An Intriguing Example of F-maximization in 3D SCFTs

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- Supersymmetric CFTs with 4 real SUSIES (N=2 in 3D, or N=1 in 4D) have a conserved U(1) R-symmetry that sits in the same supermultiplet as the stress-energy tensor.

  In a general interacting SCFT this symmetry receives quantum corrections and the quantum numbers associated with it become non-trivial functions of the parameters of the theory.

- The computation of the exact **non-perturbative** form of this symmetry is an important problem in field theory.

  Such exact knowledge can be used to determine the anomalous scaling dimensions of chiral operators, trace SUSY RG flows (hence a significant part of the topology/geometry of field theory space), test dualities, etc...

- In 4D this problem was solved by ***Intriligator and Wecht '05*** with the use of a-maximization:

  The exact U(1) R-symmetry in 4D N=1 SCFTs maximizes `a'

  (a = the coefficient of the Euler density in the conformal anomaly).

  In terms of 't Hooft anomalies

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right)$$

- Alternatives to a-maximization:

  (a) τ_{RR}-minimization:

  the exact U(1) R-symmetry minimizes the coefficient of the 2-point function

$$\langle J_R^\mu(x) J_R^\nu(y)\rangle = \frac{\tau_{RR}}{(2\pi)^d}\left(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\right)\frac{1}{(x-y)^{2(d-2)}}$$

  applies to any dimension, but hard to compute exactly...

(b) Z-minimization: applies to AdS/CFT (**Martelli, Sparks and Yau '05**)

- dual AdS space: $AdS_{d+1} \times Y_{2n-1}$, Y_{2n-1} Sasaki-Einstein manifold

the exact U(1) R-symmetry minimizes the Einstein-Hilbert action on $\mathbf{Y_{2n-1}}$

It can be shown that ``τ_{RR}-maximization = Z-minimization"
(**Barnes et al'05**)

Applies to general spacetime dimension, but requires a weakly curved AdS dual.

- In the last couple of years large classes of 3D N=2 SCFTs have been identified (constructed as Chern-Simons-Matter (CSM) theories). Until recently it was unclear practically how to determine the exact U(1) R-symmetry in these theories.

F-maximization in 3D SCFTs
& recent developments

- The proposal of Jafferis (**1012.3210**)

the exact U(1) R-symmetry in 3D SCFTs maximizes the free energy F of the theory on S^3

$$Z_{S^3} = e^{-\mathcal{F}} \ , \quad F = \frac{1}{2}(\mathcal{F} + \overline{\mathcal{F}})$$

F is extensive in the dof of the system.

It can be computed exactly using localization techniques (**Kapustin-Willett-Yaakov '09, Hama-Hosomichi-Lee '10, Jafferis '10**)

Some care needs to be taken when coupling the SUSY theory with curvature. Doing things properly requires the introduction of extra couplings between the matter fields and curvature. These couplings are determined by the choice of R-symmetry.

- In this way F becomes a function of the trial R-charges.

- Assuming that the R-symmetry does not mix with accidental flavor symmetries we can use the weak coupling formulation of the theory to compute F using localization techniques. For a CSM theory with gauge group G and chiral superfields in reps $\mathrm{R_i}$ one finds (after localization and appropriate regularization) a matrix integral:

$$Z_{S^3} = \int \prod_{\mathrm{Cartan}} du \; e^{i\pi \mathrm{Tr} u^2} \det_{Adj} \left(\sinh(\pi u)\right) \prod_{\mathrm{Chirals\ in\ rep\ R_i}} \det_{R_i} \left(e^{\ell(1-\Delta_i)+iu)} \right)$$

$$\ell(z) := -z \log\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi i z}\right)\right) - \frac{i\pi}{12}$$

- The argument for F-maximization uses 2 steps:

  (i) Z depends holomorphically on the combinations $\Delta_j - im_j$ (m_j are real mass parameters). $\longrightarrow \quad \partial_{\Delta_j} |Z_{S^3}|^2 = 0$

  (ii) The one-point function $\frac{1}{Z}\partial_{m_j} Z\Big|_{m=0}$ is a real number.

