The fermion sign problem and large N orbifold equivalence

Aleksey Cherman

AC, M. Hanada, D. Robles-Llana, PRL106,2011
AC, B. Tiburzi, 1103.1623

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Galileo Galilei Institute, Florence
How does matter behave at extreme densities?

\[ n_B \gtrsim \Lambda_{QCD}^{-3} \]

Critical for understanding e.g. neutron stars

Clean way to study finite \( n_B \) is to introduce a chemical potential for quark number (\( \sim \) baryon number):

\[ \mathcal{L}_{QCD} \rightarrow \mathcal{L}_{QCD} + \mu_B \bar{\psi} \gamma^0 \psi \]

Asymptotic freedom of QCD allows controlled calculations when \( \mu_B / \Lambda_{QCD} \rightarrow \infty \).

Only known way to study QCD away from the perturbative regime is using lattice Monte Carlo methods.

Unfortunately, Monte Carlo methods do not work at finite \( \mu_B \)!
\[
\langle O \rangle = \frac{\int dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} O[A_\mu, \psi, \bar{\psi}]}{\int dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]}}
\]

Monte Carlo method: generate random \( A_\mu \) configurations using \( \det(D) e^{-S[A_\mu]} \) as a probability distribution, then evaluate the integral. Works fine as long as distribution is \( > 0! \)
Monte Carlo

\[ \langle O \rangle = \frac{\int dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} O[A_\mu, \psi, \bar{\psi}]}{\int dA_\mu d\psi d\bar{\psi} e^{-S[A_\mu, \psi, \bar{\psi}]} } \]

\[ = \frac{1}{Z} \int dA_\mu \text{det}(\mathcal{D}) e^{-S[A_\mu]} O[A_\mu] \]

Monte Carlo method: generate random \( A_\mu \) configurations using \( \text{det}(\mathcal{D}) e^{-S[A_\mu]} \)

as a probability distribution, then evaluate the integral. Works fine as long as distribution is > 0!

QCD at \( \mu_B=0 \): \( \gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}^\dagger \)

Eigenvalues of \( \mathcal{D} \) come in \( \lambda, \lambda^* \) pairs

So then \( \text{det}(\mathcal{D}) = \prod_i \lambda_i > 0 \)
Once $\mu_B > 0$, $\gamma^5$ symmetry breaks, and $\det(\mathcal{D})$ becomes complex, with a rapidly fluctuating phase.

Can’t use importance sampling anymore!

No known way to generically dodge this kind of problem.

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int dA_\mu e^{-S[A_\mu]} \det(\mathcal{D}) \mathcal{O}[A_\mu]$$

If $\det(\mathcal{D})$ is part of the observable, but then answer is result of many cancellations between phases, difficulty $\sim e^{\# \text{d.o.f.}}$.
The sign phase problem

Once $\mu_B > 0$, $\gamma^5$ symmetry breaks, and $\det(\slashed{D})$ becomes complex, with a rapidly fluctuating phase.

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If $\det(\slashed{D})$ is part of the observable, but then answer is result of many cancellations between phases, difficulty $\sim e^{\#d.o.f.}$

But maybe one just needs a clever algorithm to sum up the fluctuating phases?

Well...
Computational Complexity and Fundamental Limitations to Fermionic Quantum Monte Carlo Simulations

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(Received 11 August 2004; published 4 May 2005)

Quantum Monte Carlo simulations, while being efficient for bosons, suffer from the “negative sign problem” when applied to fermions—causing an exponential increase of the computing time with the number of particles. A polynomial time solution to the sign problem is highly desired since it would provide an unbiased and numerically exact method to simulate correlated quantum systems. Here we show that such a solution is almost certainly unattainable by proving that the sign problem is nondeterministic polynomial (NP) hard, implying that a generic solution of the sign problem would also solve all problems in the complexity class NP in polynomial time.

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So how to make progress?

(1) Do **not** look for general solutions: exploit specific features of QCD.
(2) Forget real-world QCD with $N = 3$ colors - too hard!

Go to the large $N$ limit!
So how to make progress?

(1) Do not look for general solutions: exploit specific features of QCD.
(2) Forget real-world QCD with $N = 3$ colors - too hard!

Go to the large $N$ limit!

At large $N$, very different-looking gauge theories sometimes have the same correlation functions:

Large $N$ orbifold equivalence!

