# QCD string and scattering amplitudes

by

**Yuri Makeenko** (ITEP, Moscow)

Based on:

Y. M. Phys. Rev. D **83**, 026007 (2011) [arXiv:1012.0708 [hep-th]]

Phys. Lett. B **699**, 199 (2011) [arXiv:1103.2269 [hep-th]]

Extending: Y. M., Poul Olesen

- Phys. Rev. Lett. **102**, 071602 (2009) [arXiv:0810.4778 [hep-th]]
- Phys. Rev. D **80**, 026002 (2009) [arXiv:0903.4114 [hep-th]]
- Phys. Rev. D **82**, 045025 (2010) [arXiv:1002.0055 [hep-th]]
- JHEP **08**, 095 (2010) [arXiv:1006.0078 [hep-th]]

P. Buividovich, Y.M. Nucl. Phys. B **834**, 453 (2010) [arXiv:0911.1083 [hep-th]]

# QCD string vs fundamental string

- QCD string is formed at distances larger than confinement scale (by fluxes of the gluon field)

- It can be described by an effective string theory (like ANO vertices)

- Perturbative QCD works at small distances (asymptotic freedom), the effective string theory works at large distances (confinement)

- Scattering of quark bound states (mesons or baryons) rather than gauge fields or gravitons

- Massive vector state ($\rho$-meson) rather than massless vector $\Longrightarrow$ the need of off-shell amplitudes

- QCD string lives in $d = 4$ rather than critical dimension 26 or 10 (consistent quantization in $d = 4$)

- QCD string should describe vast amount of experimental data (linear Regge trajectories)

# Motivations since 1979

Y.M., Migdal (1979)

QCD string is not Nambu–Goto but the asymptote of large Wilson loops is universal:

classical string

$$W(C) \overset{\text{large } C}{\propto} \mathrm{e}^{-K S_{\min}(C)} \qquad \Longrightarrow \quad \text{the area law = confinement}$$

semiclassical correction (bosonic string) Lüscher, Symanzik, Weisz (1980)

$$W(C) \overset{\text{plane } C}{\propto} \mathrm{e}^{-K S_{\min}(C) + \frac{\#}{24\pi}\int \mathrm{d}^2 w \left(\partial_a \ln\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right|\right)^2} \qquad w(z) : \text{UHP} \Rightarrow D$$

$$\overset{\text{rectangle}}{\propto} \mathrm{e}^{-KRT + \frac{\#}{24}\frac{\pi T}{R}} \qquad \Longrightarrow \quad \text{the Lüscher term}$$

for rectangle with $T \gg R$

$\# = d - 2$ for (bosonic string)

Ambjorn, Olesen, Peterson (1984)

De Forcrand, Schierholz, Schneider, Teper (1985)

Universality of the $1/R$ term owing to Lüscher's roughening

$$\left\langle x_\perp^2 \right\rangle \propto \alpha' \log(R^2/\alpha') \gg \alpha'$$

Next orders in $1/R$ are *not* universal.

# Consistent noncritical string

Arvis (1983)

String quantization is consistent for long string: Light-cone (Virasoro) quantization of open string with the Dirichlet b.c. (of the length $R$):

$$E_n = \sqrt{(KR)^2 + \left(n - \frac{d-2}{24}\right) 2\pi K} \qquad K = \frac{1}{2\pi\alpha'}$$

The spectrum is consistent for

$$R > R_{\text{critical}} = \sqrt{\frac{d-2}{12K}\pi}$$

Otherwise $\Longrightarrow$ tachyonic instability.

Lorentz anomaly

$$\left[L_{1i}, L_{1j}\right] = -\mathrm{i}L_{ij} + \frac{(d-26)}{6(p^0 + KR)^2} \sum_{m=1}^{\infty} m \left(a^i_{-m} a^j_m - a^j_{-m} a^i_m\right)$$

vanishes in $d = 26$.

Olesen (1985): 1) the Lorentz anomaly vanishes for long strings.
2) $\frac{d-2}{24} \longrightarrow \frac{d-2}{16}$ for NS superstring ($\longrightarrow 0$ for R superstring).

# Effective string theory

Polchinski, Strominger (1991)

Closed string winding along a compact direction of large radius $R$ is described by nonpolynomial action

$$S_{\text{eff}} = 2K \int \mathrm{d}^2 z \, \partial X \cdot \bar{\partial} X - \frac{\beta}{2\pi} \int \mathrm{d}^2 z \, \frac{\partial^2 X \cdot \bar{\partial}^2 X}{\partial X \cdot \bar{\partial} X} + \dots \qquad \beta = \frac{26-d}{12}$$

It can be analyzed order by order in $1/R$ by expanding around the classical solution

$$X^\mu_{\text{cl}} = (e^\mu z + \bar{e}^\mu \bar{z})\, R \qquad e \cdot e = \bar{e} \cdot \bar{e} = 0 \quad e \cdot \bar{e} = -1/2$$

It looks like the Liouville action in the Polyakov formulation expressed (modulo total derivatives and the constraints) via an induced metric

$$\mathrm{e}^{\varphi_{\text{ind}}} = 2\, \partial X \cdot \bar{\partial} X$$

(in the conformal gauge), which is not treated independently.

This effective string theory has been analyzed using the conformal field theory technique order by order in $1/R$, revealing the Arvis spectrum of the Nambu–Goto string in $d$-dimensions.

# Effective string theory (cont.)

Conformal symmetry is maintained in noncritical dimension:

$$\delta X^\mu = \epsilon(z)\partial X^\mu - \frac{\beta\alpha'}{4}\partial^2\epsilon(z)\frac{\bar\partial X^\mu}{\partial X\cdot\bar\partial X} + \text{c.c.}$$

It transforms $X^\mu$ nonlinearly and the corresponding conserved energy-momentum tensor is

$$T_{zz} = -\frac{1}{\alpha'}\partial X\cdot\partial X + \frac{\beta}{2}\frac{\partial^3 X\cdot\bar\partial X}{\partial X\cdot\bar\partial X} + \mathcal{O}(R^{-2})$$

Expanding around the classical solution $X^\mu = X^\mu_{\text{cl}} + Y^\mu_{\text{q}}$, we obtain

$$T_{zz} = -\frac{2R}{\alpha'}e\cdot\partial Y_{\text{q}} - \frac{1}{\alpha'}\partial Y_{\text{q}}\cdot\partial Y_{\text{q}} - \frac{\beta}{R}\bar e\cdot\partial^3 Y_{\text{q}} + \mathcal{O}(R^{-2})$$

The central charge is determined by

$$\langle T_{zz}(z_1)T_{zz}(z_2)\rangle = \frac{d+12\beta}{2(z_1-z_2)^4} + \mathcal{O}\left((z_1-z_2)^{-2}\right)$$

to be $d+12\beta = 26$ and is cancelled by ghosts at any $d$.

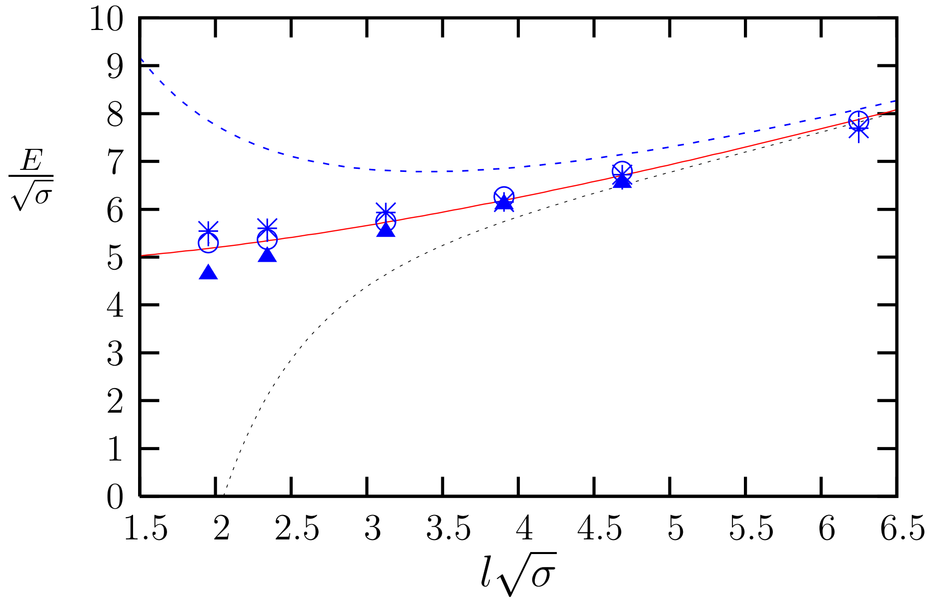
# Monte-Carlo data for energy levels

Athenodorou, Bringoltz, Teper (2010)

3+1 SU(3) LGT: closed winding string (flux tube) of length $l = 2\pi R$

$$E_n = \sqrt{(Kl)^2 + \left(n - \frac{d-2}{24}\right) 8\pi K} \qquad 8 \Longrightarrow 2 \quad \text{for open string}$$

absolutely beautifully describes the lattice calculations




$$\sqrt{K} l_{\text{crit}} = \sqrt{\frac{d-2}{3}\pi} \approx 1.44$$

Figure 20: Energies of the lightest $q = 0$ excited states in SU(3) at $\beta = 6.0625$. States are $J^{P_t P_l} = 0^{++}$, ▲, $2^{++}$, $\circ$, $2^{-+}$, $\star$. Solid line is Nambu-Goto, dotted is universal expansion to $O(1/l^3)$, and dashed is expansion to $O(1/l)$ (Lüscher correction).

