building blocks for amps: \( \tilde{R} \) invariants and path matrix \( \Xi_n^{\text{path}} \)

\[
\tilde{R}_{n; \{I\};ab} := \frac{1}{x_{ab}^2} \frac{\langle a(a-1) \rangle}{\langle n \{I\} ba|a \rangle \langle n \{I\} ba|a - 1 \rangle} \frac{\langle b(b-1) \rangle}{\langle n \{I\} ab|b \rangle \langle n \{I\} ab|b - 1 \rangle}
\]

\[
\Xi_n^{\text{path}} := \left( \begin{array}{ccc}
\langle nc_0 \rangle & \langle nc_1 \rangle & \ldots & \langle nc_p \rangle \\
(\Xi_n)^{c_0}_{a_1 b_1} & (\Xi_n)^{c_1}_{a_1 b_1} & \ldots & (\Xi_n)^{c_p}_{a_1 b_1} \\
(\Xi_n)^{c_0}_{I_2};a_2 b_2 & (\Xi_n)^{c_1}_{I_2};a_2 b_2 & \ldots & (\Xi_n)^{c_p}_{I_2};a_2 b_2 \\
\vdots & \ldots & \vdots & \vdots \\
(\Xi_n)^{c_0}_{I_p};a_p b_p & (\Xi_n)^{c_1}_{I_p};a_p b_p & \ldots & (\Xi_n)^{c_p}_{I_p};a_p b_p
\end{array} \right)
\]
All gluon-gluino trees in $\mathcal{N} = 4$ SYM [Dixon, Henn, Plefka, Schuster]

- **MHV gluon amplitudes**

  $$ A_n^{\text{MHV}}(c_0^-, c_1^-) = \delta^{(4)}(p) \frac{\langle c_0 \ c_1 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \ldots \langle n \ 1 \rangle} $$

- **$N^{\text{P}}$MHV gluon amplitudes**:

  $$ A_n^{\text{N}^{\text{P}}\text{MHV}}(c_0^-, \ldots, c_{p+1}^-) = \frac{\delta^{(4)}(p)}{\langle 1 \ 2 \rangle \ldots \langle n \ 1 \rangle} \sum_{\text{all paths of length } p} \left( \prod_{i=1}^{p} \tilde{R}_{n; \{I_i\}; a_i b_i} \right) (\det \Xi)^4 $$

- **MHV gluon-gluino amplitudes (single flavor)**

  $$ A_n^{\text{MHV}}(a^-, b_q, c_{\bar{q}}) = \delta^{(4)}(p) \frac{\langle a \ c \rangle^3 \langle a \ b \rangle}{\langle 1 \ 2 \rangle \ldots \langle n \ 1 \rangle} $$

- **$N^{\text{P}}$MHV gluon-gluino amplitudes**:

  $$ A_{(q\bar{q})^{k,n}}(\ldots, c_k^-, \ldots, (c_{\alpha_i})_{q'}, \ldots, (c_{\beta_j})_{\bar{q}'}, \ldots) = \frac{\delta^{(4)}(p) \text{sign}(\tau)}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \ldots \langle n \ 1 \rangle} \times \sum_{\text{all paths of length } p} \left( \prod_{i=1}^{p} \tilde{R}_{n; \{I_i\}; a_i b_i} \right) \left( \det \Xi \right|^q_1 \frac{3}{\langle q \leftrightarrow \bar{q} \rangle} $$
All gluon-gluino trees in $\mathcal{N} = 4$ SYM [Dixon, Henn, Plefka, Schuster]

- MHV gluon amplitudes
  \[ A_{n}^{\text{MHV}}(c_0^-, c_1^-) = \delta^{(4)}(p) \frac{\langle c_0\ c_1 \rangle^4}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \ldots \langle n\ 1 \rangle} \]

- $N^{\text{PMHV}}$ gluon amplitudes:
  \[ A_{n}^{N^{\text{PMHV}}}(c_0^-, \ldots, c_{p+1}^-) = \delta^{(4)}(p) \frac{\langle 1\ 2 \rangle \ldots \langle n\ 1 \rangle}{\sum_{\text{all paths of length } p} \left( \prod_{i=1}^{p} \tilde{R}_{L_i; R_i}^{L_i; R_i; \{I_i\}; a_i} \right) \left( \det \Xi \right)^4} \]

- MHV gluon-gluino amplitudes (single flavor)
  \[ A_{n}^{\text{MHV}}(a^-, b_q, c_{\bar{q}}) = \delta^{(4)}(p) \frac{\langle a\ c \rangle^3 \langle a\ b \rangle}{\langle 1\ 2 \rangle \ldots \langle n\ 1 \rangle} \]

- $N^{\text{PMHV}}$ gluon-gluino amplitudes:
  \[ A_{(q\bar{q})^k, n}^{N^{\text{PMHV}}} (\ldots, c_k^-, \ldots, (c_{\alpha_i})_q, \ldots, (c_{\beta_j})_{\bar{q}}, \ldots) = \frac{\delta^{(4)}(p) \text{sign}(\tau)}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \ldots \langle n\ 1 \rangle} \times \sum_{\text{all paths of length } p} \left( \prod_{i=1}^{p} \tilde{R}_{L_i; R_i}^{L_i; R_i; \{I_i\}; a_i} \right) \left( \det \Xi |_q \right)^3 \det \Xi(q \leftrightarrow \bar{q}) |_{\bar{q}} \]
From $\mathcal{N} = 4$ to massless QCD trees