Famous examples:

Eguchi-Kawai reduction and related ideas

Large $N$ equivalence between $\mathcal{N} = 1$ SYM and $N_f = 1$ QCD

Armoni, Shifman, Veneziano

Goal: find a sign-problem-free theory which is orbifold-equivalent to large $N$ QCD at $\mu_B > 0$. 
Do sign-problem-free theories exist?

Yes!

1. QCD with N=2 colors, and
2. QCD with adjoint representation quarks.

\[ \gamma_5 D/ \gamma_5 = D/\dagger \] still broken when \( m_B > 0 \)

But now fermion representation is (pseudo)-real...

additional symmetry:

\[ C\gamma_5 D/ (C\gamma_5)^{-1} = D^* \]

even when \( m_B > 0 \)!

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even when \( \mu_B > 0 \! \)!

No sign problem!

But 1 & 2 have a number of major differences from N=3 QCD...

Goal is to use large N to get something equivalent to QCD.
‘t Hooft large N limit: \( N \to \infty \), keeping \( g^2 N \) fixed, \( N_f \) fixed

Non-planar diagrams and quark loops suppressed

\[ \sim \frac{1}{N^{1/2}} \quad \sim \frac{1}{N} \quad \sim \frac{1}{N} \left( \frac{N_f}{N} \right) \sim \frac{1}{N^2} \]

Mesons are stable, weakly-interacting; meson loops suppressed.

Good (10-30%) approx. to real world for many observables at \( \mu_B = 0 \).

Extent to which large N is good for \( \mu_B > 0 \) is an interesting question.
Large N in one slide

‘t Hooft large N limit: \( N \to \infty \), keeping \( g^2 N \) fixed, \( N_f \) fixed

Non-planar diagrams and quark loops suppressed

\[
\begin{align*}
\sim \frac{1}{N^{1/2}} & \quad & \sim \frac{1}{N} & \quad & \sim \frac{1}{N} \frac{N_f}{N} & \sim \frac{1}{N^2}
\end{align*}
\]

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Sign problem still present at large N:

Suppression of quark loops may allow dropping of \( \det(\mathcal{D}) \) for generation of \( A_\mu \) configurations

But \( \det(\mathcal{D}) \) appears inside fermionic observables!
The proposal

1. SO(2N) gauge theory with $N_f$ flavors of fundamental Dirac fermions

2. SU(N) gauge theory with $N_f$ flavors of fundamental Dirac fermions

Orbifold equivalence

Large N QCD

Equivalence can be made to hold even when $\mu_B > 0$.

Use deformation approach due to Unsal+Yaffe

3. The SO(2N) theory does not have a sign problem at finite $\mu_B$.

Make sure D has enough symmetry, e.g.

$$C\gamma_5 D (C\gamma_5)^{-1} = D^*$$
A quick look at SO gauge theories

\[ \mathcal{L} = \frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^{N_f} \bar{\psi}_a (i\not{D} + m + \mu_B \gamma^4) \psi_a \]

Looks a lot like QCD: has both mesons and baryons

Still have \( SU(N_f)_L \times SU(N_f)_R \times U(1)_B \) symmetry.

But SO is real, so all fermion reps are real

\[ \langle \bar{\psi} \psi \rangle \neq 0 \]

\( SU(2N_f) \rightarrow SO(2N_f) \)

\[ N_f^2 - 1 + N_f(N_f - 1) \]

NG bosons

Witten & Coleman, Peskin, 1980
An embarrassment of riches: two ways to make color singlets

\[ N_f^2 - 1 \]

\[ \overline{\psi}_a \gamma^5 \psi_b \quad P = -1 \]

Pions!

+ all the other usual-looking mesons
## An embarrassment of riches: two ways to make color singlets

<table>
<thead>
<tr>
<th>$N_f^2 - 1$</th>
<th>$N_f (N_f - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\psi}_a \gamma^5 \psi_b$ $P = -1$</td>
<td>$\psi^T_a C \gamma^5 \psi_b$ $P=+1$</td>
</tr>
</tbody>
</table>

**Pions!**

NGBs with $U(1)_B$ charge!

Baryonic pions...

Want simple way to refer to these particles:

+ all the other usual-looking mesons
An embarrassment of riches: two ways to make color singlets

\[ N_f^2 - 1 \]
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\[ N_f(N_f - 1) \]
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barions? pyons?
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NGBs with \( U(1)_B \) charge!

