Scattering amplitudes in AdS/CFT integrability

Jan Plefka

Humboldt-Universität zu Berlin

based on work with

Fernando Alday, Lance Dixon, James Drummond,
Johannes Henn and Theodor Schuster

Institute for Theoretical Physics, Universiteit van Amsterdam

The setting

AdS/CFT correspondence: Fascinating link between conformal quantum field theories without gravity and string theory a theory with gravity (both classical and quantized)

Two major (recent) developments in the maximal susy AdS_5/CFT_4 system:

4d max. susy Yang-Mills theory $\Leftrightarrow$ Superstring theory on $AdS_5 \times S^5$

1. Integrability in AdS/CFT:
   - Scaling dimensions alias string spectrum from Bethe equations
     $\Rightarrow$ (close) to solution of the spectral problem
2. Scattering amplitudes in maximally susy Yang-Mills
   - Generalized unitarity methods and recursion relations
     $\Rightarrow$ all tree-level amplitudes and many high-loop/high-multiplicity results available
   - Relation to light-like Wilson loops/strongly coupled string description
     $\Rightarrow$ emergence of dual superconformal or Yangian symmetry

This talk: Review some of the progress and show how to connect the two

Outline

1. Introduction
2. Trees: Complete analytic result and relation to massless QCD

   [Dixon, Henn, JP, Schuster; JHEP 1101, arXiv:1012]
3. Symmetries: Superconformal, dual conformal and Yangian invariance

   [Drummond, Henn, JP; JHEP 0905, arXiv:0902]
4. Loops: Overview and novel Higgs regulator

   [Alday, Henn, JP, Schuster; JHEP 1001, arXiv:0908]

$\mathcal{N}=4$ super Yang Mills: The simplest interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: A_μ, $\mu = 0, 1, 2, 3$, $\Delta = 1$
  - 6 real scalars: Φ_I, $I = 1, \ldots, 6$, $\Delta = 1$
  - 4×4 real fermions: $\Psi_{\alpha\, A}$, $\bar{\Psi}^{\dot{\alpha}}_{A}$,$\alpha, \dot{\alpha} = 1, 2$. $A = 1, 2, 3, 4$, $\Delta = 3/2$
  - Covariant derivative: $\mathcal{D}_\mu = \partial_\mu - i[A_\mu, *]$, $\Delta = 1$
- Action: Unique model completely fixed by SUSY

$$S = \frac{1}{{g_{\mathrm{YM}}}^2} \int d^4x \, \mathrm{Tr}\Big[\tfrac{1}{4} F_{\mu\nu}^2 + \tfrac{1}{2}(D_\mu \Phi_I)^2 - \tfrac{1}{4}[\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \bar{\Psi}^A_{\dot{\alpha}} \sigma^{\dot{\alpha}\beta}_\mu \mathcal{D}^\mu \Psi_{\beta\, A} - \tfrac{i}{2} \Psi_{\alpha\, A} \sigma^{AB}_I \epsilon^{\alpha\beta} [\Phi^I, \Psi_{\beta\, B}] - \tfrac{i}{2} \bar{\Psi}_{\dot{\alpha}\, A} \sigma^{AB}_I \epsilon^{\dot{\alpha}\dot{\beta}} [\Phi^I, \bar{\Psi}_{\dot{\beta}\, B}]\Big]$$

- $\boxed{\beta_{g_{\mathrm{YM}}} = 0}$: Quantum Conformal Field Theory, 2 parameters: N & $\lambda = {g_{\mathrm{YM}}}^2 N$
- Shall consider 't Hooft planar limit: $N \to \infty$ with λ fixed.
- Is the 4d interacting QFT with highest degree of symmetry!

$\Rightarrow$ **"H-atom of gauge theories"**

Superconformal symmetry

- Symmetry: $\mathfrak{so}(2,4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$

Poincaré: $p^{\alpha\dot\alpha} = p_\mu\,(\sigma^\mu)^{\dot\alpha\beta}, \quad m_{\alpha\beta}, \quad \bar m_{\dot\alpha\dot\beta}$

Conformal: $k_{\alpha\dot\alpha}, \quad d \qquad (c$: central charge$)$

R-symmetry: r_{AB}

Poncaré Susy: $q^{\alpha\,A}, \bar q^{\dot\alpha}_A$ Conformal Susy: $s_{\alpha\,A}, \bar s^A_{\dot\alpha}$

- $4+4$ Supermatrix notation $\bar A = (\alpha, \dot\alpha | A)$

$$J^{\bar A}{}_{\bar B} = \begin{pmatrix} m^\alpha{}_\beta - \frac{1}{2}\,\delta^\alpha_\beta\,(d + \frac{1}{2}c) & k^\alpha{}_{\dot\beta} & s^\alpha{}_B \\ p^{\dot\alpha}{}_\beta & \overline{m}^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\,\delta^{\dot\alpha}_{\dot\beta}\,(d - \frac{1}{2}c) & \bar q^{\dot\alpha}{}_B \\ q^A{}_\beta & \bar s^A{}_{\dot\beta} & -r^A{}_B - \frac{1}{4}\delta^A_B\,c \end{pmatrix}$$

- Algebra:

$$[J^{\bar A}{}_{\bar B}\,,\, J^{\bar C}{}_{\bar D}\} = \delta^{\bar C}_{\bar B}\,J^{\bar A}{}_{\bar D} - (-1)^{(|\bar A|+|\bar B|)(|\bar C|+|\bar D|)}\delta^{\bar A}_{\bar D}\,J^{\bar C}{}_{\bar B}$$

Gauge Theory Observables

- **Scaling dimensions:**
  Local operators $\mathcal{O}_n(x) = \mathrm{Tr}[\mathcal{W}_1\, \mathcal{W}_2 \ldots \mathcal{W}_n]$ with $\mathcal{W}_i \in \{\mathcal{D}^k\Phi, \mathcal{D}^k\Psi, \mathcal{D}^k F\}$

$$\langle \mathcal{O}_a(x_1)\, \mathcal{O}_b(x_2)\rangle = \frac{\delta_{ab}}{(x_1 - x_2)^{2\,\Delta_a(\lambda)}} \qquad \Delta_a(\lambda) = \sum_{l=0}^{\infty} \lambda^l\, \Delta_{a,l}$$

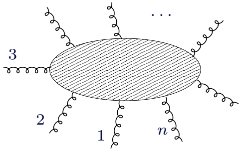
- **Wilson loops:**

$$\mathcal{W}_C = \left\langle \mathrm{Tr}\, P \exp i \oint_C ds\, (\dot{x}^\mu A_\mu + i|\dot{x}|\, \theta^I\, \Phi_I) \right\rangle$$