- Differences in color: SU(N) vs. SU(3); Fermions: adjoint vs. fundamental
- Irrelevant for color ordered amplitudes, as color d.o.f. stripped off anyway. E.g. single quark-anti-quark pair

$$A_{n}^{\text{tree}}(1\bar{q}, 2q, 3, \ldots, n) = g^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma}(3)} \cdots T^{a_{\sigma}(n)}) \bar{i}_{1}^{i_{2}}$$

Color ordered $A_{n}^{\text{tree}}(1\bar{q}, 2q, \sigma(3), \ldots, \sigma(n))$ from two-gluino-$(n - 2)$-gluon amplitude.

- For more than one quark-anti-quark pair needs to accomplish:
  1. Avoid internal scalar exchanges (due to Yukawa coupling)
  2. Allow all fermion lines present to be of different flavor

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From $\mathcal{N} = 4$ to massless QCD trees

Also worked out explicitly for 4 quark-anti-quark pairs.

**Conclusion:** Obtained all (massless) QCD trees from the $\mathcal{N} = 4$ SYM trees
GGT: Mathematica package for analytic gluon-gluino tree amplitudes

[Dixon, Henn, Plefka, Schuster, 2010] qft.physik.hu-berlin.de

\[ \text{In[9]} := \text{GGTgluon}[7, \{3, 5\}] \]
\[ \text{Out[9]} = \frac{\langle 3 | 5 \rangle^4}{\langle 1 | 2 \rangle \langle 2 | 3 \rangle \langle 3 | 4 \rangle \langle 4 | 5 \rangle \langle 5 | 6 \rangle \langle 6 | 7 \rangle \langle 7 | 1 \rangle} \]

\[ \text{In[12]} := \text{GGTgluon}[6, \{3, 5, 6\}] \]
\[ \text{Out[12]} = \left( \frac{\langle 2 | 1 \rangle \langle 4 | 3 \rangle \langle s_{2,4} \langle 6 | 3 \rangle \langle 6 | 5 \rangle + \langle 6 | 5 \rangle \langle 6 | x_{6,4} | x_{4,2} \rangle \langle 3 \rangle^4}{s_{2,4} \langle 6 | x_{6,2} | x_{2,4} \rangle \langle 6 | x_{6,2} | x_{2,4} \rangle \langle 6 | x_{6,4} | x_{4,2} | 1 \rangle \langle 6 | x_{6,4} | x_{4,2} | 2 \rangle} + \frac{\langle 2 | 1 \rangle \langle 5 | 4 \rangle \langle s_{2,5} \langle 6 | 3 \rangle \langle 6 | 5 \rangle + \langle 6 | 5 \rangle \langle 6 | x_{6,5} | x_{5,2} \rangle \langle 3 \rangle^4}{s_{2,5} \langle 6 | x_{6,2} | x_{2,5} \rangle \langle 6 | x_{6,2} | x_{2,5} \rangle \langle 6 | x_{6,5} | x_{5,2} | 1 \rangle \langle 6 | x_{6,5} | x_{5,2} | 2 \rangle} \right) + \frac{\langle 3 | 2 \rangle \langle 5 | 4 \rangle \langle s_{3,5} \langle 6 | 3 \rangle \langle 6 | 5 \rangle + \langle 6 | 5 \rangle \langle 6 | x_{6,5} | x_{5,3} \rangle \langle 3 \rangle^4}{s_{3,5} \langle 6 | x_{6,3} | x_{3,5} \rangle \langle 6 | x_{6,3} | x_{3,5} \rangle \langle 6 | x_{6,5} | x_{5,3} | 2 \rangle \langle 6 | x_{6,5} | x_{5,3} | 3 \rangle} \right) / \langle 1 | 2 \rangle \langle 2 | 3 \rangle \langle 3 | 4 \rangle \langle 4 | 5 \rangle \langle 5 | 6 \rangle \langle 6 | 1 \rangle \]

