Burgers universality in four-dimensional $SU(N)$ Yang-Mills theory at large N

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Outline

About a week ago I viewed a simulcast of Richard Strauss's Capriccio at the Met with Renee Fleming in the role of the countess.

The opera was written in 1942, a quite interesting year in Europe.

Tone and word came up already in 1786 in the title of an one opera act by Antonio Salieri [the victim of "Amadeus" by Peter Shaffer, a play first performed in 1979 and later made into a movie]:

Prima la musica e poi le parole

I'll take my cue from Salieri's title.

My obsession I

In lieu of an introduction I shall take the couch and tell you about my personal obsession.

In the 60's one tried to guess the S-matrix of strong interactions on the basis of unitarity and analyticity. One required a maximal form of analyticity, based on the principle that all singularities of the scattering amplitudes in the on-shell, analytically continued, Lorentz invariants be either a direct consequence of a physical channel or, else, removable by extracting some kinematic factor.

No non-trivial guess was found in space-time dimensions higher than 2 [scattering on a line is too constrained kinematically]. Progress was made by finding an iterative scheme, the starting point of which still was highly non-trivial and needed a guess. From that starting point the S-matrix could be argued to emerge after an infinite number of iterations organized in the procedure of dual unitarization.

My obsession II: Large N

One still needed a guess to get started, but this time there was a success in the form first found by Veneziano, and soon generalized. The unitarization program became the topological expansion in “critical” string theory.

Consider, for concreteness, pure $SU(N)$ YM. Believe Mike Teper and others who tell you that confinement holds in the large N limit in the 't Hooft sense, in the continuum limit as constructed from lattice field theory. Take all correlation functions of all local gauge singlet observables. For each such correlation function take the leading, non-trivial term in the 't Hooft expansion in $1/N^2$.

Collect all this information and make it a starting point for a topological expansion as explained by 't Hooft at the Feynman perturbative level, but accept it beyond that.

Question: Does this starting point obey the maximal properties that were postulated of the starting point (ZWR formerly the NRA) in the 60's ?

My obsession III: Large N phase transitions.

Zero width of resonances, residue factorization and, perhaps, Regge asymptotics, hold in the leading $1/N^2$ limit.
Is everything else the *S*-matricists postulated for the ZWR an exact property of the leading terms in $1/N^2$? Careful, a simple argument leads to the conclusion that at $N = \infty$ all Regge trajectories are *exactly* linear: Unlikely to be true.
Charles Thorn, for example, has offered an answer: NO.
I am too young (!) to have earned the right to make a guess but tend more to a NO then to a YES. My reason is that the large N limit often produces new singularities, present only as a result of the expansion in 1/N. It has become common to call these singularities "large N phase transitions". Typically, the singularity occurs at some intermediate scale.
For finite but large N, the dependence on N near a would-be singularity involves unusual powers, N^μ, albeit that μ often is rational: The standard 1/N expansion is at best asymptotic and may miss some information about the full finite N theory.

My obsession IV: Large N and the RG

The theory is constructed by a continuum limit from a well defined lattice model. The limiting behavior is explained postulating a RG. The central assumption of the RG is that individual infinitesimal coarsening steps ("slice integrals") can be defined so as to preserve generic analyticity step by step. Non-analyticities arise only as a consequence of infinite iteration of infinitesimal steps. However, at $N = \infty$, the number of integration variables in the slice integrals diverges and this central assumption can easily fail then. So, one more source of potential nonanalyticity is added when $N = \infty$.
This may even be good news: the $N = \infty$ non-analyticity could happen at just one point in the iteration and be of a simple, "random matrix" type. Maybe one just needs to add a "large N" universality, operating alongside with ordinary RG universality, governing the planar continuum limit.

My obsession V: A scenario

It is possible then that a large N phase transition separates particle-like 't Hooft – planar scattering processes from Regge-like processes in a nonanalytic way.

I do not have evidence for such a large N nonanalyticity in an analytically continued, on-shell, scattering amplitude. I do have evidence for such a large N nonanalyticity in some basic Euclidean-space, non-local observable.