- These are plausible arguments but do not constitute a rigorous proof.

- Happily, it has been observed that this proposal passes a number of impressive non-trivial tests:

  (1) Reproduces known perturbative results (**Jafferis '10, Amariti '11**).
  (2) Reproduces Z-minimization (**Herzog et al '10, Martelli-Sparks '11, Cheon-Kim^2 '11, Jafferis et al '11**)

(3) Verifies proposed Seiberg-like dualities
(**Kapustin '11, Willett-Yaakov '11**)

We will now discuss a qualitatively different situation.

One of its characteristic features will be that as we increase the coupling a number of fields hit the unitarity bound and successively decouple from the rest of the theory.

F-maximization can in principle fail in such situations. It is interesting to explore if it does and how...

1-adjoint CSM theory

The theory of interest is:

$\hat{\mathbf{A}}$ CSM theory

N=2 Chern-Simons at level *k* with gauge group *G=U(N)* coupled to one chiral superfield X in the adjoint representation (NO superpotential)

- Important information about this theory can be obtained by studying the superpotential deformations

$\mathbf{A_{n+1}}$ CSM theory

$$\hat{\mathbf{A}}\ CSM\ theory \ \oplus\ W_{n+1} = \mathrm{Tr} X^{n+1} \ , \quad n = 1, 2, \ldots$$

It is convenient to study these theories in the large-N ‘t Hooft limit

$$N, k \to \infty \ , \quad \lambda = \frac{N}{k} = \text{fixed}$$

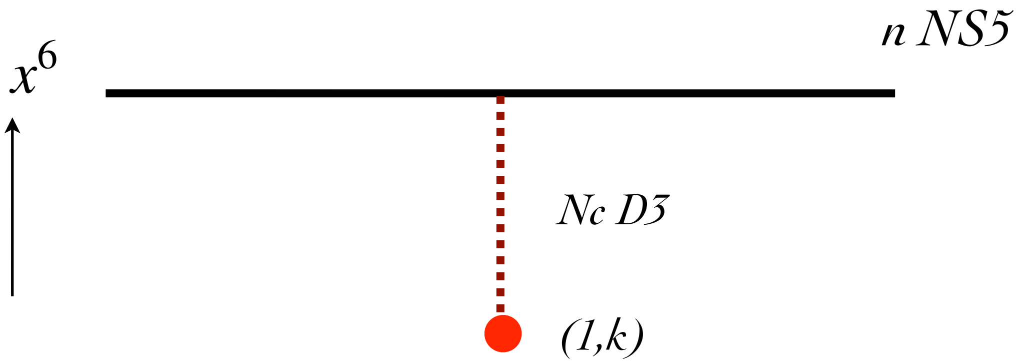
• It is believed (**Gaiotto, Yin ’07**) that the $\hat{\mathbf{A}}$ theory is exactly superconformal at the quantum level at any value of the coupling λ.

At weak coupling the R-symmetry can be determined perturbatively and assigns R-charge

$$R(\lambda) \simeq \frac{1}{2} - 2\lambda^2 + \mathcal{O}(\lambda^4) \ , \quad \lambda \ll 1$$

to the chiral superfield X.

• No holographic description of this theory in supergravity is expected. Cannot appeal to AdS/CFT for any information about this theory. We would like to know the full (non-perturbative) dependence of R on λ.