\[ \text{In[11]} := \text{GGTfermionS}[7, \{1, 7\}, \{3, 4\}, \{5, 6\}] \]
\[ \text{Out[11]} = - \left( \langle 2 | 1 \rangle \langle 4 | 3 \rangle \langle 6 | 5 \rangle \langle 7 | 1 \rangle \langle 4 | x_{2,4} | x_{7,2} \rangle \langle 7 | 7 \rangle \langle 7 | x_{7,4} | x_{4,2} \rangle \langle 3 \rangle \right) \right) / \left( s_{2,4} s_{4,6} \langle 7 | 1 \rangle \langle 7 | 5 \rangle \langle 7 | 6 \rangle + s_{2,4} \langle 7 | 1 \rangle \langle 7 | 6 \rangle \langle 7 | x_{7,6} | x_{6,4} | 5 \rangle \right)^3 \]
\[ \left( s_{2,4} s_{4,6} \langle 7 | x_{7,2} | x_{2,4} \rangle \langle 7 | x_{7,2} | x_{2,4} \rangle \langle 7 | x_{7,4} | x_{4,2} | 1 \rangle \langle 7 | x_{7,4} | x_{4,2} | 2 \rangle \langle 7 | x_{7,4} | x_{4,6} | 5 \rangle \langle 7 | x_{7,4} | x_{4,6} | 6 \rangle \langle 7 | x_{7,6} | x_{6,4} | x_{2,4} | x_{7,2} | 7 \rangle \right) + \left( s_{2,6}^2 \langle 2 | 1 \rangle \langle 3 | 2 \rangle \langle 6 | 5 \rangle \langle 7 | 1 \rangle \langle 3 \rangle \right) \right) \left( - \langle 7 | 1 \rangle \langle 7 | x_{7,6} | x_{6,2} \rangle \langle 4 \rangle \langle 7 | x_{7,6} | x_{6,2} | x_{2,3} | x_{3,6} | 3 \rangle + \langle 7 | 1 \rangle \langle 7 | x_{7,6} | x_{6,2} | 3 \rangle \langle 7 | x_{7,6} | x_{6,2} | x_{2,3} | x_{3,6} | 4 \rangle \right) \right) \left[ 16/33 \right] \left( 7 \right) \left( x_{7,2} \right) \left( x_{7,4} \right) \left( x_{4,2} \right) \left( 6 \right) \]
GGT: Mathematica package for analytic gluon-gluino tree amplitudes

[Dixon, Henn, Plefka, Schuster, 2010]  
quft.physik.hu-berlin.de

- Similar solutions for all gluon-gluino-scalar trees in $\mathcal{N} = 4$ SYM also available from the Mathematica package BCFW
- Makes use of Grassmannian approach and momentum twistors

[Bourjaily, 2010]  
[Arkani-Hamed et al]
Symmetries
\( \mathfrak{su}(2,2|4) \) invariance

- Superamplitude: \((i = 1, \ldots, n)\)

\[
\mathbb{A}_n^{\text{tree}}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}) \delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle} P_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})
\]

- Realization of \( p\mathfrak{su}(2,2|4) \) generators in \textit{on-shell superspace}, e.g. [Witten]

\[
p^{\alpha \dot{\alpha}} = \sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \quad q^{\alpha A} = \sum_{i=1}^n \lambda_i^\alpha \eta_i^A \quad \Rightarrow \text{obvious symmetries}
\]

\[
k_{\alpha \dot{\alpha}} = \sum_{i=1}^n \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \quad s_{\alpha A} = \sum_{i=1}^n \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \eta_i^A} \quad \Rightarrow \text{less obvious sym}
\]

- Invariance:

\[
\{ p, k, m, m, d, r, q, \bar{q}, s, \bar{s}, c_i \} \mathbb{A}_n^{\text{tree}}(\{\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A\}) = 0
\]

- N.B.: \textbf{Local} invariance \( h_i \mathbb{A}_n = 1 \cdot \mathbb{A}_n \)

Helicity operator:

\[
h_i = -\frac{1}{2} \lambda_i^\alpha \partial_i \alpha + \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \partial_i \dot{\alpha} + \frac{1}{2} \eta_i^A \partial_i A = 1 - c_i
\]
The $\mathfrak{su}(2, 2|4)$ generators acting in on-shell superspace $(\lambda^\alpha_i, \tilde{\lambda}^{\dot{\alpha}}_i, \eta^A_i)$:

\[
\begin{align*}
    p^{\dot{\alpha}\alpha} &= \sum_i \tilde{\lambda}^{\dot{\alpha}}_i \lambda^\alpha_i, \\
    \overline{m}_{\dot{\alpha}\beta} &= \sum_i \tilde{\lambda}^{\dot{\alpha}}_i (\partial_{i\dot{\beta}}), \\
    d &= \sum_i \left[ \frac{1}{2} \lambda^\alpha_i \partial_{i\alpha} + \frac{1}{2} \tilde{\lambda}^{\dot{\alpha}}_i \partial_{i\dot{\alpha}} + 1 \right], \\
    q^{\alpha A} &= \sum_i \lambda^\alpha_i \eta^A_i, \\
    s_{\alpha A} &= \sum_i \partial_{i\alpha} \partial_{iA}, \\
    \bar{s}^A_{\dot{\alpha}} &= \sum_i \eta^A_i \partial_{i\dot{\alpha}}.
\end{align*}
\]

\[
c = \sum_i \left[ 1 + \frac{1}{2} \lambda^\alpha_i \partial_{i\alpha} - \frac{1}{2} \tilde{\lambda}^{\dot{\alpha}}_i \partial_{i\dot{\alpha}} - \frac{1}{2} \eta^A_i \partial_{iA} \right].
\]

**N.B:** For collinear momenta picks up important additional length changing terms, due to holomorphic anomaly \( \frac{\partial}{\partial \lambda^{\dot{\alpha}}} \frac{1}{\langle \lambda, \mu \rangle} = 2\pi \tilde{\mu}_{\dot{\alpha}} \delta^2(\langle \lambda, \mu \rangle) \)

