AdS/CFT and the Quark-Gluon Plasma

(Beyond CFT & SUGRA)

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Basic AdS/CFT correspondence:

gauge theory		string theory
$\mathcal{N} = 4\ SU(N)\,\mathrm{SYM}$	$\Longleftrightarrow$	N-units of 5-form flux in type IIB string theory
g^2_{YM}	$\Longleftrightarrow$	g_s

$\Rightarrow$ In the simplest case, the SYM theory is in the 't Hooft (planar limit), $N \to \infty$, $g^2_{YM} \to 0$ with $N g^2_{YM}$ kept fixed. SUGRA is valid $N g_s \to \infty$. In which case the background geometry is

$$AdS_5 \times S^5$$

$\Rightarrow$ The main message is that AdS/CFT sets up a framework that could be used in analyzing the dynamics of strongly coupled gauge theories, in particular, sQGP

To make a closer link to realistic systems we need to go beyond the basic AdS/CFT:

- *Beyond the conformal approximation* (but still in a 'SUGRA'-land — non-conformal theories in the planar limit and infinite 't Hooft coupling)
  $\Rightarrow$ deform a conformal gauge theory by a relevant operator (introduce a mass term)
  $\Rightarrow$ consider gauge theories without explicit mass terms, but with a non-vanishing β-function(s) for the gauge coupling(s)

- *Beyond the SUGRA approximation* (but still in a 'CFT'-land — conformal theories with leading deviations from planar limit/infinite t' Hooft coupling)

$$\frac{1}{N}\text{-corrections} \quad \Longleftrightarrow \quad g_s\text{-corrections}$$
$$\frac{1}{N g_{YM}^2}\text{-corrections} \quad \Longleftrightarrow \quad \alpha'\text{-corrections}$$

$\Rightarrow$ Ideally, we would like to combine the both *beyonds...*, but the technology is still not there yet.

Outline of the talk:

- Thermodynamics of strongly coupled non-conformal plasma:
  - Susy/non-susy mass deformations of $\mathcal{N}=4$ in QFT/supergravity ($\mathcal{N}=2^*$ model)
  - Gauge theories with $\beta_{g_{YM}}\neq 0$ in QFT/supergravity (Klebanov-Strassler model)

- Bulk viscosity of gauge theory plasma at strong coupling:
  - Sound modes in plasma and the corresponding quasinormal modes of the holographic dual — $\mathcal{N}=4$ and $\mathcal{N}=2^*$ gauge theory
  - Bulk viscosity bound

- Hydrodynamic relaxation time of holographic models:
  - Why should be care about the relaxation time — fundamental $\&$ practical perspective
  - Relaxation time bound

- Hydrodynamics of conformal plasma beyond SUGRA approximation:
  - finite 't Hooft coupling corrections — $1/(Ng^2_{YM})$
  - finite $1/N$ corrections
  - is there a bound on η/s?
- sQGP as hCFT

$\mathcal{N} = 2^*$ gauge theory (a QFT story)

$\Longrightarrow$ Start with $\mathcal{N} = 4$ $SU(N)$ SYM. In $\mathcal{N} = 1$ 4d susy language, it is a gauge theory of a vector multiplet V, an adjoint chiral superfield Φ (related by $\mathcal{N} = 2$ susy to V) and an adjoint pair $\{Q, \tilde{Q}\}$ of chiral multiplets, forming an $\mathcal{N} = 2$ hypermultiplet. The theory has a superpotential:

$$W = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr}\left(\left[Q, \tilde{Q}\right] \Phi\right)$$

We can break susy down to $\mathcal{N} = 2$, by giving a mass for $\mathcal{N} = 2$ hypermultiplet:

$$W = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr}\left(\left[Q, \tilde{Q}\right] \Phi\right) + \frac{m}{g_{YM}^2} \left(\mathrm{Tr}Q^2 + \mathrm{Tr}\tilde{Q}^2\right)$$

This theory is known as $\mathcal{N} = 2^*$ gauge theory

When $m \neq 0$, the mass deformation lifts the $\{Q, \tilde{Q}\}$ hypermultiplet moduli directions; we are left with the $(N-1)$ complex dimensional Coulomb branch, parametrized by

$$\Phi = \mathrm{diag}\left(\mathrm{a}_1, \mathrm{a}_2, \cdots, \mathrm{a}_\mathrm{N}\right) \,, \qquad \sum_\mathrm{i} \mathrm{a}_\mathrm{i} = 0$$

We will study $\mathcal{N} = 2^*$ gauge theory at a particular point on the Coulomb branch moduli space:

$$a_i \in [-a_0, a_0] \,, \qquad a_0^2 = \frac{m^2 g_{YM}^2 N}{\pi}$$

with the (continuous in the large N-limit) linear number density

$$\rho(a) = \frac{2}{m^2 g_{YM}^2} \sqrt{a_0^2 - a^2} \,, \qquad \int_{-a_0}^{a_0} da \; \rho(a) = N$$

Reason: we understand the dual supergravity solution only at this point on the moduli space.

$\mathcal{N} = 2^*$ gauge theory (a supergravity story — a.k.a Pilch-Warner flow)

Consider 5d gauged supergravity, dual to $\mathcal{N} = 2^*$ gauge theory. The effective five-dimensional action is

$$S = \frac{1}{4\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \left(\tfrac{1}{4} R - (\partial\alpha)^2 - (\partial\chi)^2 - \mathcal{P} \right) ,$$

where the potential $\mathcal{P}$ is

$$\mathcal{P} = \frac{1}{16} \left[\left(\frac{\partial W}{\partial \alpha} \right)^2 + \left(\frac{\partial W}{\partial \chi} \right)^2 \right] - \frac{1}{3} W^2 ,$$

with the superpotential

$$W = -\frac{1}{\rho^2} - \frac{1}{2} \rho^4 \cosh(2\chi) , \qquad \alpha \equiv \sqrt{3} \ln \rho$$

$\Longrightarrow$ The 2 supergravity scalars $\{\alpha, \chi\}$ are holographic dual to dim-2 and dim-3 operators which are nothing but (correspondingly) the bosonic and the fermionic mass terms of the $\mathcal{N} = 4 \to \mathcal{N} = 2$ SYM mass deformation.

PW geometry ansatz:

$$ds_5^2 = e^{2A}\left(-dt^2 + d\vec{x}^{\,2}\right) + dr^2$$

solving the Killing spinor equations, we find a susy flow:

$$\frac{dA}{dr} = -\frac{1}{3}W\,, \qquad \frac{d\alpha}{dr} = \frac{1}{4}\frac{\partial W}{\partial \alpha}\,, \qquad \frac{d\chi}{dr} = \frac{1}{4}\frac{\partial W}{\partial \chi}$$

Solutions to above are characterized by a single parameter k:

$$e^A = \frac{k\rho^2}{\sinh(2\chi)}\,, \qquad \rho^6 = \cosh(2\chi) + \sinh^2(2\chi)\,\ln\frac{\sinh(\chi)}{\cosh(\chi)}$$

In was found (Polchinski,Peet,AB) that

$$k = 2m$$

Introduce

$$\hat{x} \equiv e^{-r/2}\,,$$

then

$$\chi = k\hat{x}\Big[1 + k^2\hat{x}^2\left(\tfrac{1}{3} + \tfrac{4}{3}\ln(k\hat{x})\right) + k^4\hat{x}^4\left(-\tfrac{7}{90} + \tfrac{10}{3}\ln(k\hat{x}) + \tfrac{20}{9}\ln^2(k\hat{x})\right) + \mathcal{O}\left(k^6\hat{x}^6\ln^3(k\hat{x})\right)\Big]\,,$$

$$\rho = 1 + k^2\hat{x}^2\left(\tfrac{1}{3} + \tfrac{2}{3}\ln(k\hat{x})\right) + k^4\hat{x}^4\left(\tfrac{1}{18} + 2\ln(k\hat{x}) + \tfrac{2}{3}\ln^2(k\hat{x})\right) + \mathcal{O}\left(k^6\hat{x}^6\ln^3(k\hat{x})\right)\,,$$

$$A = -\ln(2\hat{x}) - \tfrac{1}{3}k^2\hat{x}^2 - k^4\hat{x}^4\left(\tfrac{2}{9} + \tfrac{10}{9}\ln(k\hat{x}) + \tfrac{4}{9}\ln^2(k\hat{x})\right) + \mathcal{O}\left(k^6\hat{x}^6\ln^3(k\hat{x})\right)$$

Or in standard Poincare-patch AdS_5 radial coordinate:

$$A \propto \ln r, \qquad \alpha \propto \frac{k^2\ln r}{r^2}\,, \qquad \chi \propto \frac{k}{r}\,, \qquad r \to \infty$$

$\Longrightarrow$ Notice that the nonnormalizable components of $\{\alpha, \chi\}$ are related — this is holographic dual to $\mathcal{N} = 2$ susy preserving condition on the gauge theory side:

$$m_b = m_f$$

But in general, we can keep $m_b \neq m_f$:

$$A \propto \ln r, \qquad \alpha \propto \frac{m_b^2 \ln r}{r^2}\,, \qquad \chi \propto \frac{m_f}{r}\,, \qquad r \to \infty$$

The precise relation, including numerical coefficients can be works out.

$\Longrightarrow$ There are no singularity-free flows (geometries) with $m_b \neq m_f$ and at zero temperature $T = 0$. However, one can study $m_b \neq m_f$ mass deformations of $\mathcal{N} = 4$ SYM at finite temperature.

$\Longrightarrow$To study holographic duality in full details, we need the full ten-dimensional background of type IIB supergravity, i.e, we need the lift of 5-dimensional gauged SUGRA solutions. This will be obvious when we discuss jet quenching in $\mathcal{N}=2^*$.

Such a lift was constructed in J.Liu,AB. Specifically, for any 5d solution, the 5d background:

$$ds_5^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \qquad \text{plus} \qquad \{\alpha, \chi\}$$

is uplifted to a solution of 10d type IIB supergravity:

$$ds^2_{10(E)} = \Omega^2 ds_5^2 + \Omega^2 \frac{4}{\rho^2} \left[\frac{1}{c} d\theta^2 + \rho^6 \cos^2(\theta) \left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2+\sigma_3^2}{X_1} \right) + \sin^2(\theta) \frac{1}{X_2} d\phi^2 \right]$$

$$\Omega^2 = \frac{(cX_1X_2)^{1/4}}{\rho} \,, \qquad X_1 = \cos^2\theta + c(r)\rho^6 \sin^2\theta \,, \qquad X_2 = c\cos^2\theta + \rho^6 \sin^2\theta$$

with

$$c \equiv \cosh 2\chi \,,$$

plus dilaton-axion, various 3-form fluxes, various 5-form fluxes.

Thermodynamics of $\mathcal{N}=2^*$ for (non-)susy mass-deformations (with J.Liu,P.Kerner,...)