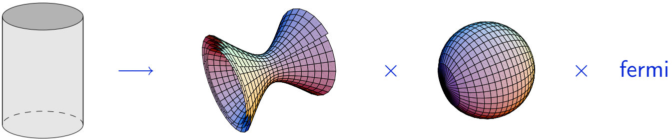
- **Scattering amplitudes:**

$$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda) = \left\{ \begin{matrix} \text{UV-finite} \\ \text{IR-divergent} \end{matrix} \right\}$$

helicities: $h_i \in \{0, \pm\frac{1}{2}, \pm 1\}$



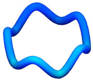
Superstring in $AdS_5 \times S^5$

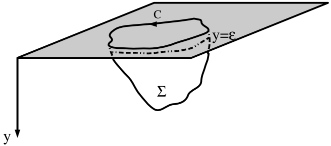


$$I = \sqrt{\lambda} \int d\tau\, d\sigma \left[G_{mn}^{(\mathrm{AdS}_5)}\, \partial_a X^m \partial^a X^n + G_{mn}^{(\mathrm{S}_5)}\, \partial_a Y^m \partial^a Y^n + \text{fermions} \right]$$

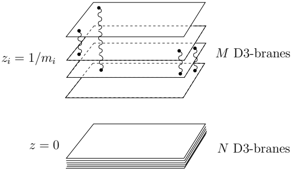
- $ds^2_{AdS} = R^2 \dfrac{dx^2_{3+1} + dz^2}{z^2}$ has boundary at $z = 0$
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, classical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $\mathcal{O}(1/\sqrt{\lambda})$
- $AdS_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- **Quantization unsolved!**
- String coupling constant $g_s = \frac{\lambda}{4\pi N} \to 0$ in 't Hooft limit
- Isometries: $\mathfrak{so}(2,4) \times \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$
- Include fermions: Formulate as $\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$ supercoset model [Metsaev,Tseytlin]

Gauge Theory - String Theory Dictionary of Observables

$\Delta_a(\lambda)$ spectrum of scaling dimensions $\Leftrightarrow$  $E(\lambda)$ string excitation spectrum — solved (?)

Wilson loop $\mathcal{W}_C$ $\Leftrightarrow$ 

minimal surface

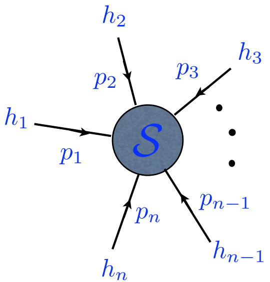
$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda)$ $(\Leftrightarrow)$ 

(a)

open string amps

Scattering amplitudes in $\mathcal{N} = 4$ SYM

- Consider n-particle scattering amplitude



- Planar amplitudes most conveniently expressed in color ordered formalism:

$$A_n(\{p_i, h_i, a_i\}) = (2\pi)^4\, \delta^{(4)}(\sum_{i=1}^{n} p_i) \sum_{\sigma \in S_n/Z_n} g^{n-2}\, \mathrm{tr}[t^{a_{\sigma_1}} \ldots t^{a_{\sigma_n}}]$$
$$\times \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\}; \lambda = g^2\, N)$$

$\mathcal{A}_n$: Color ordered amplitude. Color structure is stripped off.

Helicity of ith particle: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm\frac{1}{2}$ gluino

Spinor helicity formalism

- Express momentum and polarizations via commuting spinors $\lambda^{\alpha},\ \tilde{\lambda}^{\dot{\alpha}}$:

$$p^{\alpha\dot{\alpha}} = (\sigma^{\mu})^{\alpha\dot{\alpha}}\, p_{\mu} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}} \quad \Leftrightarrow \quad p_{\mu}\, p^{\mu} = \det p^{\alpha\dot{\alpha}} = 0$$

- Choice of helicity determines polarization vector ε^{μ} of external gluon

$$h = +1 \qquad \varepsilon^{\alpha\dot{\alpha}} = \frac{\lambda^{\alpha}\tilde{\mu}^{\dot{\alpha}}}{[\tilde{\lambda}\,\tilde{\mu}]} \qquad [\tilde{\lambda}\,\tilde{\mu}] := \epsilon_{\dot{\alpha}\dot{\beta}}\,\tilde{\lambda}^{\dot{\alpha}}\tilde{\mu}^{\dot{\beta}}$$

$$h = -1 \qquad \bar{\varepsilon}^{\alpha\dot{\alpha}} = \frac{\mu^{\alpha}\tilde{\lambda}^{\dot{\alpha}}}{\langle\lambda\,\mu\rangle} \qquad \langle\lambda\,\mu\rangle := \epsilon_{\alpha\beta}\,\lambda^{\alpha}\mu^{\beta}$$

  $\mu, \bar{\mu}$ arbitrary reference spinors.

- E.g. scalar products: $2\,p_1 \cdot p_2 = \langle\lambda_1, \lambda_2\rangle\,[\tilde{\lambda}_2, \tilde{\lambda}_1] = \langle 1, 2\rangle\,[2, 1]$

Trees

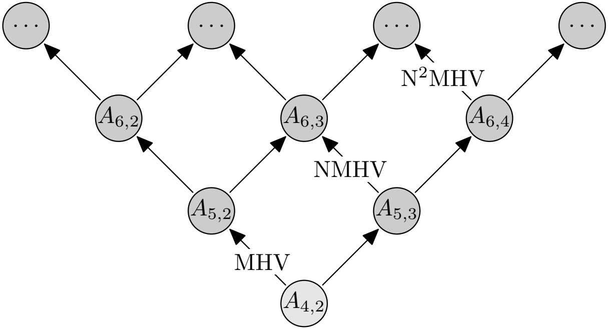
Gluon Amplitudes and Helicity Classification

Classify gluon amplitudes by # of helicity flips

- By SUSY Ward identities: $\mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+)$ true to all loops
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \delta^{(4)}(\sum_i p_i) \frac{\langle i, j \rangle^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle}$$ [Parke, Taylor]

- Next-to-maximally helicity amplitudes (N^{k}MHV) have more involved structure!



$A_{n,m} : g_+^{n-m} g_-^m$

[Picture from T. McLoughlin]

On-shell superspace

- Augment λ_i^α and $\tilde{\lambda}_i^{\dot\alpha}$ by Grassmann variables η_i^A $\quad A = 1, 2, 3, 4$ [Nair]
- On-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}^{\dot\alpha}, \eta_i^A)$ with on-shell superfield:

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\,\epsilon_{ABCD}G^-(p)$$

- Superamplitudes: $\left\langle \Phi(\lambda_1,\tilde{\lambda}_1,\eta_1)\,\Phi(\lambda_2,\tilde{\lambda}_2,\eta_2)\ldots\Phi(\lambda_n,\tilde{\lambda}_n,\eta_n)\right\rangle$

  Packages all n-parton gluon$^{\pm}$-gluino$^{\pm 1/2}$-scalar amplitudes

- General form of **tree superamplitudes**:

$$\mathbb{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i\,\tilde{\lambda}_i)\,\delta^{(8)}(\sum_i \lambda_i\,\eta_i)}{\langle 1,2\rangle\,\langle 2,3\rangle\ldots\langle n,1\rangle}\,\mathcal{P}_n(\{\lambda_i,\tilde{\lambda}_i,\eta_i\})$$