[Bargheer, Beisert, Galleas, Loebbert, McLoughlin]

[Korchemsky, Sokatchev] [Skinner, Mason] [Arkani-Hamed, Cachazo, Kaplan]
**su(2, 2|4) invariance**

- The su(2, 2|4) generators acting in on-shell superspace \((\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)\):

\[
p^{\dot{\alpha}\alpha} = \sum_i \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha ,
\]

\[
\bar{m}_{\dot{\alpha}\beta} = \sum_i \tilde{\lambda}_i(\alpha \partial_i \beta) ,
\]

\[
d = \sum_i [\frac{1}{2} \lambda_i^\alpha \partial_i \alpha + \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \partial_i \dot{\alpha} + 1] ,
\]

\[
q^{\alpha A} = \sum_i \lambda_i^\alpha \eta_i^A ,
\]

\[
s_{\alpha A} = \sum_i \partial_i \alpha \partial_i A ,
\]

\[
c = \sum_i [1 + \frac{1}{2} \lambda_i^\alpha \partial_i \alpha - \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \partial_i \dot{\alpha} - \frac{1}{2} \eta_i^A \partial_i A] .
\]

- **N.B.:** For collinear momenta picks up important additional length changing terms, due to holomorphic anomaly \(\frac{\partial}{\partial \lambda^{\dot{\alpha}}} \frac{1}{\langle \lambda, \mu \rangle} = 2\pi \tilde{\mu}_{\dot{\alpha}} \delta^2(\langle \lambda, \mu \rangle)\)

[Bargheer, Beisert, Galleas, Loebbert, McLoughlin]

[Korchemsky, Sokatchev]

[Skinner, Mason]

[Arkani-Hamed, Cachazo, Kaplan]
Dual Superconformal symmetry

- Planar MHV amplitudes are **dual conformal** $SO(2,4)$ invariant in dual space $x_i$
  [Drummond,Korchemsky,Sokatchev]
- Derives from Scattering amplitude/Wilson Loop duality
  [Alday, Maldacena;Drummond,Korchemsky,Sokatchev]
- May be extended to **dual superconformal** invariance of tree-level superamplitudes: Introduce dual on-shell superspace
  [Drummond, Henn, Korchemsky, Sokatchev]
  \[(x_i - x_{i+1})^{\alpha \dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \quad (\theta_i - \theta_{i+1})^{\alpha A} = \lambda_i^\alpha \eta_i^A\]
- Dual special conformal generator:
  \[K^{\alpha \dot{\alpha}} = \sum_i x_i^{\alpha \dot{\beta}} x_i^{\dot{\beta} \dot{\alpha}} \frac{\partial}{\partial x_i^{\beta \dot{\beta}}} + x_i^{\dot{\alpha} \beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}}\]
- Translate to on-shell superspace:
  \[x_i^{\alpha \dot{\alpha}} = \sum_{j=1}^{i-1} \lambda_j^\alpha \tilde{\lambda}_j^{\dot{\alpha}} \quad \text{and} \quad \theta_i^{\alpha A} = \sum_{j=1}^{i-1} \lambda_j^\alpha \eta_j^A\]
  \[K^{\alpha \dot{\alpha}} = \sum_{i=1}^n x_i^{\dot{\alpha} \beta} \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\beta} + x_i^{\alpha \dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\beta}} + \tilde{\lambda}_i^{\dot{\alpha}} \theta_i^{\alpha B} \frac{\partial}{\partial \eta_i^B} + x_i^{\alpha \dot{\alpha}} + \ldots\] 
  Nonlocal structure!
Dual Superconformal symmetry

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  [Drummond,Korchemsky,Sokatchev]

- Derives from Scattering amplitude/Wilson Loop duality
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- May be extended to dual superconformal invariance of tree-level superamplitudes: Introduce dual on-shell superspace
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\[
(x_i - x_{i+1})^{\alpha\dot{\alpha}} = \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \quad \left(\theta_i - \theta_{i+1}\right)^{\alpha A} = \lambda_i^{\alpha} \eta_i^A
\]

- Dual special conformal generator:

\[
K^{\alpha\dot{\alpha}} = \sum_i x_i^{\alpha\beta} x_i^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + x_i^{\dot{\alpha}\beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}}
\]

- Translate to on-shell superspace:

\[
x_i^{\alpha\dot{\alpha}} = \sum_{j=1}^{i-1} \lambda_j^{\alpha} \tilde{\lambda}_j^{\dot{\alpha}} \quad \text{and} \quad \theta_i^{\alpha A} = \sum_{j=1}^{i-1} \lambda_j^{\alpha} \eta_j^A
\]

\[
K^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} x_i^{\dot{\alpha}\beta} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\dot{\alpha}} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{beta}}} + \tilde{\lambda}_i^{\dot{alpha}} \theta_{i+1}^{\alpha B} \frac{\partial}{\partial \eta_i^B} + x_i^{\alpha\dot{\alpha}}
\]

Nonlocal structure!
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\[(x_i - x_{i+1})^{\alpha\dot{\alpha}} = \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \quad (\theta_i - \theta_{i+1})^{\alpha A} = \lambda_i^{\alpha} \eta_i^{A}\]

