Hydrodynamic fluctuations

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University of Victoria

GGI, May 3, 2011

Outline

1. Why hydro?
2. Hydro fluctuations
3. A simple calculation
4. Fluctuations: Brownian motion
5. Fluctuations: Diffusion equation
6. Fluctuations: Linear hydrodynamics
7. Fluctuations: Non-linear hydrodynamics
8. Conclusions

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- Thermodynamics $=$ simplest possible effective theory at $T > 0$

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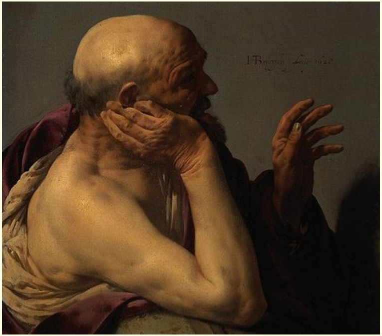
- Real-time dynamics is much harder
- Thermodynamics = simplest possible effective theory at $T>0$
- Hydrodynamics = next simplest effective theory at $T>0$

Next simplest effective theory



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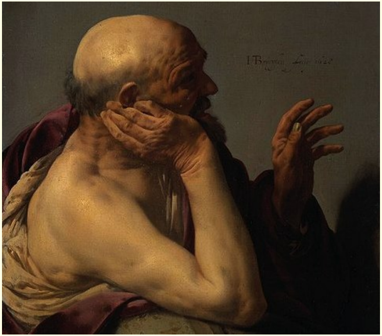

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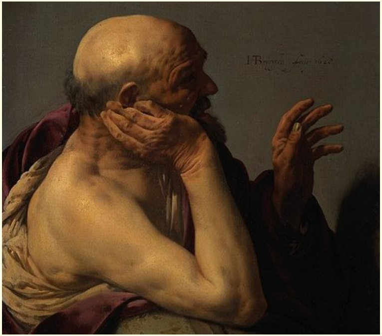

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How well a given substance flows depends on its viscosity

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However, QCD at $T \gtrsim T_c$ is a nearly-erfect fluid *not* because $\eta = 0$, but because η is small compared to something.

Kinematic viscosity of QCD

A natural measure of viscosity at a given T is

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- Hydro fits to data: $\dfrac{\eta}{s} = (0.1 \pm 0.1 \pm 0.08)\,\hbar$ — Luzum+Romatschke, 2008
- $g\to 0$, pure glue $SU(3)$: $\dfrac{\eta}{s} = \dfrac{3.87}{g^4 \ln 1/g}\hbar$ — Arnold+Moore+Yaffe, 2000
- $N\to\infty$, $\lambda\to\infty$ gauge-gravity: $\dfrac{\eta}{s} = \dfrac{\hbar}{4\pi}$ — PK+Son+Starinets, 2004

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Can we say anything about the viscosity of QCD without making the above approximations?

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Linearized relativistic hydro

Relativistic hydro with $\mu = 0$:

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0\,, \qquad \frac{\partial \pi_i}{\partial t} + \partial_j T_{ij} = 0\,.$$

$$T_{ij} = P\delta_{ij} - \gamma_\eta \left(\partial_i \pi_j + \partial_j \pi_i - \frac{2}{d}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\right) - \gamma_\zeta \delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{\pi} + \ldots$$

$\gamma_\eta \equiv \eta/\bar{w}$, $\gamma_\zeta \equiv \zeta/\bar{w}$, and $\bar{w} = \bar{\epsilon}+\bar{P}$.

$$\text{Fluctuations of } \boldsymbol{\pi}_\perp : \;\; \omega = -i\gamma_\eta \boldsymbol{k}^2\,,$$

$$\text{Fluctuations of } \pi_\parallel, \epsilon : \;\; \omega = \pm v_s |\boldsymbol{k}| - i\frac{\gamma_s}{2}\boldsymbol{k}^2\,, \quad \gamma_s \equiv \gamma_\zeta + \frac{2d-2}{d}\gamma_\eta\,.$$

Simple picture for viscosity

Viscosity measures rate of momentum transfer between layers of fluid

$$\eta = \rho v_{\rm th} \ell_{\rm mfp}$$

Maxwell, 1860

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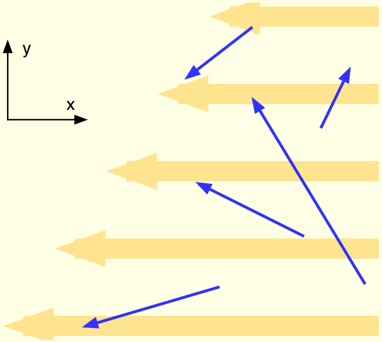

