Holographic three-point functions of semiclassical states

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K.Z.,1008.1059

“Large-N Gauge Theories”, GGI, Firenze, 27.04.11

AdS/CFT correspondence

Yang-Mills theory with N=4 supersymmetry

$\Updownarrow$

Maldacena'97

String theory on $AdS_5 x S^5$ background

$$\lambda = g_{\mathrm{YM}}^2 N \qquad\qquad \frac{\sqrt{\lambda}}{2\pi}$$

't Hooft coupling

string tension

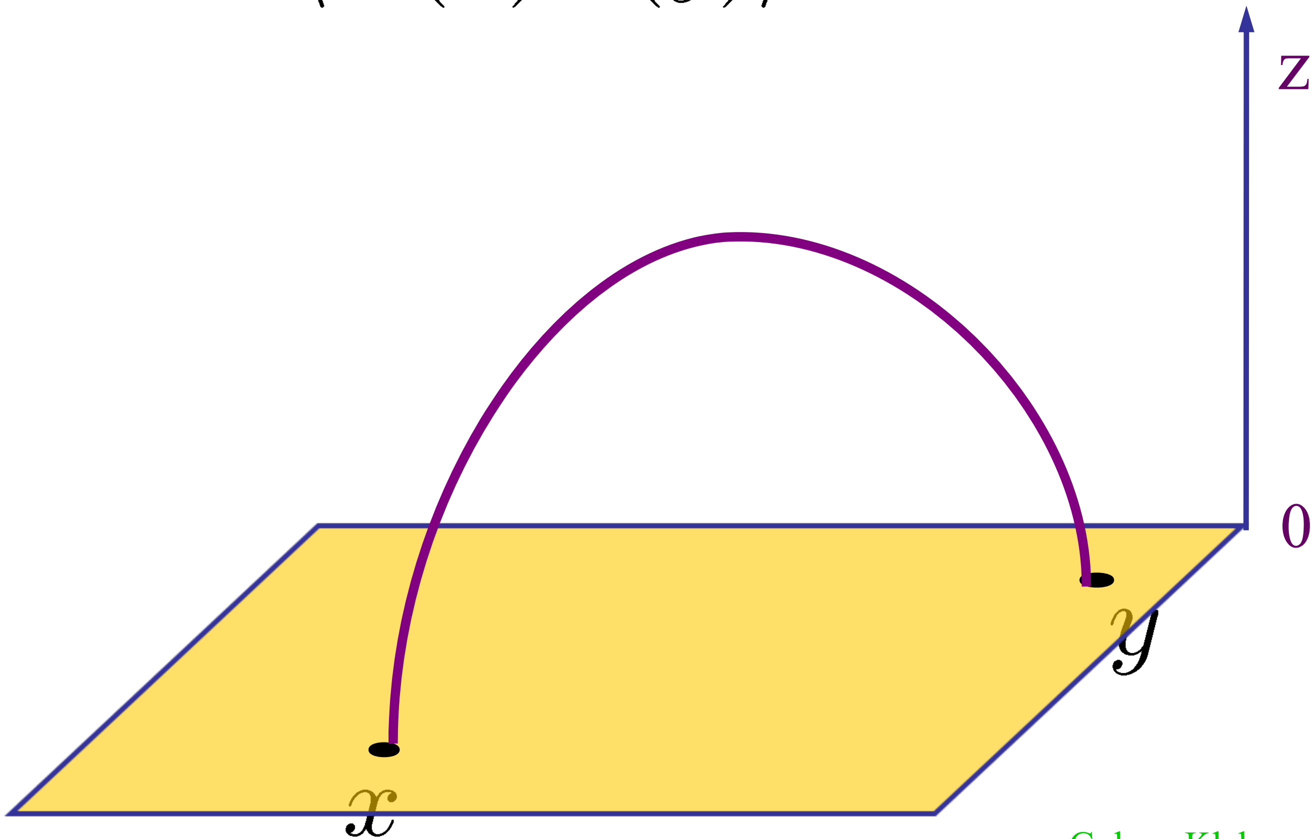
$$N \to \infty$$

planar / no quantum gravity

$$\lambda \gg 1$$

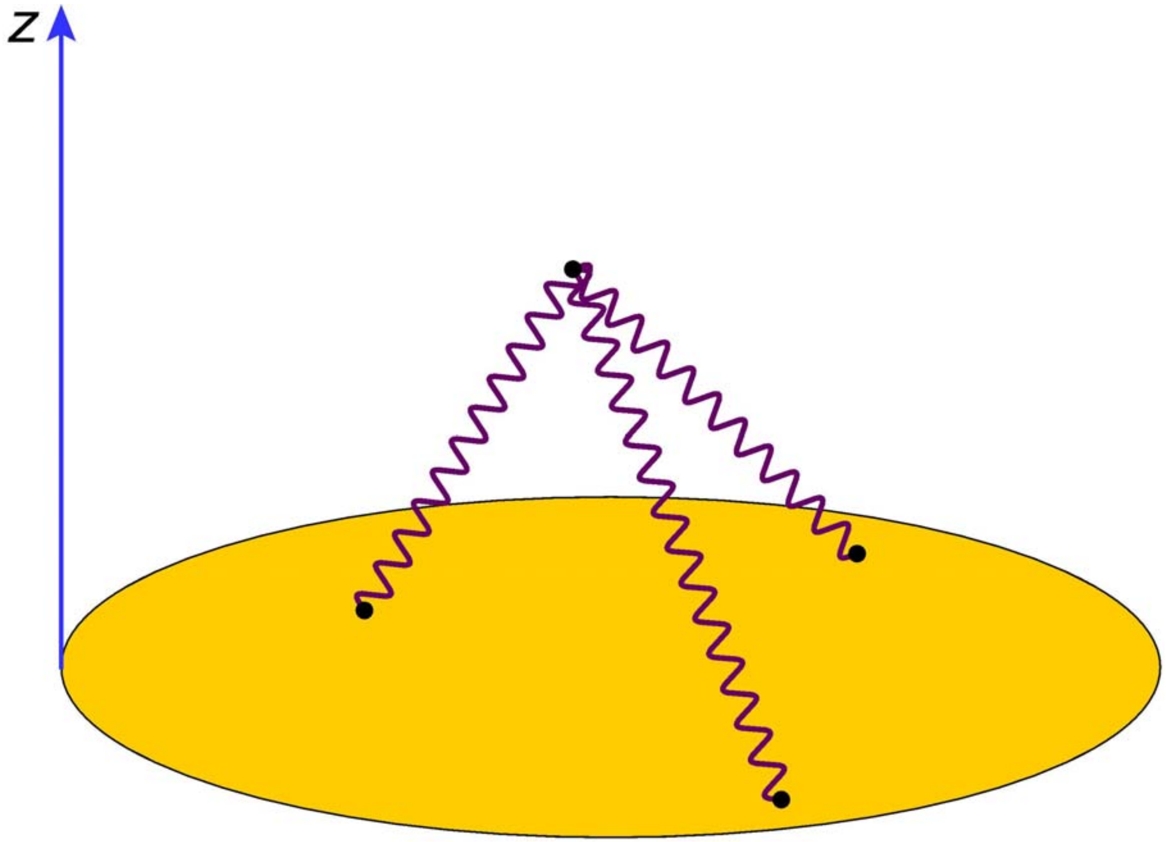
string theory - classical

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$$

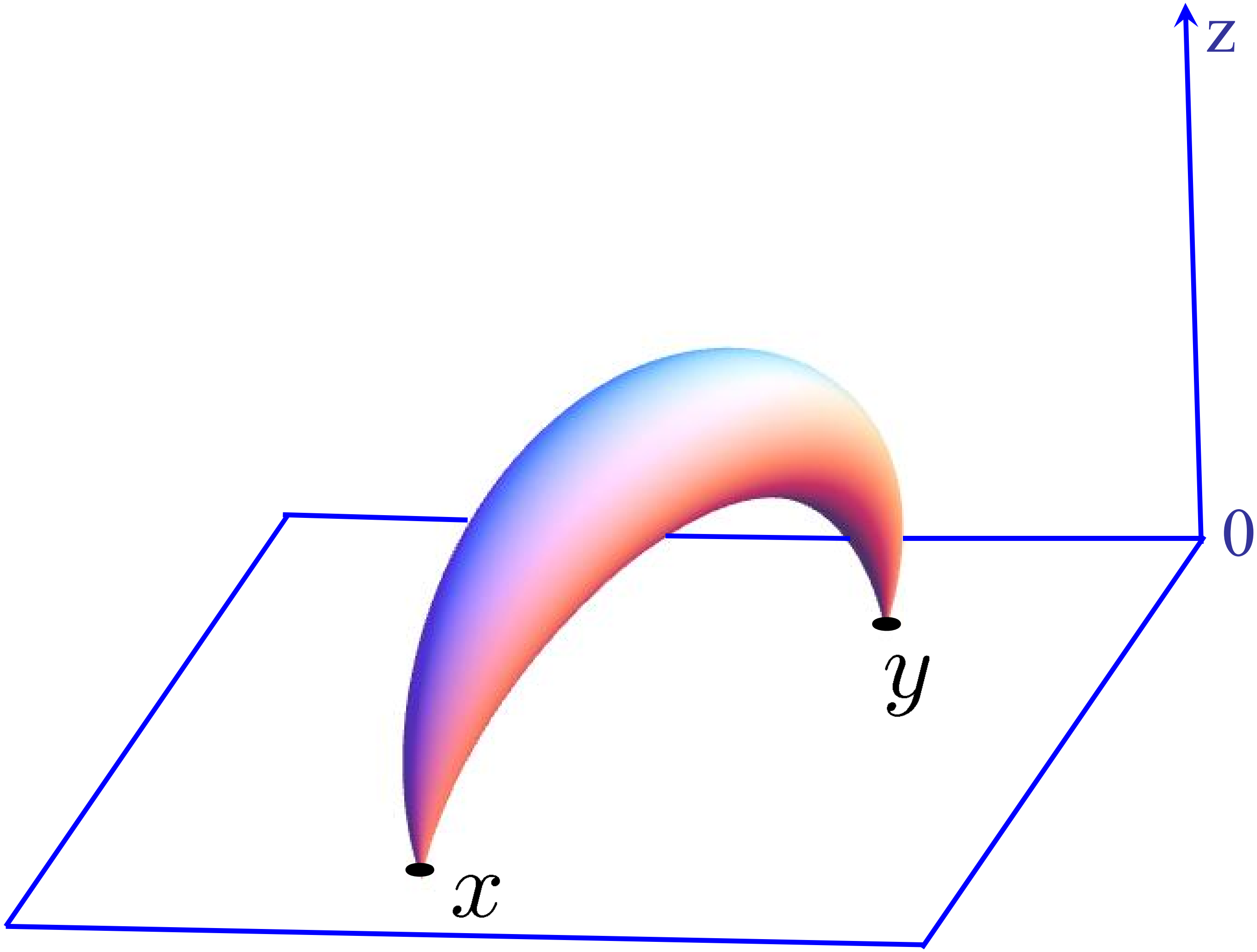


Gubser,Klebanov,Polyakov'98
Witten'98