VN '08

NS5	:	0 1 2 3 4 5
(1,k)	:	0 1 2 [3,7] 8 9
D3	:	0 1 2 6

- There are regimes along the λ-line where the superpotential deformations

$$W_{n+1} = \frac{g_{n+1}}{n+1} \mathrm{Tr} X^{n+1}$$

are relevant and drive the theory to a new IR fixed point: the A_{n+1} theory.

can be argued also directly in field theory

- From a D-brane construction we learn that:

(i) the superpotential deformation W_{n+1} lifts the supersymmetric vacuum when

$$N > nk \quad (\text{equivalently in 't Hooft limit } \lambda > n)$$

(ii) the theory exhibits a Seiberg-like duality:

$$\boxed{\lambda \leftrightarrow n - \lambda}$$

$$U(N)_k \text{ with } W_{n+1} \;\sim\; U(nk-N)_k \text{ with } W_{n+1}$$

• This information has important implications for the undeformed $\hat{\mathbf{A}}$ theory.

(1) The fact that W_{n+1} can lift the supersymmetric vacuum at arbitrarily large integer values of λ implies that the R-charge decreases (with increasing λ) towards 0.

(2) More specifically, there has to be a sequence of critical couplings

$$0 = \lambda_2^* = \lambda_3^* = \lambda_4^* < \lambda_5^* < \cdots < \lambda_n^* < \lambda_{n+1}^* < \cdots$$

where each time one of the chiral operators $\mathrm{Tr}X^{n+1}$ becomes marginal. By definition

$$R(\lambda_{n+1}^*) = \frac{2}{n+1}$$

(3) The generic operator $\mathrm{Tr}X^{n+1}$ must become marginal ***before*** it becomes capable of lifting the SUSY vacuum at $\lambda=n$. This implies

$$\lambda^*_{n+1} < n \ , \quad R(n) < \frac{2}{n+1}$$

(4) The existence of a "conformal window" for Seiberg-like duality implies

$$\lambda^*_{n+1} < \frac{n}{2} \ , \quad R\left(\frac{n}{2}\right) < \frac{2}{n+1}$$

assuming $R(\lambda)$ is monotonic

(5) At $\lambda=\lambda^*_{4(n+1)}$ the operator $\mathrm{Tr}X^{n+1}$ hits the unitarity bound, becomes **free** and decouples from the rest of the theory. At that point we can no longer use it to deform the theory without destabilizing the SUSY vacuum (F-term SUSY breaking). Hence, spontaneous SUSY breaking must occur before this point:

$$n < \lambda^*_{4(n+1)} \ , \quad \frac{1}{2(n+1)} < R(n)$$

Summary

In the $\hat{\mathbf{A}}$ theory the following inequalities are expected to hold:

$$\left[\frac{n-3}{4}\right] \leq \lambda^*_{n+1} < \frac{n}{2}$$

Seiberg-like duality imposes more constraints (see below)

$$\frac{1}{2(\lambda+1)} \leq R(\lambda) < \frac{2}{\lambda+1} \ , \quad \lambda = 1, 2, \ldots$$

$$R(\lambda) < \frac{2}{2\lambda+1} \ , \quad \lambda = \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

At weak coupling TrX is already free and decoupled. As we further increase the coupling more and more of the chiral ring operators TrX^{n+1} hit the unitarity bound and decouple. At strong coupling there is a sequential decommissioning of the bottom part of the chiral ring.

F-maximization puzzles

What does F-maximization have to say about all this?

- We are instructed to maximize the free energy

$$F = -\log \left| \int \left(\prod_{j=1}^{N} e^{\frac{i\pi N}{\lambda} t_j^2} dt_j \right) \prod_{i<j}^{N} \sinh^2\left(\pi(t_{ij})\right) \prod_{i,j=1}^{N} e^{\ell(1-R+it_{ij})} \right|$$

- We computed this function (and maximized) in the large-N limit using the saddle point approximation. This entails solving the algebraic equations

$$\mathcal{I}_i \equiv \frac{i}{\lambda} t_i + \frac{1}{N} \sum_{j \neq i} \left[\coth(\pi t_{ij}) - \frac{(1-R)\sinh(2\pi t_{ij}) + t_{ij}\sin(2\pi R)}{\cosh(2\pi t_{ij}) - \cos(2\pi R)} \right] = 0\ , \quad i = 1, 2, \ldots, N$$

at a saddle point configuration

$$-\mathcal{F}(\lambda, N) = \sum_{i=1}^{N} \frac{i\pi N}{\lambda} t_i^2 + \sum_{i<j}^{N} \log\sinh^2(\pi t_{ij}) + \sum_{i,j=1}^{N} \ell(1-R+it_{ij})$$

• In general, the t_i's that solve these equations are complex numbers.