  Conservation of super-momentum: $\quad \delta^{(8)}(\sum_i \lambda^\alpha\eta_i^A) = (\sum_i \lambda^\alpha\eta_i^A)^8$

- η-expansion of $\mathcal{P}_n$ yields N^{k}MHV-classification of superamps as $h(\eta) = -1/2$

$$\mathcal{P}_n = \mathcal{P}_n^{\text{MHV}} + \eta^4\,\mathcal{P}_n^{\text{NMHV}} + \eta^8\,\mathcal{P}_n^{\text{NNMHV}} + \ldots + \eta^{4n-8}\,\mathcal{P}_n^{\overline{\text{MHV}}}$$

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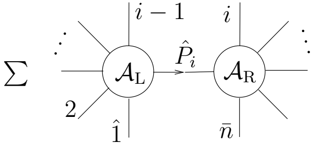
Superamplitudes and BCFW recursion

- Efficient way of constructing tree-level amplitudes via BCFW recursion

  [Britto,Cachazo,Feng+Witten '04,05]

$$A_n = \sum_i A^h_{i+1} \frac{1}{P_i^2} A^{-h}_{n-i+1}$$



  - N-point amplitudes are obtained recursively from lower-point amplitudes
  - All amplitudes are on-shell
  - Special cases can be solved analytically, e.g. split-helicity amplitudes $\mathcal{A}(-,\ldots,-,+,\ldots,+)$ [Roiban,Spradlin,Volovich]

- Reformulation of recursion relations in on-shell superspace via shift in $(\lambda_i, \tilde{\lambda})$ and η_i [Elvang et al, Arkani-Hamed et al, Brandhuber et al]

- Super BCFW recursion is much simpler and can be solved analytically!

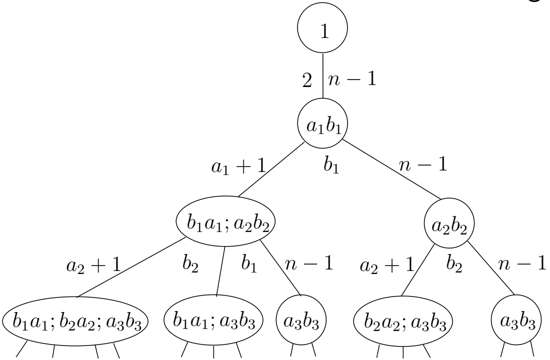
$\Rightarrow$ $\boxed{\mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) \text{ known in closed analytical form at tree-level}}$ [Drummond,Henn]

The Drummond-Henn solution

$\mathcal{P}_n$ expressed as sums over R-invariants determined by paths on rooted tree

$$\mathcal{P}_n^{\text{N}^k\text{MHV}} = \sum_{\substack{\text{all paths}\\ \text{of length } k}} 1 \cdot R_{n,a_1 b_1} \cdot R_{n,\{I_2\},a_2 b_2}^{\{L_2\};\{U_2\}} \cdot \ldots \cdot R_{n,\{I_p\},a_p b_p}^{\{L_p\};\{U_p\}}$$

E.g.



$$\mathcal{P}^{\text{NMHV}} = \sum_{1<a_1,b_1<n} R_{n,a_1 b_1}$$

$$\mathcal{P}_n^{\text{N}^2\text{MHV}} = \sum_{1<a_1,b_1<n} R_{n;a_1 b_1} \times \left[\sum_{a_1<a_2,b_2\leq b_1} R_{n;b_1 a_1;a_2 b_2}^{0;a_1 b_1} + \sum_{b_1\leq a_2,b_2<n} R_{n;a_2 b_2}^{a_1 b_1;0} \right]$$

Goal: Project onto component field amplitudes [Dixon, Henn, Plefka, Schuster]

Region momenta or dual coordinates

$$x_i - x_{i+1} = p_i \qquad x_{ij} := x_i - x_j \stackrel{i<j}{=} p_i + p_{i+1} + \cdots + p_{j-1}$$

- All amplitudes expressed via momentum invariants x_{ij}^2 and the scalar quantities:

$$\begin{aligned}\langle n a_1 a_2 \ldots a_k | a \rangle &:= \langle n | x_{na_1} x_{a_1 a_2} \ldots x_{a_{k-1} a_k} | a \rangle \\ &= \lambda_n^{\alpha} (x_{na_1})_{\alpha\dot\beta} (x_{a_1 a_2})^{\dot\beta\gamma} \ldots (x_{a_{k-1}a_k})^{\dot\delta\rho} \lambda_{a\,\rho}\end{aligned}$$

- Building blocks for amps: $\tilde{R}$ invariants and path matrix Ξ_n^{path}

$$\tilde{R}_{n;\{I\};ab} := \frac{1}{x_{ab}^2} \frac{\langle a(a-1)\rangle}{\langle n\,\{I\}\,ba|a\rangle\,\langle n\,\{I\}\,ba|a-1\rangle} \frac{\langle b(b-1)\rangle}{\langle n\,\{I\}\,ab|b\rangle\,\langle n\,\{I\}\,ab|b-1\rangle}\,;$$

$$\Xi_n^{\text{path}} := \begin{pmatrix} \langle n c_0 \rangle & \langle n c_1 \rangle & \ldots & \langle n c_p \rangle \\ (\Xi_n)^{c_0}_{a_1 b_1} & (\Xi_n)^{c_1}_{a_1 b_1} & \ldots & (\Xi_n)^{c_p}_{a_1 b_1} \\ (\Xi_n)^{c_0}_{\{I_2\};a_2 b_2} & (\Xi_n)^{c_1}_{\{I_2\};a_2 b_2} & \ldots & (\Xi_n)^{c_p}_{\{I_2\};a_2 b_2} \\ \vdots & \vdots & & \vdots \\ (\Xi_n)^{c_0}_{\{I_p\};a_p b_p} & (\Xi_n)^{c_1}_{\{I_p\};a_p b_p} & \ldots & (\Xi_n)^{c_p}_{\{I_p\};a_p b_p} \end{pmatrix}$$

All gluon-gluino trees in $\mathcal{N} = 4$ SYM [Dixon, Henn, Plefka, Schuster]

- MHV gluon amplitudes [Parke, Taylor]

$$A_n^{\text{MHV}}(c_0^-, c_1^-) = \delta^{(4)}(p) \frac{\langle c_0\; c_1\rangle^4}{\langle 1\;2\rangle\langle 2\;3\rangle \dots \langle n\;1\rangle}$$

- N^pMHV gluon amplitudes:

$$A_n^{\text{N}^\text{p}\text{MHV}}(c_0^-, \dots, c_{p+1}^-) = \frac{\delta^{(4)}(p)}{\langle 1\;2\rangle \dots \langle n\;1\rangle} \sum_{\substack{\text{all paths} \\ \text{of length } p}} \left(\prod_{i=1}^{p} \tilde{R}^{L_i;R_i}_{n;\{I_i\};a_i b_i} \right) (\det \Xi)^4$$