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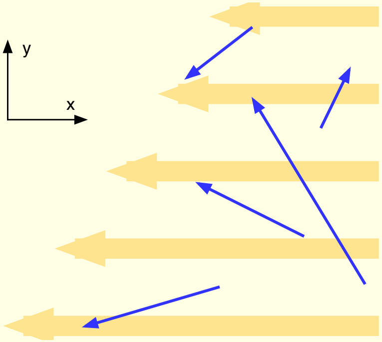

$$\ell_{\rm mfp} \sim \frac{1}{n\sigma} \sim \frac{T}{\lambda^2}$$

$$\eta_0 \sim \frac{N^2 T^3}{\lambda^2}$$

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Elementary excitations are not the only way to transfer momentum. Momentum can also be transfered by collective excitations.

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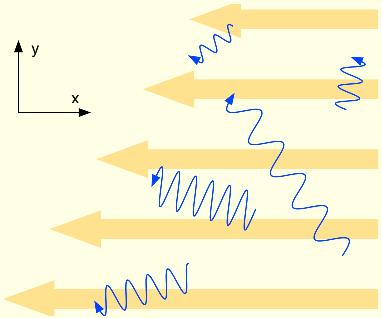

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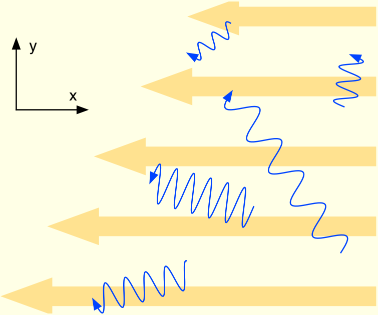

$$\ell_{\rm mfp} \sim \frac{1}{\frac{\eta}{\epsilon+P}\boldsymbol{k}^2}$$

$$\eta_1 \sim \int^{k_{\rm max}} d^3k \frac{T}{\frac{\eta_0}{\epsilon+P}\boldsymbol{k}^2} \sim \frac{k_{\rm max}T^2}{\eta_0/s}$$

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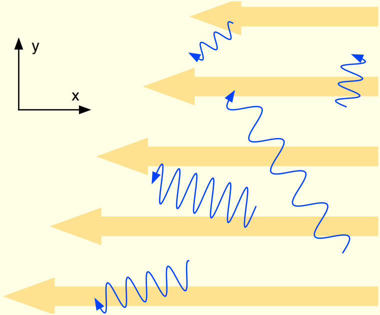

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- Total viscosity $\eta_{\rm total} = \eta_0 + \eta_1$ is bounded from below
- This integral IR finite in $d = 3{+}1$, but IR divergent in $d = 2{+}1$

Forster+Nelson+Stephen, 1977

The rest of the talk will expand on these points

Namely

- How do hydro fluctuations change viscosity in $d = 3{+}1$?
- How do hydro fluctuations change second-order hydrodynamics?
- How do hydro fluctuations change viscosity in $d = 2{+}1$?

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Interaction of hydro modes

- In hydro, there are no arbitrary "coupling constants" like g
- Coefficients of non-linear terms are fixed by symmetry (Galilean or Lorentz) E.g.

$$J^\mu = n u^\mu + \nu^\mu \,, \qquad T^{\mu\nu} = (\epsilon{+}P) u^\mu u^\nu + P\eta^{\mu\nu} + \tau^{\mu\nu} \,.$$

  All transport coefs η, ζ, κ are present already in linearized hydro
- Interaction of modes will change hydro correlation functions
- Was known since late 1960's – "mode-mode coupling"

Long-time tails

Start with $\boldsymbol{J} = -D\boldsymbol{\nabla} n + n\boldsymbol{v}$, take $\boldsymbol{k} = 0$. Schematically:

$$\begin{aligned}
\langle \boldsymbol{J}(t)\boldsymbol{J}(0)\rangle &\supset \int d^d x\, \langle n(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})n(0)\boldsymbol{v}(0)\rangle \\
&= \int d^d x\, \langle n(t,\boldsymbol{x})n(0)\rangle\langle \boldsymbol{v}(t,x)\boldsymbol{v}(0)\rangle \\
&\sim \int d^d k\, e^{-D\boldsymbol{k}^2 t} e^{-\gamma_\eta \boldsymbol{k}^2 t} \\
&\sim \left[\frac{1}{(D+\gamma_\eta)t}\right]^{d/2}
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See e.g. Arnold+Yaffe, PRD 1997
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When FT, the convective contribution to $S(\omega)$ is

$$S(\omega) \sim \omega^{1/2}\,, \quad d=3$$
$$S(\omega) \sim \ln(\omega)\,, \quad d=2$$