• In lack of a better strategy we solved these equations numerically.

• Practically we introduce a ficticious time coordinate τ and solve the differential equations

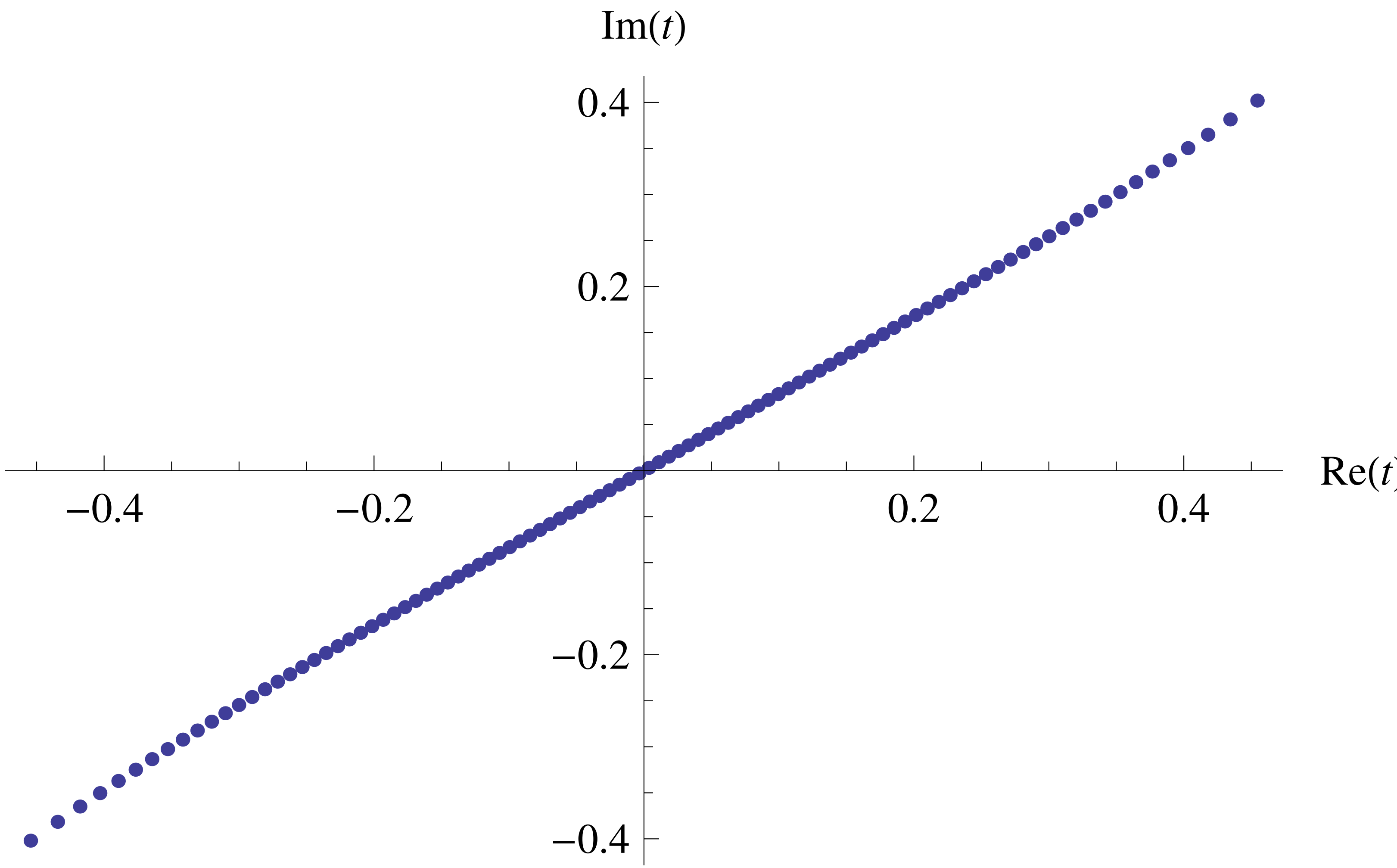
$$a\frac{dt_i(\tau)}{d\tau} = \mathcal{I}_i$$