- MHV gluon-gluino amplitudes (single flavor)

$$A_n^{\text{MHV}}(a^-, b_q, c_{\bar{q}}) = \delta^{(4)}(p) \frac{\langle a\; c\rangle^3 \langle a\; b\rangle}{\langle 1\;2\rangle \dots \langle n\;1\rangle}$$

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$$A_{(q\bar{q})^k,n}^{\text{N}^\text{p}\text{MHV}}(\dots, c_k^-, \dots, (c_{\alpha_i})_q, \dots, (c_{\bar{\beta}_j})_{\bar{q}}, \dots) =$$

$$\frac{\delta^{(4)}(p)\text{sign}(\tau)}{\langle 1\;2\rangle\langle 2\;3\rangle \dots \langle n\;1\rangle} \times \sum_{\substack{\text{all paths} \\ \text{of length } p}} \left(\prod_{i=1}^{p} \tilde{R}^{L_i;R_i}_{n;\{I_i\};a_i b_i} \right) \left(\det \Xi\big|_q \right)^3 \det \Xi(q \leftrightarrow \bar{q})\big|_{\bar{q}}$$

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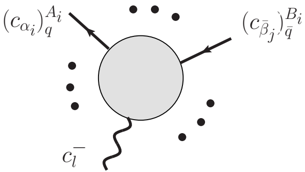

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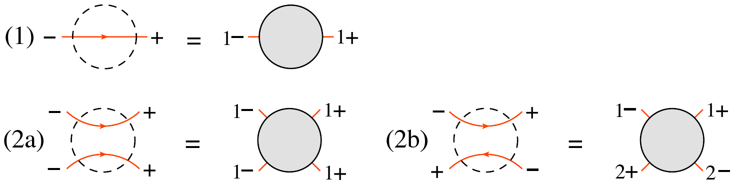
From $\mathcal{N} = 4$ to massless QCD trees

- Differences in color: SU(N) vs. SU(3); Fermions: adjoint vs. fundamental
  Irrelevant for color ordered amplitudes, as color d.o.f. stripped off anyway. E.g. single quark-anti-quark pair

$$\mathcal{A}_n^{\text{tree}}(1_{\bar{q}}, 2_q, 3, \ldots, n) = g^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}})_{i_2}^{\bar{i}_1} A_n^{\text{tree}}(1_{\bar{q}}, 2_q, \sigma(3), \ldots, \sigma(n))$$

  Color ordered $A_n^{\text{tree}}(1_{\bar{q}}, 2_q, 3, \ldots, n)$ from two-gluino-$(n-2)$-gluon amplitude.

- For more than one quark-anti-quark pair needs to accomplish:
  (1) Avoid internal scalar exchanges (due to Yukawa coupling)
  (2) Allow all fermion lines present to be of different flavor



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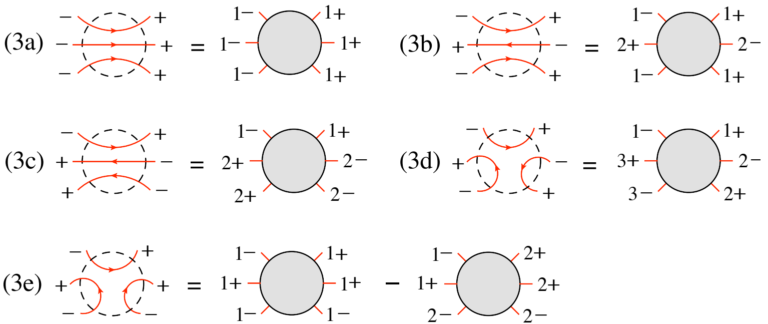

- Also worked out explicitly for 4 quark-anti-quark pairs.
- Conclusion: Obtained all (massless) QCD trees from the $\mathcal{N} = 4$ SYM trees

GGT: Mathematica package for analytic gluon-gluino tree amplitudes [Dixon, Henn, Plefka, Schuster, 2010] qft.physik.hu-berlin.de

In[9]:= **GGTgluon[7, {3, 5}]**

Out[9]=
$$\frac{\langle 3\mid 5\rangle^4}{\langle 1\mid 2\rangle\langle 2\mid 3\rangle\langle 3\mid 4\rangle\langle 4\mid 5\rangle\langle 5\mid 6\rangle\langle 6\mid 7\rangle\langle 7\mid 1\rangle}$$

In[12]:= **GGTgluon[6, {3, 5, 6}]**

Out[12]=
$$\left(\frac{\langle 2\mid 1\rangle\langle 4\mid 3\rangle\left(s_{2,4}\langle 6\mid 3\rangle\langle 6\mid 5\rangle+\langle 6\mid 5\rangle\langle 6\mid x_{6,4}\mid x_{4,2}\mid 3\rangle\right)^4}{s_{2,4}\langle 6\mid x_{6,2}\mid x_{2,4}\mid 3\rangle\langle 6\mid x_{6,2}\mid x_{2,4}\mid 4\rangle\langle 6\mid x_{6,4}\mid x_{4,2}\mid 1\rangle\langle 6\mid x_{6,4}\mid x_{4,2}\mid 2\rangle}+\frac{\langle 2\mid 1\rangle\langle 5\mid 4\rangle\left(s_{2,5}\langle 6\mid 3\rangle\langle 6\mid 5\rangle+\langle 6\mid 5\rangle\langle 6\mid x_{6,5}\mid x_{5,2}\mid 3\rangle\right)^4}{s_{2,5}\langle 6\mid x_{6,2}\mid x_{2,5}\mid 4\rangle\langle 6\mid x_{6,2}\mid x_{2,5}\mid 5\rangle\langle 6\mid x_{6,5}\mid x_{5,2}\mid 1\rangle\langle 6\mid x_{6,5}\mid x_{5,2}\mid 2\rangle}+\frac{\langle 3\mid 2\rangle\langle 5\mid 4\rangle\left(s_{3,5}\langle 6\mid 3\rangle\langle 6\mid 5\rangle+\langle 6\mid 5\rangle\langle 6\mid x_{6,5}\mid x_{5,3}\mid 3\rangle\right)^4}{s_{3,5}\langle 6\mid x_{6,3}\mid x_{3,5}\mid 4\rangle\langle 6\mid x_{6,3}\mid x_{3,5}\mid 5\rangle\langle 6\mid x_{6,5}\mid x_{5,3}\mid 2\rangle\langle 6\mid x_{6,5}\mid x_{5,3}\mid 3\rangle}\right)\Big/\left(\langle 1\mid 2\rangle\langle 2\mid 3\rangle\langle 3\mid 4\rangle\langle 4\mid 5\rangle\langle 5\mid 6\rangle\langle 6\mid 1\rangle\right)$$