Correction to Kubo formulas

Recall Kubo formula for the diffusion constant:

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$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const}\ \omega^{1/2}\right) , \quad d = 3$$

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Same applies to shear viscosity:

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In 2+1 dimensional hydro, transport coefficients blow up

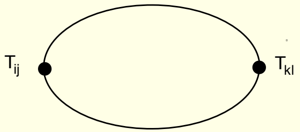
Comment

- In AdS/CFT, the $\ln(\omega)$ correction is $1/N^{3/2}$ suppressed
- Transport coefficients come out finite in $3+1$ dimensional *classical* gravity
- Long-time tails come from quantum corrections to classical gravity

Kovtun+Yaffe, 2003
Caron-Huot + Saremi, 2009

- This is an example where long-time limit does not commute with large-N limit

Can do the same calculation in momentum space



One-loop diagram with sound and/or shear waves in the loop

$$S_{xy,xy}(\omega, \boldsymbol{k}{=}0) = (\epsilon{+}P)^2 \int \frac{d\omega'}{2\pi} \frac{d^3k}{(2\pi)^3}$$
$$\Big(\Delta_{xx}(\omega', \boldsymbol{k})\Delta_{yy}(\omega{-}\omega', -\boldsymbol{k}) + \Delta_{xy}(\omega', \boldsymbol{k})\Delta_{yx}(\omega{-}\omega', -\boldsymbol{k})\Big)$$

where Δ_{ij} = FT of $\langle u_i(x)u_j(0)\rangle$

When the dust settles...

$$G^R_{xy,xy}(\omega \ll k_{\max}, \boldsymbol{k}{=}0) = -i\omega\eta_0$$
$$-i\omega\frac{17Tk_{\max}}{120\pi^2\gamma_{\eta_0}} + (1+i)\omega^{3/2}\frac{(7+(3/2)^{3/2})T}{240\pi\gamma_{\eta_0}^{3/2}} + O\left((k_{\max}\gamma_{\eta_0})^2, \omega^2\right)$$

PK+Moore+Romatschke, 2011

The contribution due to hydro fluctuations is suppressed at either small coupling, or large N

Implications for the shear viscosity

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  at $T \gtrsim T_c$
- Current hydro simulations of QGP are blind to these effects because they simply solve the classical hydro equations and ignore the fluctuations

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Take diffusion equation, add higher-derivative terms

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DeSchepper + Van Beyeren + Ernst, 1974

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- Alternatively, the dispersion of hydro modes has no analytic expansion in powers of $|\boldsymbol{k}|$, i.e. $\omega \neq c_1|\boldsymbol{k}| + c_2\boldsymbol{k}^2 + c_4\boldsymbol{k}^4 + \dots$
  Interaction of hydro modes produces ∞ many fractional powers
  $\omega = c_1|\boldsymbol{k}| + c_2\boldsymbol{k}^2 + a_1|\boldsymbol{k}|^{5/2} + a_2|\boldsymbol{k}|^{11/4} + \dots$

Ernst + Dorfman, 1975

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In linearized second order hydro:

$$G^R_{xy,xy}(\omega, \boldsymbol{k}) = P - i\omega\eta + \left(\eta\tau_\Pi - \frac{\kappa}{2}\right)\omega^2 - \frac{\kappa}{2}\boldsymbol{k}^2 + \dots$$

Baier+Romatschke+Son+Starinets+Stephanov, 2007

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But this gets seriously modified by 1-loop hydro fluctuations,

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This means τ_Π does not exist

Can we save second-order hydro?

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- 2nd-order hydro breaks down below some ω_* depends on η_0/s

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- Can estimate when $\omega^{3/2}$ term becomes comparable to ω^2 term
- 2nd-order hydro breaks down below some ω_* depends on η_0/s
- If $\eta_0/s \sim 0.16$, then $\omega_* \sim T/20$,
  2nd-order hydro OK for heavy-ion collisions
- If $\eta_0/s \sim 0.08$, then $\omega_* \sim 2.5T$,
  2nd order hydro makes no sense for heavy-ion collisions

PK+Moore+Romatschke, 2011

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Rather, viscosity, conductivity etc become scale-dependent "running masses" in the low-energy effective hydro theory