# Objective of the research

Motivation: Consistency of the effective string theory approach in noncritical dimension $d < 26$ for long strings.

Question: Whether or not (and if yes then when) is it possible to apply the effective string theory ideology to consistent calculations of meson scattering amplitudes in QCD?

Answer: Regge behavior of scattering amplitudes at high energy and fixed momentum transfer under a few controllable approximations (nonperturbative stringy effects dominate).

The results are then reliable but cease to be applicable for larger momentum transfer $-t \lesssim -2\text{GeV}^2$ (where perturbative QCD dominates).

# Contents of the talk

- Classical string
  - stepwise momentum-space loops
  - Douglas' minimization
  - polygonal light-like contour
  - induced metric and minimal surface
- Semiclassical fluctuations as the Lüscher term
  - mapping UHP onto rectangle
  - effective open-string theory and conformal invariance in $d<26$
  - scattering amplitudes in effective string theory
  - Regge behavior and semiclassical Reggeon intercept $(d-2)/24$
  - mean field in effective string theory is exact
- Generalization to off-shell
  - reparametrization path integral
  - measure on Diff $\mathbb{R}$ and Lévy flights
  - consistent off-shell amplitudes
- Application to large-$N$ QCD
  - meson scattering amplitudes via Wilson loops
  - perturbative QCD vs QCD string: energy dependence of $\alpha_{\text{eff}}(t)$
- Conclusion and Outlook

# Reparametrization path integral

Polyakov (1997)

Wilson loop of large size in large-$N$ QCD = string disk amplitude integrated over reparametrizations of the boundary contour:

$$W[x(\cdot)] = \int \mathcal{D}_{\text{diff}} t(s)\, \mathrm{e}^{-KS[x(t)]}$$

i.e. over functions $t(s)$ with $t'(s) \geq 0$ (string tension $K = 1/2\pi\alpha'$).

Douglas algorithm for solving the Plateau problem Douglas (1931) (finding the minimal surface) is to minimize the boundary functional

$$S[x(t)] = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_1 \mathrm{d}s_2}{(s_1 - s_2)^2} \left[x(t(s_1)) - x(t(s_2))\right]^2$$

with respect to reparametrizations $t(s)$ ($\dot{t}(s) \geq 0$).

This representation can be derived for critical strings (bosonic string in $d = 26$ or superstring in $d = 10$).

Area law for asymptotically large $C$ (or very large $K$) $\Longrightarrow$ a saddle point in the reparametrization path integral at $t(s) = t_*(s)$.

Zig-zag or backtracking symmetry holds for the minimal area.

# Large loops and minimal area

Gaussian fluctuations around the saddle-point $t_*(s)$ result in a pre-exponential factor

$$W\left[x(\cdot)\right] \stackrel{\text{large loops}}{=} F\left[\sqrt{K}x(\cdot)\right] \mathrm{e}^{-KS_{\text{min}}[x(\cdot)]}\left[1+\mathcal{O}\left((KS_{\text{min}})^{-1}\right)\right],$$

which is contour dependent

Asymptotic area law is recovered modulo the pre-exponential which is not essential for large loops.

More subtle effects (such as the Lüscher term) resides in the pre-exponential factor, coming from fluctuations around $t_*(s)$

$$t(s) = t_*(s) + \frac{\beta(s)}{\sqrt{KS_{\text{min}}}}$$

For a $R \times T$ rectangle Y.M., Olesen (2010)

$$F\left[\text{rectangle}\right] \propto \mathrm{e}^{\pi T/R} \qquad \text{for } T \gg R$$

reproducing the Lüscher term for bosonic string in $d = 26$.

# Functional Fourier transformation

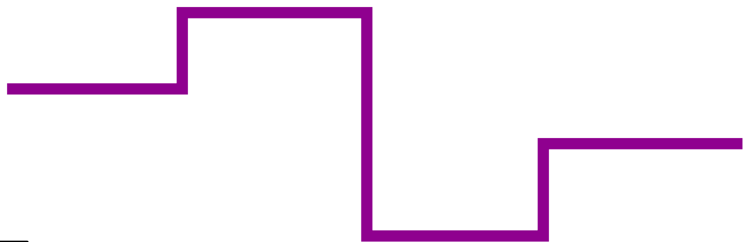
Migdal (1986)

Scattering amplitudes are given by a reparametrization-invariant functional Fourier transformation

$$A[p(\cdot)] = \int \mathcal{D}x \; \mathrm{e}^{\mathrm{i}\int p\cdot \mathrm{d}x} \, W[x(\cdot)]$$

of disk amplitude (to be identified with the Wilson loop) for piecewise constant momentum-space loop $p(t)$:

$$p(t) = p_i \qquad \text{for } t_i < t < t_{i+1}$$



$$\dot{p}(t) = -\sum_i \Delta p_i \delta(t - t_i) \qquad \text{with} \quad \Delta p_i \equiv p_{i-1} - p_i$$

representing $M$ momenta of (all incoming) particles.
Then momentum conservation is automatic while an (infinite) volume $V$ is produced, say, by integration over $x_0 = x_M$.

The Fourier transformation of string vertex operators is reproduced:

$$\int \mathrm{d}t \, p(t) \cdot \dot{x}(t) = -\int \mathrm{d}t \, \dot{p}(t) \cdot x(t) = \sum_i \Delta p_i \cdot x_i$$

# Momentum-space disk amplitude

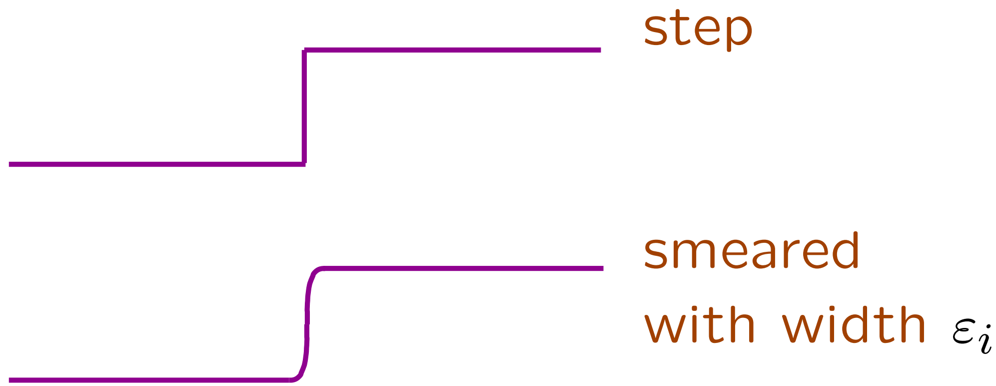
After the Gaussian path integration (with $s$-independent determinant)

$$A[p(\cdot)] = \int \mathcal{D}s(t) \exp\left(\alpha' \int_{-\infty}^{+\infty} \mathrm{d}t_1 \int_{-\infty}^{+\infty} \mathrm{d}t_2\, \dot{p}(t_1)\cdot\dot{p}(t_2) \ln|s(t_1)-s(t_2)|\right)$$

It looks like the disk amplitude: scattering amplitude $=$ Wilson loop

provided that

$$C_* := \;\; x^\mu(t) = \frac{1}{K} p^\mu(t)$$

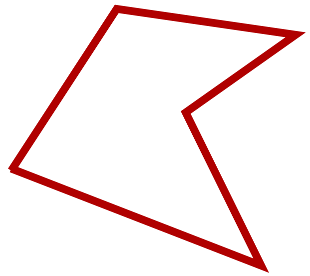



Smeared step-wise (with boundary Liouville field $\varphi(t_i)$ for covariance)

$$p^\mu(t) = \frac{1}{\pi}\sum_i \Delta p_i^\mu \arctan\frac{(t-t_i)}{\varepsilon_i} \xrightarrow{\varepsilon_i\to 0} \frac{1}{2}\sum_i \Delta p_i^\mu \,\mathrm{sign}\,(t-t_i) \qquad \varepsilon_i = \varepsilon\, \mathrm{e}^{-\varphi(t_i)}$$

$\Longrightarrow$ polygon with vertices

$$x_i^\mu = \frac{1}{K} p_i^\mu \qquad\qquad x_i^\mu - x_{i-1}^\mu = \frac{1}{K}\Delta p_i^\mu$$



where $K = 1/2\pi\alpha'$ is string tension.