Witten diagrams



$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle$$



$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$$

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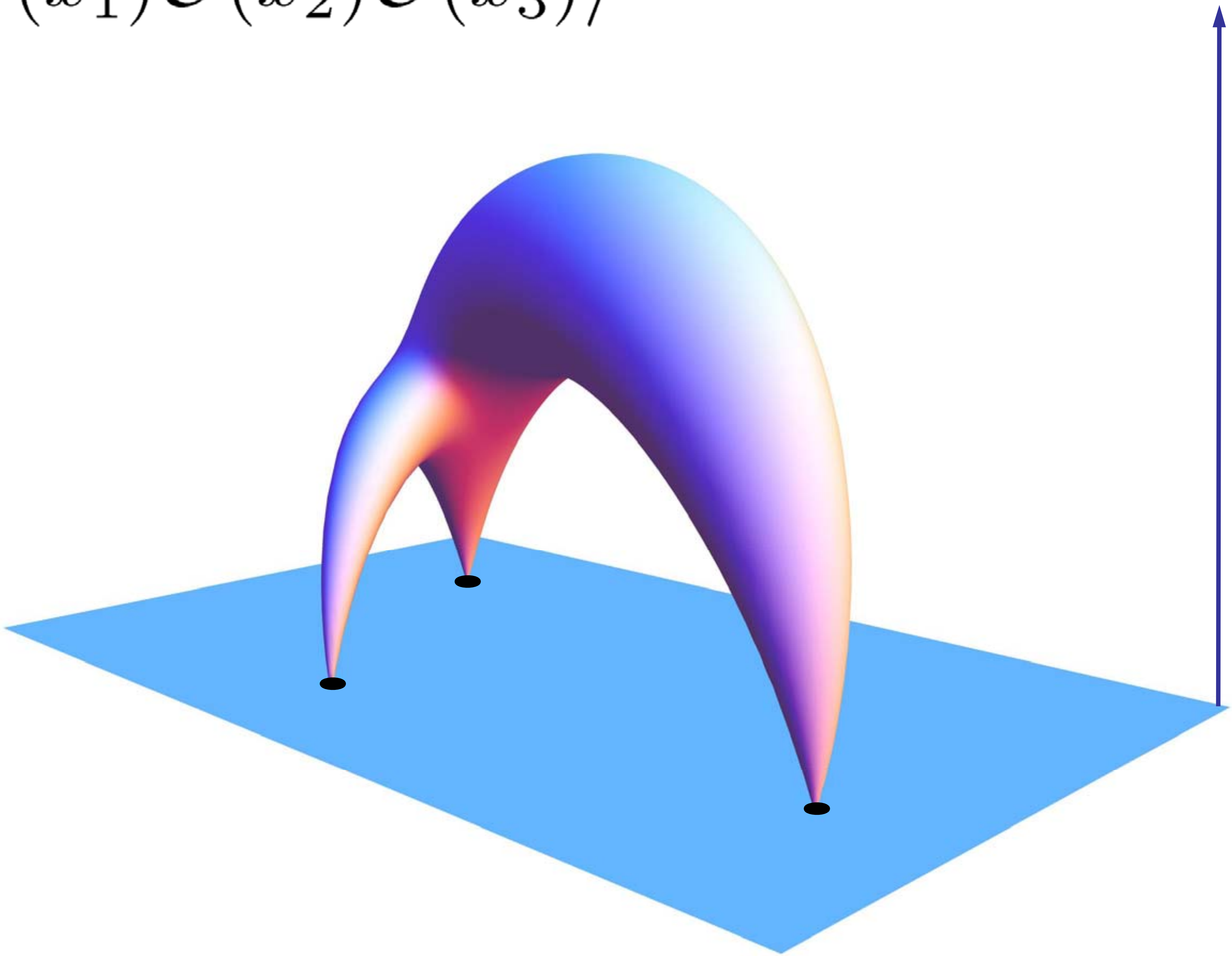


Idea: consider "semiclassical" operators with large quantum numbers

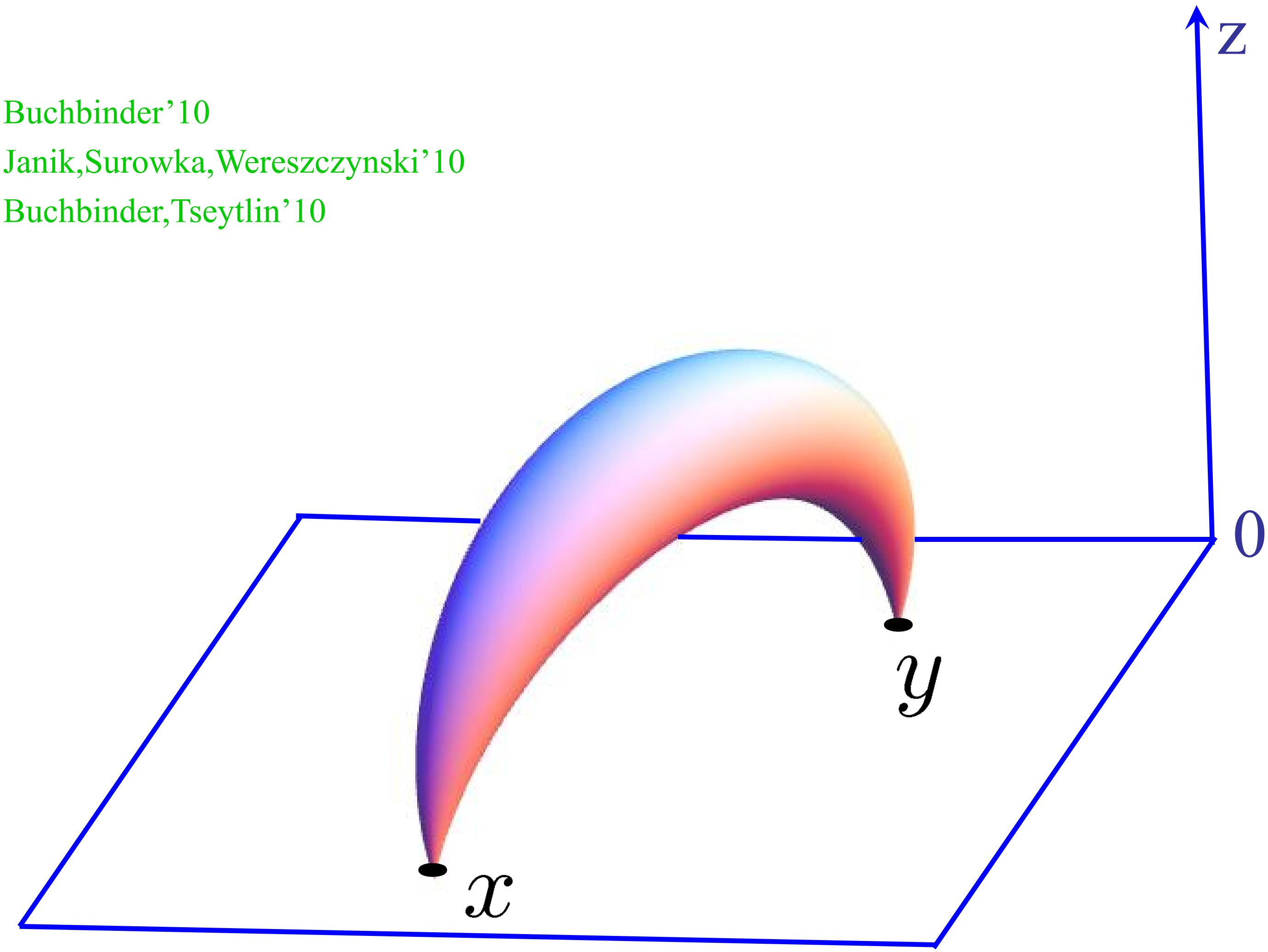
Berenstein,Maldacena,Nastase'02
Gubser,Klebanov,Polyakov'02

$$\mathcal{O} = \mathrm{tr}\ \Phi_{I_1} \ldots \Phi_{I_L}$$

$$L \sim \sqrt{\lambda} \gg 1$$

- described by classical strings

Buchbinder'10
Janik,Surowka,Wereszczynski'10
Buchbinder,Tseytlin'10



Two-point functions

$$\left\langle \mathcal{O}_I^\dagger(x)\mathcal{O}_J(y)\right\rangle = \frac{\delta_J^I}{|x-y|^{2\Delta_I}}$$