To find this low-energy effective hydro theory, need both dissipation (transport coefficients) and fluctuations (thermally excited modes)

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Langevin equation

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Brownian particle:

$$m\frac{d^2x}{dt^2} = -(6\pi\eta a)\frac{dx}{dt} + f(t)\,,$$

$(6\pi\eta a)$ = friction coefficient (Stokes law)
$f(t)$ = random force

Langevin equation

Brownian particle:

$$m\frac{d^2x}{dt^2} = -(6\pi\eta a)\frac{dx}{dt} + f(t)\,,$$

$(6\pi\eta a) =$ friction coefficient (Stokes law)
$f(t) =$ random force

Take $q \equiv \frac{dx}{dt}$, $\Rightarrow$ Langevin equation:

$$\boxed{\dot{q}(t) + \gamma q(t) = \xi(t)}$$

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Noise properties:

$$\langle \xi(t) \rangle = 0\,, \qquad \langle \xi(t)\xi(t') \rangle = C\delta(t-t')\,.$$

C determines the strength of the noise

Correlation function of $q(t)$

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- Take the Langevin equation $\dot{q}(t) + \gamma q(t) = \xi(t)$
- Solve for $q(t)$ in terms of $\xi(t)$
- Find $\langle q(t)q(t')\rangle$ by averaging over $\xi(t)$
- When $\gamma t, \gamma t' \gg 1$, find

$$\langle q(t)q(t')\rangle = \frac{C}{2\gamma} e^{-\gamma|t-t'|}$$

- Fourier transform:

$$S(\omega) = \frac{C}{\omega^2 + \gamma^2}$$

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- Langevin equation gives $G_R(\omega) = \frac{i}{\omega + i\gamma}$

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- Demand FDT:

$$\boxed{C = 2T}$$

Path integral for Brownian particle

Let us now represent the Brownian motion as Quantum Mechanics
(0+1 dimensional quantum field theory)

Path integral for Brownian particle

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Step 3 Recall $\delta(f(x)) \sim \delta(x - x_0)$, where x_0 solves $f(x_0) = 0$. So

$$\int \mathcal{D}q \, J \, \delta(EoM) \, q(t_1) \, q(t_2) ... = \underbrace{q_\xi(t_1)} \; \underbrace{q_\xi(t_2)} ...$$

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Step 4 Write $\delta(EoM) = \int \mathcal{D}p \, e^{i \int p \, EoM}$, do the integral over $\xi(t)$.

Path integral for Brownian particle (2)

When the dust settles:

$$\langle\, q(t_1) \ldots q(t_n) \,\rangle = \int \mathcal{D}q \, \mathcal{D}p \, J \, e^{iS[q,p]} \, q(t_1) \ldots q(t_n)$$

where

$$S[q,p] = \int dt \, \left(p\dot{q} + p\frac{\partial F}{\partial q} + \frac{iC}{2} p^2 \right) \, .$$

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For the simple Langevin equation $F(q) = \frac{1}{2}\gamma q^2$,

$$S(\omega) = \text{FT of } \langle q(t) q(t')\rangle = \frac{C}{\omega^2 + \gamma^2}\, ,$$

as expected.

Bottomline:

In the stochastic model

$$\dot{q}(t) + \underbrace{\frac{\partial F(q)}{\partial q}}_{\text{relaxation term}} = \underbrace{\xi(t)}_{\text{noise term}}$$

correlation functions can be derived from field theory with

$$S[q,p] = \int dt \, \left(p\dot{q} + p\frac{\partial F}{\partial q} + \frac{iC}{2}p^2 \right)$$

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Fields

Many variables: $q_i(t) \to \phi(\boldsymbol{x}, t)$

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Langevin equation:

$$\dot{q}(t) = -\frac{\partial F(q)}{\partial q} + \xi(t) \quad \to \quad \frac{\partial}{\partial t}\phi(\boldsymbol{x}, t) = -\Gamma \frac{\delta F[\phi]}{\delta \phi} + \xi(\boldsymbol{x}, t)$$

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Functional $F[\phi]$ depends on the problem e.g.

$$F[\phi] = \int d^d x \left(\frac{a}{2}\phi^2 + \frac{b}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{\lambda}{24}\phi^4 \right)$$

is "model A" in the classification of dynamic critical phenomena by

Hohenberg and Halperin, RMP, 1977

Also called "time-dependent Landau-Ginzburg theory"