# Wilson-loop/scattering-amplitude duality

Similar with WL/SA duality in $\mathcal{N}=4$ SYM — Alday, Maldacena (2007); Drummond, Korchemsky, Sokatchev (2008)

for Regge kinematical regime with

$$s \gg -t \gtrsim -\Delta p_i^2$$

For $-t \gg -\Delta p_i^2$ it looks like light-like edges
(otherwise simplifications for tachyonic $\Delta p_i^2 = 1/\alpha'$).

$2 \to 2$ kinematics (center-mass frame, $u$-channel scattering for both $s<0$ and $t<0$)

$$\begin{aligned}
\Delta p_1^\mu &= (E, p, 0, 0) \\
\Delta p_2^\mu &= (-E, -p\cos\theta, -p\sin\theta, 0) \\
\Delta p_3^\mu &= (E, -p, 0, 0) \\
\Delta p_4^\mu &= (-E, p\cos\theta, p\sin\theta, 0)
\end{aligned}$$

scattering angle $\theta$: $\quad \cos\theta = \dfrac{t}{s+t}, \qquad (1-\cos\theta) = \dfrac{s}{s+t}.$

# Douglas' minimization

The minimum of Douglas' integral is reached for $t(s)$ obeying

$$\int \mathrm{d}t_1 \frac{\dot{x}(t)\cdot\dot{x}(t_1)}{[s(t)-s(t_1)]} = 0,$$

where $s(t)$ denotes inverse to $t(s)$.

For smeared stepwise $p^\mu(t)$ (resulting in polygonal $x^\mu(t)$) Douglas' minimization is trivially satisfied at intermediate points, when $t$ is not close to $t_i$'s, because then $\dot{x}^\mu(t)=0$. For $t=t_i$ we have

$$\sum_{j\neq i} \frac{2\Delta p_i \cdot \Delta p_j}{s_i - s_j} + \pi \sum_j \left(\Delta p_j^2 - \frac{1}{\alpha'}\right) \left\langle \frac{\partial G\left(s_j, s_j\right)}{\partial s_i} \right\rangle = 0$$

$$\left\langle G\left(s_j, s_j\right)\right\rangle = \frac{\int \mathcal{D}_{\mathrm{diff}} s \, G\left(s_j, s_j\right)}{\int \mathcal{D}_{\mathrm{diff}} s} \overset{\text{Lovelace}}{=} \frac{1}{\pi} \ln \frac{(s_{j+1}-s_{j-1})}{(s_{j+1}-s_j)(s_j-s_{j-1})\varepsilon}$$

with the reparametrization path integral going over functions obeying $s(t_i)=s_i$ which are zero modes of the Douglas minimization.

Great simplification occurs for tachyonic edges $\Delta p_i^2 = 1/\alpha'$, but similar formulas appear for $\Delta p_i^2 \gg 1/\alpha'$ with $t = -\Delta p_1^2 - 2\Delta p_1 \cdot \Delta p_4 - \Delta p_4^2$.

# Douglas' minimization (cont.)

For $\Delta p_i^2$ light-like (or tachyonic), Douglas' minimization gives

$$\sum_{j\neq i} \frac{\Delta p_i \cdot \Delta p_j}{s_i - s_j} = 0$$

Only $M-3$ from these $M$ equations are independent because of the $PSL(2;\mathbb{R})$ projective invariance. The minimal surface does not depend on three values.

For $M = 4$ we can set $s_1 = 0$, $s_3 = 1$, $s_4 = \infty$ in the usual way $\Longrightarrow$

$$s_{2\,*} = \frac{s}{s+t}$$

— the saddle point of the Veneziano amplitude at large $-s, -t$.

$C_*$ bounds the minimal surface of the area

$$KS_{\text{min}} = \alpha' s \ln \frac{s}{s+t} + \alpha' t \ln \frac{t}{s+t} \stackrel{s \gg t}{\rightarrow} -\alpha' t \ln \frac{s}{t}$$

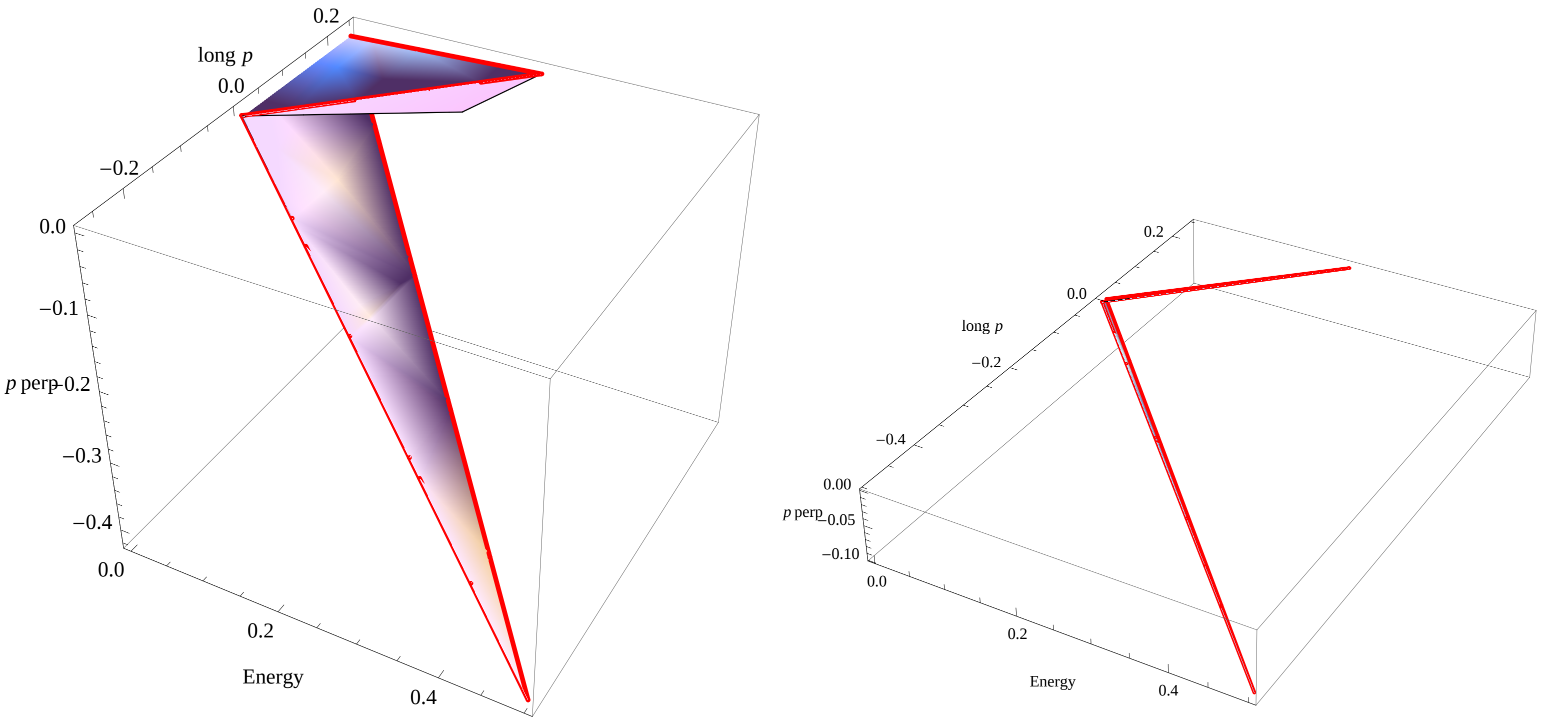
whose exponential reproduces the classical Regge behavior of the scattering amplitude:

$$A(s,t) = \mathrm{e}^{-KS_{\text{min}}} \propto s^{\alpha' t}$$

# Minimal surface

The minimal surface spanned by the rectangle with stepwise $p^\mu(t)$ is given by the harmonic function

$$X^\mu(x,y) = \frac{1}{\pi K} \sum_i \Delta p_i^\mu \arctan \frac{(x - s_i)}{y} \qquad s_i = s(t_i)$$




This minimal surface is depicted for $\theta = 1.0$ (left) and $\theta = 0.2$ (right).