- Dual special conformal generator:

\[K^{\alpha\dot{\alpha}} = \sum_i x_i^{\alpha\dot{\beta}} \lambda_i^{\dot{\alpha}} \frac{\partial}{\partial \lambda_i^{\beta}} + x_i^{\alpha\dot{\beta}} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}}\]

- Translate to on-shell superspace:

\[x_i^{\alpha\dot{\alpha}} = \sum_{j=1}^{i-1} \lambda_j^{\alpha} \tilde{\lambda}_j^{\dot{\alpha}}\text{ and } \theta_i^{\alpha A} = \sum_{j=1}^{i-1} \lambda_j^{\alpha} \eta_j^{A}\]

\[K^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} x_i^{\alpha\dot{\beta}} \lambda_i^{\dot{\alpha}} \frac{\partial}{\partial \lambda_i^{\beta}} + x_i^{\alpha\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\beta}} + \tilde{\lambda}_i^{\dot{\alpha}} \theta_i^{\alpha B} \frac{\partial}{\partial \eta_i^{B}} + x_i^{\alpha\dot{\alpha}}\]

Nonlocal structure!
Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- **Superconformal + Dual superconformal** algebra = Yangian algebra

\[ Y[p\mathfrak{su}(2, 2|4)] \]

\[
\begin{align*}
[J_a^{(0)}, J_b^{(0)}] &= f_{ab}^\ c \ J_c^{(0)} & \text{conventional superconformal symmetry} \\
[J_a^{(0)}, J_b^{(1)}] &= f_{ab}^\ c \ J_c^{(1)} & \text{from dual conformal symmetry with nonlocal generators}
\end{align*}
\]

\[
J_a^{(1)} = f^{cb} \sum_{1<j<i<n} J_{i,b}^{(0)} J_{j,c}^{(0)}
\]

and super Serre relations (representation dependent).

- To define “inverted” $f^{cb} \ a$ needs to extend to $u(2, 2|4)$ for nondegen. metric

- In particular: Bosonic invariance $p_{\alpha\dot{\alpha}} \ A_n = 0$ with

\[
p_{\alpha\dot{\alpha}}^{(1)} = K_{\alpha\dot{\alpha}} + \Delta K_{\alpha\dot{\alpha}}
\]

\[
= \frac{1}{2} \sum_{i<j} (m_{i,\alpha} \gamma^{\dot{\gamma}} \delta^{\gamma}_{\dot{\alpha}} + \bar{m}_{i,\dot{\alpha}} \gamma^{\dot{\gamma}} \delta^{\gamma}_{\alpha} - d_i \delta^{\gamma}_{\alpha} \delta^{\gamma}_{\dot{\alpha}}) p_{j,\gamma^{\dot{\gamma}}} + \bar{q}_{i,\dot{\alpha}} C q_{j,\alpha} - (i \leftrightarrow j)
\]
Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- In supermatrix notation: $\bar{A} = (\alpha, \dot{\alpha}|A)$

$$
J\bar{A}^\alpha_B = \begin{pmatrix}
  m^{\alpha\beta} - \frac{1}{2} \delta^{\alpha\beta} (d + \frac{1}{2} c) & k^{\alpha\beta} & s^{\alpha B} \\
  p^{\dot{\alpha}\beta} & \bar{m}^{\dot{\alpha}\beta} + \frac{1}{2} \delta^{\dot{\alpha}\beta} (d - \frac{1}{2} c) & \bar{q}^{\dot{\alpha} B} \\
  q^{A}_{\beta} & \bar{q}^{A}_{\dot{\beta}} & -r^A_B - \frac{1}{4} \delta^A_B c
\end{pmatrix}
$$

$$
\Rightarrow \quad J^{(1)}\bar{A}^\alpha_B := -\sum_{i>j} (-1)^{|\tilde{C}|} (J_i\bar{A}^\alpha_{\tilde{C}} J_{\tilde{C}}^\alpha_B - J_j\bar{A}^\alpha_{\tilde{C}} J_{\tilde{C}}^\alpha_B)
$$

- Implies an infinite-dimensional symmetry algebra for tree-level $\mathcal{N} = 4$ SYM scattering amplitudes! $\iff$ spin chain picture

\[ J\bar{A}^\alpha_B \circ \mathbb{A}_n = 0 \quad J^{(1)}\bar{A}^\alpha_B \circ \mathbb{A}_n = 0 \]

- Including correction terms arising from collinear momenta this symmetry is constructive: Unambiguously fixes tree-level amplitudes.