Effective action

- Gaussian noise: $\langle\xi(\boldsymbol{x}_1,t_1)\xi(\boldsymbol{x}_2,t_2)\rangle = C\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\delta(t_1-t_2)$
- Correlation functions:

$$\langle\phi(\boldsymbol{x}_1,t_1)...\phi(\boldsymbol{x}_n,t_n)\rangle = \int\mathcal{D}\phi\,\mathcal{D}\chi\;Je^{iS[\phi,\chi]}\phi(\boldsymbol{x}_1,t_1)...\phi(\boldsymbol{x}_n,t_n)\,,$$

where

$$S[\phi,\chi] = \int dt\,d^dx\,\left(\chi\partial_t\phi + \chi\Gamma\frac{\delta F}{\delta\phi} + i\frac{C}{2}\chi^2\right)\,.$$

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$$S[\phi, \chi] = \int dt\, d^d x\, \left(\chi \partial_t \phi + \chi \Gamma \frac{\delta F}{\delta \phi} + i \frac{C}{2} \chi^2 \right)\,.$$

- In model A $(\lambda = 0)$:

$$S_{\phi\phi}(\omega, \boldsymbol{k}) = \Big(\text{FT of } \langle \phi(\boldsymbol{x}_1, t_1) \phi(\boldsymbol{x}_2, t_2) \rangle \Big) = \frac{C}{\omega^2 + \Gamma^2 (a + b\boldsymbol{k}^2)^2}$$

Retarded function

- Effective action for model A (Langevin eqn for fields) :

$$S[\phi,\chi] = \int dt\, d^d x \, \left(\chi \partial_t \phi + \chi \Gamma \frac{\delta F}{\delta \phi} + i \frac{C}{2} \chi^2 \right) .$$

- Add source as $F[\phi] \to F[\phi] - \int dt\, d^d x\, h\, \phi$
- Response of the field:

$$\delta \langle \phi(\boldsymbol{x},t) \rangle = -i\Gamma \int dt'\, d^d x'\, G(t{-}t', \boldsymbol{x}{-}\boldsymbol{x}') \delta h(\boldsymbol{x}', t')$$

where $G(t{-}t', \boldsymbol{x}{-}\boldsymbol{x}') \equiv \langle \phi(\boldsymbol{x},t) \chi(\boldsymbol{x}',t') \rangle$.

- Can identify

$$G_R(t,\boldsymbol{x}) = -i\Gamma \langle \phi(\boldsymbol{x},t)\chi(0)\rangle \,, \qquad G_A(t,\boldsymbol{x}) = -i\Gamma \langle \phi(0)\chi(\boldsymbol{x},t)\rangle \,.$$

Fluctuation-dissipation theorem

- Note: $S_{\phi\phi}(\boldsymbol{x},t) \equiv \langle \phi(\boldsymbol{x},t)\phi(0)\rangle$ and $G(\boldsymbol{x},t) \equiv \langle \phi(\boldsymbol{x},t)\chi(0)\rangle$ are not independent.

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$$C = 2T\Gamma$$

- In model A ($\lambda = 0$)

$$G_R(\omega,\boldsymbol{k}) = \frac{-\Gamma}{i\omega - \Gamma(a+b\boldsymbol{k}^2)}\,, \qquad S_{\phi\phi}(\omega,\boldsymbol{k}) = \frac{C}{\omega^2+\Gamma^2(a+b\boldsymbol{k}^2)^2}$$

Model A

Nice singulatities of correlation functions,
but still not quite hydrodynamics

Diffusion

- Note that model A (Langevin eqn for fields) does **not** describe diffusion of a conserved density
- Field ϕ is referred to as a "non-conserved order parameter"
- Diffusion equation $\partial_t n(t,\boldsymbol{x}) = D\boldsymbol{\nabla}^2 n(t,\boldsymbol{x})$ predicts

$$G_R(\omega,\boldsymbol{k}) = \frac{-D\chi\boldsymbol{k}^2}{i\omega - D\boldsymbol{k}^2}\,, \qquad S_{nn}(\omega,\boldsymbol{k}) = \frac{2DT\chi\boldsymbol{k}^2}{\omega^2 + (D\boldsymbol{k}^2)^2}$$

  where $\chi \equiv \partial\langle n\rangle/\partial\mu$ is static susceptibility
- Guess: take model A, with $\Gamma \to D\chi\boldsymbol{k}^2$. This is "model B" in the classification of Hohenberg and Halperin, RMP, 1977