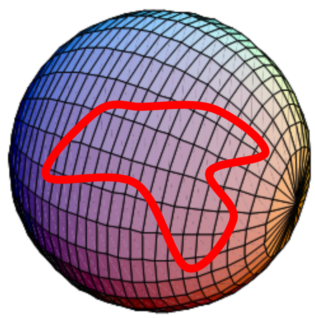
Spectrum: Δ_I

Known from integrability

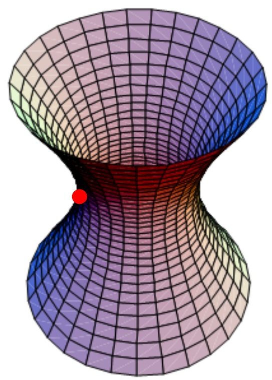
exactly at large-N

Bombardelli,Fioravanti,Tateo'09
Gromov,Kazakov,Vieira'09
Arutyunov,Frolov'09

Semiclassical states



S^5



global AdS_5

Gubser,Klebanov,Polyakov'02
Frolov,Tseytlin'03
…

$$\mathcal{O} = \operatorname{tr} Z^{L-M} W^M + \text{perm}$$

Periodic solutions in sigma-model $\longleftrightarrow$ Long operators in SYM

Energy: $\Delta \sim \sqrt{\lambda}$

Angular momenta: $L, M \sim \sqrt{\lambda}$

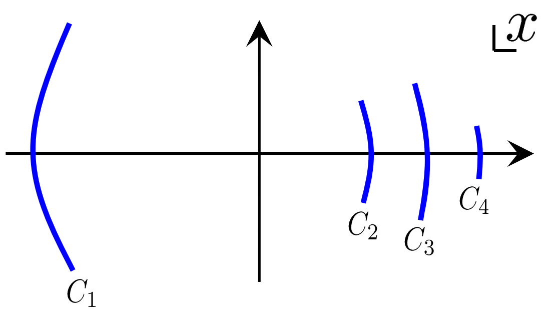
…

Finite-gap solutions

$$2 \fint \frac{dy\,\rho(y)}{x-y} = \frac{x}{x^2 - \frac{\lambda}{16\pi^2 L^2}}\,\frac{\Delta}{L} + 2\pi n_k, \qquad x \in C_k$$

Kazakov,Marshakov,Minahan,Z.'04

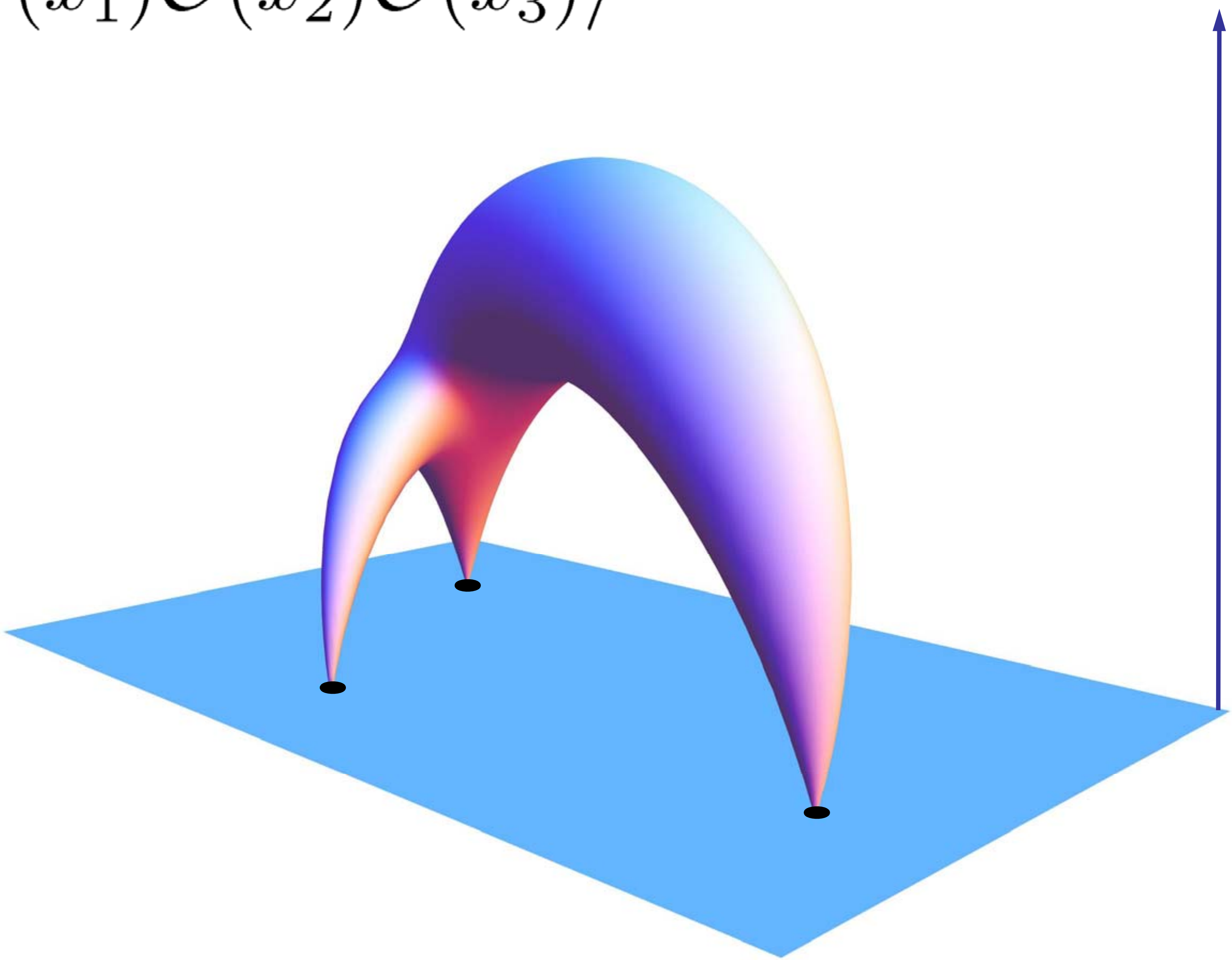
Normalization: $\displaystyle \int dx\,\rho(x) = \frac{M}{L} + \frac{\Delta - L}{2L}$



Level matching: $\displaystyle \int \frac{dx\,\rho(x)}{x} = 2\pi m$

Scaling dimension: $\displaystyle \Delta = L + \frac{\lambda}{8\pi^2 L} \int \frac{dx\,\rho(x)}{x^2}$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle$$



Three-point functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

OPE coefficients:

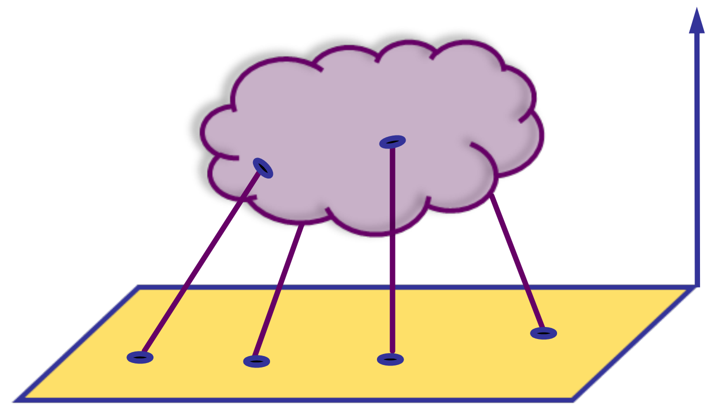
$$\mathcal{O}_J(x)\mathcal{O}_K(0) = \sum_I C^I_{JK}|x|^{\Delta_I-\Delta_J-\Delta_K}\mathcal{O}_I(0) + \text{descendants}$$

Simplest 1/N observables:

$$C^I_{JK} = O\left(\frac{1}{N}\right)$$

Correlation functions in string theory

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_{\mathrm{SYM}} = \int d^{2n-6}\sigma_i \ \langle V_1(\sigma_1;x_1)\ldots V_n(\sigma_n;x_n)\rangle_{\mathrm{world-sheet}}$$



Vertex operators:

- (1,1) operators in the sigma-model

Semiclassically:

$$\left(\Delta_{AdS^5} + \Delta_{S^5} + M^2\right) V = 0$$

Callan,Gan'86

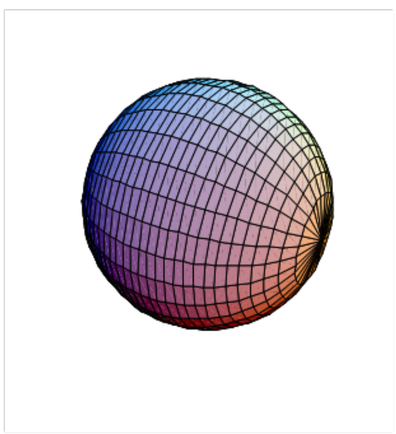
Vertex operators in AdS$_5$xS^5

$$V(Z, X, \mathbf{N}; x) \sim Y_I(\mathbf{N}) \times \frac{Z^{\Delta}}{\left[Z^2 + (X-x)^2\right]^{\Delta}}$$

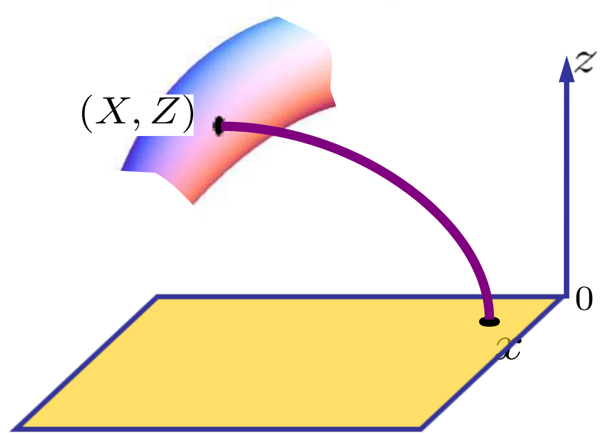
Polyakov'01

Tseytlin'03

Spherical functions



S^5



AdS$_5$

Semiclassical limit

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \, G_{MN} \partial_a X^M \partial^a X^N - \sum_i \ln V_i(\sigma_i)$$

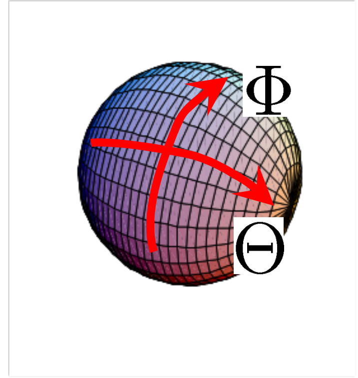
Semiclassical states: $\ln V_i \sim J_i, \Delta_i \sim \sqrt{\lambda}$

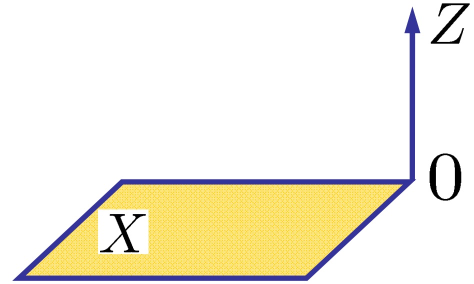
$$\Downarrow$$

Sources in classical equations of motion:

$$\partial_a \partial^a X^M + \Gamma^M_{NL} \partial_a X^N \partial^a X^L = -\frac{2\pi}{\sqrt{\lambda}} G^{MN} \sum_i \frac{\partial \ln V_i}{\partial X^N} \delta(\sigma - \sigma_i)$$

Example:





$$V \sim (\sin\Theta)^k \; \mathrm{e}^{\,ik\Phi} \, \frac{Z^k}{\left[Z^2 + (X-x)^2\right]^k} \;\xrightarrow{X(\sigma)\to x}\; (\sin\Theta)^k \; \mathrm{e}^{\,ik\Phi} Z^{-k}$$

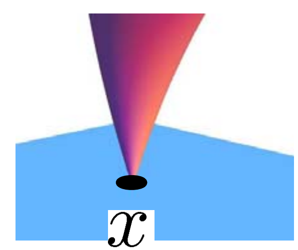
10d massless $\rightarrow$ creates a BPS state

$$\begin{aligned}
-\partial^2 \ln Z - \frac{(\partial X)^2}{Z^2} &= -\frac{2\pi k}{\sqrt{\lambda}}\delta(\sigma) \\
-\partial^2\Theta + \sin\Theta\,\cos\Theta\,(\partial\Phi)^2 &= \frac{2\pi k}{\sqrt{\lambda}}\cot\Theta\,\delta(\sigma) \\
-\partial_{\mathbf{a}}\left(\sin^2\Theta\,\partial^{\mathbf{a}}\Phi\right) &= \frac{2\pi k}{\sqrt{\lambda}}\delta(\sigma),
\end{aligned}$$

$\Longrightarrow$

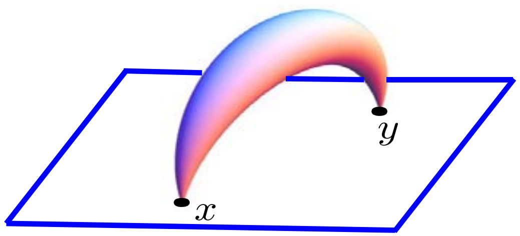
boundary conditions

$$\begin{aligned}
\ln Z &\rightarrow -\frac{k}{\sqrt{\lambda}}\ln|\sigma| \\
\Theta &\rightarrow \frac{\pi}{2} \\
\Phi &\rightarrow -i\frac{k}{\sqrt{\lambda}}\ln|\sigma|
\end{aligned}$$



de Boer,Ooguri,Robins,Tannenhauser'98

Holographic two-point functions



Buchbinder'10
Janik,Surowka,Wereszczynski'10
Buchbinder,Tseytlin'10

Two-point functions $\leftrightarrow$ Spectrum $\leftrightarrow$ Periodic solutions in global AdS

- start with time-periodic (finite-gap) solution in global AdS
- Wick-rotate
- transform to Poincaré patch

- solution in general complex
- does not necessarily shrink to a point on the boundary
- vertex operators $\leftrightarrow$ finite-gap solutions (?)

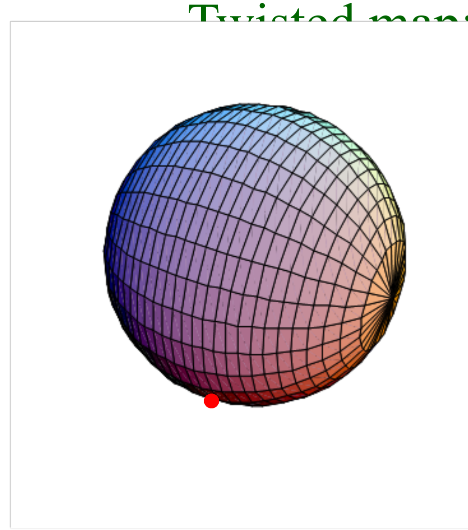
Example: BMN string: $t = \kappa\tau, \rho = 0, \varphi = \kappa\tau \iff \mathcal{O} = \operatorname{tr} Z^L$

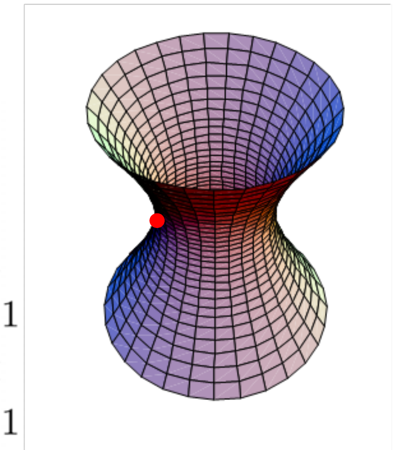
Standard global-Poincaré map (AdS_3):

Twisted map:

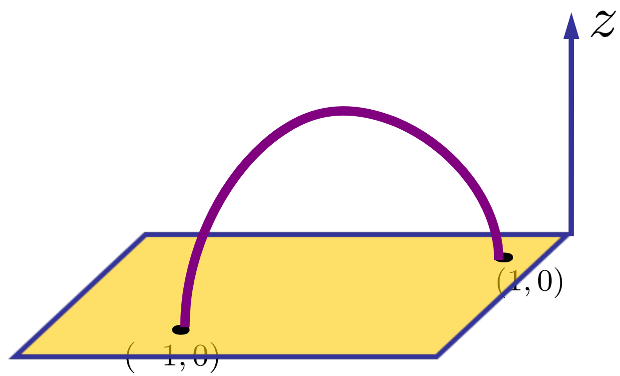
Cartesian coordinates on $R^{3,1}$

$$\left\{\begin{array}{rcl}
\sinh\rho\cos\varphi &=& \dfrac{x}{z} \\
\sinh\rho\sin\varphi &=& \dfrac{y}{z} \\
\cosh\rho\sinh t &=& \dfrac{z}{2}\left(1+\dfrac{x^2+y^2-1}{z^2}\right) \\
\cosh\rho\cosh t &=& \dfrac{z}{2}\left(1+\dfrac{x^2+y^2+1}{z^2}\right)
\end{array}\right.$$