Consider metric ansatz:

$$ds_5^2 = -c_1^2(r)\, dt^2 + c_2^2(r)\ \left(dx_1^2 + dx_2^2 + dx_3^2\right) + dr^2$$

Introducing a new radial coordinate

$$x \equiv 1 - \frac{c_1}{c_2}\,,$$

with $x \to 0_+$ being the boundary and $x \to 1_-$ being the horizon, we find:

$$c_2'' + 4c_2\,(\alpha')^2 - \frac{1}{x-1}c_2' - \frac{5}{c_2}(c_2')^2 + \frac{4}{3}c_2\,(\chi')^2 = 0$$

$$\alpha'' + \frac{1}{x-1}\,\alpha' - \frac{\frac{\partial \mathcal{P}}{\partial \alpha}}{12\,\mathcal{P}c_2^2(x-1)}\left[(x-1)\left(6(\alpha')^2 + 2(\chi')^2\right)c_2^2 - 3c_2'c_2 - 6(c_2')^2(x-1)\right] = 0$$

$$\chi'' + \frac{1}{x-1}\,\chi' - \frac{\frac{\partial \mathcal{P}}{\partial \chi}}{4\,\mathcal{P}c_2^2(x-1)}\left[(x-1)\left(6(\alpha')^2 + 2(\chi')^2\right)c_2^2 - 3c_2'c_2 - 6(c_2')^2(x-1)\right] = 0$$

We look for a solution to above subject to the following (fixed) boundary conditions:

$\Longrightarrow$ near the boundary, $x \propto r^{-4} \to 0_+$

$$\left\{ c_2(x), \alpha(x), \chi(x) \right\} \to \left\{ x^{-1/4}, \qquad \frac{m_b^2}{T^2}\, x^{1/2} \ln x, \qquad \frac{m_f}{T}\, x^{1/4} \right\}$$

of course, we need a precise coefficients here relating the non-normalizable components of the sugra scalars to the gauge theory masses

$\Longrightarrow$ near the horizon, $x \to 1_-$ (to have a regular, non-singular Schwarzschild horizon)

$$\left\{ c_2(x), \alpha(x), \chi(x) \right\} \to \left\{ \text{constant}\,, \qquad \text{constant}\,, \qquad \text{constant} \right\}$$

System of above equations can be solved analytically when $\frac{m_b}{T} \ll 1$ and $\frac{m_f}{T} \ll 1$ With the help of the holographic renormalization (in this model AB) we can independently compute the free energy density $\mathcal{F} = -P$, the energy density $\mathcal{E}$, and the entropy density s of the resulting black brane solution:

$$-\mathcal{F} = P = \frac{1}{8}\pi^2 N^2 T^4 \left[1 - \frac{192}{\pi^2}\, \ln(\pi T)\, \delta_1^2 - \frac{8}{\pi}\, \delta_2^2\right]$$

$$\mathcal{E} = \frac{3}{8}\pi^2 N^2 T^4 \left[1 + \frac{64}{\pi^2}\, \left(\ln(\pi T) - 1\right)\, \delta_1^2 - \frac{8}{3\pi}\, \delta_2^2\right]$$

$$s = \frac{1}{2}\pi^2 N^2 T^3 \left(1 - \frac{48}{\pi^2}\, \delta_1^2 - \frac{4}{\pi}\, \delta_2^2\right)$$

with

$$\delta_1 = -\frac{1}{24\pi}\, \left(\frac{m_b}{T}\right)^2, \qquad \delta_2 = \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^2}{2\pi^{3/2}}\, \frac{m_f}{T}$$

A highly nontrivial consistency test on the analysis, as well as on the identification of gauge theory/supergravity parameters are the basic thermodynamics identities:

$$\mathcal{F} = \mathcal{E} - sT$$

$$d\mathcal{E} = T ds$$

$\Longrightarrow$ For finite (not small) m_b/T and m_f/T we need to do numerical analysis. However, we always check the consistency of the thermodynamic relations. In our numerics

$$\frac{d\mathcal{E} - T ds}{d\mathcal{E}} \sim 10^{-3}$$

The phase diagram of the model depends on

$$\Delta \equiv \frac{m_f^2}{m_b^2}\ :$$

- when $\Delta \geq 1$ there is no phase transition in the system;
- when $\Delta < 1$ there is a critical point in the system with the divergent specific heat. The corresponding critical exponent is $\alpha = 0.5$:

$$c_V \sim |1 - T_c/T|^{-\alpha}$$

  where $T_c = T_c(\Delta)$.

For concreteness we discuss below 2 cases:

(a) $\Delta = 1$ ('susy' flows at finite temperature)

(b) $\Delta = 0$ ('bosonic' flows at finite temperature)

Before we discuss the flows, recall the lattice data for the QCD:

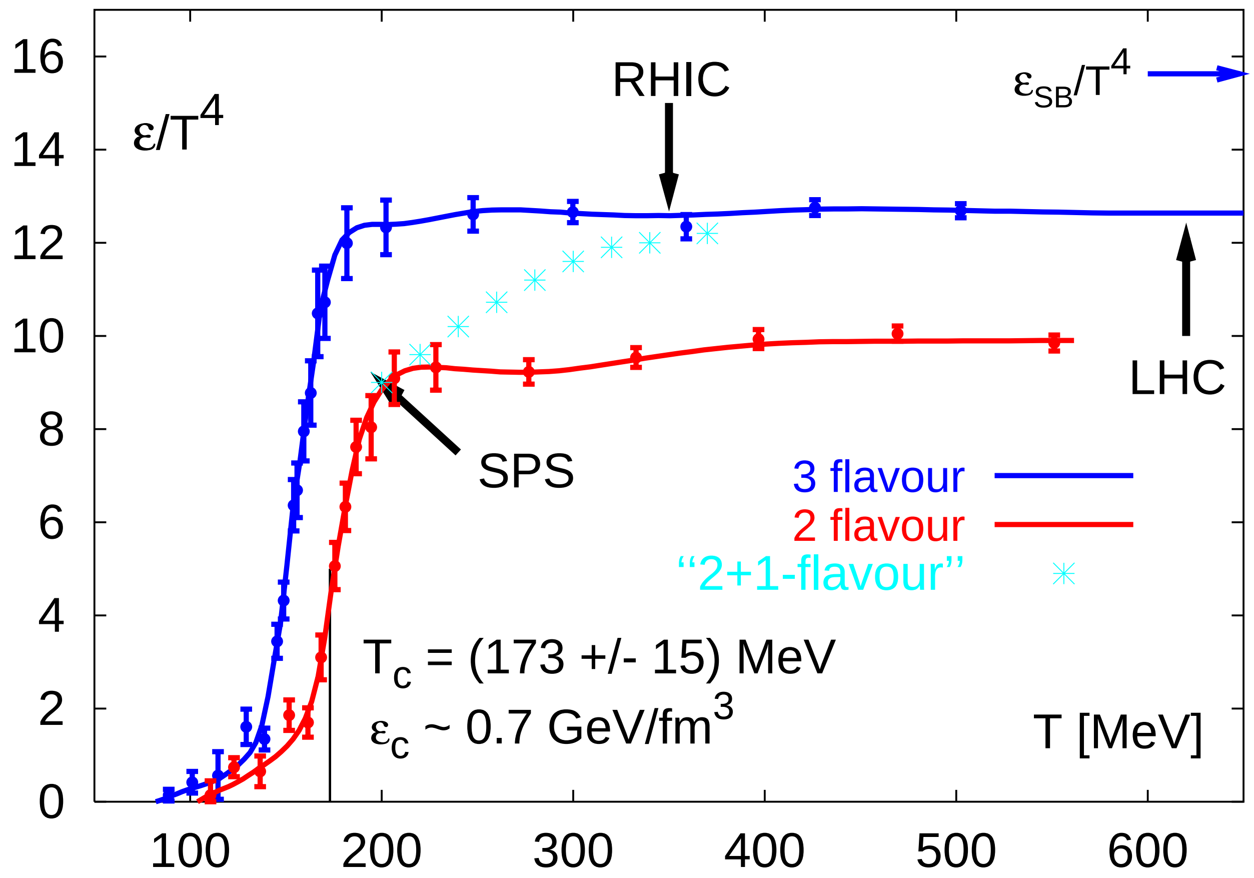


Figure 1: QCD thermodynamics from lattice; F.Karsch and E.Laermann, hep-lat/0305025.

- RHIC QGP is strongly coupled because equilibrium plasma temperature is roughly the QCD deconfinement temperature,

$$T_{plasma} \sim T_{deconfinement} \sim \Lambda_{QCD}$$

- Thus scale invariance is strongly broken and it is not clear why conformal $\mathcal{N} = 4$ plasma or near-conformal plasma thermodynamics/hydrodynamics should be relevant...

Surprisingly...

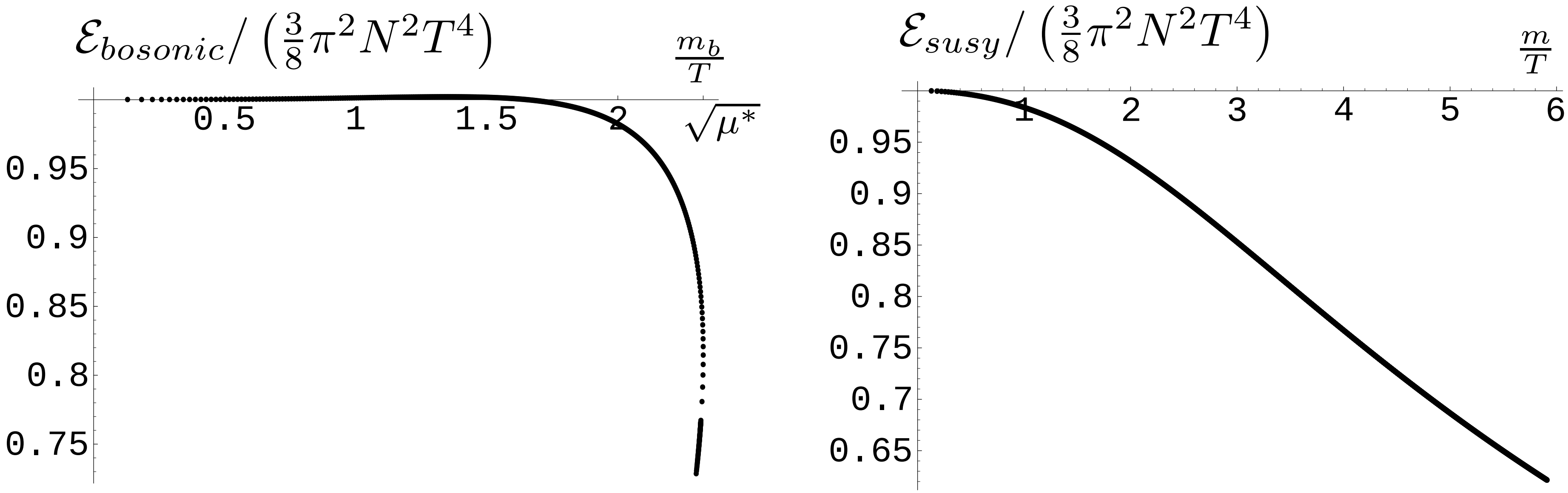


Figure 2: Equation of state of the mass deformed $\mathcal{N}=4$ gauge theory plasma. At $T\sim m$ the deviation from the conformal thermodynamics is $\sim 2\%$. For the ideal gas approximation the deviation is about 40%. (S.Deakin, P.Kerner, J.Liu, AB, hep-th/0701142.)

$\Longrightarrow \mathcal{N}=2^*$ model appears to share a 'thermodynamic plateau' with QCD!

$\Rightarrow$ Compare with yesterday Chris Altes's talk:

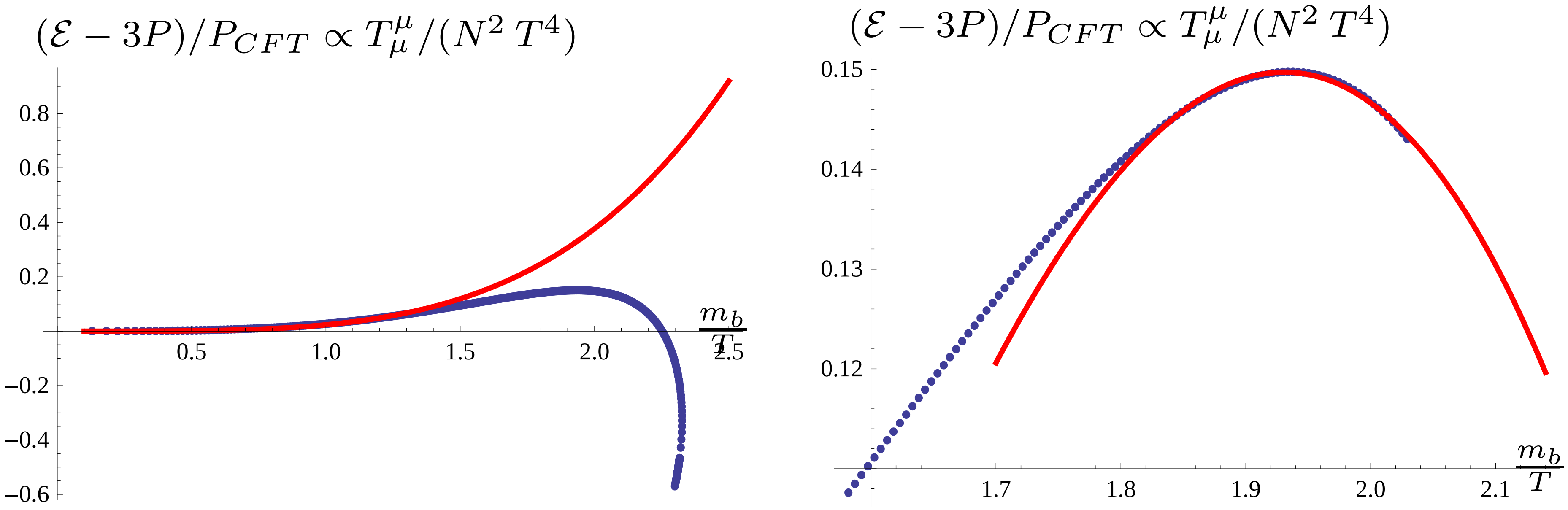


Figure 3: Blue dots are the data points. The red curve fit on the left is $\propto \left(\frac{m_b}{T}\right)^4$ — from the analytic high-T result we actually expect $\propto \left(\frac{m_b}{T}\right)^4 \ln \frac{T}{m_b}$. The red curve on the right is the fit of the 'top' of the stress-energy trace with $c_0 + c_1 \left(\frac{m_b}{T}\right)^2 + c_2 \left(\frac{m_b}{T}\right)^4$.