In[11]:= **GGTfermionS[7, {1, 7}, {3, 4}, {5, 6}]**

Out[11]=
$$-\Big(\Big(\langle 2\mid 1\rangle\langle 4\mid 3\rangle\langle 6\mid 5\rangle\langle 7\mid 1\rangle\langle 4\mid x_{2,4}\mid x_{7,2}\mid 7\rangle\langle 7\mid x_{7,4}\mid x_{4,2}\mid 3\rangle$$
$$\left(s_{2,4}\,s_{4,6}\langle 7\mid 1\rangle\langle 7\mid 5\rangle\langle 7\mid 6\rangle+s_{2,4}\langle 7\mid 1\rangle\langle 7\mid 6\rangle\langle 7\mid x_{7,6}\mid x_{6,4}\mid 5\rangle\right)^3\Big)\Big/$$
$$\big(s_{2,4}\,s_{4,6}\langle 7\mid x_{7,2}\mid x_{2,4}\mid 3\rangle\langle 7\mid x_{7,2}\mid x_{2,4}\mid 4\rangle\langle 7\mid x_{7,4}\mid x_{4,2}\mid 1\rangle\langle 7\mid x_{7,4}\mid$$
$$x_{4,2}\mid 2\rangle\langle 7\mid x_{7,4}\mid x_{4,6}\mid 5\rangle\langle 7\mid x_{7,4}\mid x_{4,6}\mid 6\rangle\langle 7\mid x_{7,6}\mid x_{6,4}\mid x_{2,4}\mid x_{7,2}\mid 7\rangle\big)+$$
$$\Big(s_{2,6}^2\langle 2\mid 1\rangle\langle 3\mid 2\rangle\langle 6\mid 5\rangle\langle 7\mid 1\rangle^3\langle 7\mid 6\rangle^3$$
$$\big(-\langle 7\mid 1\rangle\langle 7\mid x_{7,6}\mid x_{6,2}\mid 4\rangle\langle 7\mid x_{7,6}\mid x_{6,2}\mid x_{2,3}\mid x_{3,6}\mid 3\rangle+$$
$$\langle 7\mid 1\rangle\langle 7\mid x_{7,6}\mid x_{6,2}\mid 3\rangle\langle 7\mid x_{7,6}\mid x_{6,2}\mid x_{2,3}\mid x_{3,6}\mid 4\rangle\big)$$

GGT: Mathematica package for analytic gluon-gluino tree amplitudes [Dixon, Henn, Plefka, Schuster, 2010] qft.physik.hu-berlin.de

- Similar solutions for all gluon-gluino-scalar trees in $\mathcal{N} = 4$ SYM also available from the Mathematica package BCFW [Bourjaily, 2010]
- Makes use of Grassmannian approach and momentum twistors [Arkani-Hamed et al]

Symmetries

$\mathfrak{su}(2,2|4)$ invariance

- Superamplitude: $(i = 1, \ldots, n)$

$$\mathbb{A}_n^{\text{tree}}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}})\, \delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

- Realization of $\mathfrak{psu}(2,2|4)$ generators in **on-shell superspace**, e.g. [Witten]

$$p^{\alpha\dot{\alpha}} = \sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \qquad q^{\alpha A} = \sum_{i=1}^n \lambda_i^\alpha \eta_i^A \qquad \Rightarrow \text{ obvious symmetries}$$

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \qquad s_{\alpha A} = \sum_{i=1}^n \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \eta_i^A} \qquad \Rightarrow \text{ less obvious sym}$$

- Invariance: $\{p, k, \bar{m}, m, d, r, q, \bar{q}, s, \bar{s}, c_i\}\, \mathbb{A}_n^{\text{tree}}(\{\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A\}) = 0$

- N.B.: **Local** invariance $\quad h_i\, \mathbb{A}_n = 1 \cdot \mathbb{A}_n$

Helicity operator: $\quad h_i = -\frac{1}{2} \lambda_i^\alpha \partial_{i\,\alpha} + \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \partial_{i\,\dot{\alpha}} + \frac{1}{2} \eta_i^A \partial_{i\,A} = 1 - c_i$

$\mathfrak{su}(2,2|4)$ invariance

- The $\mathfrak{su}(2,2|4)$ generators acting in on-shell superspace $(\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A)$:

$$p^{\dot\alpha\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^\alpha\,, \qquad k_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha}\partial_{i\dot\alpha}\,,$$

$$\overline{m}_{\dot\alpha\dot\beta} = \sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}, \qquad m_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)}\,,$$

$$d = \sum_i [\tfrac{1}{2}\lambda_i^\alpha\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + 1], \qquad r^A{}_B = \sum_i [-\eta_i^A\partial_{iB} + \tfrac{1}{4}\delta_B^A\eta_i^C\partial_{iC}]\,,$$

$$q^{\alpha A} = \sum_i \lambda_i^\alpha\eta_i^A\,, \qquad \bar q_A^{\dot\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\partial_{iA}\,,$$

$$s_{\alpha A} = \sum_i \partial_{i\alpha}\partial_{iA}, \qquad \bar s_{\dot\alpha}^A = \sum_i \eta_i^A\partial_{i\dot\alpha}\,,$$

$$c = \sum_i [1 + \tfrac{1}{2}\lambda_i^\alpha\partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} - \tfrac{1}{2}\eta_i^A\partial_{iA}]\,.$$

- N.B: For collinear momenta picks up important additional length changing terms, due to holomorphic anomaly $\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}\frac{1}{\langle\lambda,\mu\rangle} = 2\pi\tilde\mu_{\dot\alpha}\,\delta^2(\langle\lambda,\mu\rangle)$

[Bargheer, Beisert, Galleas, Loebbert, McLoughlin]

[Korchemsky, Sokatchev] [Skinner, Mason][Arkani-Hamed, Cachazo, Kaplan]

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Dual Superconformal symmetry

- Planar MHV amplitudes are dual conformal $SO(2,4)$ invariant in dual space x_i

  [Drummond,Korchemsky,Sokatchev]
- Derives from Scattering amplitude/Wilson Loop duality

  [Alday, Maldacena;Drummond,Korchemsky,Sokatchev]
- May be extended to dual superconfromal invariance of tree-level superamplitudes: Introduce dual on-shell superspace [Drummond, Henn, Korchemsky, Sokatchev]

$$(x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha \, \tilde\lambda_i^{\dot\alpha} \qquad (\theta_i - \theta_{i+1})^{\alpha\, A} = \lambda_i^\alpha \, \eta_i^A$$

- Dual special conformal generator:

$$K^{\alpha\dot\alpha} = \sum_i x_i^{\alpha\dot\beta} \, x_i^{\dot\alpha\beta} \, \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta} \, \theta_i^{\alpha\, B} \, \frac{\partial}{\partial \theta_i^{\beta\, B}}$$

- Translate to on-shell superspace: $x_i^{\alpha\dot\alpha} = \sum_{j=1}^{i-1} \lambda_j^\alpha \, \tilde\lambda_j^{\dot\alpha}$ and $\theta_i^{\alpha A} = \sum_{j=1}^{i-1} \lambda_j^\alpha \, \eta_j^A$

$$K^{\alpha\dot\alpha} = \sum_{i=1}^{n} x_i^{\dot\alpha\beta} \, \lambda_i^\alpha \, \frac{\partial}{\partial \lambda_i^\beta} + x_{i+1}^{\alpha\dot\beta} \, \tilde\lambda_i^{\dot\alpha} \, \frac{\partial}{\partial \tilde\lambda_i^{\dot\beta}} + \tilde\lambda_i^{\dot\alpha} \, \theta_{i+1}^{\alpha B} \, \frac{\partial}{\partial \eta_i^B} + x_i^{\alpha\dot\alpha}$$

Nonlocal structure!