Baryonic pions...

Want simple way to refer to these particles:

barions? No.
pyons? No.
An embarrassment of riches: two ways to make color singlets

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Want simple way to refer to these particles:

barions? \[ \text{No.} \]
pyons? \[ \text{No.} \]
bpions!
An embarrassment of riches: two ways to make color singlets

\[ N_f^2 - 1 \]
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Pions!
+ all the other usual-looking mesons

\[ N_f (N_f - 1) \]
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Baryonic pions...

NGBs with \( U(1)_B \) charge!

Want simple way to refer to these particles:

bpions!
+ bmeson relatives of all the usual mesons

Ex.: b\( \rho \) mesons

In what sense can such a weird theory be ‘equivalent’ to QCD?
Orbifold Projection: the formal picture

Pick “mother” theory with a global symmetry $G$.

Pick a discrete cyclic subgroup $\mathbb{Z}_\Gamma \subset G$

The orbifold projection: $\mathbb{Z}_\Gamma$ orbifold “daughter theory”

Set to zero all degrees of freedom in the mother not invariant under $\mathbb{Z}_\Gamma \subset G$

Looks violent, but...
Large N orbifold equivalence

If two theories are related by an $\mathbb{Z}_\Gamma$ orbifold projection

and

$\mathbb{Z}_\Gamma$ symmetry is not spontaneously broken

then...

Correlation functions of neutral operators in mother and daughter theories will coincide in the large N limit.

Neutral operators in mother are the ones with non-zero image in daughter.

Kachru, Silverstein 1998

Kovtun, Unsal, Yaffe, 2003-4
If two theories are related by an $\mathbb{Z}_\Gamma$ orbifold projection and $\mathbb{Z}_\Gamma$ symmetry is not spontaneously broken then...

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Truth in advertising: Existing proofs of large N equivalence require some generalizations for this application!
From SO(2N) to SU(N) QCD in one slide

**SO theory has** $SO(2N) \times U(1)_B$ **symmetry**

Pick group generators:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO(2N)$$

$$\omega = e^{i\pi/2} \in U(1)_B$$

**Group action:**

$$A_\mu \rightarrow JA_\mu J^T$$

$$\psi \rightarrow \omega J\psi$$

**Result of orbifold:**

$$\mathcal{L}^{SO} \rightarrow \mathcal{L}^{SU}$$
Survivors of projection

All gauge-invariant operators in pure-glue sector of SO theory

All mesons

Victims of projection

All bmesons

Reason bmesons are killed by the orbifold projection:

Operators of the form $\psi^T \psi$ have $\mathbb{Z}_2$ charge -1

Projection sets to zero all degrees of freedom not invariant under $\mathbb{Z}_2$

Baryons: orbifold projections get subtle, work in progress.
Cartoon picture of orbifold equivalence

Daughter:

\[ m \sim 1/N \]

+\( m \) tree diagrams

\[ m, m, m = \text{meson} \]

\[ b = b\text{meson} \]
Cartoon picture of orbifold equivalence

Daughter:

Mother:

\[ m \sim \frac{1}{N} \]

\[ +m \text{ tree diagrams} \]

\[ m = \text{meson} \]

\[ b = \text{bmeson} \]

\[ \sim \]

\[ \frac{1}{N^2} \]

\[ \text{Not allowed if } U(1)_B \text{ unbroken} \]

\[ \text{Allowed} \]
The good news

No bmeson condensation at $\mu_B=0$. \hspace{1cm} \text{Vafa-Witten theorem}

In fact, can show that there is no bmeson condensation at least for $\mu_B < m_\pi/2$.

So at least up to $\mu_B < m_\pi/2$, expect equivalence to hold.
The good news

No bmeson condensation at $\mu_B=0$.  
Vafa-Witten theorem

In fact, can show that there is no bmeson condensation at least for $\mu_B < m_\pi/2$.

So at least up to $\mu_B < m_\pi/2$, expect equivalence to hold.

But large N QCD has a sign problem for any $\mu_B > 0$!

So orbifold equivalence gives a way to dodge the sign problem at least for $\mu_B < m_\pi/2$.