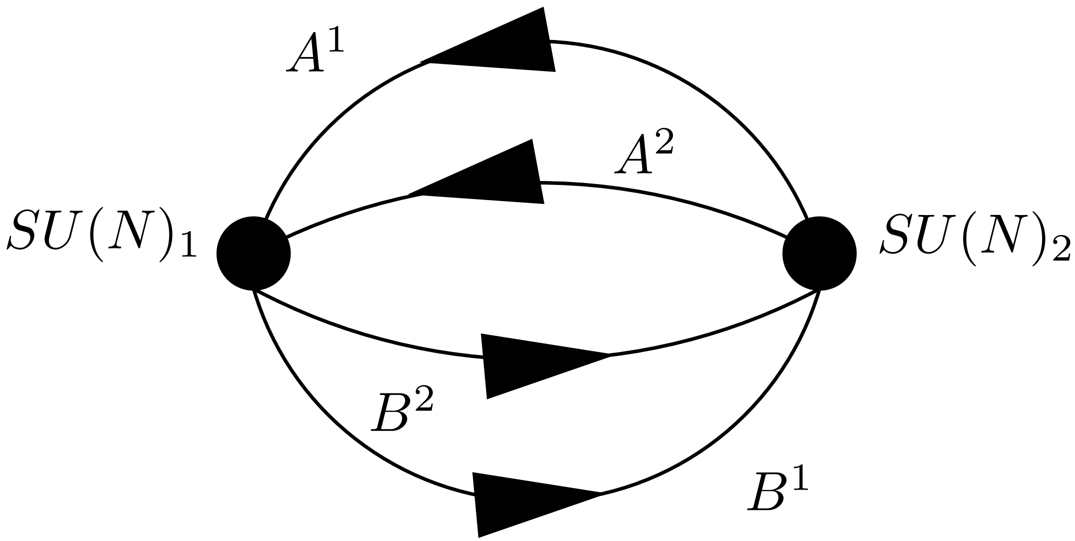
$\Rightarrow$ This model, however, does not have confinement/deconfinement phase transition...

Klebanov-Strassler model (a QFT story)

$\Longrightarrow$ The staring point again is $\mathcal{N}=4$ SU(N) SYM.

- Consider a $\mathbb{Z}_2$ orbifold of above SYM:

$$\mathcal{N}=4 \qquad \rightarrow \qquad \mathcal{N}=2$$



	$SU(N)_1 \times SU(N)_2$
A^i :	$(\bar{N}, N)$
B^i :	$(N, \bar{N})$

$$\mathcal{W}_{\mathcal{N}=2} = g_1 \ \mathrm{Tr}\,\Phi_1 \left[A^1 B^1 + A^2 B^2\right] + g_2 \ \mathrm{Tr}\,\Phi_2 \left[B^1 A^1 + B^2 A^2\right]$$

Note: $\beta_i = 0 \Rightarrow g_1, g_2$ are exactly marginal couplings

- Turn on the mass term that breaks SUSY $\mathcal{N}=2 \rightarrow \mathcal{N}=1$

$$\mathcal{W}_{\mathcal{N}=2} \rightarrow \mathcal{W}_{\mathcal{N}=1} = \mathcal{W}_{\mathcal{N}=2} + m \; \mathrm{Tr}\left(\Phi_1^2 - \Phi_2^2\right)$$

$\Rightarrow$ Integrating out the massive fields we find

$$\mathcal{W}_{eff} = \lambda \, \mathrm{Tr}\, A^i B^j A^k B^\ell \epsilon^{ik} \epsilon^{j\ell}$$

$\Rightarrow$ Klebanov and Witten argued that at energy scales $\ll m$ the theory flows to a strongly interactive superconformal field theory; the coupling λ is exactly marginal, and thus the fields A^i, B^j develop large anomalous dimensions

$$[A^i]^{UV} = 1 \quad \rightarrow \quad [A^i]^{IR} = \frac{3}{4} \qquad \Rightarrow \; \gamma_{A^i} = -\frac{1}{4}$$

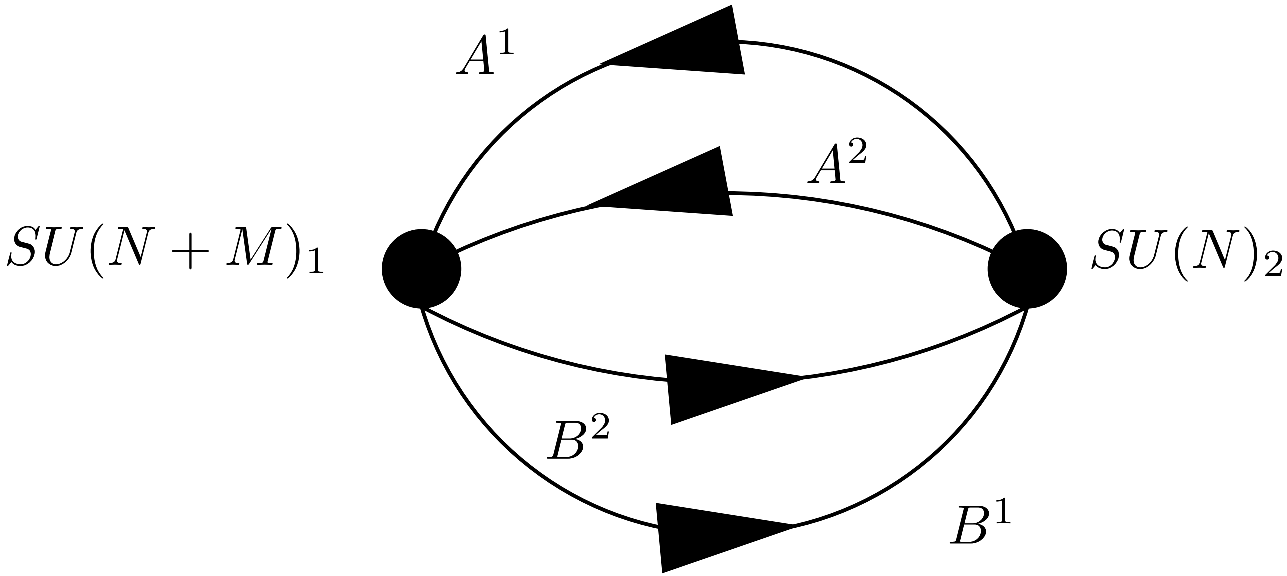
$$[B^i]^{UV} = 1 \quad \rightarrow \quad [B^i]^{IR} = \frac{3}{4} \qquad \Rightarrow \; \gamma_{B^i} = -\frac{1}{4}$$

$\Rightarrow$ From the exact NSWZ gauge β-functions (accounting for the anomalous dim of fields) we find

$$\beta_i = 0$$

- Consider a discrete deformation

$$SU(N)_1 \ \rightarrow \ SU(N+M)_1 \,, \qquad M \ll N$$



$$\beta_1 \sim 3(N+M) - 2N(1-\gamma_{A^i}-\gamma_{B^j}) = 3M + \mathcal{O}(M^3/N^2)$$

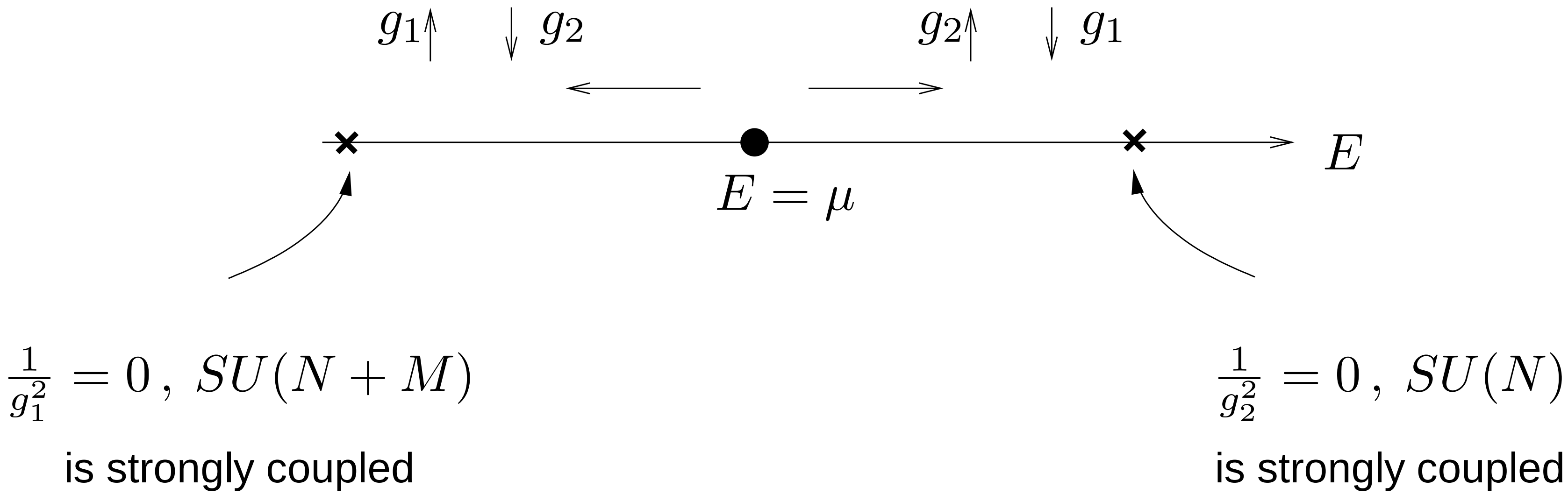
$$\beta_2 \sim 3N - 2(N+M)(1-\gamma_{A^i}-\gamma_{B^j}) = -3M + \mathcal{O}(M^3/N^2)$$

From the β-functions:

$$\frac{4\pi}{g_1^2(\mu)} + \frac{4\pi}{g_2^2(\mu)} = \mathrm{const}$$

$$\frac{4\pi}{g_1^2(\mu)} - \frac{4\pi}{g_2^2(\mu)} \sim M \ \ln \frac{\mu}{\Lambda}$$

where Λ is the strong coupling scale of the theory



What is the effective description of the theory past the Landau poles?

$\Rightarrow$ Using Seiberg duality for $\mathcal{N}=1$ SUSY gauge theory, the extension of the model past the Landau poles results in self-similarity cascade (Klebanov and Strassler):

$$N \to N(\mu) \sim 2M^2 \, \ln\frac{\mu}{\Lambda}$$

$$\mathrm{UV}: \qquad N \to N + M\,, \qquad\qquad\qquad \mathrm{IR}: \qquad N \to N - M$$

$\Rightarrow$ If N is a multiple of M, the theory in the deep infrared is $\mathcal{N}=1$ $SU(M)$ SYM; this theory confines with the spontaneous chiral $U(1)_R$ symmetry breaking

Klebanov-Strassler model (a supergravity story)

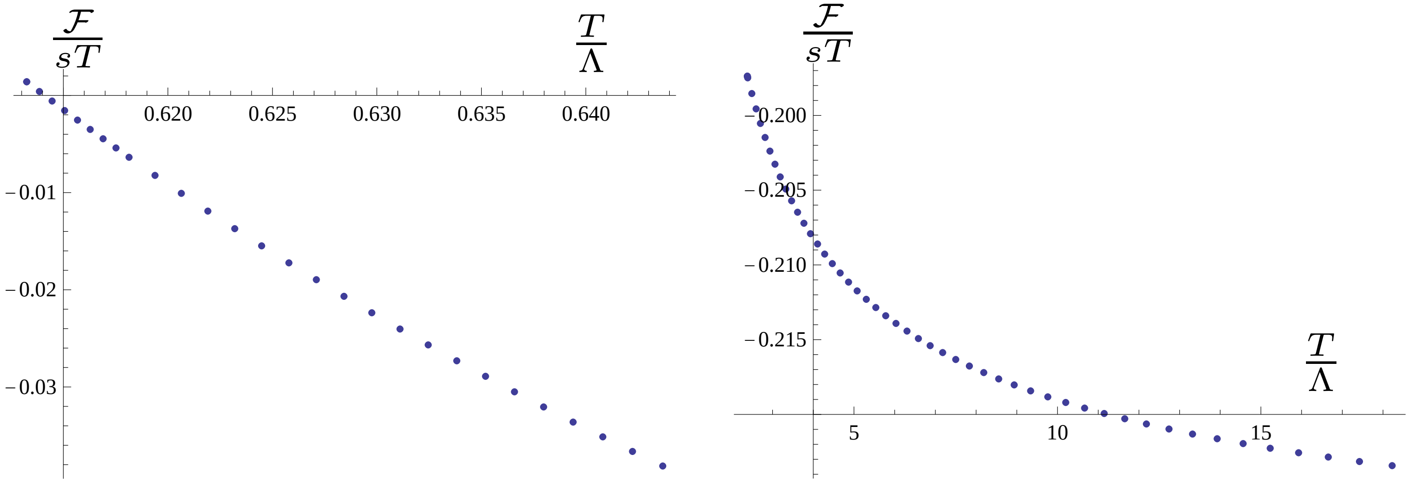
It is possible to derive an effective 5d action from string theory dual to KS model in the deconfined phase with unbroken chiral symmetry:

$$S = \frac{1}{16\pi G_5}\int_{\mathcal{M}_5} d\xi^5 \sqrt{-g}\biggl(R - \frac{40}{3}(\partial f)^2 - 20(\partial w)^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{4M^2}(\partial K)^2 e^{-\Phi-4f-4w} - \mathcal{P}\biggr)$$

$$\mathcal{P} = -24 e^{-\frac{16}{3}f-2w} + 4e^{-\frac{16}{3}f-12w} + M^2 e^{\Phi-\frac{28}{3}f+4w} + \frac{1}{2}K^2 e^{-\frac{40}{3}f}$$

$\Rightarrow$ The 4 supergravity scalars $\{\Phi, f, w, K\}$ encode operators of dim=$\{4,4,6,8\}$.