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Nonlocal structure!

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- Superconformal + Dual superconformal algebra = Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$ [Drummond, Henn, Plefka]

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^c\, J_c^{(0)} \qquad \text{conventional superconformal symmetry}$$

$$[J_a^{(0)}, J_b^{(1)}\} = f_{ab}{}^c\, J_c^{(1)} \qquad \text{from dual conformal symmetry}$$

with nonlocal generators

$$\boxed{J_a^{(1)} = f^{cb}{}_a \sum_{1<j<i<n} J_{i,b}^{(0)}\, J_{j,c}^{(0)}}$$

and super Serre relations (representation dependent). [Dolan,Nappi,Witten]

- To define "inverted" $f^{cb}{}_a$ needs to extend to $\mathfrak{u}(2,2|4)$ for nondegen. metric
- In particular: Bosonic invariance $\boxed{p^{(1)}_{\alpha\dot\alpha}\, \mathbb{A}_n = 0}$ with

$$p^{(1)}_{\alpha\dot\alpha} = K_{\alpha\dot\alpha} + \Delta K_{\alpha\dot\alpha}$$

$$= \frac{1}{2}\sum_{i<j}(m_{i,\alpha}{}^{\gamma}\delta^{\dot\gamma}_{\dot\alpha} + \bar m_{i,\dot\alpha}{}^{\dot\gamma}\delta^{\gamma}_{\alpha} - d_i\, \delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha})\, p_{j,\gamma\dot\gamma} + \bar q_{i,\dot\alpha C}\, q^{C}_{j,\alpha} - (i \leftrightarrow j)$$

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- In supermatrix notation: $\bar{A} = (\alpha, \dot{\alpha} | A)$

$$J^{\bar{A}}{}_{\bar{B}} = \begin{pmatrix} m^{\alpha}{}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\,(d + \frac{1}{2}c) & k^{\alpha}{}_{\dot{\beta}} & s^{\alpha}{}_{B} \\ p^{\dot{\alpha}}{}_{\beta} & \overline{m}^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{1}{2}\,\delta^{\dot{\alpha}}_{\dot{\beta}}\,(d - \frac{1}{2}c) & \bar{q}^{\dot{\alpha}}{}_{B} \\ q^{A}{}_{\beta} & \bar{s}^{A}{}_{\dot{\beta}} & -r^{A}{}_{B} - \frac{1}{4}\delta^{A}_{B}\,c \end{pmatrix}$$

$$\Rightarrow \quad \boxed{J^{(1)\,\bar{A}}{}_{\bar{B}} := -\sum_{i>j} (-1)^{|\bar{C}|} (J_i^{\bar{A}}{}_{\bar{C}}\, J_j^{\bar{C}}{}_{\bar{B}} - J_j^{\bar{A}}{}_{\bar{C}}\, J_i^{\bar{C}}{}_{\bar{B}})}$$

- Implies an infinite-dimensional symmetry algebra for tree-level $\mathcal{N}=4$ SYM scattering amplitudes! $\Leftrightarrow$ spin chain picture

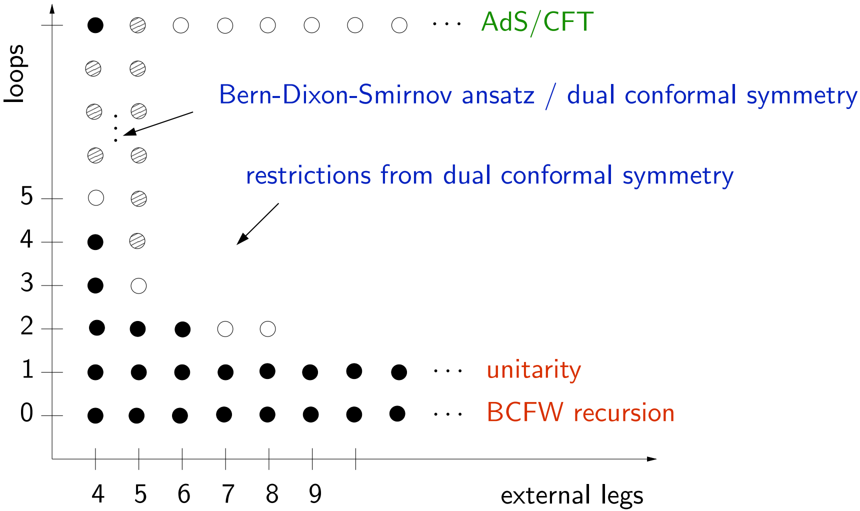
$$\boxed{J^{\bar{A}}{}_{\bar{B}} \circ \mathbb{A}_n = 0 \qquad J^{(1)\bar{A}}{}_{\bar{B}} \circ \mathbb{A}_n = 0}$$

- Including correction terms arising from collinear momenta this symmetry is constructive: Unambiguously fixes tree-level amplitudes.

[Bargheer, Beisert, Galleas, Loebbert,McLoughlin; Korchemsky, Sokatchev]

Loops

Status of higher loop/leg calculations in $\mathcal{N} = 4$ SYM



- Diagram has three important ingredients:
  analytic properties, symmetries (+IR structure), AdS/CFT

Higher loops and Higgs regulator

- Beyond tree-level: Conformal and dual conformal symmetry is broken by IR divergencies $\Rightarrow$ $\{\not{s}, \not{\bar{s}}, \not{k}, \not{K}, \not{S}, \not{\bar{Q}}\}$
- Need for regularization: Standard method Dim reduction $10 \to 4 - \epsilon$
- Alternative method: Higgs regulator $U(N+M) \to U(N) \times U(1)^M$

  [Alday, Henn, Plefka, Schuster]

  Best way to understand dual conformal symmetry in the field theory:

  $\Rightarrow$ Inspired by AdS/CFT [Alday, Maldacena; Schabinger, 2008; Sever, McGreevy]

  $\Rightarrow$ IR divergences regulated by masses, at least for large N

  $\Rightarrow$ Conjecture: Existence of an extended dual conformal symmetry

  [Alday, Henn, Plefka, Schuster]

  $\Rightarrow$ Lots of supporting evidence [Naculich, Henn, Schnitzer, Spradlin; Boels, Bern, Dennen, Huang]

  $\Rightarrow$ Now essentially proven through 6D SYM [Caron-Huot, OConnel; Dennen, Huang, 2010]

  $\Rightarrow$ Heavily restricts the loop integrand/integrals!
- Related development: (Unregulated) planar integrand has Yangian symmetry