Model B

Stochastic equation

$$\frac{\partial}{\partial t} n(\boldsymbol{x}, t) = \gamma \boldsymbol{\nabla}^2 \frac{\delta F[n]}{\delta n} + \xi(\boldsymbol{x}, t)$$

with the free energy

$$F[n] = \int d^d x \left(\frac{a}{2} n^2 + \frac{b}{2} (\boldsymbol{\nabla} n)^2 + \ldots \right)$$

and Gaussian noise

$$\langle \xi(\boldsymbol{x}, t) \xi(\boldsymbol{x}', t') \rangle = -2T\gamma \boldsymbol{\nabla}^2 \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t')$$

Bottomline

Correlation functions for the simple diffusion equation:

$$\langle n(\boldsymbol{x},t)n(\boldsymbol{x}',t')...\rangle = \int \mathcal{D}n\,\mathcal{D}\psi\, e^{iS[n,\psi]} n(\boldsymbol{x},t)n(\boldsymbol{x}',t')...$$

$$\boxed{S[n,\psi] = \int_{t,\boldsymbol{x}} \left(\psi \frac{\partial n}{\partial t} - \psi D \boldsymbol{\nabla}^2 n + iD\chi T(\boldsymbol{\nabla}\psi)^2\right)}$$

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Can integrate out ψ, get a non-local effective action for n only

$$S_{\text{eff}}[n] = \frac{1}{2}\int_{t,\boldsymbol{x},\boldsymbol{x}'} E(\boldsymbol{x},t)D(\boldsymbol{x},\boldsymbol{x}')E(\boldsymbol{x}',t)$$

where $E(\boldsymbol{x},t) \equiv (\frac{\partial n}{\partial t} - D\boldsymbol{\nabla}^2 n)$, and $\boldsymbol{\nabla}^2 D(\boldsymbol{x},\boldsymbol{x}') = -\frac{1}{2D\chi T}\delta(\boldsymbol{x}-\boldsymbol{x}')$.

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- We have an effective action for simple diffusion
- This effective action is **not** meant to reproduce the classical diffusion equation
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  i) at low energies

  ii) in real time

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Now that we know how to construct the effective action for diffusion, can do the same for hydrodynamics

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Stochastic model for linearized hydro

$$\frac{\partial \epsilon}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{\pi} \, ,$$

$$\frac{\partial \pi_i}{\partial t} = -v_s^2 \partial_i \epsilon + M_{ij} \pi_j + \xi_i(\boldsymbol{x}, t) \, .$$

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Dissipative terms:

$$M_{ij} \equiv \gamma_\eta (\boldsymbol{\nabla}^2 \delta_{ij} - \partial_i \partial_j) + \gamma_s \partial_i \partial_j$$

Noise correlations:

$$\langle \xi_i(\boldsymbol{x}, t) \xi_j(\boldsymbol{x}', t') \rangle = -2\bar{w} T M_{ij} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t')$$

Note the same M_{ij} must appear both in the hydro equations, and in the noise correlations

Functional integral for hydro

Correlation functions in linearized hydro:

$$\langle \epsilon(\boldsymbol{x},t)\pi_k(\boldsymbol{x}',t')...\rangle = \int \mathcal{D}\epsilon\, \mathcal{D}\boldsymbol{\pi}\, \mathcal{D}\eta\, \mathcal{D}\boldsymbol{\lambda}\, e^{iS} \epsilon(\boldsymbol{x},t)\pi_k(\boldsymbol{x}',t')...$$

$$S = \int_{t,\boldsymbol{x}} \left(\eta \left(\tfrac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} \right) + \lambda_i \left(\tfrac{\partial \pi_i}{\partial t} + v_s^2 \partial_i \epsilon - M_{ij}\pi_j \right) - i\bar{w}T\, \lambda_i M_{ij} \lambda_j \right)$$

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Can integrate out the auxiliary field $\boldsymbol{\lambda}$:

$$S_{\text{eff}}[\epsilon,\boldsymbol{\pi}] = \frac{1}{2}\int_{t,\boldsymbol{x},\boldsymbol{x}'} E_i(t,\boldsymbol{x})D_{ij}(\boldsymbol{x},\boldsymbol{x}')E_j(t,\boldsymbol{x}')$$

where $E_i \equiv (\frac{\partial\pi_i}{\partial t}+v_s^2\partial_i\epsilon-M_{ij}\pi_j)$, and $M_{ij}D_{jk} = -\frac{1}{2\bar{w}T}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta_{ik}$

Note the action $S_{\text{eff}}[\epsilon,\boldsymbol{\pi}]$ is time-reversal invariant, as it should be

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This effective action produces the correct hydro response functions