$\Rightarrow$ It is possible (though quite technical) to repeat thermodynamic analysis analogous to those of the $\mathcal{N}=2^*$ model.



The free energy density $\mathcal{F}$, divided by sT, as a function of $\frac{T}{\Lambda}$. On the left we plot temperatures at and slightly above the deconfinement transition, and on the right much higher temperatures. Note:

$$\left(\frac{T}{\Lambda}\right)_{deconfinement} = 0.614111(3)$$

$$\left.\frac{\mathcal{F}}{sT}\right|_{conformal} = -\frac{1}{4}$$

which is different from the cascading model at $\frac{T}{\Lambda} = 10$ result by $\sim 12\%$

$\Rightarrow$ No 'thermodynamic plateau' near the deconfinement transition!

$\Rightarrow$ Compare with yesterday Chris Altes's talk:

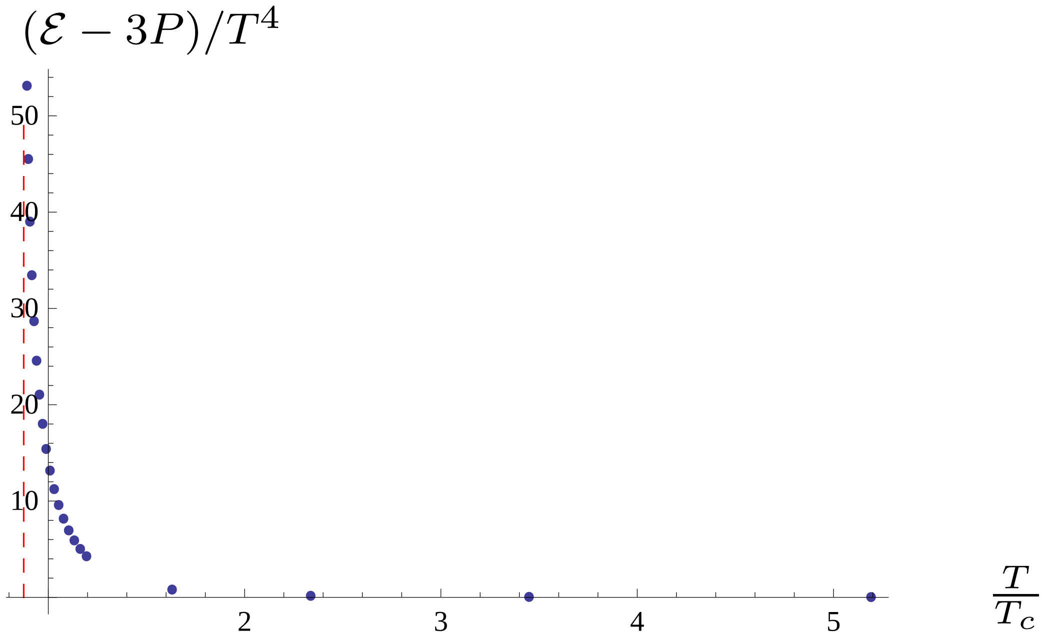


Figure 4: Blue dots are the data points. The dashed red vertical line denotes second-order transition. There is chiral symmetry breaking transition prior to the red line. At high temperatures the trace drops as $1/(T^4 \ln \frac{T}{T_c})$

Summary of holographic non-conformal thermodynamics:

- Most important: by deforming appropriately AdS/CFT correspondence we can produce examples of nontrivial renormalization group flows of gauge theories. It does not make sense to **believe** in AdS/CFT, but **question** holographic dualities for nonconformal models. Hence:

$$\text{AdS/CFT} \qquad \Longrightarrow \qquad \text{gauge/string duality}$$

Of cause, if does not mean that 'anything' goes: each realistic holographic duality must be derivable (in a sense of $\mathcal{N}=4$ SYM) from string theory.

- Resulting nonconformal plasma have rich thermodynamics — first and second order phase transitions, (de)confinement, chiral symmetry breaking, etc.
- In QCD plasma there is a distinctive 'thermodynamic plateau' in the vicinity of the phase transition (crossover) — some holographic models share it, the other do not $\Rightarrow$ one has to be careful of blindly applying holographic results appropriate for conformal theories to strongly coupled QGP.

Sound modes in plasma and in its holographic dual

Hydrodynamics is an effective theory describing near-equilibrium phenomena in (relativistic) QFT:

$$\nabla_\nu T^{\mu\nu} = 0$$

The stress-energy tensor includes both an equilibrium part ($\mathcal{E}$ and $\mathcal{P}$ terms) and a dissipative part $\Pi^{\mu\nu}$

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + \Pi^{\mu\nu} \,.$$

where u^μ is a local 4-velocity of the fluid and

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \,, \quad \Pi^\mu{}_\nu u^\nu = 0$$

Effective hydrodynamic description is equivalent to a derivative expansion of $\Pi^{\mu\nu}$ in local velocity gradients

Thus, to linear order in the derivative expansion

$$\Pi^{\mu\nu} = \Pi_1^{\mu\nu}(\eta, \zeta) = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}(\nabla_\alpha u^\alpha)$$

($\sigma^{\mu\nu} \propto \nabla_\nu u^\mu$) with $\{\eta, \zeta\}$ being the viscosity coefficients.

$\Rightarrow$ It is straightforward to study dispersion relation of the linearized fluctuations in above theory: there are sound and shear modes.

The dispersion relation of a sound mode is given by

$$\omega = \pm c_s k - i\Gamma k^2 + \mathcal{O}(k^3)\,,$$

where c_s is the speed of the sound waves (obtained from the equation of state), and Γ is the sound wave attenuation (determined by the shear and the bulk viscosities)

$$c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}}\,, \qquad \Gamma = \left(\frac{2}{3}\frac{\eta}{\mathcal{E}+\mathcal{P}} + \frac{1}{2}\frac{\zeta}{\mathcal{E}+\mathcal{P}}\right) = \frac{2}{3T}\,\frac{\eta}{s}\left(1 + \frac{3}{4}\,\frac{\zeta}{\eta}\right)$$

$\Rightarrow$ Once we know the sound wave dispersion relation, and the shear viscosity, we can extract the bulk viscosity of non-conformal plasma.

To understand how to describe sound waves in dual holographic models, recall that the on-shell fluctuations in plasma will show up as poles of the energy-stress tensor 2-point correlation functions.

$\Rightarrow$ A plasma stress-energy tensor is holographically dual to a 5d graviton, and thus the energy-momentum fluctuations in plasma are dual to the graviton quasinormal modes (Kovtun and Starinets)

$\Rightarrow$ I explain the notion of the quasinormal modes using $\mathcal{N}=4$ SYM plasma as an example:

$$S=\frac{1}{16\pi G_5}\int_{\mathcal{M}_5} d\xi^5\sqrt{-g}\,(R+12)$$

where I set the AdS radius to 1

⇒ The equilibrium sate of $\mathcal{N}=4$ plasma is described by a BH solution:

$$ds_5^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\pi^2 T^2}{r}\left(-(1-r^2)dt^2 + d\vec{x}^3\right) + \frac{dr^2}{4(1-r^2)r^2}$$

where T is the Hawking temperature of the BH (to be identified with the plasma temperature)

⇒ Consider the graviton $h_{\mu\nu}$ fluctuations

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$$

⇒ We can always choose the gauge

$$h_{\mu r} = 0$$

⇒ At a linearized level, we can assume that

$$h_{\mu\nu} = h_{\mu\nu}(t, x_3, r) \sim e^{-i\omega t + iqx_3}$$

⇒ Above fluctuations preserve $O(2)$ symmetry — rotations in $x_1 - x_2$ plane

$\Rightarrow$ Because of the symmetry, fluctuations of different helicities would decouple from each other:

$$\text{helicity}-2: \qquad \{h_{x_1x_2}\}\,,\ \{h_{x_1x_1}-h_{x_2x_2}\}$$

$$\text{helicity}-1: \qquad \{h_{tx_1},\ h_{x_1x_3}\}\,,\ \{h_{tx_2},\ h_{x_2x_3}\}$$

$$\text{helicity}-0: \qquad \{h_{tt},\ h_{aa}\equiv h_{x_1x_1}+h_{x_2x_2},\ h_{tx_3},\ h_{x_3x_3}\}$$

$\Rightarrow$ The shear modes correspond to helicity-1 fluctuations; the sound modes are encoded in helicity-0 fluctuations; helicity-2 fluctuations are not hydrodynamic

$\Rightarrow$ Introduce

$$h_{tt}=e^{-i\omega t+iqx_3}\ \frac{\pi^2T^2(1-r^2)}{r}\ H_{tt}(r)$$

$$h_{tz}=e^{-i\omega t+iqx_3}\ \frac{\pi^2T^2}{r}\ H_{tz}(r)$$

$$h_{aa}=e^{-i\omega t+iqx_3}\ \frac{\pi^2T^2}{r}\ H_{aa}(r)$$

$$h_{zz}=e^{-i\omega t+iqx_3}\ \frac{\pi^2T^2}{r}\ H_{zz}(r)$$

From the Einstein equations

$$R_{\mu\nu}\Big[g_{\mu\nu} + h_{\mu\nu}\Big] = -4(g_{\mu\nu} + h_{\mu\nu})$$

we obtain

- 4 second-order differential equations for

$$\{H_{tt}, H_{tz}, H_{aa}, H_{zz}\}$$

- 3 first order differential constraints associated with fixing the gauge

$$h_{tr} = h_{zr} = h_{rr} = 0$$

$\Rightarrow$ Had the constraints been algebraic, it could have been used to eliminate 3 fields, and produce a single (4-3=1) second order differential equation

$\Rightarrow$ The differential elimination is possible if one uses a gauge-invariant variables!