I repeat: my scenario is not defeatist. On the contrary, I hope that the large N transition is simple and has a universal character. I hope this universality produces an approximate starting point of the 1/N expansion, valid for all scales, on both sides of the transition. Having the freedom of two different regimes, connected in a well understood manner, might be a simplification. These are dreams.

My talk is about reality: one example of a large N phase transition in large N QCD, involving Euclidean space-time Wilson loops, whose universality I claim we understand.

Large N transition in Wilson loops I

Because of asymptotic freedom, parallel transport round a closed curve in $SU(N)$ pure gauge theory [with the θ-parameter set to zero] is believed to be close to identity for small curves, and far from identity for large curves.

Parallel transport is identified by a set of N angles, constrained to sum to a multiple of 2π. These angles are the phases of the eigenvalues of the parallel transport matrix. In the context of Euclidean field theory this is a fluctuating object, constrained to $SU(N)$.

The set of eigenvalues fluctuates and individual eigenvalues repel kinematically. When we imagine a simple smooth curve being shrunk, the eigenvalues associated with it all feel a dynamical force pushing them toward unity. In the infinite N limit one expects that the balance between these two forces would produce a nonanalytic single eigenvalue density.

Large N transition in Wilson loops II

For small loops the density has support on a small arc centered at unity, while for large loops the entire unit circle is covered, almost uniformly. For a fixed loop shape, there will be a sharply defined size at which a large N phase transition occurs. It is plausible to view the critical size as identifying a crossover between short distance and long distance dynamics.
It also seems plausible that in the vicinity of that size, the fixed shape loop would have a universal dependence on scale and N for $N \gg 1$. Thus, one may be able to make some specific exact statements about basic gauge invariant observables in the short distance – long distance crossover regime of $SU(N)$ four dimensional gauge theory at N large enough !

The observable

We need an observable that is sensitive to more than a Wilson loop operator trace in the fundamental, W_f; this means more representations must enter, W_r.
A minimal set of representations containing complete confinement information consists of all totally antisymmetric representations of $SU(N)$. The W_r's for these representations are collected into a generating function given by:

$$\mathcal{O}_N(\mathcal{C}, y) \equiv \langle \det \left(e^{y/2} + e^{-y/2} \Omega_f(\mathcal{C}; x) \right) \rangle$$

where the Wilson loop operator matrices are defined by

$$\mathcal{P} e^{i \oint_{\mathcal{C}}^{x} A_r \cdot dx} \equiv \Omega_r(\mathcal{C}; x)$$

$\mathcal{O}_N$ does not depend on the point x. $W_r = \frac{1}{d(r)} \langle \mathrm{tr} \Omega_r \rangle$, with $d(r)$ the dimension of r.

A convenient representation of the observable

Define a "partition function", $\mathcal{Z}(\mathcal{C}, y)$, by

$$\mathcal{Z}(\mathcal{C}, y) = \int [dA_\mu][d\bar{\psi} d\psi] e^{-\frac{1}{2g^2}\int d^4x \mathrm{tr}[F_{\mu\nu}^2(x)]} e^{\int_0^l d\sigma \bar{\psi}(\partial - \mathcal{A} - \mu)\psi}$$

The parametrization of the curve is fixed by $(dx_\mu/d\sigma)^2 = 1$ and l is the length of the curve. Further, $\mathcal{A} = iA_\mu(x(\sigma))(dx_\mu/d\sigma)$, $\mu = -y/l$ and the Grassmann variables obey anti-periodic boundary conditions when going round the curve.

$$\mathcal{O}_N(\mathcal{C}, y) = \mathcal{Z}(\mathcal{C}, y)/\mathcal{Z}$$

$$\langle \det(e^{y/2} + e^{-y/2}\Omega_f)\rangle = \mathcal{Z}(\mathcal{C}, y)/\mathcal{Z} = a_0 + a_1 y^2 + a_2 y^4 + ...$$

$\mathcal{Z}$ is the partition function in the absence of the curve and the fermions on it.