[Bargheer, Beisert, Galleas, Loebbert, McLoughlin; Korchemsky, Sokatchev]
Loops
Status of higher loop/leg calculations in $\mathcal{N} = 4$ SYM

Diagram has three important ingredients:
- analytic properties
- symmetries (+IR structure)
- AdS/CFT

Bern-Dixon-Smirnov ansatz / dual conformal symmetry

restrictions from dual conformal symmetry

unitarity

BCFW recursion

Diagram has three important ingredients:
- analytic properties
- symmetries (+IR structure)
- AdS/CFT
Higher loops and Higgs regulator

- **Beyond tree-level:** Conformal and dual conformal symmetry is broken by IR divergencies \( \Rightarrow \{ \mathcal{J}, \mathcal{J}, \mathcal{K}, \mathcal{K}, \mathcal{J}, \mathcal{Q} \} \)

- **Need for regularization:** Standard method **Dim reduction** \( 10 \rightarrow 4 - \epsilon \)
  - Alternative method: Higgs regulator \( U(N + M) \rightarrow U(N) \times U(1)^M \)

  [Alday, Henn, Plefka, Schuster]

  Best way to understand dual conformal symmetry in the field theory:
  - **Inspired by AdS/CFT**
  - **IR divergences regulated by masses, at least for large \( N \)**
  - **Conjecture:** Existence of an extended dual conformal symmetry

  [Alday, Maldacena; Schabinger, 2008; Sever, McGreevy]

  - Lots of supporting evidence
  - Now essentially proven through 6D SYM
  - Heavily restricts the loop integrand/integrals!

  [Naculich, Henn, Schnitzer, Spradlin; Boels, Bern, Dennen, Huang]

  [Caron-Huot, OConnel; Dennen, Huang, 2010]

- **Related development:** (Unregulated) planar integrand has Yangian symmetry

  [Arkani-Hamed et al, 2010]

  Higgs regulator and its exact dual conformal symmetry is used to justify transition to regulated integrand
**Higher loops and Higgs regulator**

- **Beyond tree-level:** Conformal and dual conformal symmetry is broken by IR divergencies $\Rightarrow \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{Q}\}$

- **Need for regularization:** Standard method Dim reduction $10 \rightarrow 4 - \epsilon$

- **Alternative method:** Higgs regulator $U(N + M) \rightarrow U(N) \times U(1)^M$

  [Alday, Henn, Plefka, Schuster]

  Best way to understand dual conformal symmetry in the field theory:
  $\Rightarrow$ Inspired by AdS/CFT [Alday, Maldacena; Schabinger, 2008; Sever, McGreevy]
  $\Rightarrow$ IR divergences regulated by masses, at least for large $N$
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- Related development: (Unregulated) planar integrand has Yangian symmetry [Arkani-Hamed et al, 2010]

  Higgs regulator and its exact dual conformal symmetry is used to justify transition to regulated integrand
Take string picture serious:

Field Theory: Higgsing $U(N + M) \to U(N) \times U(1)^M$. One brane for every scattered particle, $N \gg M$.

Renders amplitudes IR finite. Have light $(m_i - m_j)$ and heavy $m_i$ fields.

Higgs regularization [Alday, Henn, Plefka, Schuster]
Consider the string description of the IR-regulated amplitude in the T-dual theory: The radial coordinates are related by

\[ \frac{1}{z} = r = m \]

The \( SO(2, 4) \) isometry of \( AdS_5 \) in T-dual theory is generated by \( J_{MN} \) with embedding coordinates \( M = -1, 0, 1, 2, 3, 4 \).

In Poincaré coordinates \( (r, x^\mu) \) we have

- \( J_{-1,4} = r\partial_r + x^\mu \partial_\mu = \hat{D} \)
- \( J_{4,\mu} - J_{-1,\mu} = \partial_\mu = \hat{P}_\mu \)
- \( J_{4,\mu} + J_{-1,\mu} = 2x_\mu (x_\nu \partial^\nu + r\partial_r) - (x^2 + r^2)\partial_\mu = \hat{K}_\mu \)

**Expectation:** Amplitudes regulated by Higgsing should be invariant exactly under extended dual conformal symmetry \( \hat{K}_\mu \) and \( \hat{D} \) with \( r \to m \)!
Higgsing $\mathcal{N} = 4$ Super Yang-Mills

- **Action**

$$\hat{S}^{U(N+M)}_{\mathcal{N}=4} = \int d^4 x \ Tr \left( -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{2} (D_\mu \hat{\Phi}_I)^2 + \frac{g^2}{4} [\hat{\Phi}_I, \hat{\Phi}_J]^2 + \text{ferms} \right),$$

- **Decompose into $N + M$ blocks**

$$\hat{A}_\mu = \left( \begin{array}{cc} (A_\mu)_{ab} & (A_\mu)_{aj} \\ (A_\mu)_{ia} & (A_\mu)_{ij} \end{array} \right), \quad \hat{\Phi}_I = \left( \begin{array}{cc} (\Phi_I)_{ab} & (\Phi_I)_{aj} \\ (\Phi_I)_{ia} & \delta_{IJ} \frac{m_i}{g} \delta_{ij} + (\Phi_I)_{ij} \end{array} \right),$$

$$a, b = 1, \ldots, N, \quad i, j = N + 1, \ldots, N + M,$$

- **Add $R_\xi$ gauge fixing and ghost terms. Quadratic terms** ($A_M := (A_\mu, \Phi_I)$)

$$\hat{S}_{\mathcal{N}=4} \bigg|_{\text{quad}} = \int d^4 x \left\{ -\frac{1}{2} \text{Tr} (\partial_\mu A_M)^2 - \frac{1}{2} (m_i - m_j)^2 (A_M)_{ij} (A_M)_{ji} 
- m_i^2 (A_M)_{ia} (A_M^*)_{ai} + \text{ferms} \right\}$$