Already something... But not as much as we want!
The bad news

Once $\mu_B > m_\pi/2$ bpions condense: $\langle \psi^T C \gamma^5 \psi \rangle \neq 0$

Equivalence is lost for $\mu_B > m_\pi/2$!
The bad news

Once $\mu_B > m_\pi / 2$ bpions condense:

$$\langle \psi^T C \gamma^5 \psi \rangle \neq 0$$

Equivalence is lost for $\mu_B > m_\pi / 2$!
The proposal

1. \( \text{SO}(2N) \) gauge theory with \( N_f \) flavors of fundamental Dirac fermions \( \cong \) \( \text{SU}(N) \) gauge theory with \( N_f \) flavors of fundamental Dirac fermions

Orbifold equivalence Large N QCD

2. Equivalence can be made to hold even when \( \mu_B > m_{\pi}/2 \).

Use deformation approach due to Unsal+Yaffe

3. The \( \text{SO}(2N) \) theory does not have a sign problem at finite \( \mu_B \).

Make sure \( D \) has enough symmetry, e.g.

\[
C \gamma_5 \not{\!D} (C \gamma_5)^{-1} = \not{\!D}^* 
\]
Can we modify the SO theory so that
(1) the modified theory still maps to QCD, and
(2) prevent bpion condensation?

Yes!

\[ \mathcal{L}_{SO} \rightarrow \mathcal{L}_{SO} + \frac{c^2}{\Lambda_{QCD}^2} S_{ab}^{\dagger} S^{ab} \]

\[ S_{ab} = \psi_a^T C \gamma^5 \psi_b \]

New term orbifolds to zero.

Cartoon picture: free energy becomes

\[ \langle F \rangle \rightarrow \langle F \rangle + \frac{c^2}{\Lambda_{QCD}} \langle S \rangle \langle S^{\dagger} \rangle \]

This makes system pay a cost for condensing.

So use deformations to discourage bpion condensation.

Next step: make sure this is more than a cartoon.
Hard to understand deformed theory analytically in general.

But if $m_q$, $\mu_B$, and $c$ are small compared to $\Lambda_{QCD}$, low-energy physics can be systematically described using effective field theory.
Deformations and Effective Field Theory

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But if $m_q$, $\mu_B$, and $c$ are small compared to $\Lambda_{QCD}$, low-energy physics can be systematically described using effective field theory.

Strategy:

1. Construct chiral perturbation theory for the SO(2Nc) theory
2. Use standard spurion analysis tricks to find new interaction terms for the pions, bpions induced by deformations.
3. Determine the phase structure as a function of $m_q$, $\mu_B$, $c$. 
Deformations and Effective Field Theory

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3. Determine the phase structure as a function of $m_q$, $\mu_B$, $c$.

$XPT$ is written in terms of $\Sigma \in U(2N_f)/SO(2N_f)$

$\Sigma$ contains all the NGBs:

$$\Sigma = \exp \left( \frac{i\eta'}{F_\Pi \sqrt{N_f}} \right) \exp \left( \frac{i\Pi}{F_\Pi} \right) \Sigma_0 \quad \Pi = \begin{pmatrix} \pi \\ b^\dagger \\ \pi_T \end{pmatrix}$$

Note: at large $N_c$, chiral anomaly is suppressed and $\eta'$ is light.
Without deformations, the EFT has the Lagrangian
\[ \mathcal{L} = \frac{F_\Pi^2}{4} \text{tr} \left[ D_\mu \Sigma D_\mu \Sigma^\dagger \right] - \frac{\lambda F_\Pi^2}{4} \text{tr} \left[ \Sigma \mathcal{M} + \Sigma^\dagger \mathcal{M}^\dagger \right] \]
\[ D_\mu \Sigma = \partial_\mu \Sigma + i B_\mu \Sigma + i \Sigma B_\mu^T \]
\[ B_\mu \sim \mu_B, \mathcal{M} \sim m_q \]

Higher-order terms are suppressed at large \( N_c \)
(also by low-energy expansion)

Deformations induces new terms in the low-energy action...
Low energy action

Without deformations, the EFT has the Lagrangian
\[
\mathcal{L} = \frac{F_{\Pi}^2}{4} \text{tr} \left[ D_\mu \Sigma \Sigma^{\dagger} \right] - \frac{\lambda F_{\Pi}^2}{4} \text{tr} \left[ \Sigma \mathcal{M} + \Sigma^{\dagger} \mathcal{M}^{\dagger} \right]
\]
\[
D_\mu \Sigma = \partial_\mu \Sigma + iB_\mu \Sigma + i\Sigma B^T_\mu, \quad B_\mu \sim \mu_B, \mathcal{M} \sim m_q
\]

Higher-order terms are suppressed at large \(N_c\) (also by low-energy expansion)

Deformations induce new terms in the low-energy action...