# Momentum-space Lüscher term

Schwarz–Christoffel map of the upper half-plane onto a rectangle:

$$\begin{aligned}\omega(z) &= \int_{s_2}^{z} \frac{\mathrm{d}x}{\sqrt{(s_4 - x)(s_3 - x)(x - s_2)(x - s_1)}} \\ &= \frac{2}{\sqrt{s_{42}s_{31}}} F\left(\sqrt{\frac{s_{31}(z - s_2)}{s_{32}(z - s_1)}}, \sqrt{\frac{s_{32}s_{41}}{s_{42}s_{31}}}\right)\end{aligned}$$

$$R = \frac{2}{\sqrt{s_{42}s_{31}}} K\left(\sqrt{1 - r}\right), \qquad T = \frac{2}{\sqrt{s_{42}s_{31}}} K\left(\sqrt{r}\right)$$

where

$$r = \frac{s_{43}s_{21}}{s_{42}s_{31}} \qquad s_{ij} = s_i - s_j$$

is the projective-invariant ratio. Therefore,

$$\frac{T}{R} = \frac{K\left(\sqrt{r}\right)}{K\left(\sqrt{1 - r}\right)}$$

is projective invariant.

# Momentum-space Lüscher term (cont.)

To calculate the momentum-space Lüscher term, we expand

$$X^\mu(\omega_1,\omega_2) = X^\mu_{\mathrm{cl}}(\omega_1,\omega_2) + Y^\mu_{\mathrm{q}}(\omega_1,\omega_2),$$

where $X^\mu_{\mathrm{cl}}$ is harmonic and obeys the boundary condition, so $Y^\mu_{\mathrm{q}}$ has the mode expansion

$$Y^\mu_{\mathrm{q}}(\omega_1,\omega_2) = \sum_{m,n} \chi^\mu_{mn} \sin\frac{\pi m \omega_1}{R} \sin\frac{\pi n \omega_2}{T}$$

to be substituted in the quadratic part of the action

$$S^{(2)} = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\omega\, \partial Y_{\mathrm{q}}\cdot\bar\partial Y_{\mathrm{q}} + \frac{\beta}{\pi R^2}\int \mathrm{d}^2\omega \left(\partial^2 Y_{\mathrm{q}}\cdot\bar e\right)\left(e\cdot\bar\partial^2 Y_{\mathrm{q}}\right) + \mathcal{O}\left((R)^{-3}\right)$$

Using the asymptotes for $-s \ll -t \Longrightarrow T \gg R$:

$$K\left(\sqrt{r}\right) \overset{r\to 1}{\to} \frac{1}{2}\ln\frac{16}{1-r}, \qquad K\left(\sqrt{1-r}\right) \overset{r\to 1}{\to} \frac{\pi}{2},$$

it is now clear that each set of modes results in the Lüscher term

$$\frac{\pi T}{24R} = \frac{1}{24}\ln\frac{16s}{t} \qquad r = r_* = 1-\frac{t}{s}$$

# Semiclassical Reggeon intercept

There are $(d-2)$ such sets of modes for bosonic string, so their contribution to the intercept of the linear Regge trajectory is

$$\alpha(0) = \frac{d-2}{24}$$

The expansion of the effective string theory goes for the scattering amplitude in the parameter

$$\left(\ln \frac{1}{1-r}\right)^{-1} = \left(\ln \frac{s}{t}\right)^{-1},$$

like it was $T^{-1}$ for the closed winding string. Therefore, the Regge behavior

$$A \propto \mathrm{e}^{\alpha(t)\ln(s/t)} \qquad \text{is similar to} \qquad W \propto \mathrm{e}^{-TV(R)}$$

The semiclassical Regge trajectory of the effective string theory in $d < 26$ can be computed for UHP like in Durhuus, Olesen, Petersen (1984) for the Polyakov string. Now the same result emerges as

$$\alpha(0) = 1 + \frac{d-26}{24} = \frac{d-2}{24}$$

# Large-$d$ limit for static potential

O. Alvarez (1981)

Path integral of the Nambu–Goto string is calculable as $d \to \infty$ by a saddle-point technique.

Mean values of the induced metric (diagonal $\sigma_{ab}$)

$$\sigma_0 = \langle \partial_0 X(z) \cdot \partial_0 X(z) \rangle , \qquad \sigma_1 = \langle \partial_1 X(z) \cdot \partial_1 X(z) \rangle ,$$

and the Lagrange multipliers (diagonal $\alpha_{ab}$)

$$\alpha_0 = \langle \alpha_0(z) \rangle , \qquad \alpha_1 = \langle \alpha_1(z) \rangle$$

The mean-field action reads for a $T \times R$ rectangle

$$S_{\text{mf}} = \frac{TR}{2\pi\alpha'} \left( \sqrt{(1+\sigma_0)(1+\sigma_1)} - \frac{1}{2}(\sigma_0\alpha_0 + \sigma_1\alpha_1) \right) - \frac{\pi(d-2)}{24} \frac{T}{R} \left( \frac{\alpha_0}{\alpha_1} \right)^{1/2}$$

The first term comes from the classical Nambu–Goto action and the second one is the semiclassical Lüscher–Symanzik–Weisz term

There are no higher-order terms as $(d-2) \sim R^2/\alpha' \to \infty$, when fluctuations of the mean-field values are suppressed.

# Large-$d$ limit for static potential (cont.)

Minimizing $S_{\text{mf}}$ with respect to $\sigma_0$, $\sigma_1$, $\alpha_0$, $\alpha_1$, one finds

$$\sigma_0 = \lambda(1-2\lambda)^{-1} \qquad \sigma_1 = -\lambda \qquad \lambda = \frac{\pi^2(d-2)\alpha'}{12R^2}$$

$$\alpha_0 = (1-2\lambda)^{1/2} \qquad \alpha_1 = (1-2\lambda)^{-1/2}$$

so at the minimum

$$S_{\text{mf}*} = \frac{T}{2\pi\alpha'}\sqrt{R^2 - \pi^2\frac{(d-2)}{6}\alpha'}.$$

Many results indicate this formula to be exact order by order in $1/R$. This apparently means that the mean field turns out to be *exact* not only for large $d$, as usual, but also for any $d > 2$.

Quite similar to scattering amplitudes after conformal mapping of rectangle onto upper half-plane.

# Mean-field approximation for static potential

Similarity with the saddle point in the large-$d$ limit by O. Alvarez (1981)

Classical configuration in world-sheet parametrization:

$$X^1_{\rm cl} = \frac{\omega_1}{\omega_R} R \qquad X^2_{\rm cl} = \frac{\omega_2}{\omega_T} T \qquad\qquad \omega_1, \omega_2 \in \omega_R \times \omega_T \text{ rectangle}$$

$\omega_R$, $\omega_T$ change under reparametrizations (variational parameters).

The mean-field action (with account for the Lüscher term)

$$S_{\rm mf} = \frac{1}{4\pi\alpha'}\left( R^2 \frac{\omega_T}{\omega_R} + T^2 \frac{\omega_R}{\omega_T} \right) - \frac{\pi(d-2)}{24}\frac{\omega_T}{\omega_R}$$

Minimization with respect to $\omega_T/\omega_R$ reproduces the square root

$$\left(\frac{\omega_T}{\omega_R}\right)_* = \frac{T}{\sqrt{R^2 - \pi^2 \frac{(d-2)}{6}\alpha'}} \qquad\qquad S_{\rm mf*} = \frac{T}{2\pi\alpha'}\sqrt{R^2 - \pi^2 \frac{(d-2)}{6}\alpha'}$$

For upper half-plane parametrization

$$\left(\frac{\omega_T}{\omega_R}\right) = \frac{K\left(\sqrt{r}\right)}{K\left(\sqrt{1-r}\right)} \qquad\qquad r = \frac{s_{43}s_{21}}{s_{42}s_{31}}$$

$=$ the Grötzsch modulus which is monotonic in $r$.