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Binder's cumulant

$$\log[\mathcal{Z}(\mathcal{C}, y)/\mathcal{Z}(\mathcal{C}, 0)] = 1 + M_2 y^2/2 + M_4 y^4/24 + \ldots$$

The nonzero cumulants are:

$$M_{2n} = \langle \left[\frac{1}{l} \int_0^l d\sigma \bar{\psi}\psi(\sigma) \right]^{2n} \rangle^c_{\mu=0}$$

We can think of $\frac{1}{l} \int_0^l d\sigma \bar{\psi}\psi(\sigma) \equiv m$ as a magnetization. Binder's cumulant is given by:

$$\mathcal{B} = \frac{M_4}{M_2^2}$$

$$\mathcal{B} = 6 \left[\omega - \frac{1}{2} \right]$$

$$\omega = \frac{a_0 a_2}{a_1^2}$$

Smearing

We need to renormalize our operator. A convenient globally applicable operator renormalization is by smearing:

$$\partial A_\mu / \partial s = D_\nu(A) F_{\nu\mu}(A)$$

with $A_\mu(s = 0, x) = A_\mu(x)$.

Motivated by many lattice field theory papers, Narayanan and I introduced continuum smearing in 2006, because it had the above elegant form, and it has proven to be a malleable tool, preserving the advantages established by many lattice practitioners over many years; apparently, some think smearing needs reinvention.

Smearing takes care of ultraviolet divergences specific to the operator, leaving the intrinsic UV divergences of the action to be dealt with by another regularization which can be a nonperturbative lattice method or any perturbative continuum method. The regularized operator is made out of $A_\mu(s, x)$ with $s > 0$. s has dimensions of length squared.

An example of smearing: circular loop I

At leading order in the YM coupling g_0^2 the smeared Wilson loop average for a circle of radius R is

$$W_r^{(1)}(R,s) = 1 - \frac{g_0^2}{2} C_2(r) \oint dx_\mu \oint dy_\mu D_0(x-y,s)$$

where $\partial_s D_0(x,s) = 2\nabla_x^2 D_0(x,s) = -\frac{1}{32\pi^2 s^2} \exp\left[-\frac{x^2}{8s}\right]$. $C_2(r)$ is the quadratic Casimir of r.

The dilatation invariance of the action ensures that the exponent is a function of the dimensionless ratio $t = \frac{s}{R^2}$ and the answer is $W_r^{(1)}(R,s) = \exp\left[-\frac{g_0^2}{2} C_2(r) f(t)\right]$ with

$$f(t) = -\frac{1}{2} + \frac{1}{4t} e^{-\frac{1}{2t}} \left[(1+2t) I_0\left(\frac{1}{2t}\right) + I_1\left(\frac{1}{2t}\right)\right]$$

An example of smearing: circular loop II

For large t: $f(t) = \frac{1}{32t^2} - \frac{1}{96t^3} + ...$
For small t: $f(t) = \frac{1}{2\sqrt{\pi t}}\left(1 - \sqrt{\pi t} + \frac{3t}{4} + \dots\right)$.
The leading term is a perimeter term, rendered finite for finite R and $s > 0$. The next term is a pure number, the single piece of the answer which is independent of R. It only depends on the shape of the loop and not on its scale.
As defined, $f(t)$ is positive for all $t > 0$ and monotonically decreasing to zero with increasing t. This makes the quantity in the exponent negative resulting in $W_r^{(1)}(R, s) \leq 1$ as befitting the average of a unitary matrix. Note however that the constant piece has the opposite sign. In general, getting rid of the perimeter divergence using a regularization based on analyticity has a tendency to violate the unitarity inequality $W_r^{(1)}(R, s) \leq 1$.

RG improved PT and Burgers' equation I

In four dimensional $SU(N)$ pure gauge theory the expectation value of a Wilson loop parallel transporter Ω_r will have, to order g_0^4 (one loop) in perturbation theory, the form $\langle \mathrm{tr}\Omega_r \rangle = d(r) e^{-\frac{C_2(r)}{N+1}\tau}$. After renormalization with the help of smearing, to order g_0^4, the above group theoretical structure still holds. An identical representation dependence holds, this time exactly, in 2D.