Correlation functions

Once know $S_{\pi_i\pi_j}(\omega, \boldsymbol{k})$, the others follow from energy conservation:

$$\omega S_{\epsilon\pi_i}(\omega, \boldsymbol{k}) = k_l S_{\pi_l\pi_i}(\omega, \boldsymbol{k})\,,$$
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Can read off correlation functions from the effective action $S_{\text{eff}}[\epsilon, \boldsymbol{\pi}]$:

$$S_{\pi_i\pi_j}(\omega, \boldsymbol{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right) \frac{2\gamma_\eta \bar{w} T \boldsymbol{k}^2}{\omega^2 + (\gamma_\eta \boldsymbol{k}^2)^2} + \frac{k_i k_j}{\boldsymbol{k}^2} \frac{2\gamma_s \bar{w} T \boldsymbol{k}^2 \omega^2}{(\omega^2 - v_s^2 \boldsymbol{k}^2)^2 + (\gamma_s \boldsymbol{k}^2 \omega)^2}$$

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Bottomline

- We have an effective action for linearized relativistic hydro
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Now that we know how to construct the effective action for linearized hydro, can look at the full non-linear hydrodynamics

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A simple toy model

- Incompressible fluid: impose $\boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0$ Forster+Nelson+Stephen, 1977
- Momentum conservation:

$$\partial_t \pi_i = -\partial_j T_{ij} + \xi_i \,, \qquad T_{ij} = P\delta_{ij} - \gamma_\eta(\partial_i\pi_j + \partial_j\pi_i) + \frac{\pi_i\pi_j}{\bar{w}}$$

- Current conservation:

$$\partial_t n = -\partial_i J_i + \theta \,, \qquad J_i = -D\partial_i n + \frac{n\pi_i}{\bar{w}}$$

- Stochastic model:

$$\partial_t \pi_i = -\partial_i P + \gamma_\eta \boldsymbol{\nabla}^2 \pi_i - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})\pi_i}{\bar{w}} + \xi_i \,,$$

$$\partial_t n = D\boldsymbol{\nabla}^2 n - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})n}{\bar{w}} + \theta \,,$$

A simple toy model

- Incompressible fluid: impose $\boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0$ — Forster+Nelson+Stephen, 1977
- Momentum conservation:

$$\partial_t \pi_i = -\partial_j T_{ij} + \xi_i \,, \qquad T_{ij} = P\delta_{ij} - \gamma_\eta(\partial_i\pi_j + \partial_j\pi_i) + \frac{\pi_i\pi_j}{\bar{w}}$$

- Current conservation:

$$\partial_t n = -\partial_i J_i + \theta \,, \qquad J_i = -D\partial_i n + \frac{n\pi_i}{\bar{w}}$$

- Stochastic model:

$$\partial_t \pi_i = -\partial_i P + \gamma_\eta \boldsymbol{\nabla}^2 \pi_i - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})\pi_i}{\bar{w}} + \xi_i \,,$$

$$\partial_t n = D\boldsymbol{\nabla}^2 n - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})n}{\bar{w}} + \theta \,,$$

Note that the convective term couples charge density fluctuations to momentum density fluctuations

Effective action for the toy model

$$S_{\text{eff}} = \int dt\, d^d x \Big(\mathcal{L}^{(2)} + \mathcal{L}^{(int)} \Big)$$

$$\begin{aligned}
\mathcal{L}^{(2)} = \ & -\frac{\sigma}{2}\rho\nabla^2\rho - \frac{\tilde{\sigma}}{2}\lambda_i\nabla^2\lambda_i - i\rho(\partial_t n - D\nabla^2 n) - i\lambda_i(\partial_t\pi_i - \Gamma\nabla^2\pi_i) \\
& + \bar{\psi}_i(\partial_t - \Gamma\nabla^2)\psi_i + \bar{\psi}_{\rm n}(\partial_t - D\nabla^2)\psi_{\rm n}\,, \\
\mathcal{L}^{(int)} = \ & -\frac{i}{w}\rho\pi_i\partial_i n - \frac{i}{w}\lambda_i\pi_j\partial_j\pi_i \\
& + \frac{1}{w}\bar{\psi}_i\partial_k\pi_i\,\psi_k + \frac{1}{w}\bar{\psi}_i\pi_k\partial_k\psi_i + \frac{1}{w}\bar{\psi}_{\rm n}\partial_i n\,\psi_i + \frac{1}{w}\bar{\psi}_{\rm n}\pi_k\partial_k\psi_{\rm n}\,,
\end{aligned}$$

plus the constraints $\partial_i\pi_i = 0$, $\partial_i\lambda_i = 0$, $\partial_i\bar{\psi}_i = 0$, $\partial_i\psi_i = 0$.
The constants are $\sigma = 2TD\chi$, $\tilde{\sigma} = 2T\Gamma w$, $\Gamma = \eta/w$.