# Mean-field approximation for scattering amplitude

For the scattering amplitude the Mandelstam variables $s$ and $t$ play the role of $T$ and $R$:

$$S_{\rm mf} = \alpha' s \ln r + \alpha' t \ln(1-r) + \frac{(d-2)}{24} \ln(1-r) \qquad \text{valid as } r \to 1$$

where we have included the associated momentum-space Lüscher term Janik (2001), Y.M. (2011). Minimizing, we have

$$r_* = 1 - \frac{\alpha' t + (d-2)/24}{\alpha' s}$$

which results in the linear Regge trajectory

$$\alpha(t) = \frac{(d-2)}{24} + \alpha' t.$$

It is obtained for large $d$ but is expected to be exact for any $d$:

- Semiclassical cancellation of reparametrization and ghosts (any $d$)
- Exact results in $d=26$ and $d=2$

Quadratic fluctuations around this mean field are stable for $\alpha(t) < 0$:

$$\alpha' t < -\frac{d-2}{24}$$

# Invariant regularization and Liouville field

The Gaussian exponent

$$
\begin{aligned}
&-\pi \int_{-\infty}^{+\infty} \mathrm{d}t_1\, \mathrm{d}t_2\, \dot{p}(t_1)\cdot\dot{p}(t_2)\, G\left(s(t_1), s(t_2)\right) \\
&\quad = \sum_{k\neq l} \Delta p_k \cdot \Delta p_l \log|s_k - s_l| - \pi \sum_j \Delta p_j^2 G(s_j, s_j)
\end{aligned}
$$

For an invariant regularization $G(s_j, s_j)$ involves the Liouville field

Polyakov (1981)

$$
G(s_i, s_j) = -\frac{1}{\pi} \ln|s_i - s_j| \qquad \text{for } |s_i - s_j| \gg \varepsilon_i, \varepsilon_j
$$

$$
G(s_j, s_j) \longrightarrow G_\varepsilon(s_j, s_j) = \frac{1}{\pi}\log\frac{1}{\varepsilon} + \frac{1}{2\pi}\varphi(s_j)
$$

For critical bosonic string (in $d = 26$): Aoyama, Dhar, Namazie (1986)

$$
A = \int \mathcal{D}\varphi(s) \int \prod_m \mathrm{d}s_m\, \mathrm{e}^{\varphi(s_m)/2 - \pi\alpha' \Delta p_m^2 G(s_m, s_m)} \prod_{j\neq m} |s_j - s_m|^{\alpha' \Delta p_j \cdot \Delta p_m}
$$

the path integration over $\varphi(s)$ — boundary Liouville field decouples only for tachyonic scalar, massless vector, etc.

# Path integrals over reparametrizations

The measure on $Diff(\mathbb{R})$

$$\int\limits_{\substack{s(\tau_0)=s_0 \\ s(\tau_f)=s_f}} \mathcal{D}_{\text{diff}} s(\tau) \cdots = \lim_{N\to\infty} \int_{s_0}^{s_f} \prod_{j=1}^{N-1} \int_{s_0}^{s_{j+1}} \mathrm{d}s_j \, \frac{1}{(s_{j+1}-s_j)} \frac{1}{(s_1-s_0)} \cdots$$

is invariant under reparametrizations

$$s \to t(s)\,, \qquad t(s_0)=s_0\,, \qquad t(s_f)=s_f\,, \qquad \frac{\mathrm{d}t}{\mathrm{d}s} \geq 0$$

Integration goes over $(N-1)$ subordinated values

$$s_0 \leq \cdots \leq s_{i-1} \leq s_i \leq \cdots \leq s_N = s_f$$

Discretizing $s' = \exp[-\varphi]$ that relates reparametrizations to the boundary value of the Liouville field $\varphi$ by $s_i - s_{i-1} = \exp[-\varphi_i] \Longrightarrow$

$$\int_{s_0}^{s_f} \mathcal{D}_{\text{diff}} s \cdots = \lim_{N\to\infty} \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}\varphi_i \, \delta^{(1)}(s_f - s_0 - \sum_{j=1}^{N} \mathrm{e}^{-\varphi_j}) \cdots$$

with the only restriction on $\varphi_i$'s given by the delta-function.

# Path integrals over reparametrizations (cont.)

Regularization of (logarithmically) divergent integral

$$\frac{1}{(s_i - s_{i-1})} \longrightarrow \frac{1}{\Gamma(\delta_i)(s_i - s_{i-1})^{1-\delta_i}} \qquad \text{all } \delta_i = \delta$$

Main integral for the integration at the intermediate point $s_i$

$$\int_{s_{i-1}}^{s_{i+1}} \mathrm{d}s_i \, \frac{\Gamma^{-1}(\delta_i)\Gamma^{-1}(\delta_{i+1})}{(s_{i+1} - s_i)^{1-\delta_{i+1}}(s_i - s_{i-1})^{1-\delta_i}} = \frac{\Gamma^{-1}(\delta_i + \delta_{i+1})}{(s_{i+1} - s_{i-1})^{1-\delta_i-\delta_{i+1}}}$$

This is an analogue of the well-known formula

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}s_i}{\sqrt{2\pi}} \frac{\mathrm{e}^{-(s_f - s_i)^2/2\nu_1}}{\sqrt{\nu_1}} \frac{\mathrm{e}^{-(s_i - s_0)^2/2\nu_2}}{\sqrt{\nu_2}} = \frac{\mathrm{e}^{-(s_f - s_0)^2/2(\nu_1+\nu_2)}}{\sqrt{(\nu_1 + \nu_2)}}$$

used for calculations with the usual Wiener measure.

The functional limit is when $N \to \infty$ with $N\delta \to 0$:

$$\int_{s_0}^{s_N = s_f} \mathcal{D}^{(N)}_{\text{diff}} s = \frac{1}{\Gamma(N\delta)} \frac{1}{(s_N - s_0)^{1-N\delta}} \stackrel{N\delta \to 0}{\longrightarrow} N\delta \frac{1}{\left(s_f - s_0\right)}$$

reproducing the projective-invariant result.

# Reparametrizations as Lévy stochastic process

Buividivich, Y.M. (2009)

What trajectories are typical in path integral over reparametrizations?

Subordinated stochastic process (gamma-subordinator) with PDF

$$P\left(\Delta s_i\right) = \frac{1}{\Gamma(\delta)\left(\Delta s_i\right)^{1-\delta}} \qquad \delta > 0 \text{ is a time step}$$

$$\mathrm{d}s_f \int_{s_0}^{s_f} \mathcal{D}^{(N)}_{\text{diff}}\, s \quad \text{— propagator from } s_0 \text{ to } \left[s_f, s_f + \mathrm{d}s_f\right]$$

during the time $\tau = N\delta$

Scaling variable

$$z = \tau \ln \frac{1}{(s_f - s_0)} \Longrightarrow \frac{\tau \mathrm{d}s_f}{\left(s_f - s_0\right)^{1-\tau}} = \mathrm{d}z\, \mathrm{e}^{-z},$$

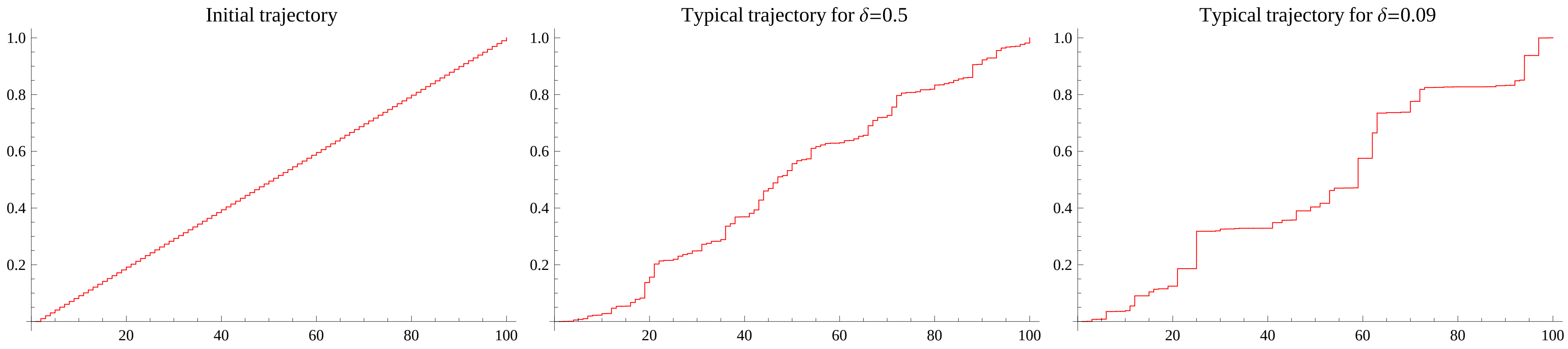
Scaling with

$$(s_f - s_0) \sim \mathrm{e}^{-1/\tau} \Longrightarrow \text{Hausdorff dimension}=0$$

supersedes $(s_f - s_0)^2 \sim \tau$ for the Brownian motion (whose $d_H = 2$).

# Sample trajectories in path integral

Typical trajectories for the gamma-subordinator (obtained by Metropolis–Hastings algorithm)




Lévy's flights are seen in the right figure.
Their origin is that $P(\Delta s_i)$ is very large at small $\Delta s_i$ $\Longrightarrow$ most of $\Delta s_i$'s are small.
Then some of $\Delta s_i$ has to be large to satisfy the boundary condition.