For a relatively small effective coupling $g^2(l)$, perturbation theory would say: $\tau = g^2(l)(N+1)F(\xi_a, l^2/s)$, where F is a function of dimensionless loop parameters: shape – ξ_a, and smearing – l^2/s. $g^2(l)N$ depends on l in the usual manner dictated by asymptotic freedom, vanishing as $l \to 0$. Common sense implies that $\tau > 0$ would increase monotonically with l at fixed ξ_a and l^2/s even beyond perturbation theory.

RG improved PT and Burgers' equation II

- We define a map from the variable l, measuring the overall scale of the loop in 4D to $\tau_N(l)$ not from the W_r, but rather by $\omega_N^{4D}[\mathcal{C}(\xi_a, S(l), l)] = \omega_N^{2D}(\tau_N(l))$. The latter definition could be used even if the r-dependence of W_r in 4D differs from 2D. $S(l)$ is a fixed function defining the amount of smearing $s = S(l)$, which goes as a constant times l^2 for small l, but violates scaling for larger l. We focus on varying l at a fixed set of ξ_a's and a fixed functional form of S.
- $\tau_N(l)$ approaches $\tau_\infty(l)$ quite rapidly, point-wise at all l in a range that includes at least a sizable neighborhood of $\tau = 4$.
- $\tau_N(l)$ and the limiting function $\tau_\infty(l)$ are all smooth in l.
- For reasonably large N, because of the uniformity of the large N limit on $\tau_N(l)$ we can replace $\tau_N(l)$ by $\tau_\infty(l)$ without altering the singular large N properties. The postulate of universality means that for the vicinities of the critical scale and $y = 0$, for $N \gg 1$, we may replace $\mathcal{O}_N^{4D}(\mathcal{C}, y)$, by $\mathcal{O}_N^{2D}(\tau_\infty(l), y)$.

RG improved PT and Burgers' equation III

We now describe the two dimensional large N singularity.
To bring out the N dependence we define

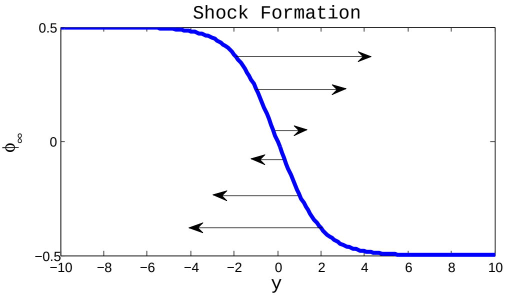
$$\phi_N(y,\tau) = -\frac{1}{N}\frac{\partial}{\partial y}\log\left[\mathcal{O}_N(\tau,y)\right]$$

We are actually interested in ϕ_N more than in $\mathcal{O}_N$.
It is straightforward to show, as a consequence of the r-dependence, that $\phi_N(y,\tau)$ satisfies Burgers' equation:

$$\frac{\partial\phi_N}{\partial\tau} + \phi_N\frac{\partial\phi_N}{\partial y} = \frac{1}{2N}\frac{\partial^2\phi_N}{\partial y^2}$$

The initial condition on $\phi_N(y,\tau)$ is $\phi_N(y,0) = -\frac{1}{2}\tanh\frac{y}{2}$.
Asymptotically, for large τ we have, $\phi_N(y,\infty) = -\frac{1}{2}\tanh\frac{Ny}{2}$.
Burgers' equation in the inviscid limit ($N=\infty$) with the $\tau = 0$ initial condition above produces a singularity as a function of y at $y = 0$, when τ reaches the value 4.