With suitably chosen coefficient *a* the solution converges very quickly to the equilibrium configuration we are looking for.

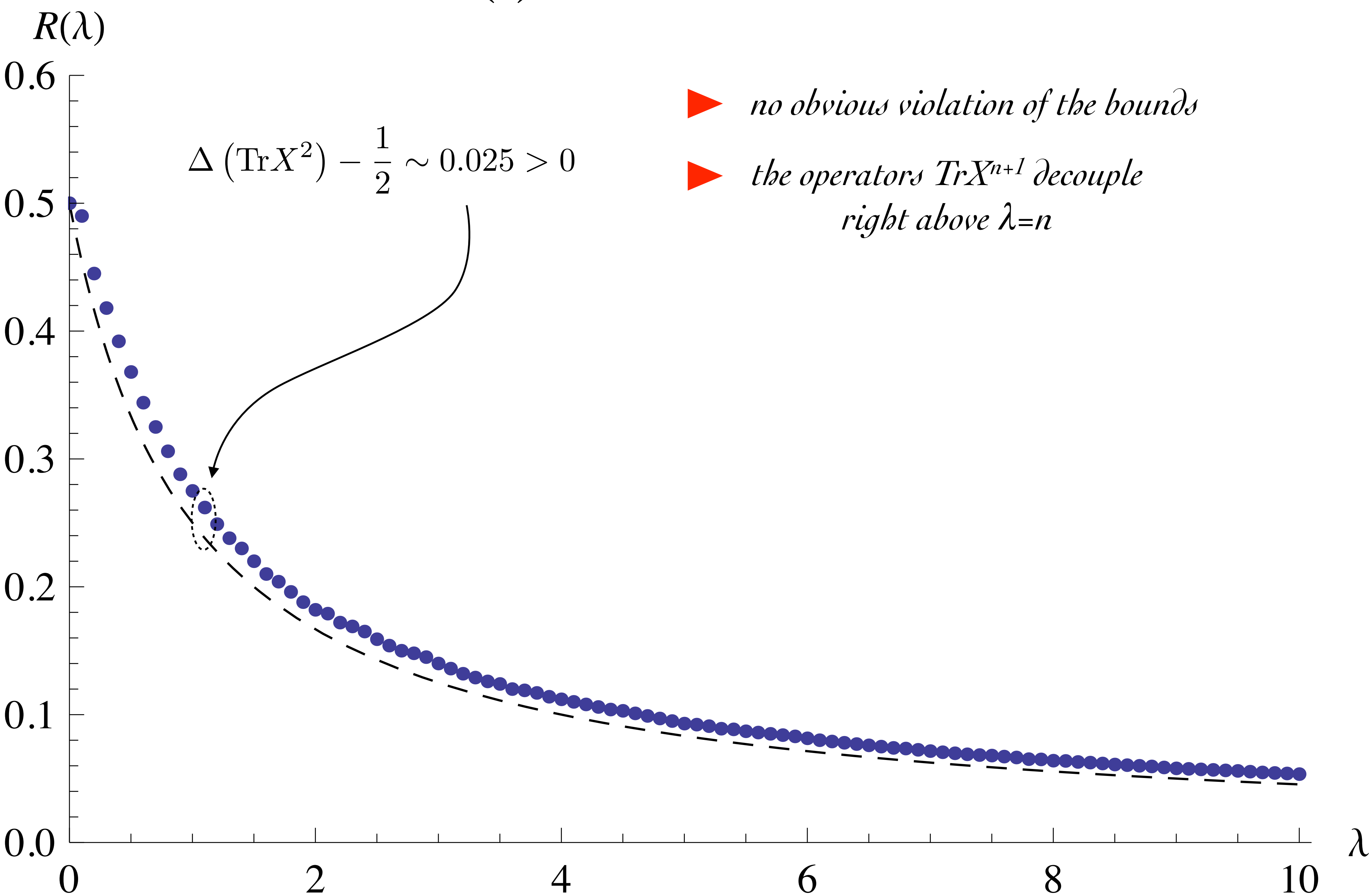
• Implemented this approach with the use of Mathematica for various values of N. At N=100 the numerical result seems to approach the large-N asymptote within a few percent.