- **Plus novel bosonic 3-point interactions proportional to $m_i$**
Higgsing $\mathcal{N} = 4$ Super Yang-Mills

- **Action**

$$\hat{S}^{(N+M)}_{\mathcal{N}=4} = \int d^4x \, \text{Tr} \left( -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{2} (D_\mu \hat{\Phi}_I)^2 + \frac{g^2}{4} [\hat{\Phi}_I, \hat{\Phi}_J]^2 + \text{ferms} \right),$$

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$$\hat{A}_\mu = \begin{pmatrix} (A_\mu)_{ab} & (A_\mu)_{aj} \\ (A_\mu)_{ia} & (A_\mu)_{ij} \end{pmatrix}, \quad \hat{\Phi}_I = \begin{pmatrix} (\Phi_I)_{ab} & (\Phi_I)_{aj} \\ (\Phi_I)_{ia} & \delta_{I9} \frac{m_i}{g} \delta_{ij} + (\Phi_I)_{ij} \end{pmatrix}$$

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- **Add $R_\xi$ gauge fixing and ghost terms. Quadratic terms** ($A_M := (A_\mu, \Phi_I)$)

$$\hat{S}_{\mathcal{N}=4} \bigg|_{\text{quad}} = \int d^4x \left\{ -\frac{1}{2} \text{Tr} (\partial_\mu A_M)^2 - \frac{1}{2} (m_i - m_j)^2 (A_M)_{ij} (A^M)_{ji} - m_i^2 (A_M)_{ia} (A^M)_{ai} + \text{ferms} \right\}$$

- **Plus novel bosonic 3-point interactions proportional to $m_i$**
Consider the (special) purely scalar amplitude:

\[
A_4 = \langle \Phi_4(p_1) \Phi_5(p_2) \Phi_4(p_3) \Phi_5(p_4) \rangle = i g_{YM}^2 \left( 1 + \lambda I^{(1)}(s, t, m_i) + O(a^2) \right)
\]

\[I^{(1)}(s, t, m_i): \text{ Massive box integral in dual variables } (p_i = x_i - x_{i+1})\]

\[
= \int d^4 x_5 \frac{(x_{13}^2 + (m_1 - m_3)^2)(x_{24}^2 + (m_2 - m_4)^2)}{(x_{15}^2 + m_1^2)(x_{25}^2 + m_2^2)(x_{35}^2 + m_3^2)(x_{45}^2 + m_4^2)}
\]

Reexpressed in 5d variables \(\hat{x}^M\): \[
\hat{x}_i^\mu := x_i^\mu, \quad \hat{x}_i^4 := m_i, \quad i = 1 \ldots 4
\]

\[
I^{(1)}(s, t, m_i) = \hat{x}_{13}^2 \hat{x}_{24}^2 \int d^5 \hat{x}_5 \frac{\delta(\hat{x}_5^{M=4})}{\hat{x}_{15}^2 \hat{x}_{25}^2 \hat{x}_{35}^2 \hat{x}_{45}^2}
\]

Indeed \(I^{(1)}(s, t, m_i)\) is extended dual conformal invariant: \(\hat{K}_\mu I^{(1)}(s, t, m_i) = 0\)
Extended dual conformal invariance

\[ \hat{K}_\mu I^{(1)}(s, t, m_i) := \sum_{i=1}^{4} \left[ 2x_{i\mu} \left( x^\nu_i \frac{\partial}{\partial x^\nu_i} + m_i \frac{\partial}{\partial m_i} \right) - (x^2_i + m^2_i) \frac{\partial}{\partial x^\mu_i} \right] I^{(1)}(s, t, m_i) = 0 \]

- \( m_i \) is the fifth coordinate \( x^M = (x^\mu, m) \).
- Triangle and bubble graphs are forbidden by extended conformal symmetry!
- Indeed an explicit one-loop calculation shows the cancelation of triangles.
- Dual conformal symmetry exists in 6d \( \mathcal{N} = (1, 1) \) SYM at tree-level. Also at loop-level for integrands with 4d momentum measure

[Caron-Huot, O'Connel; Dennen, Huang, 2010]

\( \Rightarrow \) Proof of extended conformal symmetry for \( \mathcal{N} = 4 \) SYM at loop level.
At 2 loops: Only one integral is allowed by extended dual conformal symmetry:

\[ \begin{array}{c}
\text{\[i_1\]} \quad \text{\[i_2\]} \\
\text{\[i_3\]} \quad \text{\[i_4\]}
\end{array} \]

Should similarly restrict possible integrals at higher loops.