Shock formation visualized



RG improved PT and Burgers' equation IV

From Burgers' equation:

$$\frac{\partial}{\partial \tau}\left[\frac{1}{\partial \phi_N/\partial y|_{y=0}}\right] = 3\omega_N(\tau) - \frac{1}{2}$$

The shock wave is easy to understand from Burgers' equation: $\omega_\infty(\tau)$ is equal to 1/2 for $0 < \tau < 4$ and to 1/6 for $4 < \tau$. As τ increases from zero, the RHS stays equal to unity. The inverse slope increases from -4 at unit slope in τ. For $\tau \to 4^-$, the inverse slope becomes 0^-, that is a discontinuous jump in ϕ_N takes place. After the jump the RHS becomes zero and the slope no longer changes. The infinite jump at $y = 0$ persists for all $\tau > 4$.

The main point is that $\phi_N^{4D}[\mathcal{C}(\xi_a, S(l), l)]$ for $l \sim l_c$ presents a similar shock wave structure.

RG improved PT and Burgers' equation V

Some consequences of Burgers' universality:

$$\lim_{N\to\infty}\left[N^{-3/2}\ \frac{a_1}{a_0}\bigg|_{\tau=4}\right] = \frac{1}{8}\sqrt{\frac{3}{2}}\frac{1}{K},$$

$$\lim_{N\to\infty}\left[N^{-3/2}\ \frac{a_2}{a_1}\bigg|_{\tau=4}\right] = \frac{1}{24}\sqrt{\frac{3}{2}}K,\ K \equiv \frac{1}{4\pi}\Gamma^2\left(\frac{1}{4}\right) \approx 1.046$$

$$\lim_{N\to\infty}\left[\omega_N|_{\tau=4}\right] = \frac{1}{3}K^2 \equiv \omega_c \approx 0.3647$$

$$\lim_{N\to\infty}\left[N^{-1/2}\ \frac{d\omega_N}{d\tau}\bigg|_{\tau=4}\right] = -\frac{1}{6}\sqrt{\frac{3}{2}}K(K^2-1) \approx -0.0201$$

The N dependence of the a-ratios reflects the large N separation between the roots of $\mathcal{O}_N(\mathcal{C}, y)$ on the imaginary y-axis near $y = 0$ and at $\tau \sim 4$ where it goes like $N^{-\frac{3}{4}}$. The N dependence of the τ derivative at $\tau \sim 4$ reflects the mean field character of the "magnetic" exponent.

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Numerical objectives

To test our hypothesis about Burgers universality in 4D we wish to:

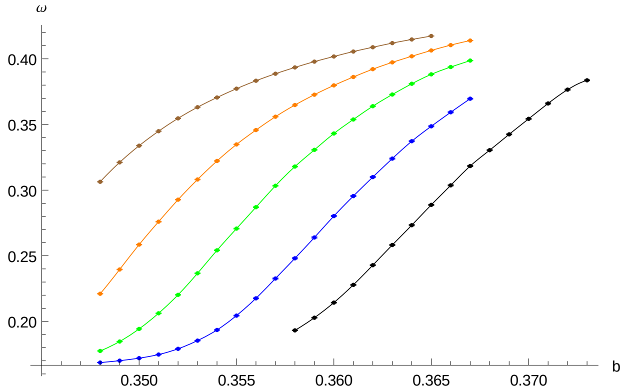
- Show that indeed $\tau_N(l)$ has a continuum limit consistent with asymptotic freedom and the critical regime $\tau_N(l) \sim 4$ naturally falls at weak to strong crossover scales.
- Confirm by independent tests that the large N dependence is governed by the exponents 3/4 and 1/2.

Our statistical errors are small, but we have not yet tested sufficiently for systematic effects.

To check for the continuum limit we studied a sequence of square Wilson loops of side $L = 3, 4, 5, 6, 7$ at $N = 19$ for inverse 't Hooft couplings $0.348 \leq b \leq 0.373$.