A typical distribution of the eigenvalues t_i in the complex plane.

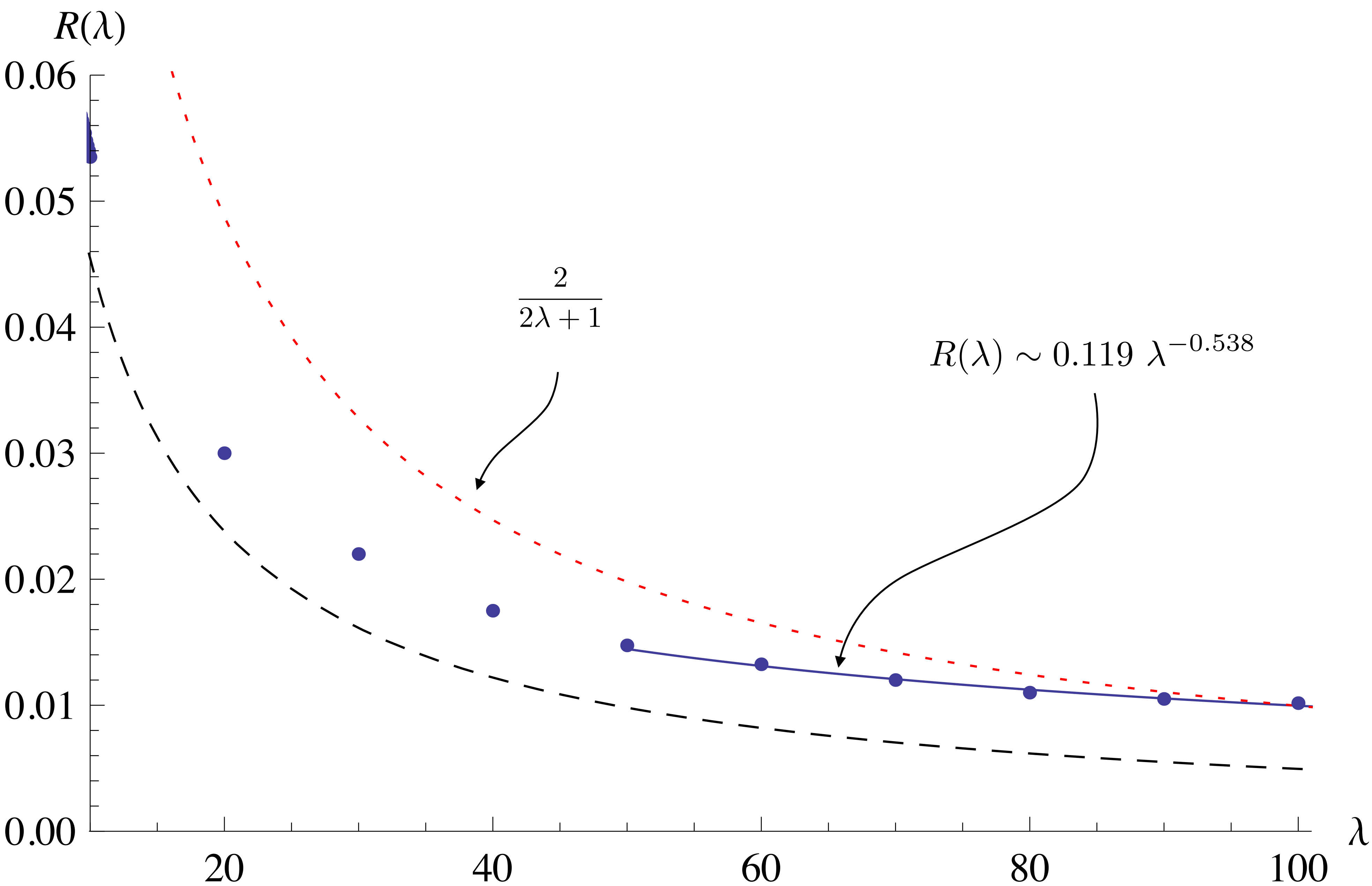
(this particular plot was obtained for N=100, λ=1, R=0.225)