  [Arkani-Hamed et al, 2010]

  Higgs regulator and its exact dual conformal symmetry is used to justify transition to regulated integrand

Higher loops and Higgs regulator

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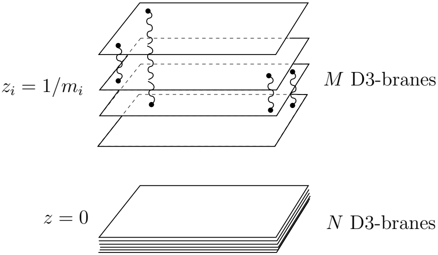
Higgs regularization [Alday, Henn, Plefka, Schuster]

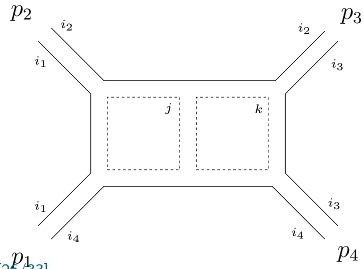
- Take string picture serious:



(a)

- Field Theory: Higgsing $U(N+M) \to U(N) \times U(1)^M$. One brane for every scattered particle, $N \gg M$.

Renders amplitudes IR finite. Have light $(m_i - m_j)$ and heavy m_i fields



Extended dual conformal symmetry: The string picture

- Consider the string description of the IR-regulated amplitude in the T-dual theory: The radial coordinates are related by

$$\boxed{1/z = r = m}$$

- The $SO(2,4)$ isometry of AdS_5 in T-dual theory is generated by J_{MN} with embedding coordinates $M = -1, 0, 1, 2, 3, 4$.
  In Poincaré coordinates (r, x^μ) we have

$$\begin{aligned} J_{-1,4} &= r\partial_r + x^\mu \partial_\mu = \hat{D} \\ J_{4,\mu} - J_{-1,\mu} &= \partial_\mu = \hat{P}_\mu \\ J_{4,\mu} + J_{-1,\mu} &= 2x_\mu (x_\nu \partial^\nu + r\partial_r) - (x^2 + r^2)\partial_\mu = \hat{K}_\mu \end{aligned}$$

- Expectation: Amplitudes regulated by Higgsing should be invariant exactly under extended dual conformal symmetry $\hat{K}_\mu$ and $\hat{D}$ with $r \to m$!

Higgsing $\mathcal{N}=4$ Super Yang-Mills

- Action

$$\hat{S}^{U(N+M)}_{\mathcal{N}=4} = \int d^4x \, \mathrm{Tr}\Big(-\frac{1}{4}\hat{F}^2_{\mu\nu} - \frac{1}{2}(D_\mu \hat{\Phi}_I)^2 + \frac{g^2}{4}[\hat{\Phi}_I, \hat{\Phi}_J]^2 + \text{ferms} \Big) \, ,$$

- Decompose into $N+M$ blocks

$$\hat{A}_\mu = \begin{pmatrix} (A_\mu)_{ab} & (A_\mu)_{aj} \\ (A_\mu)_{ia} & (A_\mu)_{ij} \end{pmatrix} , \qquad \hat{\Phi}_I = \begin{pmatrix} (\Phi_I)_{ab} & (\Phi_I)_{aj} \\ (\Phi_I)_{ia} & \delta_{I9} \frac{m_i}{g} \delta_{ij} + (\Phi_I)_{ij} \end{pmatrix}$$

$$a, b = 1, \ldots, N \, , \quad i, j = N+1, \ldots, N+M \, ,$$

- Add R_ξ gauge fixing and ghost terms. Quadratic terms ($A_M := (A_\mu, \Phi_I)$)

$$\hat{S}_{\mathcal{N}=4}\Big|_{\text{quad}} = \int d^4x \Big\{ -\tfrac{1}{2}\mathrm{Tr}(\partial_\mu A_M)^2 - \tfrac{1}{2}(m_i - m_j)^2 (A_M)_{ij} (A^M)_{ji} - m_i^2 (A_M)_{ia} (A^M)_{ai} + \text{ferms} \Big\}$$

- Plus novel bosonic 3-point interactions proportional to m_i

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$$a,b = 1,\ldots,N\,, \quad i,j = N+1,\ldots,N+M\,,$$

- Add R_ξ gauge fixing and ghost terms. Quadratic terms $(A_M := (A_\mu,\Phi_I))$

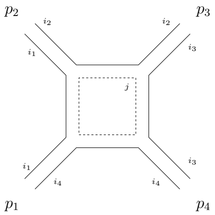
$$\hat{S}_{\mathcal{N}=4}\Big|_{\text{quad}} = \int d^4x\,\Big\{ -\tfrac{1}{2}\mathrm{Tr}(\partial_\mu A_M)^2 - \tfrac{1}{2}(m_i-m_j)^2\,(A_M)_{ij}\,(A^M)_{ji} - m_i^2\,(A_M)_{ia}\,(A^M)_{ai} + \text{ferms}\Big\}$$

- Plus novel bosonic 3-point interactions proportional to m_i

- Consider the (special) purely scalar amplitude:

$$\boxed{A_4 = \langle \Phi_4(p_1)\, \Phi_5(p_2)\, \Phi_4(p_3)\, \Phi_5(p_4) \rangle = i g^2_{\rm YM} \Big(1 + \lambda\, I^{(1)}(s,t,m_i) + O(a^2) \Big)}$$

$I^{(1)}(s,t,m_i)$: Massive box integral in dual variables ($p_i = x_i - x_{i+1}$)



$$= \int d^4 x_5 \frac{(x_{13}^2 + (m_1 - m_3)^2)(x_{24}^2 + (m_2 - m_4)^2)}{(x_{15}^2 + m_1^2)(x_{25}^2 + m_2^2)(x_{35}^2 + m_3^2)(x_{45}^2 + m_4^2)}$$

- Reexpressed in 5d variables $\hat{x}^M$: $\boxed{\hat{x}_i^\mu := x_i^\mu\,, \quad \hat{x}_i^4 := m_i\,,}$ $\quad i = 1 \ldots 4$

$$\boxed{I^{(1)}(s,t,m_i) = \hat{x}_{13}^2 \hat{x}_{24}^2 \int d^5 \hat{x}_5 \frac{\delta(\hat{x}_5^{M=4})}{\hat{x}_{15}^2 \hat{x}_{25}^2 \hat{x}_{35}^2 \hat{x}_{45}^2}}$$

Indeed $I^{(1)}(s,t,m_i)$ is extended dual conformal invariant: $\hat{K}_\mu I^{(1)}(s,t,m_i) = 0$

Extended dual conformal invariance

- Extended dual conformal invariance

$$\hat{K}_\mu I^{(1)}(s,t,m_i) := \sum_{i=1}^{4} \left[2x_{i\mu} \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + m_i \frac{\partial}{\partial m_i} \right) - (x_i^2 + m_i^2) \frac{\partial}{\partial x_i^\mu} \right] I^{(1)}(s,t,m_i) = 0$$

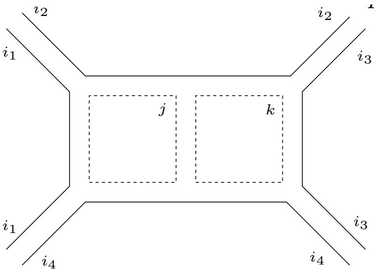
- m_i is the fifth coordinate $x^M = (x^\mu, m)$.
- Triangle and bubble graphs are forbidden by extended conformal symmetry!
- Indeed an explicit one-loop calculation shows the cancelation of triangles.
- Dual conformal symmetry exists in 6d $\mathcal{N} = (1,1)$ SYM at tree-level. Also at loop-level for integrands with 4d momentum measure

  [Caron-Huot, OConnel; Dennen, Huang, 2010]

  $\Rightarrow$ Proof of extended conformal symmetry for $\mathcal{N} = 4$ SYM at loop level.