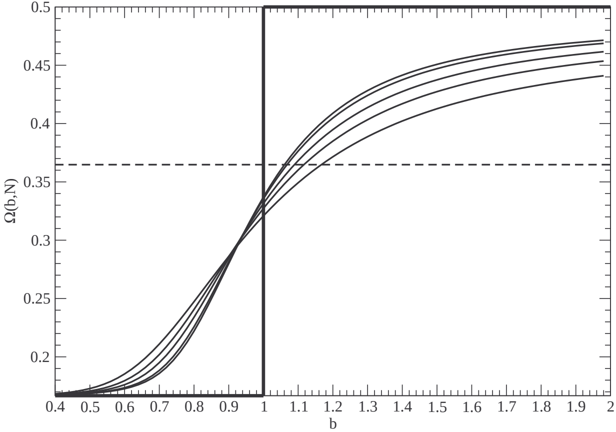
To check for the large N exponents we looked at the eigenvalue spacing at -1 close to criticality and at ω_N^{4D} there for $N = 19, 29, 47$.

Approach to continuum limit I: raw data



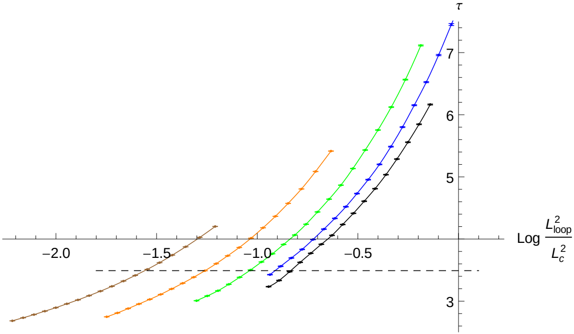
Largest loops rightmost. *y*-axis is ω. *x* axis is *b*. $N = 19$. Data points plus cubic spline interpolation.

Approach to infinite N: analytical 2D



$N = 17, 23, 29, 37, 41, 47, \infty$. Need a very large N to see the jump. b is the 2D 't Hooft gauge coupling. Plot shows analytically known functions. Picking the same N and locally shifting and stretching or compressing the x-axis produces, for each line of raw data, a fragment of the analytical curve.

Approach to continuum limit II

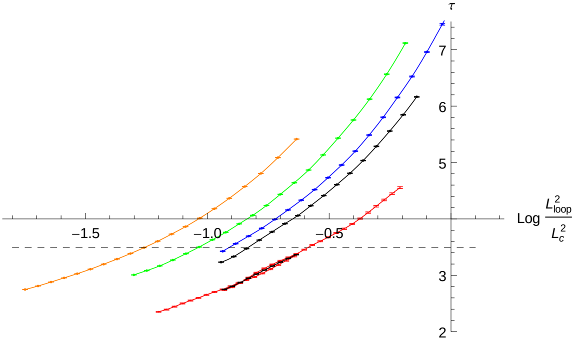


Largest loops rightmost. y-axis is τ. $L_c(b)$:

$$L_c(b) = 0.26 \left(\frac{11}{48\pi^2 b_i(b)} \right)^{51/121} e^{\frac{24\pi^2 b_i(b)}{11}}$$

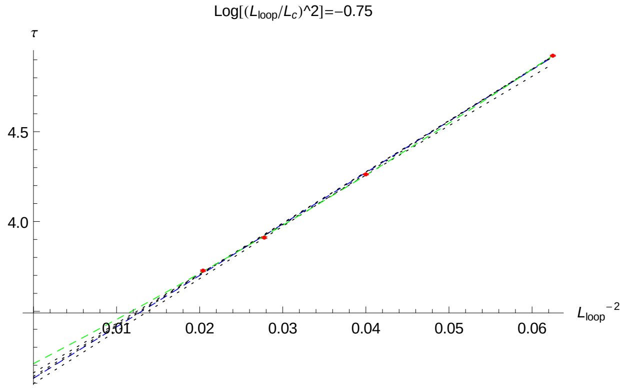
where $b_i(b) = b\frac{\langle tr\Omega_{\text{plaq}}\rangle}{N}$. Here we mapped $b \to L_c(b)$ where $L_c(b) \sim 3/4\ \text{fm}$ in QCD units. We also mapped, for each fixed lattice loop size L_{loop}, $\omega_N^{4D} \to \tau_N(\log \frac{L^2_{loop}}{L^2_c(b)})$, setting $N = 19$.