R(λ) after F-maximization



- *no obvious violation of the bounds*
- *the operators $\mathrm{Tr}X^{n+1}$ decouple right above λ=n*


$R(\lambda)$
0.06
0.05
0.04
0.03
0.02
0.01
0.00
$\frac{2}{2\lambda+1}$
$R(\lambda) \sim 0.119\ \lambda^{-0.538}$
20
40
60
80
100
λ

- **Checks**

► The numerical code reproduces very nicely the perturbative result. In fact, at leading order in λ the eigenvalue distribution is (**Minwalla et al '11**)

$$t = e^{\frac{\pi i}{4}}\sqrt{\lambda}y \ , \quad \rho(y) = \sqrt{\frac{2}{\pi} - y^2}$$

The numerical result verifies this behavior.

► We have written independently two different numerical codes (in Mathematica and Fortran) that reproduce the same result.

► We have explored a wide range of initial conditions for the τ-differential equations and parameters a.

In all cases we obtain numerically similar results.

• There are no obvious violations of the bounds for $\lambda \sim \mathcal{O}(1)$.

• However, the results are inconsistent with the Seiberg-like duality of **VN '08**. That duality predicts the following set of equations

$$\mathcal{F}\left(\frac{2}{n+1}, \lambda\right) = \mathcal{F}\left(\frac{2}{n+1}, n-\lambda\right) \ , \quad \lambda \in (\lambda^*_{n+1}, n-\lambda^*_{n+1}) \ , \quad n = 1, 2, \ldots ,$$

These equations imply an oscillatory feature that is not observed.

• Possible implications:

(*a*) More dominant saddle points modify this result.

(*b*) The F-maximization recipe should be modified.

(*c*) The Seiberg-like duality of **VN '08** does not hold.

- If such violations are really present should they be taken seriously?

As we increase the coupling more and more operators hit the unitarity bound and decouple creating new accidental symmetries. The first operator that decouples non-perturbatively is $\mathrm{Tr}X^2$ at $\lambda\sim1$. It is not surprising that F-maximization in its current form can fail in such cases.

Recall what happens in 4D with a-maximization. In similar cases (e.g. in 4D 1-adjoint SQCD) when fields decouple one is instructed to subtract the anomalies of the decoupling fields from a and maximize the remaining contributions (**Kutasov, Parnachev, Sahakyan '05**).

Puzzle: In 4D 1-adjoint SQCD N^2 dof decouple (mesons). In our CSM example order 1 dof decouple. *How can these have a sizable effect in the large-N limit at small enough 't Hooft coupling?*

Amusing speculation

The R-charge obtained from F-maximization remains in the vicinity of the curve $\frac{1}{2(\lambda+1)}$.