In XPT turns it is easier to work with the deformations

\[
V_{\pm} = \frac{c^2}{\Lambda_{QCD}^2} \sum_{a,b=1}^{N_f} \left( S_{ab}^{\dagger} S_{ab} \pm P_{ab}^{\dagger} P_{ab} \right) P_{ab} = \psi_a^T C \psi_b
\]
\[
S_{ab} = \psi_a^T C \gamma^5 \psi_b
\]

\(V_+\) is a chiral singlet, while \(V_-\) is not.

Will see that both deformations have identical effect on neutral-sector physics, so long as \(U(1)_B\) not broken.
Two deformations

To capture effects of deformations, use spurion analysis.

Same approach as used in XPT to understand lattice-spacing effects, inclusion of weak interaction effects, etc

\[ V_+ \text{ produces just one new term in the EFT} \]

\[ c_+ F^2_{\Pi} \sum_{a,b=1}^{N_f} \left( \text{tr} \left[ \Sigma L^{(ab)} \right] \text{tr} \left[ \Sigma^\dagger L^{(ab)^\dagger} \right] + \text{tr} \left[ \Sigma R^{(ab)} \right] \text{tr} \left[ \Sigma^\dagger R^{(ab)^\dagger} \right] \right) \]

\[ V_- \text{ produces two new terms in the EFT} \]

\[ c_- F^2_{\Pi} \sum_{a,b=1}^{N_f} \left( \text{tr} \left[ \Sigma L^{(ab)} \right] \text{tr} \left[ \Sigma R^{(ab)} \right] + \text{tr} \left[ \Sigma^\dagger L^{(ab)^\dagger} \right] \text{tr} \left[ \Sigma^\dagger R^{(ab)^\dagger} \right] \right) \]

\[ + d_- F^2_{\Pi} \sum_{a,b=1}^{N_f} \left( \text{tr} \left[ \Sigma L^{(ab)} \Sigma R^{(ab)} \right] + \text{tr} \left[ \Sigma^\dagger L^{(ab)^\dagger} \Sigma^\dagger R^{(ab)^\dagger} \right] \right) \]

New low-energy constants \[ c_+, c_-, d_- \]
Spectrum of the deformed theory

Without symmetry breaking:

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mass with $V_-$ deformation</th>
<th>Mass with $V_+$ deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>$(m_\pi^2 + 4d_-)^{1/2}$</td>
<td>$m_\pi$</td>
</tr>
<tr>
<td>$\eta'$</td>
<td>$(m_\pi^2 + 4d_-)^{1/2}$</td>
<td>$m_\pi$</td>
</tr>
<tr>
<td>$b$</td>
<td>$(m_\pi^2 + 4c_-)^{1/2} + 2\mu$</td>
<td>$(m_\pi^2 + 4c_+)^{1/2} + 2\mu$</td>
</tr>
<tr>
<td>$b^\dagger$</td>
<td>$(m_\pi^2 + 4c_-)^{1/2} - 2\mu$</td>
<td>$(m_\pi^2 + 4c_+)^{1/2} - 2\mu$</td>
</tr>
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</table>

Matching to microscopic theory gives $N_c$ scaling of the new LECs:

$c_-, c_+ \sim N_c^0, d_- \sim N_c^{-1}$

Can also show that the sign of $c$ in microscopic theory controls the signs of the LECs in the EFT.

So both deformations raise the bpion mass, while leaving neutral-sector stuff alone.