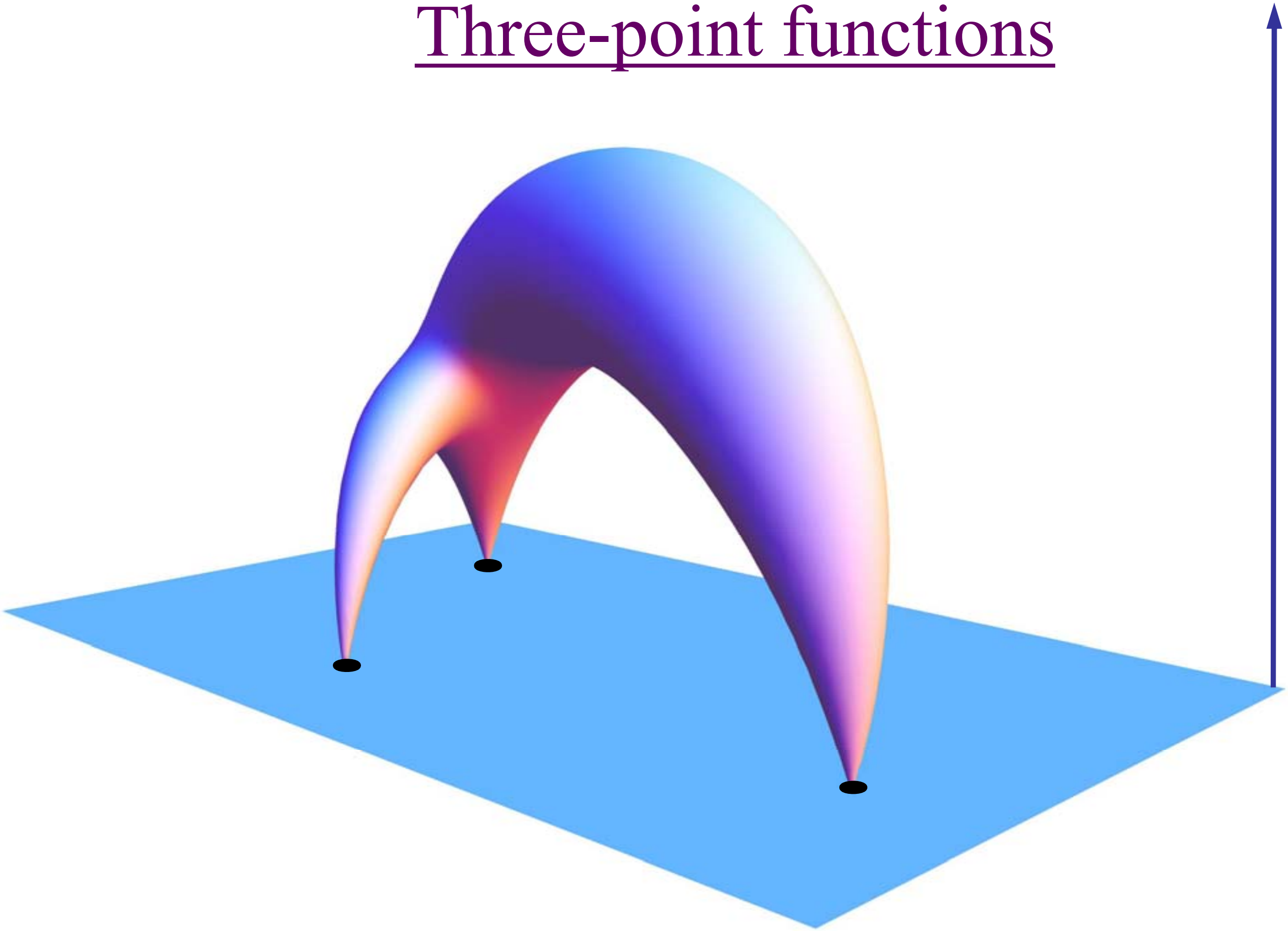
$$\begin{array}{rcl}
x &=& \tanh\kappa\tau \\
z &=& \dfrac{1}{\cosh\kappa\tau} \\
\varphi &=& i\kappa\tau
\end{array}$$



Tsuji'06

Janik,Surowka,Wereszczynski'10

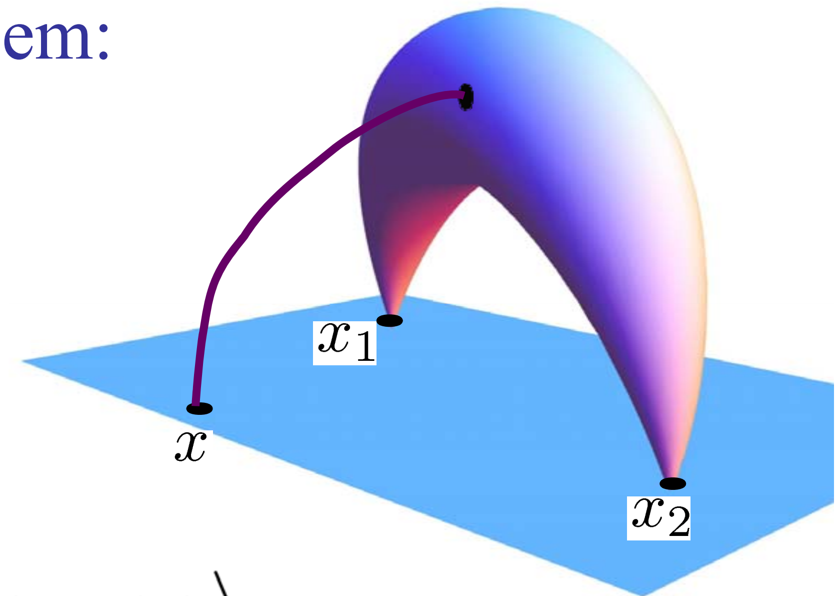
Three-point functions



- No solutions known

Z.'10
Costa,Monteiro,Santos,Zoakos'10
Roiban,Tseytlin'10
Hernandez'10
Arnaudov,Rashkov'10
Georgiou'10
Park,Lee'10
Buchbinder,Tseytlin'10
Bak,Chen,Wu'11
Bissi,Kristjansen,Young,Zoubos'11
Arnaudov,Rashkov,Vetsov'11
Bai,Lee,Park'11

Simpler problem:



$$\left\langle \mathcal{O}_J^\dagger(x_1)\mathcal{O}_K(x_2)\mathcal{O}_I(x)\right\rangle$$

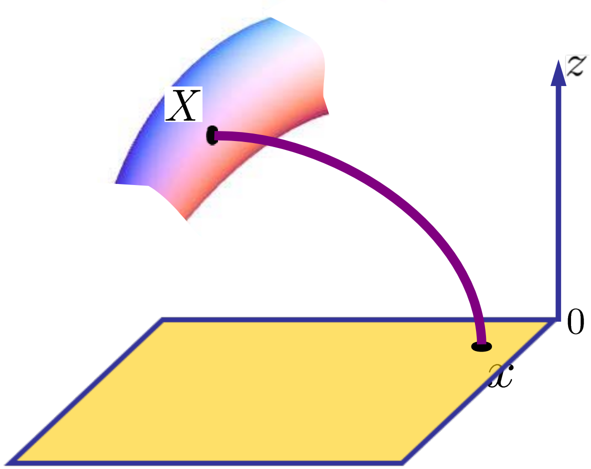
$$\left.\begin{array}{ll} \mathcal{O}_J : & \Delta_J \sim \sqrt{\lambda} \\ \mathcal{O}_K : & \Delta_K \sim \sqrt{\lambda} \end{array}\right\} \text{ create fat string}$$

$$\mathcal{O}_I : \quad \Delta_I \sim 1 \qquad \text{creates slim string}$$

General formalism

$\mathcal{W}$ big non-local operator that creates classical string

$$\frac{\langle \mathcal{W}\mathcal{O}_I(x)\rangle}{\mathcal{W}} = \lim_{\varepsilon\to 0}\frac{\pi}{\varepsilon^{\Delta_I}}\sqrt{\frac{2}{\Delta_I - 1}}\left\langle \phi_I(x,\varepsilon)\,\frac{1}{Z_{\rm str}}\int \mathcal{D}X\ \mathrm{e}^{-S_{\rm str}[X]}\right\rangle_{\rm bulk}.$$



Berenstein,Corrado,Fischler,Maldacena'98

$$S_{\rm str}=\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\left[\frac{\left(\partial X^\mu\right)^2+\left(\partial Z\right)^2}{Z^2}+\left(\partial\mathbf{N}\right)^2+\gamma_{MN}\partial_a X^M\partial^a X^N\right]$$

metric perturbation

due to operator insertion

$$\gamma_{MN}=V^I_{MN}\left(X,\frac{\partial}{\partial X}\right)\phi_I$$

$$\frac{\langle\mathcal{W}\mathcal{O}_I(x)\rangle}{\langle\mathcal{W}\rangle}=-\frac{\sqrt{2\left(\Delta_I-1\right)\lambda}}{8\pi^2}\int d^2\sigma\,\partial_a X^M\partial^a X^N V^I_{MN}\frac{Z^{\Delta_I}}{\left[Z^2+\left(X-x\right)^2\right]^{\Delta_I}}$$

vertex operator

OPE coefficient:

$$C_I[\mathcal{W}]=\lim_{x\to\infty}|x|^{2\Delta_I}\frac{\langle\mathcal{W}\mathcal{O}_I(x)\rangle}{\langle\mathcal{W}\rangle}=-\frac{\sqrt{2\left(\Delta_I-1\right)\lambda}}{8\pi^2}\int d^2\sigma\,\partial_a X^M\partial^a X^N V^I_{MN}Z^{\Delta_I}$$