Hausdorff dimension decreases from 1 to 0 (left to right)

(Horowitz, 1968)

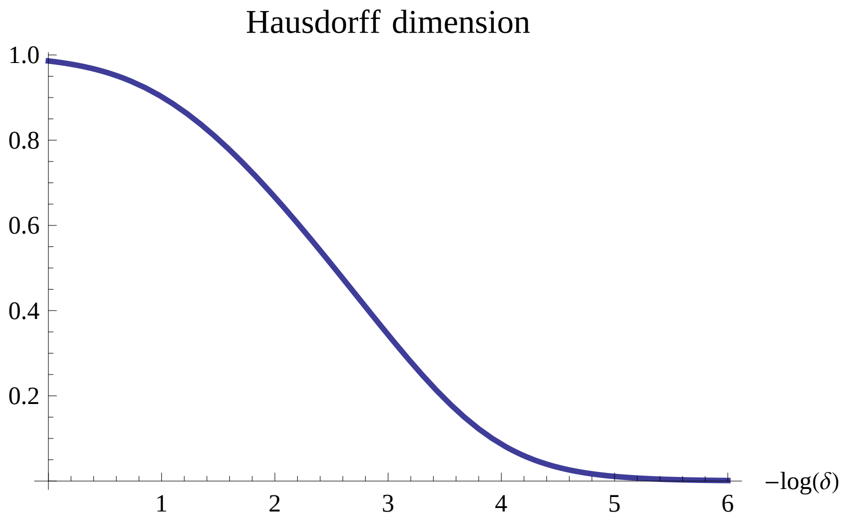
# Hausdorff dimension of sample trajectories

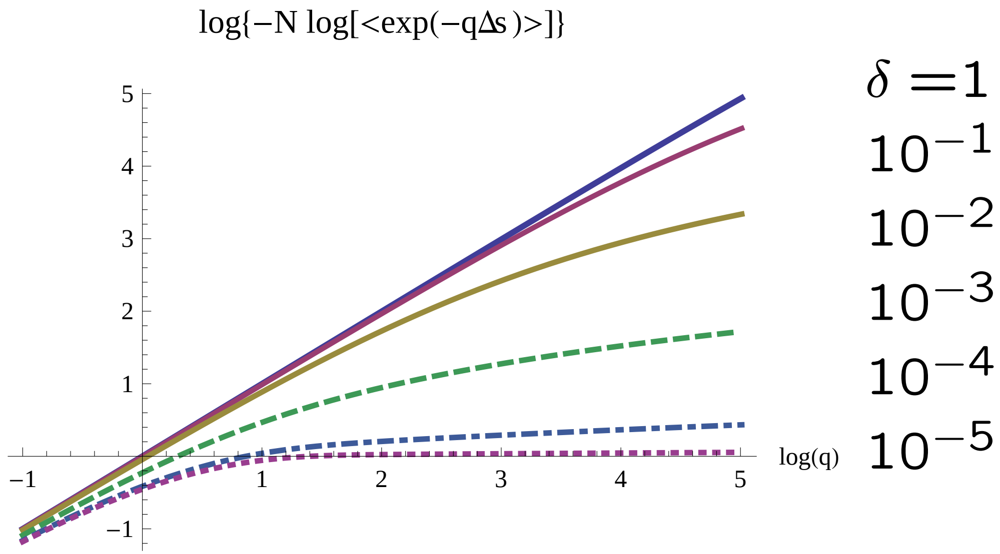
Hausdorff dimension of the discretized process is determined by its characteristic function (Lévy–Khintchine)

$$\left\langle e^{-q\Delta s_i} \right\rangle = {}_1F_1(\delta, \delta N; -q)$$

as

$$d_H = \lim_{q\to\infty} \frac{\ln\left(-N \ln \left\langle e^{-q\Delta s_i}\right\rangle\right)}{\ln q}$$




Hausdorff dimension versus $\ln(1/\delta)$ (left) from the slope of the lines (right).

It decreases from 1 for $\delta \gtrsim 1$ to 0 for $\delta N \to 0$

# Ambiguities of the measure

More symmetric discretization of the measure

$$\mathcal{D}_{\text{diff}} s = \prod_i \mathrm{d} s_i \frac{(s_{i+1} - s_{i-1})}{(s_{i+1} - s_i)(s_i - s_{i-1})}$$

Lovelace choice

(every multiplier is projective invariant).

It is the one which results in the consistent off-shell (Lovelace) amplitudes with $\alpha(0) = (d-2)/24$.

The results do not change if next-to-neighbor discretization

$$(s_{i+1} - s_i) \Longrightarrow (s_{i+n} - s_i)/n$$

$\Longrightarrow$ a continuum limit in spite of the discontinuities.

It looks like different universality classes (with different $\alpha(0)$).

# Consistent off-shell amplitudes

Y.M., Olesen (2010)

Reparametrization path integral results in

$$\left\langle G\left(s_j, s_j\right)\right\rangle = \frac{\int \mathcal{D}_{\text{diff}} s \, G\left(s_j, s_j\right)}{\int \mathcal{D}_{\text{diff}} s} \overset{\text{Lovelace}}{=} \frac{1}{\pi} \ln \frac{(s_{j+1} - s_{j-1})}{(s_{j+1} - s_j)(s_j - s_{j-1})\varepsilon}$$

With the proper choice of discretization of the measure (each multiplier is projective invariant)

$$\mathcal{D}^{(N)}_{\text{diff}} s = \prod_{i=1}^{N} \frac{\mathrm{d} s_i \, (s_{i+1} - s_{i-1})}{(s_{i+1} - s_i)(s_i - s_{i-1})}$$

this gives the scattering amplitude

$$A(\Delta p_1, \ldots, \Delta p_M)$$
$$= \int\limits_{s_{i-1} < s_i} \prod_i \mathrm{d} s_i \prod_{k \neq l} |s_k - s_l|^{\alpha' \Delta p_k \cdot \Delta p_l} \prod_j \left( \frac{|s_j - s_{j-1}| \, |s_{j+1} - s_j|}{|s_{j+1} - s_{j-1}|} \right)^{\alpha' \Delta p_j^2 - 1}$$

where the integration over $s_i$ (Koba–Nielsen variables) is inherited from the path integral over reparametrizations.

# Consistent off-shell amplitudes (cont.)

Di Vecchia, Frau, Lerda, Sciuto (1988)

This is a known off-shell tree string amplitude originally obtained from the Lovelace string vertex operator (instead of the usual one).
It is consistent off-shell and invariant under $PSL(2;\mathbb{R})$ projective transformations (subgroup of reparametrizations)

$$s \Rightarrow \frac{as+b}{cs+d} \qquad \text{with} \quad ad-bc=1$$

For 4 scalars this reproduces projective-invariant off-shell amplitude

$$A(\Delta p_1, \Delta p_2, \Delta p_3, \Delta p_4) = \int_0^1 \mathrm{d}x\, x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1},$$

where $\alpha(t) = 1 + \alpha' t$ – linear Regge trajectory – and

$$s = -(\Delta p_1 + \Delta p_2)^2, \qquad t = -(\Delta p_2 + \Delta p_3)^2$$

are usual Mandelstam's variables (for Euclidean metric).

The tachyonic condition $\alpha' \Delta p_j^2 = 1$ has not to be imposed. The on-shell Veneziano tachyon amplitudes is obtained by setting $\alpha' \Delta p_j^2 = 1$

# Application to QCD

Y.M., Migdal (1981)

Green's functions of $M$ colorless composite quark operators

$$\bar q(x_i)q(x_i) \qquad \bar q(x_i)\gamma_5 q(x_i) \qquad \bar q(x_i)\gamma_\mu q(x_i) \qquad \bar q(x_i)\gamma_\mu\gamma_5 q(x_i)$$

are given by the sum over Wilson loops passing via $x_i$ $(i=1,\ldots,M)$

$$G \equiv \left\langle \prod_{i=1}^{M} \bar q(x_i)q(x_i) \right\rangle_{\text{conn}} = \sum_{\text{paths} \ni \{x_1,\ldots,x_M\equiv x_0\}} J[x(\tau)]\, W[x(\tau)]$$

The weight for the path integration is

$$J[x(\tau)] = \int \mathcal{D}k(\tau)\ \text{sp P}\, \mathrm{e}^{\mathrm{i}\int_0^{\mathcal{T}} \mathrm{d}\tau\, [\dot x(\tau)\cdot k(\tau) - \gamma\cdot k(\tau)]}$$

for spinor quarks of mass $m$ and scalar operators or

$$J[x(\tau)] = \mathrm{e}^{-\frac{1}{2}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot x^2(\tau)} = \int \mathcal{D}k(\tau)\ \mathrm{e}^{\int_0^{\mathcal{T}} \mathrm{d}\tau\, [\mathrm{i}\dot x(\tau)\cdot k(\tau) - k^2(\tau)/2]}$$

for scalar quarks. $\tau$ is the proper time.

The Wilson loop $W(C)$ is in pure Yang–Mills at large $N$ (or quenched). For finite $N$, correlators of several Wilson loops are present.