(Kovtun-Starinets):

- first, identify residual diffeomorphisms

$$x^\mu \to x^\mu + \xi^\mu \qquad \Longrightarrow \qquad g_{\mu\nu} \to g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$$

such that

$$g_{\mu r} \to g_{\mu r} \qquad \Longrightarrow \qquad 0 = \nabla_\mu \xi_r + \nabla_r \xi_\mu$$

- under above transformations

$$\{H_{tt}, H_{tz}, H_{aa}, H_{zz}\} \to \{H'_{tt}, H'_{tz}, H'_{aa}, H'_{zz}\}$$

- second, introduce a linear combination of metric fluctuations that stays invariant

$$Z \equiv 4\frac{q}{w} H_{tz} + 2H_{zz} - H_{aa}\left(1 - (1 + r^2)\frac{q^2}{w^2}\right) + 2(1 - r^2)\frac{q^2}{w^2} H_{tt} \to Z' = Z$$

$\Rightarrow$ The equation of motion for Z we completely decouple:

$$0 = Z'' + \mathcal{A}_z \, Z' + \mathcal{B}_z \, Z$$

$$\mathcal{A}_z = -\frac{3\mathfrak{q}^2 r^4 - 2\mathfrak{q}^2 r^2 + 3\mathfrak{q}^2 - 3\mathfrak{w}^2 r^2 - 3\mathfrak{w}^2}{(-1+r^2) r (-3\mathfrak{q}^2 + 3\mathfrak{w}^2 + \mathfrak{q}^2 r^2)}$$

$$\mathcal{B}_z = \frac{4\mathfrak{q}^2 r^5 + \mathfrak{q}^4 r^4 - 4\mathfrak{q}^2 r^3 + 4\mathfrak{w}^2 \mathfrak{q}^2 r^2 - 4\mathfrak{q}^4 r^2 + 3\mathfrak{q}^4 + 3\mathfrak{w}^4 - 6\mathfrak{w}^2 \mathfrak{q}^2}{r(-3\mathfrak{q}^2 + 3\mathfrak{w}^2 + \mathfrak{q}^2 r^2)(-1+r^2)^2}$$

where

$$\mathfrak{w} = \frac{w}{2\pi T}\,, \qquad \mathfrak{q} = \frac{q}{2\pi T}$$

$\Rightarrow$ Let's analyze the asymptotic behavior of above equation near the horizon, i.e., as $r \to 1$

$$Z \sim (1-r^2)^{\alpha}\,, \qquad \Longrightarrow \qquad \alpha = \pm i\frac{\mathfrak{w}}{2}$$

Thus, near the horizon,

$$z(t, x_3, r) = e^{-iwt+iqx^3} Z(r) \sim \exp\left[-i\mathfrak{w}\left(2\pi T t \mp \frac{1}{2}\ln(1-r)\right) + iqx_3\right]$$

So, the modes with $\alpha = -i\frac{\mathfrak{w}}{2}$ moves *into the horizon* and modes with $\alpha = +i\frac{\mathfrak{w}}{2}$ moves *away from the horizon*

$\Rightarrow$ Near the boundary, $r \to 0$,

$$Z \sim \#\, 1 + \# r^2$$

The leading asymptotic actually changes the background metric, thus, to determine physical fluctuations in $\mathcal{N} = 4$ SYM plasma in flat space-time we must insist that

$$Z(r \to 0) = 0$$

leading to a *Dirichlet* condition at the boundary

$\Rightarrow$ Notice that equation for Z is homogeneous, so imposing

$$\alpha = +i\frac{\mathfrak{w}}{2} \qquad + \qquad Dirichlet$$

or

$$\alpha = -i\frac{\mathfrak{w}}{2} \qquad + \qquad Dirichlet$$

would determine the dispersion relation for the quasinormal mode Z:

$$\mathfrak{w} = \mathfrak{w}(\mathfrak{q})$$

A careful analysis of the quasinormal equation show that

$$\alpha = \pm i\frac{\mathfrak{w}}{2} \qquad \Longrightarrow \qquad \pm\mathrm{Im}\left[\mathfrak{w}(\mathfrak{q})\right] > 0$$

$\Rightarrow$ Poles in the retarded (advanced) correlation function of the stress-energy tensor correspond to the gravitational fluctuations with $\alpha = -i\frac{\mathfrak{w}}{2}$ ($\alpha = +i\frac{\mathfrak{w}}{2}$)

$\Rightarrow$ It is not possible to solve equation for Z analytically; in the hydrodynamic limit

$$\mathfrak{w} \ll 1\,, \qquad \mathfrak{q} \ll 1\,, \qquad \mathfrak{w} \sim \mathfrak{q}$$

we find (up to an overall constant)

$$Z(r) = (1-r^2)^{-i\mathfrak{w}/2}\left(z_0(r) + i\,\mathfrak{w}\,z_1(r) + \mathcal{O}(\mathfrak{w}^2, \mathfrak{q}^2)\right)$$

with

$$z_0 = \frac{\mathfrak{q}^2(1+r^2) - 3\mathfrak{w}^2}{2\mathfrak{q}^2 - 3\mathfrak{w}^2}\,, \qquad z_1 = \frac{2\mathfrak{q}^2(r^2-1)}{2\mathfrak{q}^2 - 3\mathfrak{w}^2}$$

The Dirichlet boundary condition $Z(0) = 0$ then determines the *sound channel* quasinormal (hydrodynamic) mode:

$$\mathfrak{w} = \pm\frac{1}{\sqrt{3}}\mathfrak{q} - \frac{i}{3}\mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3)$$

Comparing with the hydro prediction we read-off

$$c_s^2 = \frac{1}{3}\,, \qquad 2\pi T\Gamma = \frac{1}{3}\,4\pi\frac{\eta}{s}\left(1 + \frac{3}{4}\frac{\zeta}{\eta}\right) = \frac{1}{3}$$

$\Rightarrow$ Given the universality of shear viscosity in holographic models,

$$\frac{\eta}{s} = \frac{1}{4\pi} \qquad \Longrightarrow \qquad \frac{\zeta}{\eta} = 0$$

Analysis of the non-conformal model, although technically more difficult, are conceptually identical

$\Rightarrow$ Consider $\mathcal{N} = 2^*$ model:

- There is a decoupled set of helicity-0 fluctuations in the background, dual to a sound wave,

$$\{H_{tt}, H_{tz}, H_{aa}, H_{zz}\} + \{\delta\alpha, \delta\chi\}$$

where $\{\delta\alpha\}$ and $\{\delta\chi\}$ are the fluctuations of the background supergravity scalars $\{\alpha, \chi\}$

- we expect 4+2-3=3 independent coupled second-order equations for the fluctuations
- for the metric ansatz

$$ds_5^2 = -c_1^2(r)\, dt^2 + c_2^2(r)\, d\vec{x}^2 + dr^2$$

the gauge-invariant combinations of the fluctuations are:

$$Z_H = 4\frac{q}{\omega}\ H_{tz} + 2\ H_{zz} - H_{aa}\left(1 - \frac{q^2}{\omega^2}\frac{c_1' c_1}{c_2' c_2}\right) + 2\frac{q^2}{\omega^2}\frac{c_1^2}{c_2^2}\ H_{tt}$$

$$Z_\phi = \delta\alpha - \frac{\alpha'}{(\ln c_2^4)'}\ H_{aa}$$

$$Z_\psi = \delta\chi - \frac{\chi'}{(\ln c_2^4)'}\ H_{aa}$$

and the equations take form:

$$A_H\ Z_H'' + B_H\ Z_H' + C_H\ Z_H + D_H\ Z_\phi + E_H\ Z_\psi = 0$$

$$A_\phi Z_\phi'' + B_\phi Z_\phi' + C_\phi Z_\phi + D_\phi Z_\psi + E_\phi Z_H' + F_\phi Z_H = 0$$

$$A_\psi Z_\psi'' + B_\psi Z_\psi' + C_\psi Z_\psi + D_\psi Z_\phi + E_\psi Z_H' + F_\psi Z_H = 0$$

where the coefficients $\{A..., B..., \cdots, F...\}$ depend on the background values c_1, c_2, α, χ.

$\Rightarrow$ The rest of analysis goes as in $\mathcal{N} = 4$ case, except numerically.

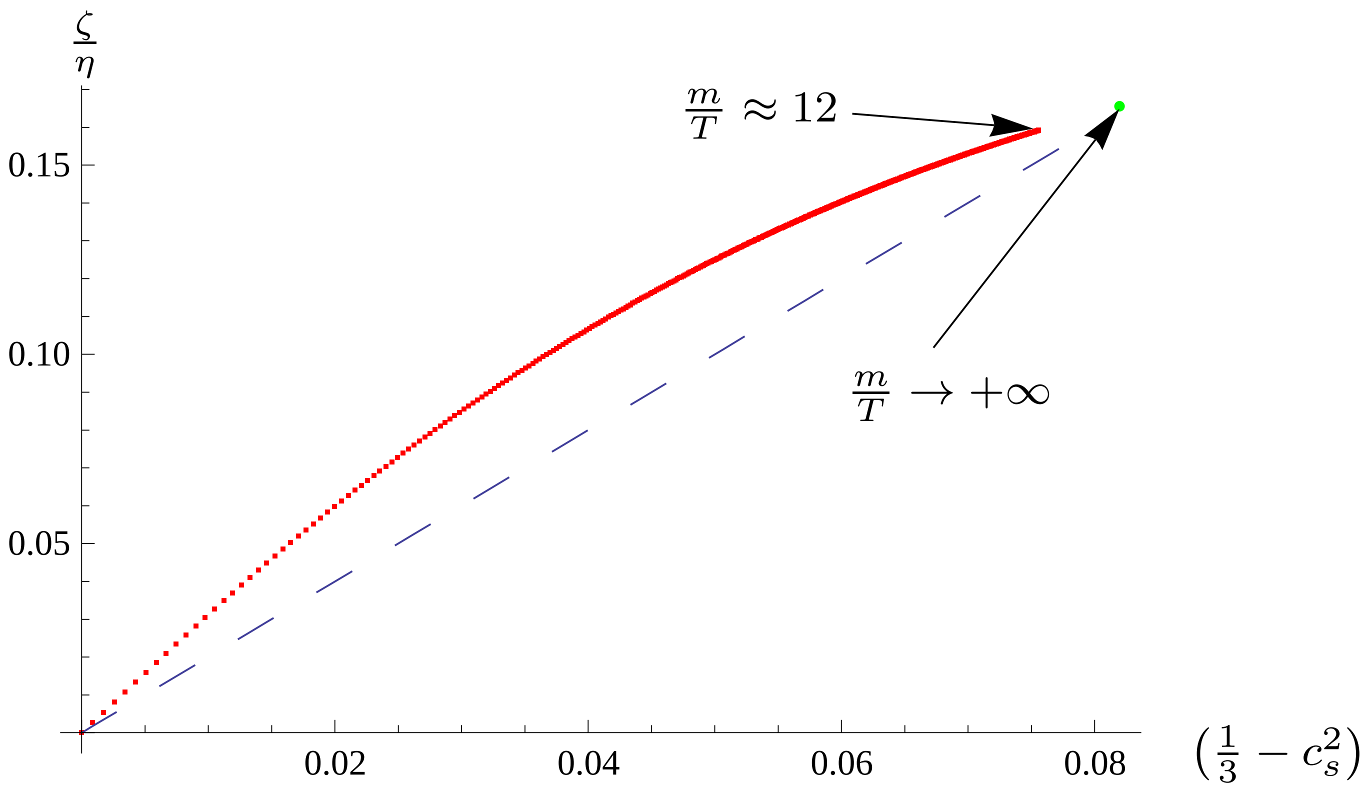


Figure 5: Ratio of viscosities $\frac{\zeta}{\eta}$ versus the speed of sound in $\mathcal{N} = 2^*$ gauge theory plasma with "supersymmetric" mass deformation parameters $m_b = m_f = m$. The dashed line represents the bulk viscosity inequality $\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right)$. We computed the bulk viscosity up to $m/T \approx 12$. A single point represents extrapolation of the speed of sound and the viscosity ratio to $T \to +0$.