Chiral Primary Operators

$$\mathcal{O}_I^{\mathrm{CPO}} = \frac{1}{\sqrt{k}} \left(\frac{8\pi^2}{\lambda} \right)^{\frac{k}{2}} K_I^{i_1 \dots i_k} \, \mathrm{tr}\, \Phi_{i_1} \dots \Phi_{i_k}$$

symmetric traceless tensor of SO(6)

Dual to scalar supergravity mode on S^5

Wavefunction on S^5:

$$Y_I(\mathbf{n}) = K_I^{i_1 \dots i_k} n_{i_1} \dots n_{i_k}$$

(spherical function of SO(6))

Kaluza-Klein reduction

$$h_{mn} = \frac{1}{\mathcal{N}_k}\,\frac{2}{k+1}\,Y_I\left[2\nabla_m\nabla_n - k(k-1)g_{mn}\right]\phi_I$$

$$h_{\alpha\beta} = \frac{2}{\mathcal{N}_k}\,k g_{\alpha\beta} Y_I \phi_I$$

Kim,Romans,van Nieuwenhuizen'85
Lee,Minwalla,Rangamani,Seiberg'98

$$\mathcal{N}_k^2 = \frac{N^2 k(k-1)}{2^{k-3}\pi^2(k+1)^2}$$

Vertex operator:

$$\boxed{C_I^{\mathrm{CPO}}[\mathcal{W}] = \frac{2^{\frac{k}{2}-3}(k+1)\sqrt{k\lambda}}{\pi N}\int d^2\sigma\, Y_I(\mathbf{N}) Z^k \left[\frac{(\partial X)^2 - (\partial Z)^2}{Z^2} - (\partial \mathbf{N})^2\right]}$$

Correlator of three chiral primaries

Superconformal highest weight:

$$\mathcal{O}_L = \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda} \right)^{\frac{L}{2}} \operatorname{tr} Z^L, \qquad Z = \Phi_1 + i\Phi_2$$

$$\left\langle \mathcal{O}^\dagger_{J+k}(x) \mathcal{O}_J(y) \mathcal{O}_k(\infty) \right\rangle$$

@

$$\begin{array}{ccc} J & \sim & \sqrt{\lambda} \\ k & \sim & 1 \end{array}$$

Spherical function:

$$Y_k = \left(\frac{n_1 + i n_2}{\sqrt{2}} \right)^k = 2^{-\frac{k}{2}} \left(\sin\theta \right)^k \, \mathrm{e}^{\, ik\varphi}$$

Classical solution:

$$\begin{aligned} x &= R \tanh \kappa\tau \\ z &= \frac{R}{\cosh \kappa\tau} \\ \varphi &= i\kappa\tau, \qquad \theta = \frac{\pi}{2}\,. \end{aligned}$$

$$\kappa = \frac{J}{\sqrt{\lambda}}$$

$$R = |x - y|$$

OPE coefficient:

$$C_{J,k}^{J+k} = \frac{1}{N}\, 2^{-k-1} J(k+1)\sqrt{k}\,\kappa \int_{-\infty}^{+\infty} d\tau\, \frac{\mathrm{e}^{-k\kappa\tau}}{\cosh^{k+2}\kappa\tau} = \frac{1}{N}\, J\sqrt{k}$$

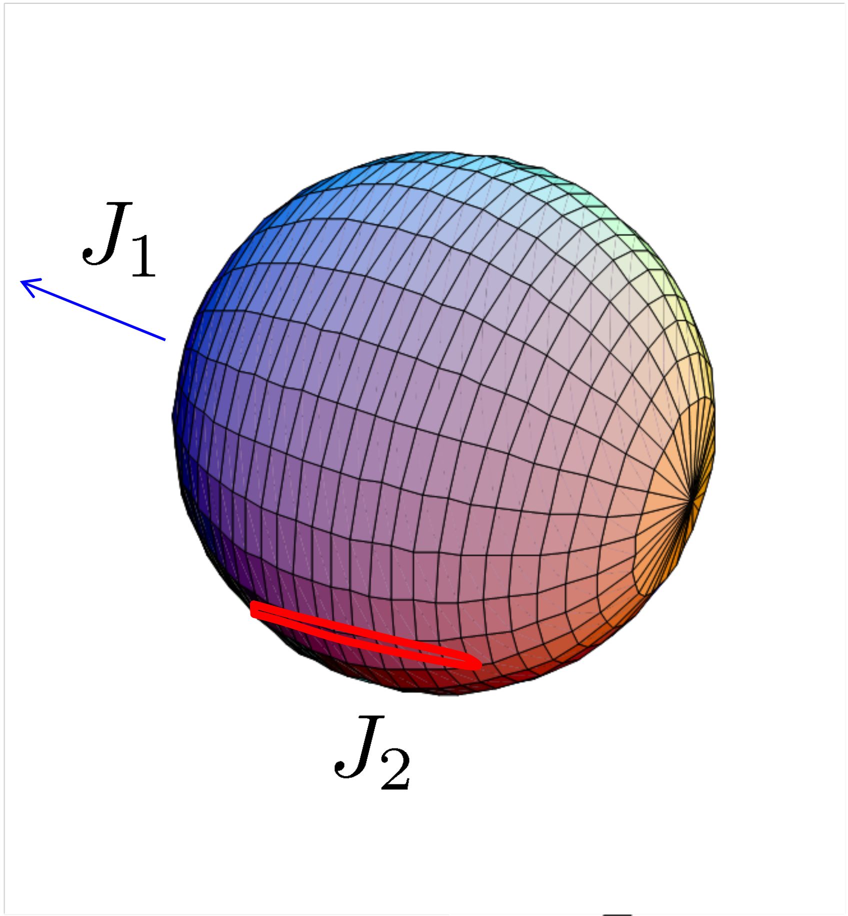
Exact OPE coefficient of three CPO's:

Agree at J>>k

$$C_{Jk}^{J+k} = \frac{1}{N}\sqrt{(J+k)Jk}$$

Lee,Minwalla,Rangamani,Seiberg'98

Spinning string on S^5



$$\Phi = i\omega_1 \tau$$

$$\Psi = i\omega_2 \tau$$

$$\Theta = \Theta(\sigma)$$

$$\acute{\Theta}^2 = \kappa^2 - \omega_1^2 \sin^2\Theta - \omega_2^2 \cos^2\Theta$$

Frolov,Tseytlin'03

Elliptic modulus:

$$s = \frac{\kappa^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}$$

$$\sqrt{\omega_2^2 - \omega_1^2} = \frac{2K(s)}{\pi}$$

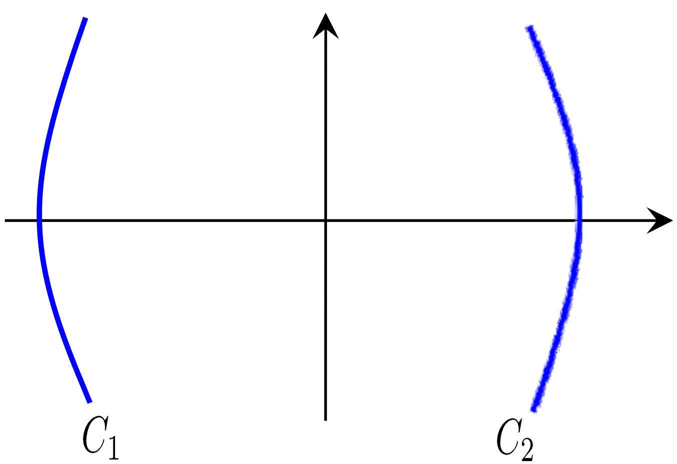
Conserved charges: $J_1 = \dfrac{\sqrt{\lambda}\omega_1 E(s)}{K(s)}\,, \qquad J_2 = \dfrac{\sqrt{\lambda}\omega_2 \left(K(s) - E(s)\right)}{K(s)}\,, \qquad \Delta = \sqrt{\lambda}\kappa$

Dual to

$$\mathcal{O}_{\text{fold.}} = \text{tr}\,\left(Z^{J_1} W^{J_2} + \text{perm.}\right)$$

The concrete operator can be identified by comparing the finite-gap curve to Bethe ansatz

Beisert,Minahan,Staudacher,Z.'03



OPE coefficient:

$$C^{\text{fold.}}_{\text{fold.},k} = \frac{1}{N}\,\frac{\pi\sqrt{k}\Delta\left(1-a^2\right)\Gamma\left(\frac{(1+a)k}{2}\right)\Gamma\left(\frac{(1-a)k}{2}\right)}{8sK(s)(k-1)!}\left[(k+1-s)_2F_1\left(-\frac{k-1}{2},\frac{1}{2};1;s\right)-(k+1)_2F_1\left(-\frac{k+1}{2},\frac{1}{2};1;s\right)\right]$$

$$a = \frac{\omega_1}{\kappa} = \frac{J_1 K(s)}{\Delta E(s)}$$

What happens when k becomes large?