# Application to QCD (cont.)

Y.M., Olesen (2008)

QCD scattering amplitude = functional Fourier transform

$$A\,(\Delta p_1,\ldots,\Delta p_M) = \sum_{\text{paths}} \mathrm{e}^{\mathrm{i}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot{x}(\tau)\cdot p(\tau)}\, J[x(\tau)]\, W[x(\tau)]$$

for piecewise constant momentum-space loop $p(\tau)$ as before.

Substituting the area-law and interchanging the integrals over $x(\tau)$ (Gaussian) and $s(\tau)$, we get

$$A\,(\{\Delta p_m\}) \propto \int_0^\infty \mathrm{d}\mathcal{T}\, \mathcal{T}^{M-1}\, \mathrm{e}^{-m\mathcal{T}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_{M-1}}{1+s_{M-1}^2} \prod_{i=1}^{M-2} \int_{-\infty}^{s_{i+1}} \frac{\mathrm{d}s_i}{1+s_i^2}$$
$$\times \int \mathcal{D}k(t)\; \mathrm{sp}\; P\, \mathrm{e}^{-\mathrm{i}\mathcal{T}\int \mathrm{d}t\,\gamma\cdot k(t)/(1+t^2)}\, W\Big[x(t) = \frac{1}{K}\,(p(t)+k(t))\Big]$$

For small $m$ and/or large $M$, the integral over $\mathcal{T}$ is dominated by large $\mathcal{T} \sim (M-1)/m$ and the path integral over $k$ factorizes:

$$A\,(\{\Delta p_m\}) \propto W\Big[x(t) = \frac{1}{K}p(t)\Big]$$

It is just the same as the Lovelace-type string amplitude!

## Justification of large $\mathcal{T}$ as $m \to 0$

Path integral over $x(\tau)$ (for scalar quarks) can be calculated via

mode expansion $$x^\mu(\tau) = x_0^\mu + \sum_{n=1}^{\infty} \left( a_n^\mu \cos \frac{2\pi\tau}{\mathcal{T}} + b_n^\mu \sin \frac{2\pi\tau}{\mathcal{T}} \right) :$$

$$\int\limits_{x(0)=x(\mathcal{T})} \mathcal{D}x(\tau)\, \mathrm{e}^{-\frac{1}{2}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot{x}^2(\tau) - \frac{K}{2} \dot{x}*G*\dot{x}} = \prod_{n=1}^{\infty} \left[ 2\pi \left( \frac{1}{\mathcal{T}} n^2 + Kn \right) \right]^{-d}$$

$\zeta$-function regularization gives

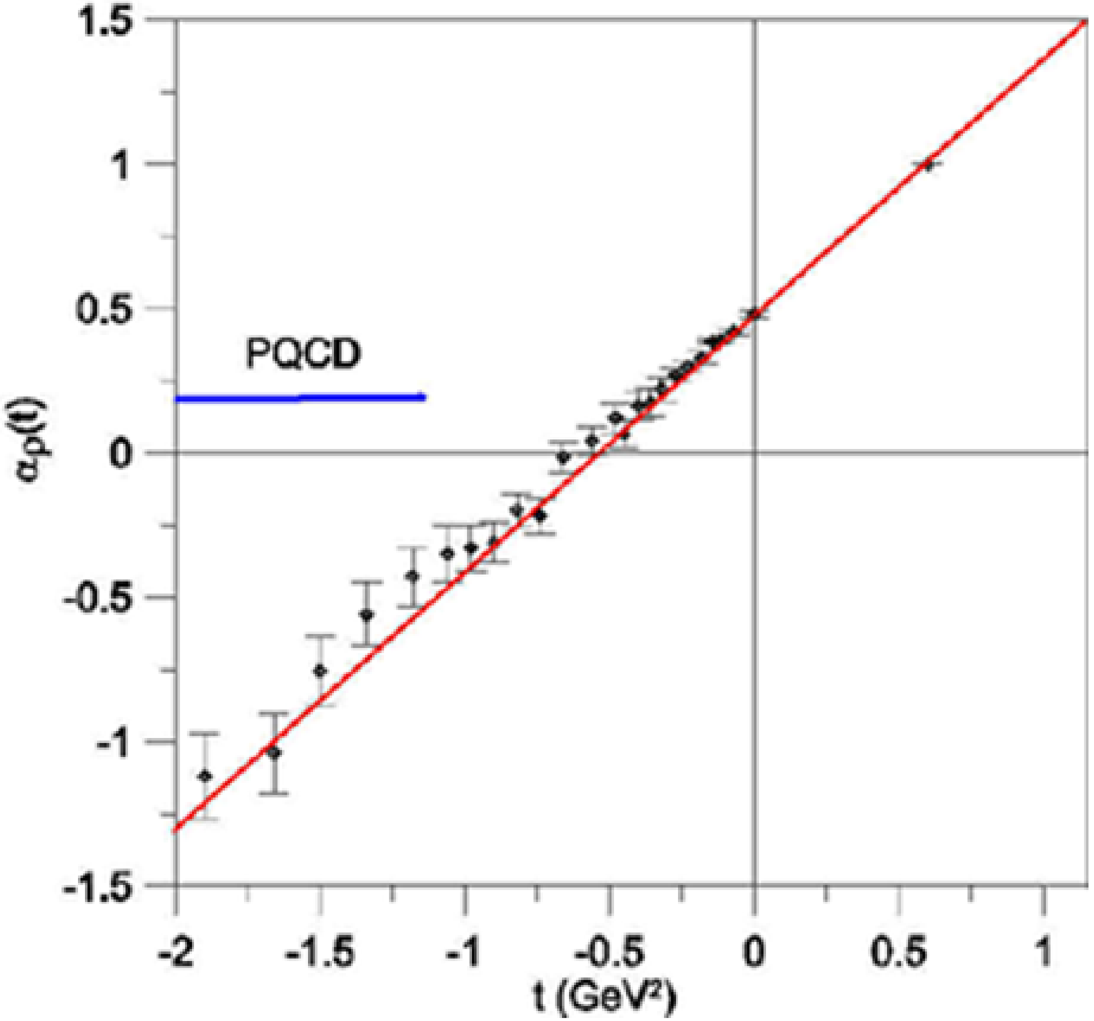
$$\prod_{n=1}^{\infty} A = A^{\zeta(0)} = A^{-1/2} \qquad \prod_{n=1}^{\infty} n = \sqrt{2\pi}$$

$$\prod_{n=1}^{\infty} \left[ 2\pi \left( \frac{1}{\mathcal{T}} n^2 + Kn \right) \right]^{-d} \quad \begin{array}{ll} \xrightarrow{\mathcal{T}\to 0} & (2\pi\mathcal{T})^{-d/2} \\ \xrightarrow{\mathcal{T}\to\infty} & K^{d/2} \end{array}$$

Miniconclusion: large $\mathcal{T}$ are essential in QCD perturbation theory only for $M > 4$, but are essential non-perturbatively: $\int \mathrm{d}\mathcal{T}\, \mathcal{T}^{M-1}\, \mathrm{e}^{-m\mathcal{T}} \ldots$

# Effective $\rho$-trajectory and pQCD prediction

The figure taken from A. B. Kaidalov, hep-ph/0612358




exclusive process

$$\pi^- p \to \pi^0 n$$

It is hard to believe that pQCD reggeization is relevant.

# Separation of pQCD and QCD string

Reggeization of $\bar{q}q$ in pQCD is due to double logarithms

Kirschner, Lipatov (1983)

$\mathcal{T}$ is restricted from above by $\tau_{\max} \sim 1/K$ to separate the contribution from small loops associated with pQCD. It plays the role of an infrared cutoff in pQCD, rather than a usual transverse mass $\mu$.

With the double logarithmic accuracy:

$$\text{pQCD ladders} = \frac{2I_1\left(\omega \ln s\tau_{\max}\right)}{\omega \ln s\tau_{\max}} - 1 \qquad \omega = \sqrt{\frac{g^2(t)C_F}{2\pi^2}} \approx .5$$

$g^2 \Rightarrow g^2(t)$ ? because of charge renormalization. Then asymptotically

$$\text{pQCD ladders} \propto \left(s\tau_{\max}\right)^{\omega(t)}$$

standard pQCD $=$ $\tau_{\max} = \infty$ $\Longrightarrow$ IR regularization by $\mu$.

Wilson loop = either QCD (small loops) or QCD string (large loops) like either $\mathcal{N} = 4$ SYM or IIB superstring in AdS/CFT.
The total amplitude = pQCD (this one) + QCD string (as before).
At finite $s$ the relative coefficient is of most importance.