- Computed this graph in \( m_i \to 0 \) limit using Mellin-Barnes techniques.
- No \( \frac{1}{\epsilon} \times \epsilon = 1 \) 'interference' as in dimred: Here \( \log(m^2) \times m^2 \to 0 \).
- Has been extended to higher loops & higher multiplicities as well as Regge limit

[Henn, Naculich, Schnitzer, Drummond]
Extracting the cusp anomalous dimension

- We have \( \mathcal{A}_4 = \mathcal{A}_4^{\text{tree}} \cdot M_4 \)

\[
M_4 \bigg|_{2\text{-loops}} = \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{array} = \exp \left[ \Gamma_{\text{cusp}}(\lambda) \right] \bigg|_{2\text{-loops}}
\]

where one splits \( M_4 \) into \( \ln m^2 \) dependent and independent pieces:

\[
\ln M_4 = D_4 + F_4 + \mathcal{O}(m^2)
\]

- Defining \( \left( \frac{\partial}{\partial \ln(m^2)} \right)^2 \ln M_4 =: -\Gamma_{\text{cusp}}(a) \) we find \( \Gamma_{\text{cusp}}(a) = 2a - 2\zeta_2 a^2 + \ldots \)

where \( a = \lambda/8\pi^2 \) in agreement with dim reg.

- Furthermore for finite piece one has

\[
F_4 = \frac{1}{2} \Gamma_{\text{cusp}}(a) \left[ \frac{1}{2} \ln^2(s/t) + \frac{1}{2} \right] + C(a)
\]

with \( C(a) = a^2 \pi^4/120 + \mathcal{O}(a^3) \).
Summary and Outlook

- **All** tree-level amplitudes in $\mathcal{N} = 4$ SYM known analytically
  - Results translate to all massless QCD trees (at least for up to 8 fermions)
  - Useful for automated evaluation of loops using unitarity (Blackhat)

- Tree level amplitudes are invariant under an infinite dimensional Yangian symmetry
  - Hint for integrability in scattering amplitudes!
  - Is form of tree amplitudes fixed by Yangian symmetry?
    $\Rightarrow$ Yes, but needs to include collinear limits $\equiv$ length changing effects

  [Bargheer, Beisert, Galleas, Loebbert, McLoughlin]

- **Challenge at weak coupling:** Does Yangian symmetry extend to the loop level?

- Breaking of dual conformal invariance at loop level under control: Best seen in Higgs regulator

- Restriction of possible integrals at higher loops.

- Can breaking of standard conformal invariance at loop level be controlled?
  **Yes!** Perturbative construction

  [Sever, Vieira] [Beisert, Henn, McLoughlin, Plefka]

- Does integrability determine the all loop planar scattering amplitudes?
**Summary and Outlook**

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**Does integrability determine the all loop planar scattering amplitudes?**
Proof of extended dual conformal invariance

- **Integral in 5d variables:** \( I^{(1)}(s, t, m_i) = \hat{x}^2_{13} \hat{x}^2_{24} \int d^5 \hat{x}_5 \frac{\delta(\hat{x}^M_{5=4})}{\hat{x}^2_{15} \hat{x}^2_{25} \hat{x}^2_{35} \hat{x}^2_{45}} \)

- **5d inversion** on all points:

  \[
  \hat{x}^\mu \to \frac{\hat{x}^\mu}{\hat{x}^2} \quad \Rightarrow \quad \hat{x}^2_{ij} \to \frac{\hat{x}^2_{ij}}{\hat{x}^2_i \hat{x}^2_j}, \quad d^5 \hat{x}_5 \to \frac{d^5 \hat{x}_5}{\hat{x}^{10}}
  \]

  Implies in particular: \( m_i \to m_i/\hat{x}^2_i \).

- Then indeed box integral covariant:

  \[
  \int d^5 \hat{x}_5 \frac{\delta(\hat{x}^M_{5=4})}{\hat{x}^2_{15} \hat{x}^2_{25} \hat{x}^2_{35} \hat{x}^2_{45}} \to \int d^5 \hat{x}_5 \frac{\delta(\hat{x}^M_{5=4}) \hat{x}^2_5}{\hat{x}^{10}_5} \hat{x}^8_5 \hat{x}^2_{15} \hat{x}^2_{25} \hat{x}^2_{35} \hat{x}^2_{45} \hat{x}^8_5 \hat{x}^2_{1} \hat{x}^2_{2} \hat{x}^2_{3} \hat{x}^2_{4}
  \]

  \( I^{(1)}(s, t, m_i) \) is also **4d translation invariant**

  \( \Rightarrow \) **Extended dual conformal invariance:** \( \hat{K}_\mu I^{(1)}(s, t, m_i) = 0 \)

- Triangles and bubbles are not invariant!
Potential problem [Beisert;Witten]: We have singled out particle 1 ⇔ Yangian-generators are not cyclic but color ordered scattering amplitudes are cyclic??

Resolution: Consider the Yangian generators produced by singling out particle 2:

\[ \tilde{J}^{(1)}_a = f^{cb} a \sum_{2<j<i<n+1} J_{i,b}^{(0)} J_{j,c}^{(0)} \]

then one shows

\[ J^{(1)} \bar{A}_B - \tilde{J}^{(1)} \bar{A}_B = \delta \bar{A}_B J^{\bar{C}}_{1,\bar{C}} = \ldots = \delta \bar{A}_B c_1 \]

Importantly \( c_i \bar{A}_n = 0 \) locally! Hence level one generators \( J^{(1)} \bar{A}_B \) are cyclic when acting on amplitudes.

Linked to vanishing Killing form of superalgebra \( (-1)^{|c|} f_{ac}^d f_{bd}^c = 0 \)

⇒ [K. Zarembo's talk]