To sort of symmetry realization pattern, minimize effective potential in deformed theory.
Phase diagram of the $V_+$-deformed theory

\[ \frac{c_+}{(m_\pi/2)^2} \]

\[ \mu_B^2 / (m_\pi/2)^2 \]

- Red: $\langle b \rangle \neq 0, \langle \eta' \rangle = 0$
- Green: $\langle b \rangle = 0, \langle \eta' \rangle = 0$
- Blue: $\langle b \rangle \neq 0, \langle \eta' \rangle \neq 0$
Phase diagram of the $V$-deformed theory

\[ \frac{c_-}{(m_\pi/2)^2} \]

\[ \mu_B^2 / (m_\pi/2)^2 \]

- $\langle b \rangle \neq 0, \langle \eta' \rangle = 0$
- $\langle b \rangle = 0, \langle \eta' \rangle = 0$
- $\langle b \rangle \neq 0, \langle \eta' \rangle \neq 0$

Exotic metastable phase!
Orbifold equivalence past $\mu_B = m_{\pi}/2$

With both deformations, the SO theory can be forced to stay in a $U(1)_B$-unbroken phase past $\mu_B = m_{\pi}/2$.

The correlation functions of neutral operators are identical with both deformations in the normal phase.

The $V$-deformed theory has an exotic phase with $\eta'$-condensation. This phase is always metastable in our analysis.
Orbifold equivalence past $\mu_B = m_\pi/2$

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The $V$-deformed theory has an exotic phase with $\eta'$-condensation. This phase is always metastable in our analysis.

At level of EFT, large N-equivalence is `obvious':

$$\frac{U(N_f)_L \times U(N_F)_R}{U(N_f)_V} \subset \frac{SU(2N_f)}{SO(2N_f)}$$

Neutral correlators in $SU(2N_f)/SO(2N_f)$ EFT with given LECs coincide with correlators computed in $SU(N_f)_V$ EFT with the same LECs, so long as $U(1)_B$ is not broken.
The proposal

1. SO(2N) gauge theory with $N_f$ flavors of fundamental Dirac fermions
   \[\cong\]
   SU(N) gauge theory with $N_f$ flavors of fundamental Dirac fermions
   Orbifold equivalence
   Large N QCD

2. Equivalence can be made to hold even when $\mu_B > m_\pi/2$.
   Use deformation approach due to Unsal+Yaffe

3. The SO(2N) theory does not have a sign problem at finite $\mu_B$.
   Make sure D has enough symmetry, e.g.
   \[C\gamma_5\slash D (C\gamma_5)^{-1} = \slash D^*\]
Sign-free implementation of deformations

Deformations are four-quark operators, so must use auxiliary fields to put them on the lattice.

Sign problem reappears if aux field implementation breaks enough symmetries!

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Integration over $f_{ij}$ gives original 4-quark terms

$$S^+_{ab} S_{ab} \longrightarrow \sum_{\Gamma} \left[ \frac{1}{2} f^\Gamma_{ij} f^i j \Gamma + i c_{\Gamma} f^\Gamma_{ij} \bar{q}^i_a \Gamma q^j_a \right]$$

+ similar terms for $P^+_{ab} P_{ab}$
Sign-free implementation of $V_-$ deformations

Result of integrating in auxiliary fields in flavor-singlet channel:

$$\frac{c^2}{\Lambda^2} (S^\dagger_{ab} S_{ab} - P^\dagger_{ab} P_{ab}) \rightarrow \frac{(f_{ij})^2}{2} + \frac{(g_{ij})^2}{2} + \frac{(h_{\mu\nu,ij})^2}{2}$$

$$+ ic_1 f_{ij} \bar{\psi}^i_a \psi^j_a + ic_2 g_{ij} \bar{\psi}^i_a \gamma^{5} \psi^j_a$$

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✓ No sign problem in the chiral limit.
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The same trick does not work for $V+$. Are there other tricks that do?
Using SO theory, we can dodge sign problem even past $m_\pi/2$.

No sign problem at $m_q = 0$  

Sign-quenching should be a good approximation for light quarks.
Summary and open questions

Using SO theory, we can dodge sign problem even past $m_\pi/2$.

No sign problem at $m_q = 0$ \hspace{2cm} Sign-quenching should be a good approximation for light quarks.

Does equivalence hold through nuclear matter transition?

- Do bmesons with charge/mass less than lightest baryons exist, even in deformed theory?
  - If so, expect condensation for big enough $\mu_B$, killing equivalence.

We need non-perturbative tests!

Lattice, AdS/CFT, ...

To do:

Work out baryon-sector matching, extend equivalence proofs, look for sign-free way to work with $V_+$, try to get away from chiral limit, try to dodge other sign problems,...