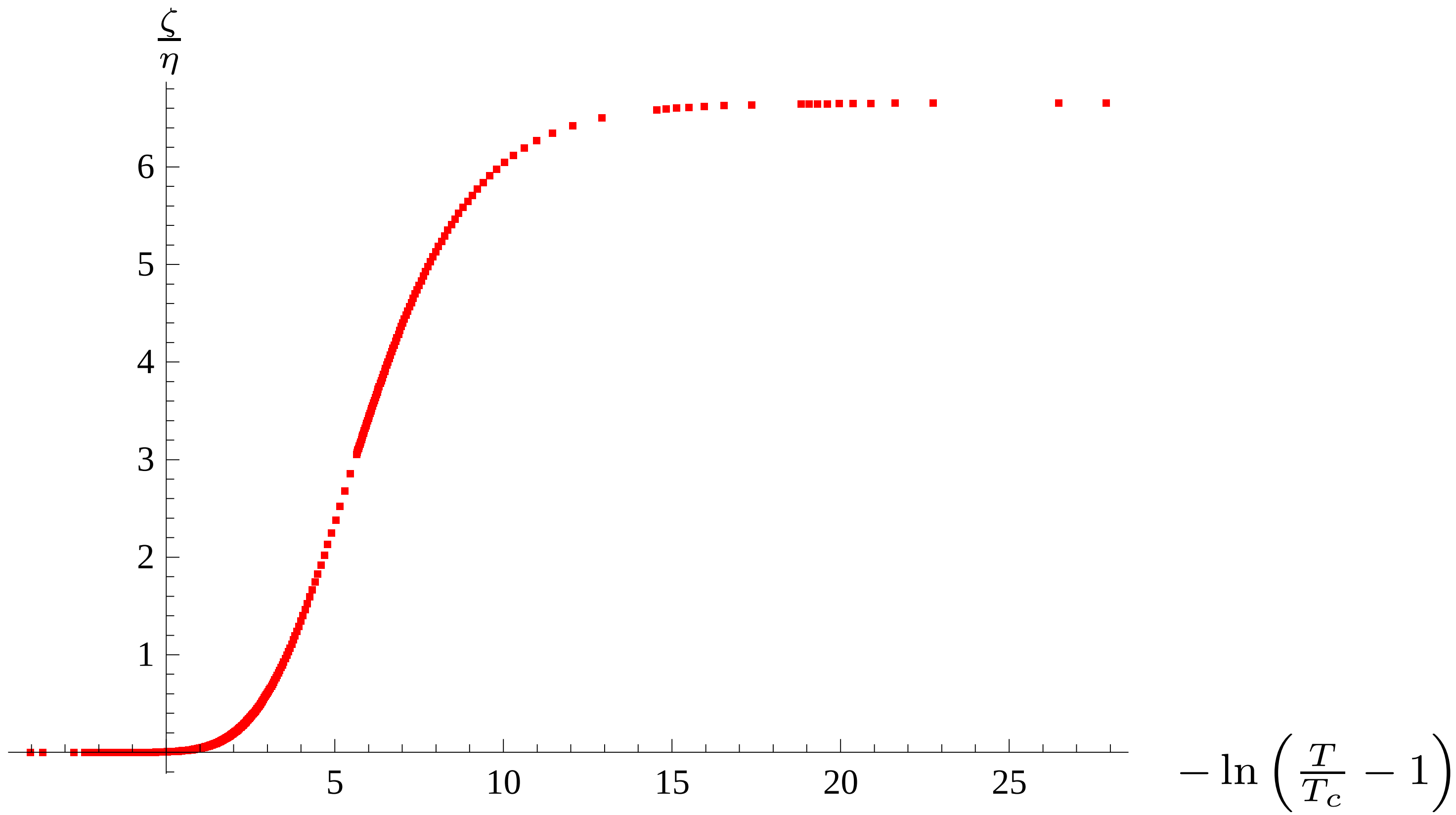


Figure 6: Ratio of viscosities $\frac{\zeta}{\eta}$ in $\mathcal{N} = 2^*$ gauge theory plasma with zero fermionic mass deformation parameter $m_f = 0$.

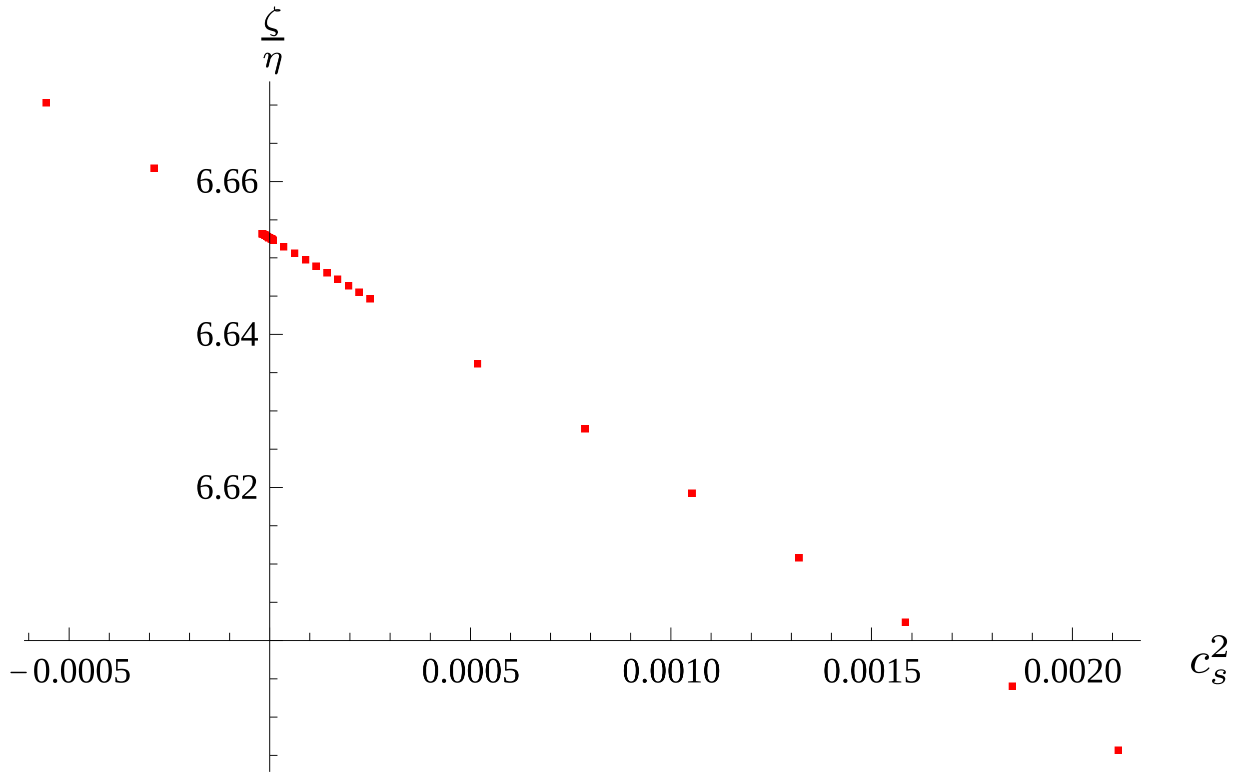


Figure 7: Ratio of viscosities $\frac{\zeta}{\eta}$ in $\mathcal{N} = 2^*$ gauge theory plasma near the critical point. Note that the critical point corresponds to $c_s^2 = 0$.

$\Longrightarrow$ Notice that the bulk viscosity is finite at the mean-field-theory critical point; the value favourably compares with Meyer's lattice simulations.

Estimates for the viscosity of QGP at RHIC. It is tempting to use the $\mathcal{N} = 2^*$ strongly coupled gauge theory plasma results to estimate the bulk viscosity of QGP produced at RHIC. For c_s^2 in the range $0.27 - 0.31$, as in QCD at $T = 1.5 T_{deconfinement}$ we find

$$\left.\frac{\zeta}{\eta}\right|_{m_f=0} \approx 0.17 - 0.61\,, \qquad \left.\frac{\zeta}{\eta}\right|_{m_b=m_f=m} \approx 0.07 - 0.15\,. \tag{1}$$

Since RHIC produces QGP close to its criticality, we believe that $m_f = 0\ \mathcal{N} = 2^*$ gauge theory model would reflect physics more accurately.

Summary on (first-order) hydrodynamic transport in non-conformal plasma

- First,

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

- In all explicit examples of gauge-string duality, for a strongly coupled plasma d spatial dimensions,

$$\frac{\zeta}{\eta} \ge 2\left(\frac{1}{d} - c_s^2\right)$$

- Bulk viscosity is finite in the vicinity of the phase transition; but is grows very rapidly:

$$\frac{d\zeta}{dT} \sim T_c^2 \left(1 - \frac{T_c}{T}\right)^{-1/2}, \qquad T \to T_c$$

Relaxation time of holographic plasma

$\Rightarrow$ Recall the effective field theory formulation of hydrodynamics:

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + \Pi^{\mu\nu} \,.$$

To simplify further discussion we consider only CFT's from now on: $\zeta = 0 \,, \mathcal{E} = 3\mathcal{P}$. To second order in the derivative expansion

$$\Pi^{\mu\nu} = \Pi_1^{\mu\nu}(\eta) + \Pi_2^{\mu\nu}(\eta, \tau_\pi, \kappa, \lambda_1, \lambda_2, \lambda_3)$$

$$= -\eta\sigma^{\mu\nu} - \eta\tau_\pi \left[\langle u \cdot \nabla \sigma^{\mu\nu} \rangle + \frac{1}{3} \left(\nabla \cdot u \right) \sigma^{\mu\nu} \right] + \mathrm{non-linear\,terms} + \cdots$$

$\Rightarrow$ It is straightforward to study dispersion relation of the linearized fluctuations in above theory

The dispersion relation of the shear channel fluctuations is given by

$$0 = -\mathfrak{w}^2\, \tau_\pi T - \frac{i\mathfrak{w}}{2\pi} + \mathfrak{q}^2\, \frac{\eta}{s}\,,$$

where $\mathfrak{w} = \omega/(2\pi T)$ and $\mathfrak{q} = q/(2\pi T)$. Now the speed with which a wave-front propagates out from a discontinuity in any initial data is governed by

$$\lim_{|\mathfrak{q}|\to\infty}\, \frac{\mathrm{Re}(\mathfrak{w})}{\mathfrak{q}}\bigg|_{[\mathrm{shear}]} = \sqrt{\frac{\eta}{s\,\tau_\pi T}} \equiv v^{front}_{[\mathrm{shear}]}\,.$$

Hence causality in this channel imposes the restriction

$$\tau_\pi T\ \geq \frac{\eta}{s}\,.$$

Notice: the first-order hydrodynamics is recovered in the limit $\tau_\pi \to 0$, so causality is always violated at this order in the derivative truncation

Similar considerations in the sound channel imposes the (more stringent) restriction

$$\tau_\pi T\ \geq 2\frac{\eta}{s}\,.$$

So, the relaxation time is *required* to restore causality of relativistic effective theory of near-equilibrium dynamics, *i.e.,* the hydrodynamics.

$\Rightarrow$ One might worry that the causality constraint on the τ_Π is obtained from the regime outside the validity of the effective hydrodynamic approximation (derivative expansion is not valid in this regime). In general, the causality of the effective hydrodynamics depends on the microscopic parameters of the theory — in the CFT case, the central charges of the theory. In some models it can be shown what once the full non-equilibrium theory is causal, it's second-order truncated (in the velocity gradients) hydrodynamic description is causal as explained above.

practical perspective:

$\Rightarrow$ Even though first-order hydrodynamics is self-consistent in its regime of applicability, the numerical hydrodynamic simulations are typically unstable. Stability is restored with the introduction of the relaxation time. In other words: the breakdown of first-order hydro arises from the modes outside its regime of applicability but the computer does not know it!

Holographic bound in τ_{eff} in supergravity approximation

$\Rightarrow$ How do we define the effective relaxation time?

The causal viscous relativistic hydrodynamics has many second order transport coefficients:

- in CFT cases - 5
- in non-CFT cases (see Romatschke) - 13

In practical simulations one usually introduces a single second-order transport coefficient (in order to limit the phenomenological parameter space). As a result, different simulations 'turn on' different combinations of the second order transport coefficients. In order to relate different hydrodynamic models, we introduce τ_{eff}, defined from the sound wave dispersion relation as follows

$$\omega = \pm c_s k - i\Gamma k^2 \pm \frac{\Gamma}{c_s}\left(c_s^2 \tau_{eff} - \frac{\Gamma}{2}\right) k^3 + \mathcal{O}(k^4)\,,$$

where c_s is the speed of the sound waves (obtained from the equation of state), and Γ is the sound wave attenuation (determined by the shear and the bulk viscosities)

$$c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}}\,, \qquad \Gamma = \left(\frac{2}{3}\frac{\eta}{\mathcal{E}+\mathcal{P}} + \frac{1}{2}\frac{\zeta}{\mathcal{E}+\mathcal{P}}\right)\,.$$

As defined, τ_{eff} is

- the relaxation time of Müller-Israel-Stewart hydrodynamics
- it coincides with τ_π of the conformal hydrodynamics
- in general non-conformal hydrodynamics of Romatschke

$$\tau_{eff} = \frac{\tau_\pi + \frac{3}{4}\frac{\zeta}{\eta}\,\tau_\Pi}{1 + \frac{3}{4}\frac{\zeta}{\eta}}\,.$$

$\Rightarrow$ It is clear how to extract the relaxation time in a particular holographic model: we simply need to compute the sound channel quasinormal dispersion relation to order $\mathcal{O}(k^3)$