One-loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2+\Big(\Gamma(\omega)\boldsymbol{k}^2\Big)^2}\,, \qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2+\Big(D(\omega)\boldsymbol{k}^2\Big)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

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In $d{=}3$ dimensions:

$$\Gamma(\omega) = \Gamma - \frac{23}{30\pi s}\frac{\sqrt{|\omega|}}{(4\Gamma)^{3/2}}\,,\qquad D(\omega) = D - \frac{1}{3\pi s}\frac{\sqrt{|\omega|}}{[2(\Gamma{+}D)]^{3/2}}\,.$$

Conventional Kubo formulas make sense:

$$D = \frac{1}{2T\chi}\lim_{\omega\to 0}\lim_{\boldsymbol{k}\to 0}\frac{\omega^2}{\boldsymbol{k}^2}G_{nn}(\omega,\boldsymbol{k})$$

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In $d=2$ dimensions:

$$\Gamma(\omega) = \Gamma(\mu) + \frac{1}{32\pi s}\frac{1}{\Gamma(\mu)}\ln\frac{\mu}{\omega}\,,\quad D(\omega) = D(\mu) + \frac{1}{8\pi s}\frac{1}{\Gamma(\mu)+D(\mu)}\ln\frac{\mu}{\omega}\,.$$

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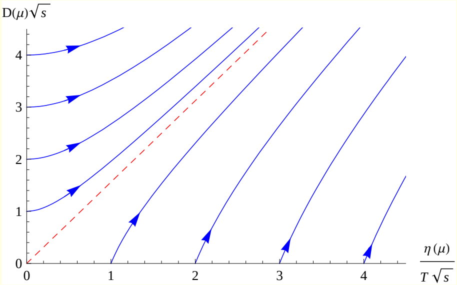
Now $\eta(\mu)$ and $D(\mu)$ are running "masses" obeying the RG equations

$$\mu\frac{\partial\Gamma}{\partial\mu} = -\frac{1}{32\pi s}\frac{1}{\Gamma}\,,\qquad \mu\frac{\partial D}{\partial\mu} = -\frac{1}{8\pi s}\frac{1}{\Gamma+D}\,.$$

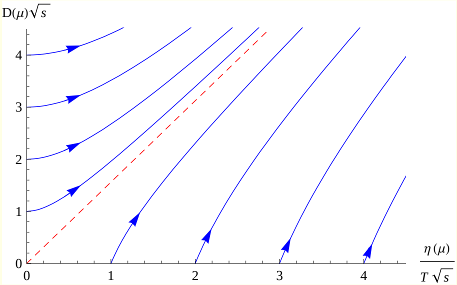
RG flow diagram in $d{=}2$



In the extreme low-frequency limit $\mu \to 0$:

$$DT = \frac{\sqrt{17}-1}{2}\,\frac{\eta}{s} \approx 1.56\,\frac{\eta}{s}$$

RG flow diagram in $d=2$



In the extreme low-frequency limit $\mu \to 0$:

$$DT = \frac{\sqrt{17}-1}{2}\,\frac{\eta}{s} \approx 1.56\,\frac{\eta}{s}$$

D and η are not independent transport coefficients in extreme IR

Outline

1. Why hydro?
2. Hydro fluctuations
3. A simple calculation
4. Fluctuations: Brownian motion
5. Fluctuations: Diffusion equation
6. Fluctuations: Linear hydrodynamics
7. Fluctuations: Non-linear hydrodynamics
8. Conclusions

Hydro fluctuations imply that

- $\dfrac{\eta}{s}$ is bounded from below in real-world QCD
- Second-order relativistic hydrodynamics stricty speaking does not exist
- However, 2nd order hydro still OK for heavy-ion collisions if η/s is sufficiently large
- Fluctuation effects disappear in the $N \to \infty$ limit

What I would like to understand

- I only showed the effective action for linearized hydro and the toy model. Can one find the covariant action for the full non-linear relativistic hydro? Work in progress with GM and PR!
- Effective action for hydro from AdS/CFT?
- Effective action for relativistic superfluids?
- How do transport coefficients in 2+1 dim flow at non-zero density?
- How do transport coefficients in 2+1 dim flow in external magnetic field?
- Other 2-nd order transport coefficients in relativistic hydro?

THE END!