We anticipate this feature to persist for any λ. Similar feature observed also in 4D 1-adjoint SQCD.

It is tempting to take this one step further. If

$$\Delta\left(\mathrm{Tr}X^{n+1}\right)\Big|_{\lambda=n}=\frac{1}{2}\quad\Leftrightarrow\quad R(n)=\frac{1}{2(1+n)}$$

is an exact property of the theory it implies that:

(i) R(λ) oscillates indefinitely in the vicinity of the curve $\frac{1}{2(\lambda+1)}$ passing through the above points at integer values of λ.

(ii) R asymptotes at large λ to the curve $\frac{1}{2\lambda}$.

Perspectives

We have identified a CSM theory with complex enough dynamics that can pose as a useful testing ground for new non-perturbative techniques in 3D QFT, like F-maximization.

A combination of field and string theory techniques can be used to pose constraints on the theory beyond the perturbative regime.

Open problems:

1) Should clarify the main message that comes from the numerical evaluation of the free energy using the matrix integral of Jafferis

Does the dominant saddle point contribution obey the constraints imposed by SUSY breaking and Seiberg-duality, analytic checks ???

2) Should the matrix integral of Jafferis be modified?
How does one implement such modifications?

3) In order to probe the effects of decoupling fields it will be interesting to consider the full CSM analog of 1-adjoint SQCD, namely

U(N_C) Chern-Simons theory at level k coupled to:
- 1 chiral superfield in the adjoint
- N_F chiral superfields in the fundamental
- N_F chiral superfields in the anti-fundamental.

To simplify things it is interesting to consider the Veneziano-like limit

$$k,\ N_C,\ N_F \to \infty\ ,\quad \lambda = \frac{N_C}{k}\ ,\quad x = \frac{N_C}{N_F}\quad \text{fixed}$$

Using information from string theory (**VN '08, '09**) one can set some constraints on the R-charge function R_X for the adjoint superfield X, e.g. now

$$\frac{\left[\frac{n-3}{4}\right] x}{x-\left[\frac{n-3}{4}\right]} < \lambda^*_{n+1} < \frac{nx}{x-n} \ , \quad n \leq \frac{[x]-3}{4}$$

$R_X(\lambda,x)$ is presumably a monotonically decreasing function of λ at fixed x that approaches at strong 't Hooft coupling a limiting lowest value

$$\frac{1}{2([x]+2)} < R_{X,\mathrm{lim}} < \frac{2}{[x]+1}$$

No corresponding information is currently available for R_Q, the R-charge functions for the quark multiplets.

4) CSM theories with 2 adjoint chiral superfields (+ additional matter) are also interesting.

In 4D a-maximization has led to an intriguing picture of 2-adjoint N=1 SCFTs that appear to admit a mysterious ADE classification. A web of RG flows connects different members of this classification.

In previous work **VN '09** we provided evidence for a similar structure in a subclass of 3D CSM SCFTs. F-maximization can help solidify and extend this picture.

It can also help find non-trivial evidence for the new Seiberg-like dualities proposed in **VN '09**.

The web of RG flows can be used to test the proposed F-theorem.

Example: Seiberg-Brodie-like duality in D_{n+2}-CSM theories **VN '09**.

Electric Theory
U(N_c) N=2 CSM theory coupled to
2-adjoints (X,X') + N_F pairs of (anti)fundamentals
+ superpotential $W=\mathrm{Tr}\, XX'^2+\mathrm{Tr}\, X^{n+1}$

↕

Magnetic Theory
U($3n(N_F+k)-N_c$) N=2 CSM theory coupled to
2-adjoints (Y,Y') + N_F pairs of (anti)fundamentals
+ superpotential $W=\mathrm{Tr}\, YY'^2+\mathrm{Tr}\, Y^{n+1}$+*mesonic contributions*