# pQCD + QCD string effective Reggeon trajectory

Simple model of pQCD (small loops) + QCD string (large loops):

$$A = \frac{2I_1\left(0.5\ln(\alpha' s)\right)}{0.5\ln(\alpha' s)} - 1 + R\left(\alpha' s\right)^{0.5+\alpha' t}$$

Plot of the effective Reggeon trajectory (for various $s$)

$$\alpha_{\text{eff}}(t) = \frac{\ln(A/R)}{\ln(\alpha' s)}$$

$R \approx 20$

$\alpha' s = 10^{40}$

$\alpha' s = 10^{16}$

$\alpha' s = 10^{9}$

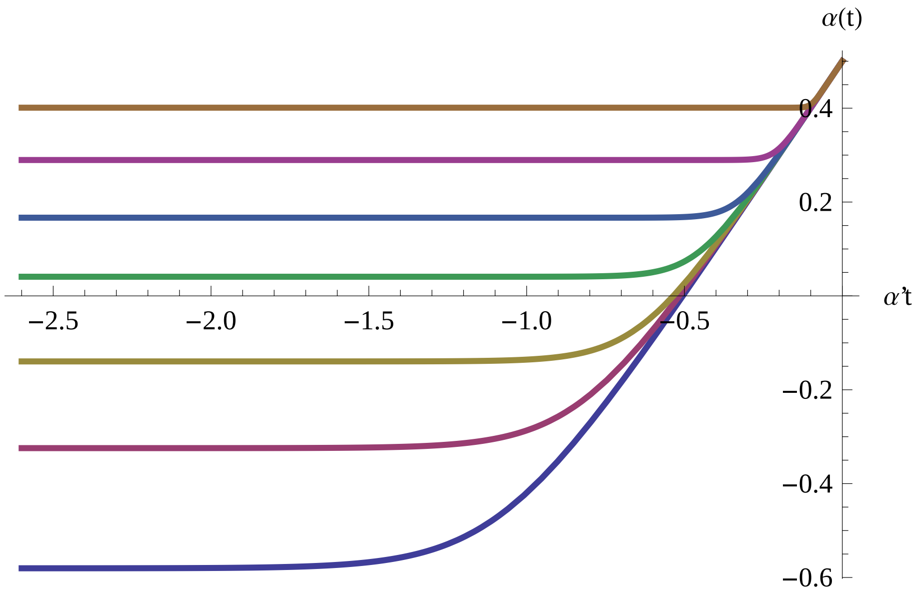
$\alpha' s = 10^{6}$

$\alpha' s = 10^{4}$

$\alpha' s = 10^{3}$

$\alpha' s = 400$

Reggeon trajectory in pQCD + QCD string



# Experimental data for $\alpha_\rho(t)$

Kennett *et al.* (1986)

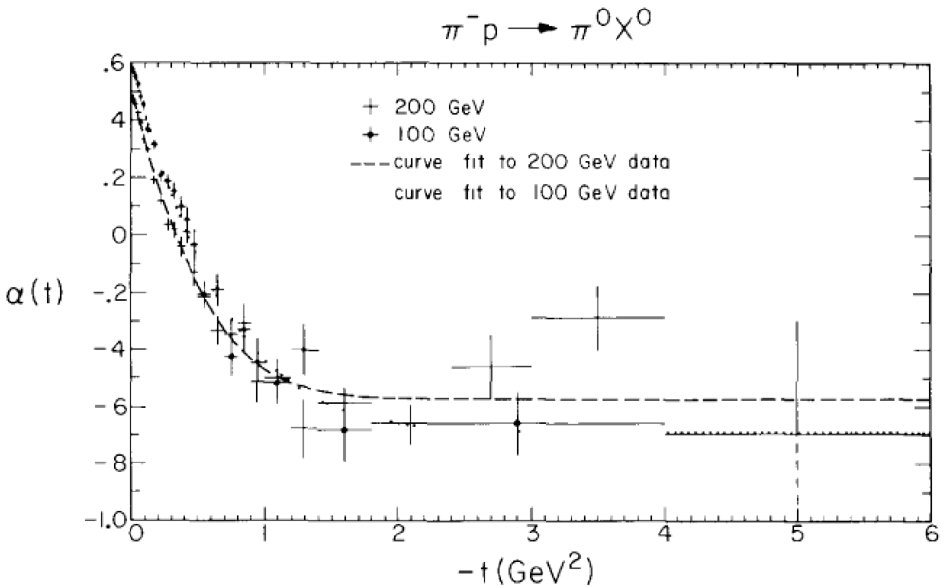
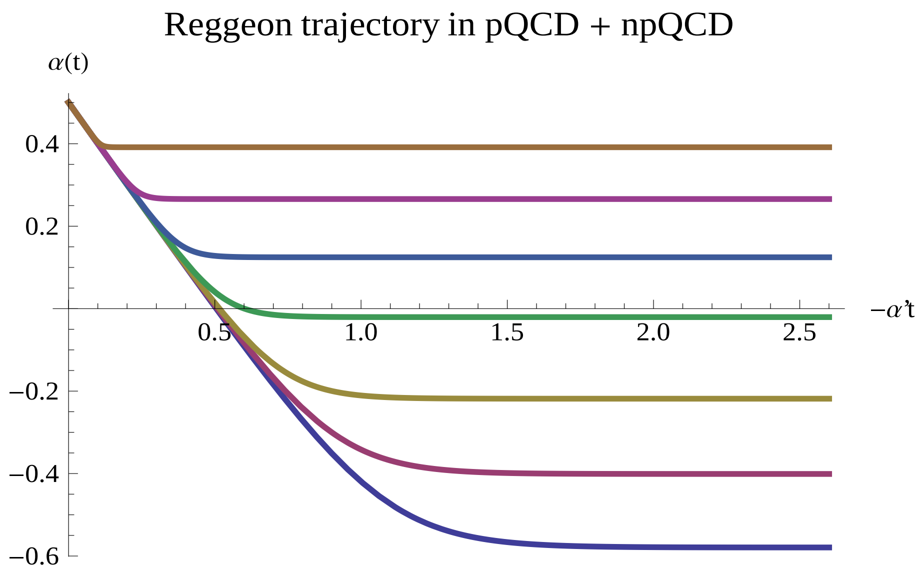



Fig 9 The $\rho$ trajectory, $\alpha(t)$, for $0 < -t < 6$ GeV$^2$ extracted from fits to the neutral final state reaction, $\pi^- p \to \pi^0 X^0$, with the $\pi^0$ energy between 140 and 192 GeV The trajectory derived at 100 GeV is shown by the dotted data points The dashed and dotted curves show the parameterizations of the trajectories from the 200 and 100 GeV beams, respectively (see text for details) The error of $\pm 0.1$ in $\alpha_\rho(t)$ coming from the uncertainty in $\tilde{\alpha}_{NFS}$ is not included in the errors shown in the figure




Brodsky, Tang, Thorn (1993)

two regimes as $s \to \infty$

# Conclusion

- Regge behavior of QCD scattering amplitudes follows from the area law. The only approximation is large $N$. Great simplification occurs for small $m$ and/or large $M$ (Lovelace-type amplitudes).

- It was crucial for the success of calculations that all integrals are Gaussian except for the one over reparametrizations which reduces to integration over the Koba–Nielsen variables.

- Derivation is legible for those momenta $\Delta p_i$ for which asymptotically large loops are essential in the sum over $C$:
$KS_{\text{min}}(C_*) = \alpha'|t| \ln \frac{s}{\max\{|t|,K\}}$ i.e. large $s$ and $|t| \ll s$.

- The classical string has intercept of the Reggeon trajectory $\alpha(0) = 0$ ($\alpha(0) \approx 0.5$ from experiment) but is applicable only for $|t| \gg 1/\alpha'$. The mean-field approximation results in $\alpha(0) = (d-2)/24$.

- 4-point scattering amplitude is valid only for asymptotically large $s$ and fixed $t$ associated with small angle or fixed momentum transfer.

- When $-t \ll s$ becomes large, there are no longer reasons to expect the contribution of large loops to dominate over perturbation theory, which comes from integration over small loops.

# Outlook

- Reggeon intercept of $\approx 0.5$ has to be obtained, most probably by accounting for spontaneous breaking of chiral symmetry like in Fedorov, Simonov (2003)

- Pomeron trajectory with the intercept of $\approx 1$ should be obtained from an annulus amplitude (to be compared with $\alpha(t) = 1 + \frac{d-2}{96} + \frac{\alpha' t}{4}$ for $d = 10$) by Janik, Peschanski (2000); Janik (2001)

- relative strength of pQCD to QCD string is to be estimated (Kaidalov (2006) estimated about 1/1000 for current energies)

- confining nonflat background in the AdS/CFT correspondence may be relevant to bending of Regge trajectories, seen in experiment, à la Brower, Polchinski, Strassler, Tan (2007)

- mean-field approximation for other strings (elfin string Migdal (1981))