Observation: in all explicit examples gauge/string duality

$$\tau_{eff}T \;\ge\; \tau_\pi^{\mathcal{N}=4}\,T \;=\; \frac{2-\ln 2}{2\pi} \equiv \tau_\pi^* T$$

where τ_π^* is the relaxation time of $\mathcal{N}=4$ plasma

τ_{eff} near the phase transition

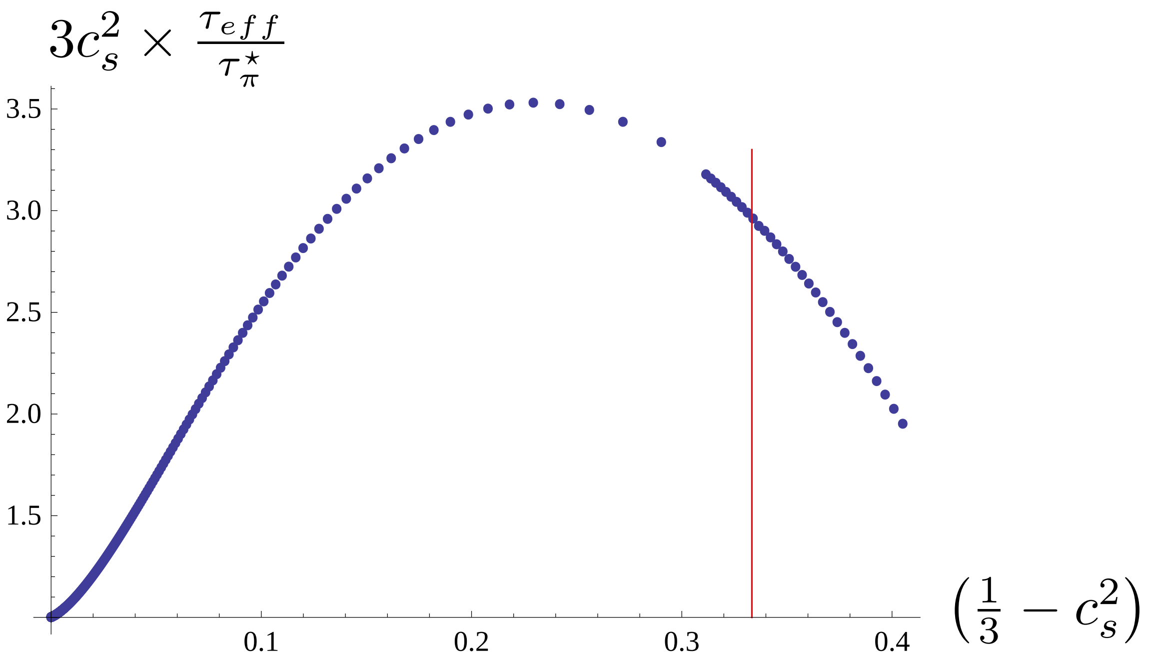


Figure 8: Effective relaxation time τ_{eff} of $\mathcal{N} = 2^*$ strongly coupled plasma. The vertical red line indicates a phase transition with vanishing speed of sound. Since $c_s^2 \propto (T - T_c)^{1/2}$ near the phase transition, $\tau_{eff} T_c \propto |1 - T_c/T|^{-1/2}$.

The critical slow-down suggested by Song and Heinz indeed happens in holographic models!

Summary on the holographic relaxation time:

- Relaxation time is necessary to reinstate causality in hydrodynamic evolution
- (I did not talk about this) Relaxation time affects (=suppresses) cavitation of the hydrodynamic evolution, in particular given that in holographic models near the phase transition

$$\tau_{eff}T_c \ \propto |1 - T_c/T|^{-1/2}$$

- Relaxation time in strongly coupled (planar) non-conformal models is longer than that in $\mathcal{N} = 4$ SYM plasma at the same temperature.
- (running ahead) the status of the relaxation time bound is exactly the same as that of the shear viscosity bound: whenever the former is violated, the latter is violated as well.

Beyond infinite 't Hooft coupling for η/s in AdS/CFT

$\Rightarrow$ Consider $\mathcal{N}=4\; SU(N)$ SYM in the planar ('t Hooft limit):

$$N\to\infty\,,\qquad g^2_{YM}\to 0\qquad \Longrightarrow \lambda\equiv N g^2_{YM}=\mathrm{const}$$

$\Rightarrow$ We would like to understand leading in $\frac{1}{\lambda}$ corrections to the (first-order) transport coefficients

- first, since the theory is conformal for all values of λ,

$$c_s^2=\frac{1}{3}\,,\qquad \zeta=0$$

and the only corrections can happen for the shear viscosity

- It can be derived from the string theory that in the planar limit, the leading $\frac{1}{\lambda}$ corrections for *any conformal plasma* (with equal central charges — see later) are described by the following effective action

$$S_5=\frac{1}{16\pi G_5}\int d^5\xi\sqrt{-g}\,\left(R+12+\gamma W+\mathcal{O}(\gamma^2)\right)$$

$$W\equiv C^{hmnk}C_{pmnq}C_h{}^{rsp}C^q{}_{rsk}+\frac{1}{2}C^{hkmn}C_{rqmn}C_h{}^{rsp}C^q{}_{rsk}$$

where C^{hmnk} is a 5d Weyl tensor, and

$$\gamma = \frac{1}{8}\xi(3)(\alpha')^3 \qquad \Longrightarrow \ (\text{for}\, \mathcal{N} = 4) \qquad \gamma = \frac{1}{8}\xi(3)\lambda^{-3/2}$$

$\Rightarrow$ We can generalize the computation of the sound channel quasinormal mode and extract shear viscosity ratio from the sound attenuation

$$w = \pm\frac{1}{\sqrt{3}} - i\Gamma k^2 + \mathcal{O}(k^3)\,, \qquad 2\pi T\Gamma = \frac{1}{3}4\pi\frac{\eta}{s}$$

$\Rightarrow$ I will outline the main steps of the computation, and emphasize some of the misconceptions that appeared in the literature recently with regards to such computations:

- First, one needs to determine the corrected equilibrium thermodynamics of the theory — it can be extracted from the α'-corrected black D3-brane solution:

$$P = \frac{1}{3}\mathcal{E} = \frac{\pi^2 N^2 T^4}{8}\ \left(1 + 15\gamma + \mathcal{O}(\gamma^2)\right)$$

- Second, we need to derive the equations for the helicity-0 metric fluctuations to order $\mathcal{O}(\gamma)$ in the $\mathcal{O}(\gamma)$ corrected black-brane solution, and set-up the gauge-invariant combination of the fluctuations. We find:

  - precisely the same combination of fluctuations as in $\mathcal{N}=2^*$ case is gauge-invariant, and decouples

$$Z = 4\frac{q}{\omega}\, H_{tz} + 2\, H_{zz} - H_{aa}\left(1 - \frac{q^2}{\omega^2}\frac{c_1' c_1}{c_2' c_2}\right) + 2\frac{q^2}{\omega^2}\frac{c_1^2}{c_2^2}\, H_{tt}$$

    where c_i are the $\mathcal{O}(g)$-correction metric warp factors

$$ds_5^2 = -c_1^2(r)\, dt^2 + c_2^2(r)\, d\vec{x}^2 + dr^2$$

  - the EOM for Z takes the form

$$A\, Z'' + B\, Z' + C\, Z = \gamma\left(D\, Z^{(iv)} + E\, Z''' + F\, Z'' + G\, Z' + H\, Z\right) + \mathcal{O}(\gamma^2)$$

    where we extracted explicit γ dependence

- boundary conditions on Z are unchanged:

$$Z(r \to 1) \sim (1-r)^{-i\mathfrak{w}/2}\,, \qquad Z(r \to 0) \sim r^2$$

where $\mathfrak{w} = w/(2\pi T)$ — it is important to include α' corrections to the BH temperature!

$\Rightarrow$ At appears that equation for Z is forth order, while I'm setting only 2 boundary conditions — this is consistent as the higher-derivative effective action derived in string theory is consistent **only** perturbatively! It simply does not make sense (in a context of effective action to study the propagation of modes which appear non-perturbatively in γ).

$\Rightarrow$ In fact, one must use lower order equations of motion for Z to eliminate the higher derivatives. Thus, using

$$A\, Z'' + B\, Z' + C\, Z = \mathcal{O}(\gamma^0)$$

we can eliminate all the derivatives on the RHS up to the first order:

$$A\, Z'' + B\, Z' + C\, Z = \gamma \left(\tilde{G}\, Z' + \tilde{H}\, Z\right) + \mathcal{O}(\gamma^2)$$

The boundary value problem for the quasinormal mode Z to order $\mathcal{G}(\gamma)$ is well defined

We find the following $\mathcal{O}(g)$ dispersion relation for the sound channel quasinormal mode:

$$\mathfrak{w} = \frac{1}{\sqrt{3}}\mathfrak{q} - i\mathfrak{q}^2\left(\frac{1}{3} + \frac{120}{3}\gamma\right) + \mathcal{O}(\mathfrak{q}^3)$$

which produces

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + 120\gamma + \mathcal{O}(\gamma^2)\right) = \frac{1}{4\pi}\left(1 + \frac{15\zeta(3)}{\lambda^{3/2}} + \mathcal{O}(\lambda^{-3})\right)$$

$\Rightarrow$ I can not cover it here, but it is possible to compute leading $\frac{1}{N}$ corrections to the shear viscosity ration for the $\mathcal{N} = 4$ SYM plasma (Myers et.al):

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{15\zeta(3)}{\lambda^{3/2}} + \frac{5}{16}\,\frac{\lambda^{1/2}}{N^2} + \cdots\right)$$

$\Rightarrow$ Notice that the KSS viscosity bound survives — does it mean that it is always true in holographic plasma?

NO! —- finite $\frac{1}{N}$ corrections

$\Longrightarrow$ A given conformal gauge theory is characterized by two different central charges c and a, defining its conformal anomaly

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4$$

where

$$E_4 = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \,, \qquad I_4 = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2$$

$\Longrightarrow$ In the planar limit

$$c = a$$

$\Longrightarrow$ In a conformal toy model of QCD we expect

$$c \neq a$$

because of the presence of fundamental matter.

Consider an effective higher-derivative model of gauge theory/string theory duality

$$S = \int d^5x\sqrt{-g}\left(\frac{1}{\kappa^2}R - \Lambda + c_1 R_{abcd}R^{abcd} + c_2 R_{ab}R^{ab} + c_3 R^2 + \mathcal{O}(R^4)\right)$$

where $\kappa^2 = 16\pi G_N$. The holographic conformal anomaly is

$$\langle T^\mu_\mu \rangle_{holographic} = \left(-\frac{l^3}{8\kappa^2} + c_2 l + 5c_3 l\right)(E_4 - I_4) + \frac{c_1 l}{2}(E_4 + I_4)$$

while Kats et.al and Brigante et.al found

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - \frac{8c_1\kappa^2}{\ell^2} + \cdots\right) = \frac{1}{4\pi}\left(1 - \frac{(c-a)}{c} + \cdots\right) = \frac{1}{4\pi}(1 - \Delta + \cdots)$$

- Notice that c_1 coefficient can come only form $R_{abcd}R^{abcd}$, and it is precisely the coefficient that corresponds to having in the dual CFT $c \neq a$. In particular R^4-terms does not effect $(c-a)$ anomaly of a CFT.

- The KSS viscosity bound is violated in a CFT whenever (c-a) is positive. The violation is under control, if $|c-a|/c \ll 1$.

Non-universal violation of the KSS bound

Consider a superconformal gauge theory. The superconformal algebra implies the existence of an anomaly-free $U(1)_R$ symmetry. It was found in Anselmi et.al that

$$c - a = -\frac{1}{16}\left(\dim G + \sum_i (\dim R_i)\,(r_i - 1)\right)$$

$$c = \frac{1}{32}\left(4\,(\dim G) + \sum_i (\dim R_i)\,(1 - r_i)\,\big(5 - 9(1 - r_i)^2\big)\right)$$

where r_i denote the R-charge of a matter chiral multiplet in the representation R_i

$\Longrightarrow$ So all we need to do is to scan through the list of available CFT's and compute $(c - a)$.