$$C^{\text{fold.}}_{\text{fold.},k} \simeq \frac{1}{N}\left(\frac{1}{s}-1\right)\sqrt{\frac{\lambda}{k}}\,\mathrm{e}^{\,k\left(\frac{1+a}{2}\,\ln\frac{1+a}{2}+\frac{1-a}{2}\,\ln\frac{1-a}{2}\right)}$$

Saddle-point approximation

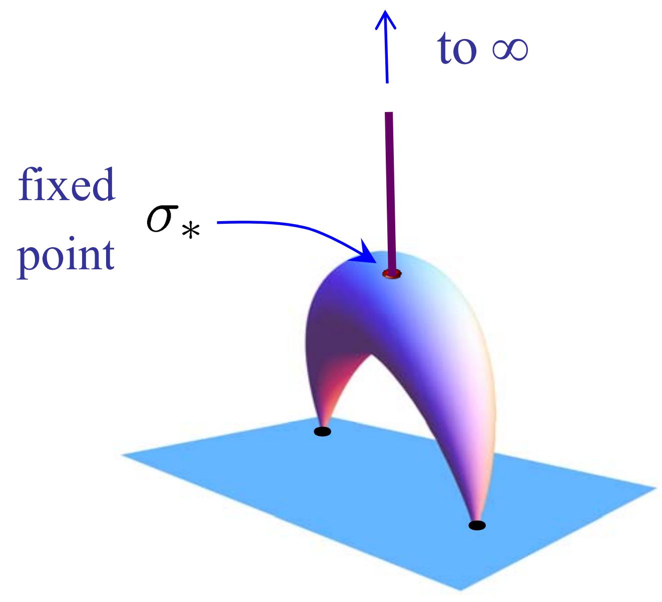
$$C^{\#}_{\#k} = \frac{(k+1)\sqrt{k\lambda}}{8\pi N} \int d^2\sigma \; (\sin\Theta)^k \; \mathrm{e}^{\,ik\Phi} Z^k \left[\frac{(\partial X)^2 - (\partial Z)^2}{Z^2} - (\partial\Theta)^2 - \sin^2\Theta \, (\partial\Phi)^2 \right]$$

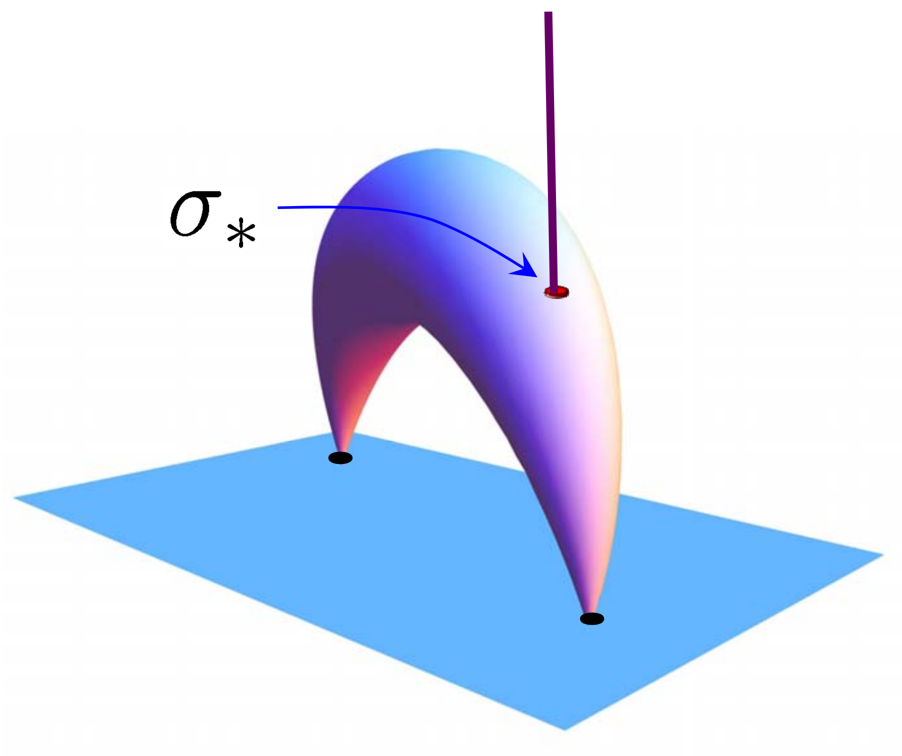
$$\Big\downarrow \; k \to \infty$$

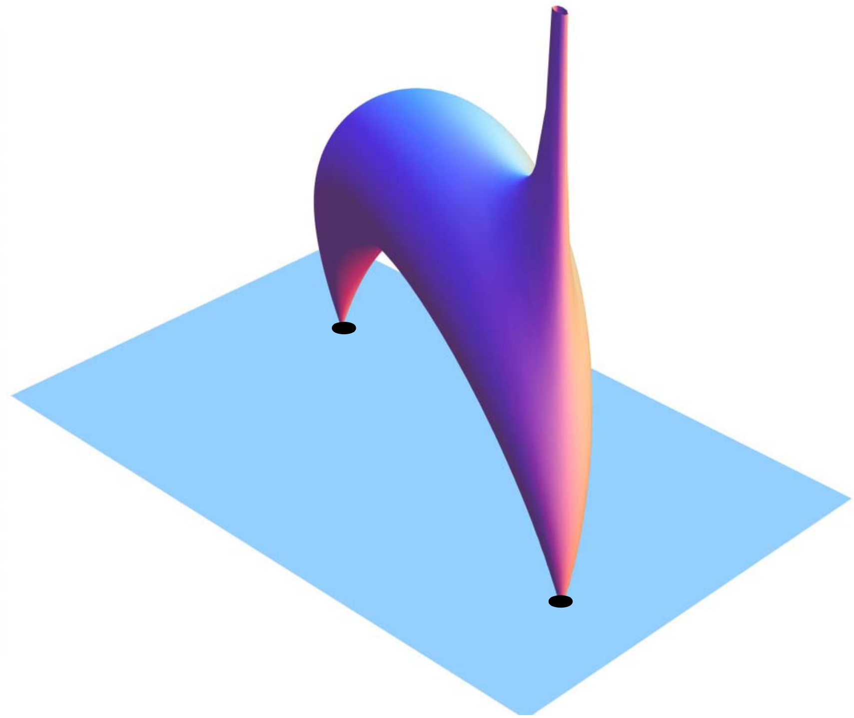
$$C^{\#}_{\#k} \simeq [\ldots] \, \det{}^{-1/2} \, \mathrm{e}^{\,-kS_{\mathrm{eff}}(\sigma_*)}$$

Saddle-point equations:

$$\cot\theta \, \partial_a \theta + i\partial_a \varphi + \frac{\partial_a Z}{Z} = 0$$







k – finite, $\quad k \to \infty$

$\frac{k}{\sqrt{\lambda}}$ – finite

Overlapping regime of validity:

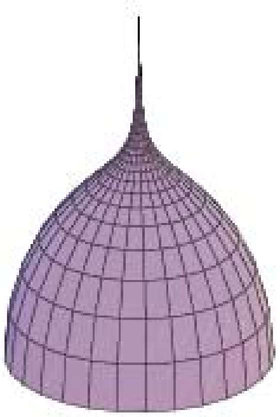
$$\sqrt{\lambda} \gg k \gg 1$$

Exact solution with a spike:

$$\begin{aligned}\Phi &= i\kappa\tau \\ Z &= \mathrm{e}^{\kappa\tau}\left[\sqrt{\kappa^2+1}\tanh\left(\sqrt{\kappa^2+1}\tau+\xi\right)-\kappa\right] \\ X^{1,2} &= \left\{\begin{matrix}\cos\sigma \\ \sin\sigma\end{matrix}\right\}\frac{\sqrt{\kappa^2+1}\,\mathrm{e}^{\kappa\tau}}{\cosh\left(\sqrt{\kappa^2+1}\tau+\xi\right)}\end{aligned}$$

$$\xi = \ln\left(\sqrt{\kappa^2+1}+\kappa\right)$$

Z.'02



Describes

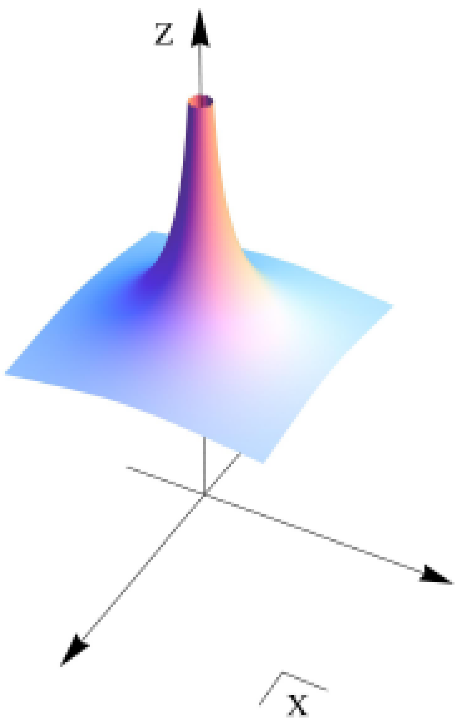
$$\lim_{x\to\infty}|x|^{2\Delta_I}\frac{\langle W(C)\mathcal{O}_I(x)\rangle}{\langle W(C)\rangle}$$

for circular Wilson loop

Solution for $\lim_{x\to\infty}|x|^{2\Delta_I}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_I(x)\rangle$?