Extended dual conformal invariance at higher loops

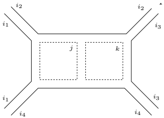
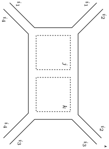
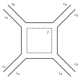
- At 2 loops: Only one integral is allowed by extended dual conformal symmetry:



  Should similarly restrict possible integrals at higher loops.
- Computed this graph in $m_i \to 0$ limit using Mellin-Barnes techniques.
- No $\frac{1}{\epsilon} \times \epsilon = 1$ 'interference' as in dimred: Here $\log(m^2) \times m^2 \to 0$.
- Has been extended to higher loops & higher multiplicities as well as Regge limit

  [Henn, Naculich, Schnitzer, Drummond]

- We have $\mathcal{A}_4 = \mathcal{A}_4^{\text{tree}} \cdot M_4$

$$M_4\Big|_{\text{2-loops}} =$$

$$+$$

$$= \exp\Big[\Gamma_{\text{cusp}}(\lambda)$$

$$\Big]\Big|_{\text{2-loops}}$$

where one splits M_4 into $\ln m^2$ dependent and independent pieces:

$$\ln M_4 = D_4 + F_4 + \mathcal{O}(m^2)$$

- Defining $\boxed{\left(\frac{\partial}{\partial \ln(m^2)}\right)^2 \ln M_4 =: -\Gamma_{\text{cusp}}(a)}$ we find $\Gamma_{\text{cusp}}(a) = 2a - 2\zeta_2\, a^2 + \dots$

  where $a = \lambda/8\pi^2$ in agreement with dim reg.
- Furthermore for finite piece one has

$$F_4 = \frac{1}{2}\Gamma_{\text{cusp}}(a)\left[\frac{1}{2}\ln^2(s/t) + \frac{1}{2}\right] + C(a)$$

with $C(a) = a^2\,\pi^4/120 + \mathcal{O}(a^3)$.

Summary and Outlook

- All tree-level amplitudes in $\mathcal{N}=4$ SYM known analytically
  - Results translate to all massless QCD trees (at least for up to 8 fermions)
  - Useful for automated evaluation of loops using unitarity (`Blackhat`)
- Tree level amplitudes are invariant under an infinite dimensional Yangian symmetry
  - Hint for integrability in scattering amplitudes!
  - Is form of tree amplitudes fixed by Yangian symmetry?

    $\Rightarrow$ Yes, but needs to include collinear limits $\equiv$ length changing effects

    [Bargheer,Beisert,Galleas,Loebbert,McLoughlin]
- Challenge at weak coupling: Does Yangian symmetry extend to the loop level?
- Breaking of dual conformal invariance at loop level under control: Best seen in Higgs regulator
- Restriction of possible integrals at higher loops.
- Can breaking of standard conformal invariance at loop level be controlled?

  Yes! Perturbative construction [Sever,Vieira] [Beisert, Henn, McLoughlin, Plefka]
- Does integrability determine the all loop planar scattering amplitudes?

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Proof of extended dual conformal invariance

- Integral in 5d variables: $I^{(1)}(s,t,m_i) = \hat{x}_{13}^2 \hat{x}_{24}^2 \int d^5\hat{x}_5 \frac{\delta(\hat{x}_5^{M=4})}{\hat{x}_{15}^2 \hat{x}_{25}^2 \hat{x}_{35}^2 \hat{x}_{45}^2}$
- 5d inversion on all points:

$$\hat{x}^\mu \to \frac{\hat{x}^\mu}{\hat{x}^2} \quad \Rightarrow \quad \hat{x}_{ij}^2 \to \frac{\hat{x}_{ij}^2}{\hat{x}_i^2 \hat{x}_j^2}\,, \quad d^5\hat{x}_5 \to \frac{d^5\hat{x}_5}{\hat{x}_5^{10}}$$

  Implies in particular: $m_i \to m_i/\hat{x}_i^2$.

- Then indeed box integral covariant:

$$\int d^5\hat{x}_5 \frac{\delta(\hat{x}_5^{M=4})}{\hat{x}_{15}^2 \hat{x}_{25}^2 \hat{x}_{35}^2 \hat{x}_{45}^2} \to \int \frac{d^5\hat{x}_5}{\hat{x}_5^{10}} \frac{\delta(\hat{x}_5^{M=4})\,\hat{x}_5^2}{\hat{x}_{15}^2 \hat{x}_{25}^2 \hat{x}_{35}^2 \hat{x}_{45}^2}\,\hat{x}_5^8\,\hat{x}_1^2 \hat{x}_2^2 \hat{x}_3^2 \hat{x}_4^2$$

  $I^{(1)}(s,t,m_i)$ is also 4d translation invariant

  $\Rightarrow$ Extended dual conformal invariance: $\hat{K}_\mu I^{(1)}(s,t,m_i) = 0$

- Triangles and bubbles are not invariant!

- Potential problem [Beisert;Witten]: We have singled out particle 1 $\Leftrightarrow$ Yangian-generators are not cyclic **but** color ordered scattering amplitudes are cyclic??
- Resolution: Consider the Yangian generators produced by singling out particle 2:

$$\tilde{J}^{(1)}_{a} = f^{cb}{}_{a} \sum_{2<j<i<n+1} J^{(0)}_{i,b}\, J^{(0)}_{j,c}$$

  then one shows

$$J^{(1)\,\bar{A}}{}_{\bar{B}} - \tilde{J}^{(1)\,\bar{A}}{}_{\bar{B}} = \delta^{\bar{A}}_{\bar{B}}\, J_{1}^{\bar{C}}{}_{\bar{C}} = \ldots = \delta^{\bar{A}}_{\bar{B}}\, c_1$$

  Importantly $c_i\, \mathbb{A}_n = 0$ locally! Hence level one generators $J^{(1)\,\bar{A}}{}_{\bar{B}}$ are cyclic when acting on amplitudes.
- Linked to vanishing Killing form of superalgebra $(-1)^{|c|} f_{ac}{}^{d}\, f_{bd}{}^{c} = 0$

  $\Rightarrow$ [K. Zarembo's talk]