Approach to continuum limit III: linear extrapolation



The bottom rightmost curve is obtained by extrapolating all the data above it linearly in $1/L^2$ at fixed physical scale $L/L_c(b)$. $1/L^2$ is an order a^2 correction where a is the lattice spacing. The coefficient of the a^2 term is roughly linear in $\log[(L/L_c(b))^2]$. On the dashed horizontal line ω_N would attain the critical value ω_∞ for $N = 19$. Note the large N correction (analytically calculated) separating the dashed line from 4, showing large $1/N$ effects (already in 2D).

Approach to continuum limit III: example of linear extrapolation



Although we see a quite linear behavior, the magnitude of the correction relative to the continuum value is quite large.

Continuum limit

In summary, there is little doubt that $\tau_N(l)$ has a nontrivial continuum smooth limit as a function of physical loop size l [$L_c(b) \sim 3/4$ fm in SU(3) terms] and this holds from small loops and through a scale where the large N phase transition occurs.
Some less reliable data (not shown) confirms this for $N = 29$.
The point-wise convergence of $\tau_N(l)$ to $\tau_\infty(l)$ is rapid, while the buildup of the shock wave singularity in $\mathcal{O}_N^{2D,4D}$ is slow.
In short, there is a large N phase transition of the type we expected in continuum $SU(N)$ gauge theory, for Wilson loops renormalized by smearing.

The critical exponents of N I: 3/4

This exponent was responsible for the large N scalings predicted for the a-coefficient ratios in the critical regime. It comes from the expected average level spacing at eigenvalues ~ -1 at criticality. We extract direct estimates for the latter from the data we generate: more precisely, we compute the average length θ_g of the arc connecting two consecutive eigenvalues on the unit circle which has -1 in its interior. For 6×6 loops and for $\omega_N \approx \omega_c$ we found $N^{3/4}\theta_g = 7.60$ for 47, 7.96 for 29 and 8.36 for 19. If we just fitted $\log \theta_g$ to $\log N$ we would have gotten a slope of 0.85. However, the theory actually predicts, at criticality, the numerical value of the zeros of $\mathcal{O}(\mathcal{C}, y)$ on the imaginary y axis that are nearest to y=0 as some number times $N^{-3/4}$. That prediction fits the number at 47 very well – we suspect that the accuracy of the prediction is accidental: It is not exactly true that the zeros give θ_g; rather this holds only in an approximate sense. In any case, the exponent of N is corroborated to be 3/4.

The critical exponents of N II: 1/2

This exponent enters the slope of ω at criticality. Taking the derivative of a cubic spline interpolation to ω_N at ω_c for 6×6 loops with respect to $\log(L/L_c)^2$ for $N = 19, 29, 47$ and least-square fitting the logarithms of these slopes to $\log N$ gave a power of 0.58; within errors this is consistent with an exponent of 1/2. Unlike in the case of the 3/4-exponent, in this case a large N non-universal amplitude enters, and there is no way to get a direct estimate for the number, except perhaps, for a situation that the smearing is chosen in such a manner that the critical regime lies in the perturbative domain. Such a choice is possible, but the necessary perturbative computation has not been yet carried out: it is fairly difficult. Our data was taken in a regime that might be within the reach of perturbation theory, but we do not know this for sure.

For both exponents, varying the definition of "criticality" by order $\frac{1}{N}$ terms (which can be motivated in various ways) one can get closer to 1/2 and 3/4.

The main question

It could be that the pre-QCD S-matrix postulates on hadronic processes do not match precisely the planar limit. There is no doubt that the latter really is well defined. One possibility is that the large N limit induces some non-analyticities of "statistical" origin, which relieve the system from the extremely tight constraints imposed by analyticity, crossing, unitarity and simultaneous Regge behavior in all channels and, at the same time, hard high momentum behavior. I have been searching for the right observables where the mechanism which relaxes the constraints can be identified for a long time, and today I described one candidate.

In general, I think it is important to identify this "violation of S-matrix basic principles" in the planar limit. This might show what exactly it is that limits the nonlinear sigma–mode–based effective string theory approach to the "soft regime" of planar QCD from extending to short distances.