- Superconformal gauge theories with exactly marginal gauge coupling

Consider $SU(N_c)$ supersymmetric gauge theory with n_{adj} χsf in the adjoint representation, n_f flavors in the fundamental representation, n_{sym} flavors in the symmetric representation and n_{asym} flavors in the anti-symmetric representation. It is easy now to enumerate all the models with $G = SU(N_c)$ and $\Delta \ll 1$ as $N_c \to \infty$:

	$(n_{adj}, n_{asym}, n_{sym}, n_f)$	$c - a$	Δ
(a)	(3,0,0,0)	0	0
(b)	(2,1,0,1)	$\frac{3N_c+1}{48}$	$\frac{1}{4N_c} + \mathcal{O}(N_c^{-2})$
(c)	(1,2,0,2)	$\frac{3N_c+1}{24}$	$\frac{1}{2N_c} + \mathcal{O}(N_c^{-2})$
(d)	(1,1,1,0)	$\frac{1}{24}$	$\frac{1}{6N_c^2} + \mathcal{O}(N_c^{-4})$
(e)	(0,3,0,3)	$\frac{3N_c+1}{16}$	$\frac{3}{4N_c} + \mathcal{O}(N_c^{-2})$
(f)	(0,2,1,1)	$\frac{N_c+1}{16}$	$\frac{1}{4N_c} + \mathcal{O}(N_c^{-2})$

For the $Sp(2N_c)$ supersymmetric gauge theories

	(n_{adj}, n_{asym}, n_f)	$c-a$	Δ
(a)	(3,0,0)	0	0
(b)	(2,1,4)	$\frac{6N_c-1}{48}$	$\frac{1}{4N_c} + \mathcal{O}(N_c^{-2})$
(c)	(1,2,8)	$\frac{6N_c-1}{24}$	$\frac{1}{2N_c} + \mathcal{O}(N_c^{-2})$
(d)	(0,3,12)	$\frac{6N_c-1}{16}$	$\frac{3}{4N_c} + \mathcal{O}(N_c^{-2})$

$\Longrightarrow$ The are no models in this class with orthogonal gauge groups

- $\mathcal{N}=2$ superconformal fixed points from F-theory

Consider N D3-branes probing an F-theory singularity generated by n_7 coincident (p,q) 7-branes, resulting in a constant dilaton. As $N \to \infty$,

$$c-a = \frac{1}{4}N\,(\delta-1) - \frac{1}{24}\,, \qquad \Delta = \frac{\delta-1}{N\delta} + \mathcal{O}(N^{-2})$$

where δ is a definite angle characterizing an F-theory singularity with a symmetry group $\mathcal{G}$

$\mathcal{G}$	H_0	H_1	H_2	D_4	E_6	E_7	E_8
n_7	2	3	4	6	8	9	10
δ	6/5	4/3	3/2	2	3	4	6

Notice that in all cases $0 < \Delta \ll 1$ as $N \to \infty$.

$\Longrightarrow$ In all examples presented the KSS bound is violated since $(c-a) > 0$

$\Longrightarrow$ There many more CFT's with $c \neq a$. For them, however, $c - a \sim c$ and so we can not say anything reliable about KSS bound. Curiously though, we did not find a single CFT with $c \neq a$ so that $(c-a) < 0$.

Is there a bound on $\frac{\eta}{s}$?

$\Rightarrow$ In realistic holographic models (derivable from string theory) corrections to η/s are due to higher derivative terms, thus these corrections are necessarily perturbative.

$\Rightarrow$ To address the bound on η/s one considers *models* of AdS/CFT correspondence:

- we use the rules of holography
- we do not care whether or not the model is embeddable in string theory

$\Rightarrow$ A nice model of this type is a Gauss-Bonnet gravity:

$$S_5 = \frac{1}{2\ell_P^3} \int d^5x \sqrt{-g} \left[\frac{12}{L^2} + R + \frac{\lambda_{GB}}{2} L^2 \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right]$$

This model is solvable for any λ_{GB} *i.e.,* , we can find exact (analytic) black hole solution and study its near-equilibrium properties.

$\Rightarrow$Computing the boundary stress-energy tensor of $\mathcal{I}$ we identify a dual gauge theory as a CFT with central charges $\{c, a\}$ given by

$$c = \frac{\pi^2}{2^{3/2}} \frac{L^3}{\ell_P^3} \left(1 + \sqrt{1 - 4\lambda_{GB}}\right)^{3/2} \sqrt{1 - 4\lambda_{GB}}$$

$$a = \frac{\pi^2}{2^{3/2}} \frac{L^3}{\ell_P^3} \left(1 + \sqrt{1 - 4\lambda_{GB}}\right)^{3/2} \left(3\sqrt{1 - 4\lambda_{GB}} - 2\right)$$

or

$$\frac{c - a}{c} = 2\left(\frac{1}{\sqrt{1 - 4\lambda_{GB}}} - 1\right)$$

$\Rightarrow$ It is straightforward to study dispersion relation of the quasinormal modes of the GB black holes — these quasinormal modes are dual to linearized fluctuations in plasma (the shear and the sound channel modes)

For the shear viscosity one finds (Brigante *et.al*)

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - 4\lambda_{GB}\right]$$

For the relaxation time:

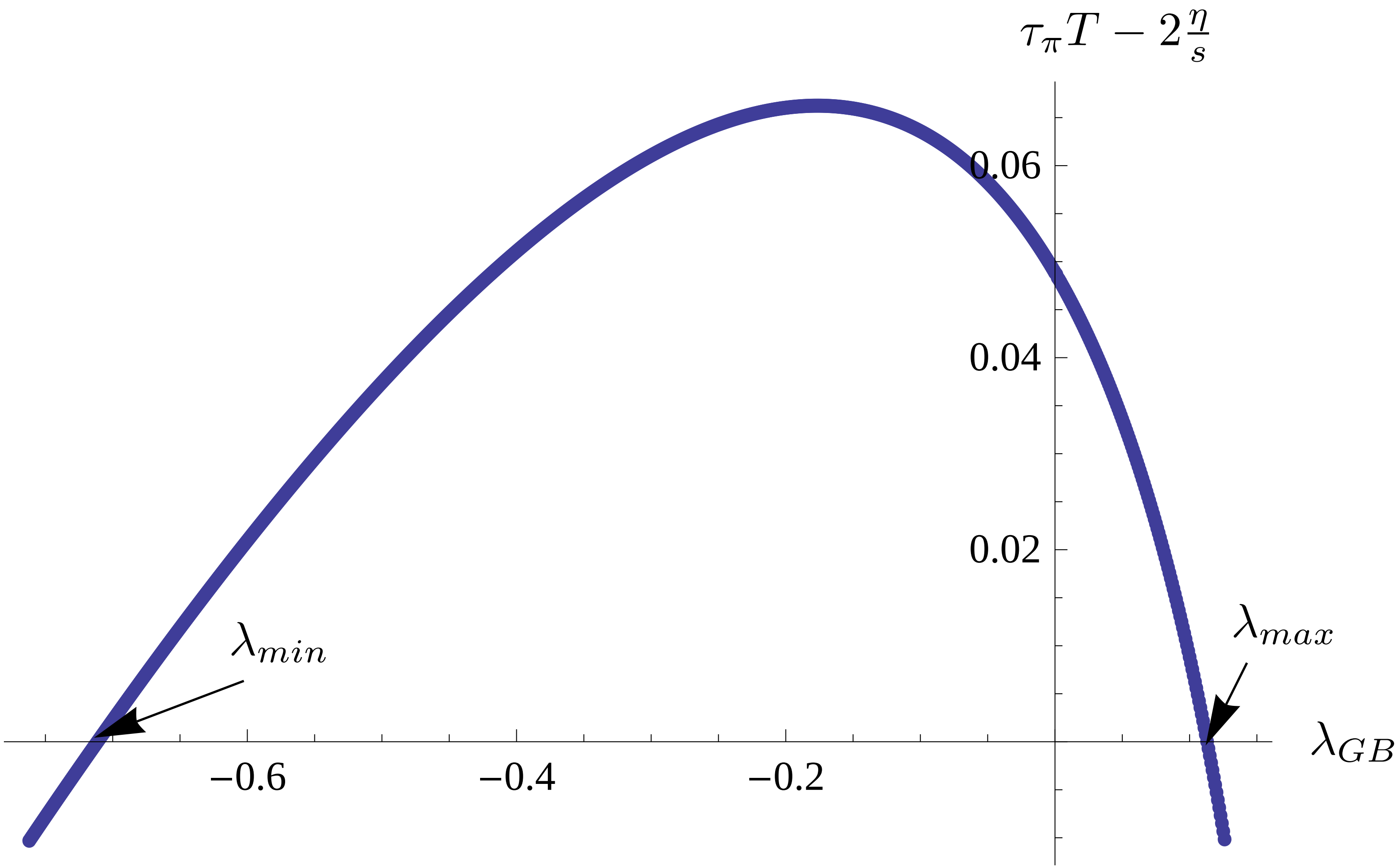


Figure 9: Causality of the second-order Gauss-Bonnet hydrodynamics is violated once $\tau_\Pi T < 2\frac{\eta}{s}$. Thus, $\lambda_{GB} \in [\lambda_{min}, \lambda_{max}]$, where $\lambda_{min} = -0.711(2)$ and $\lambda_{max} = 0.113(0)$.

$\Rightarrow$ Previous analysis were based on the effective theory of the successive local-velocity derivative expansion in GB plasma

$\Rightarrow$ However, given the gravity dual we can study dispersion relation of quasinormal modes in GB plasma without any reference to a hydrodynamic expansion!

In this way we find that causality is violated in GB CFT plasma, unless

$$-\frac{7}{36} \leq \lambda_{GB} \leq \frac{9}{100}$$

which translates into the following constrain on the CFT central charges

$$-\frac{1}{2} < \frac{c-a}{c} < \frac{1}{2}$$

and correspondingly, introduces the lower bound on the shear viscosity

$$\frac{\eta}{s} \geq \frac{1}{4\pi}\,\frac{16}{25}$$

$\Rightarrow$ Unfortunately, it appears that making a holographic model more complicated, it is possible to lower η/s even further — the existence of the low bound on η/s in holographic models is still an open question

Summary on corrections to shear viscosity in holographic conformal models

- There are generically 2 types of corrections:
  - finite (exactly) marginal coupling corrections — t' Hooft coupling corrections
  - nonplanar corrections due to $(c-a) \neq 0$

- The former (universally) satisfy KSS bound, while the latter (universally) violate it
- In some simple holographic models there is a lower bound on η/s, induced by the violation of the microcausality in the theory
- It is possible to engineer (seemingly consistent) holographic models with arbitrarily low η/s

⇒ Suppose we want to model the hydrodynamics of strongly coupled QGP by a holographic model of a CFT plasma

⇒ Consider boost-invariant hydro simulations either within MIS framework or conformal hydro framework by Baier et.al.:

$$\partial_\tau \epsilon = -\frac{4\epsilon}{3\tau} + \frac{\Phi}{\tau}$$

$$\tau_\Pi \partial_\tau \Phi = \frac{4\eta}{3\tau} - \Phi - \frac{4\tau_\Pi}{3\tau} \Phi$$

or

$$\partial_\tau \epsilon = -\frac{4\epsilon}{3\tau} + \frac{\Phi}{\tau}$$

$$\tau_\Pi \partial_\tau \Phi = \frac{4\eta}{3\tau} - \Phi - \frac{4\tau_\Pi}{3\tau} \Phi - \frac{\lambda_1}{2\eta^2} \Phi^2$$

To specify the flow uniquely (up to initial conditions) we need 3 (or 4) independent parameters, $\mathcal{A}_i$:

$$P = P_{SB}\, \frac{3}{4} \mathcal{A}_1 \,, \qquad \frac{\eta}{s} = \frac{1}{4\pi}\, \mathcal{A}_2 \,, \qquad \tau_\Pi T = \frac{1}{2\pi}\, \mathcal{A}_3 \,, \qquad \frac{\lambda_1 T}{\eta} = \frac{1}{2\pi}\, \mathcal{A}_4$$

Question: does there exist a (computationally reliable) model of holographic CFT plasma what would reproduce $\mathcal{A}_i$?

Assume that this hCFT has:

- an exactly marginal coupling — analog of t'Hooft couping — γ
- different central charges — $\delta = (c-a)/a$
- a stress-energy 3-point parameter — t_4 (typically $t_4 \propto \delta^2$)

$\Rightarrow$ We expect that γ or t_4 is zero — t_4 vanishes identically for SUSY plasma (at zero temperature), and when the plasma is non-SUSY, we don't expect any marginal couplings — thus we have two tunable parameters, which for consistency all must be small.

Then...

$$\mathcal{A}_1 = \left\{1 + \frac{9}{4}\delta + \frac{3}{8}\delta^2 + \frac{1}{180}t_4 + 15\gamma + \mathcal{O}\left(\delta^3, \delta t_4, t_4^2, \gamma^2\right)\right\}$$

$$\mathcal{A}_2 = \left\{1 - \delta + \frac{7}{4}\delta^2 - \frac{4}{45}t_4 + 120\gamma + \mathcal{O}\left(\delta^3, \delta t_4, t_4^2, \gamma^2\right)\right\}$$

$$\mathcal{A}_3 = \left\{2 - \ln 2 - \frac{11}{8}\delta - \frac{125}{64}\delta^2 - \frac{13}{540}t_4 + \frac{375}{2}\gamma + \mathcal{O}\left(\delta^3, \delta t_4, t_4^2, \gamma^2\right)\right\}$$

$$\mathcal{A}_4 = \frac{1}{2\pi}\left\{1 - \frac{1}{4}\delta - \frac{73}{32}\delta^2 - \frac{1}{135}t_4 + 215\gamma + \mathcal{O}\left(\delta^3, \delta t_4, t_4^2, \gamma^2\right)\right\}$$

’sQGP = hCFT’ becomes a falsifiable test!