Boundary conditions at the spike

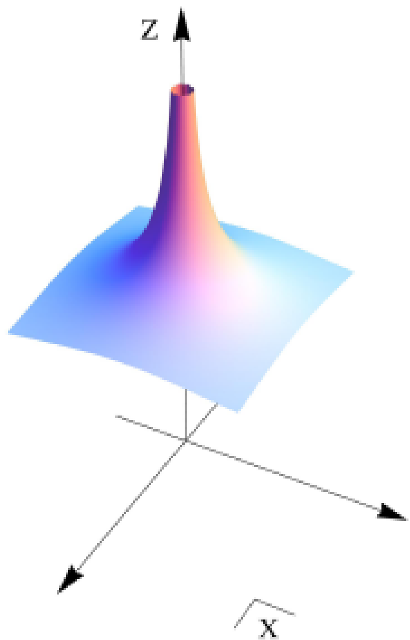
$$
\begin{aligned}
\ln Z &\rightarrow -\chi \ln|w| \\
\Theta &\rightarrow \frac{\pi}{2} \\
\Phi &\rightarrow -i\chi \ln|w|
\end{aligned}
$$



$$w = \mathrm{e}^{-\tau + i\sigma}$$

$$\chi = \frac{k}{\sqrt{\lambda}}$$

Fine structure of the spike



Regular solution without the spike

$$
\begin{aligned}
Z &= |w|^{-\chi}\overbrace{\left(z_0 + \partial z_0 w + \bar{\partial} z_0 \bar{w}\right)} + \ldots \\
X^\mu &= x_0^\mu + |w|^{\sqrt{1+\chi^2}-1-\chi}\underbrace{\left(\partial x_0^\mu w + \bar{\partial} x_0^\mu \bar{w}\right)} + \ldots
\end{aligned}
$$

Solution on S^5:

$$\mathrm{e}^{i\Phi} = \frac{|w|^{\chi}\,\mathrm{e}^{i\varphi_0}\sin\theta_0}{\sqrt{1-|w|^{2\chi}\cos^2\theta_0}}\left[1+i\,\frac{1-|w|^{\chi}\cos^2\theta_0}{1-|w|^{2\chi}\cos^2\theta_0}\left(\partial\varphi_0 w+\bar{\partial}\varphi_0\bar{w}\right)+\frac{(1-|w|^{\chi})\cot\theta_0}{1-|w|^{2\chi}\cos^2\theta_0}\left(\partial\theta_0 w+\bar{\partial}\theta_0\bar{w}\right)\right]+\ldots$$

$$\cos\Theta = |w|^{\chi}\cos\theta_0-|w|^{\sqrt{1+\chi^2}-1}\left[\frac{1-|w|^{\chi}\cos^2\theta_0}{\sin\theta_0}\left(\partial\theta_0 w+\bar{\partial}\theta_0\bar{w}\right)+i\,(1-|w|^{\chi})\cos\theta_0\left(\partial\varphi_0 w+\bar{\partial}\varphi_0\bar{w}\right)\right]+\ldots$$

Virasoro constraints:

$$0=T_{ww}\equiv\frac{(\partial Z)^2+(\partial X)^2}{Z^2}+(\partial\Theta)^2+\sin^2\Theta\,(\partial\Phi)^2$$

$\chi\to 0$ limit:

$$\frac{\partial z_0}{z_0}+\cot\theta_0\,\partial\theta_0+i\partial\varphi_0 = 0$$

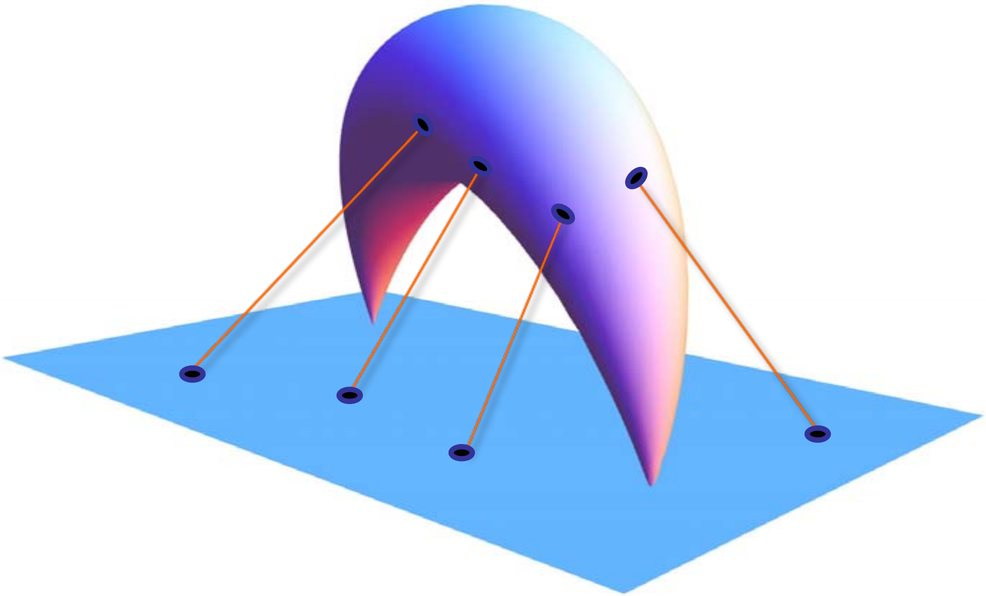
$$\frac{\bar{\partial} z_0}{z_0}+\cot\theta_0\,\bar{\partial}\theta_0+i\bar{\partial}\varphi_0 = 0$$

Determine the position σ_* on the worldsheet, where the spike can be attached.

The same as the saddle-point equation for the vertex operator!

Factorization

Roiban,Tseytlin'10

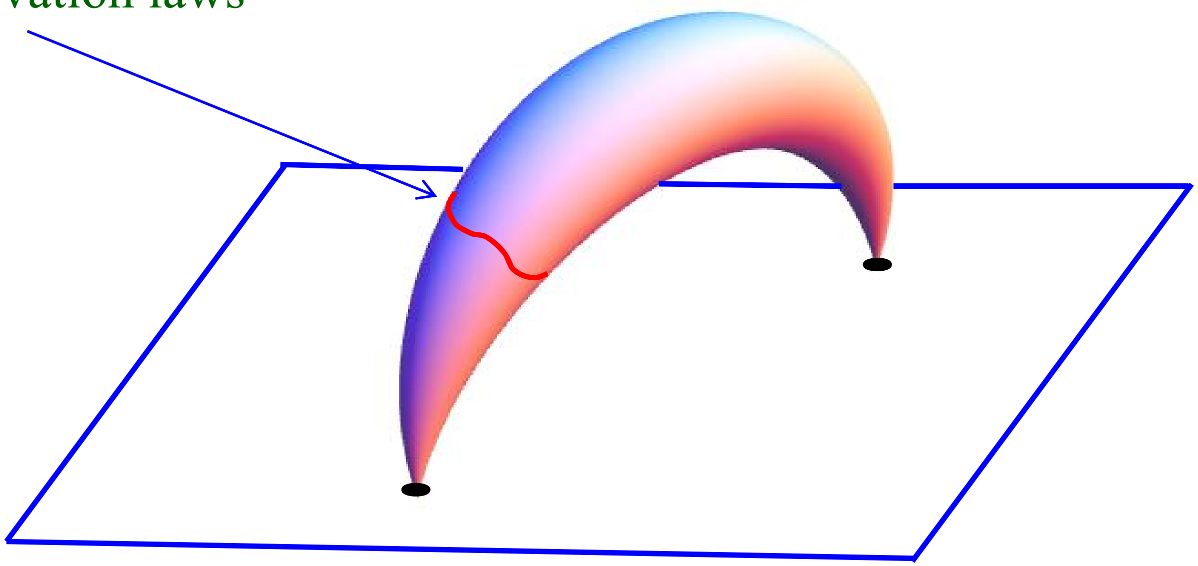


Integration over σ_i independent:

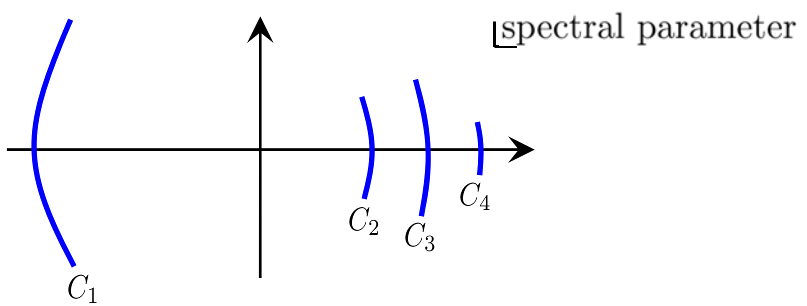
$$\langle HHL_1 \dots L_n \rangle \sim \langle HHL_1 \rangle \dots \langle HHL_n \rangle$$

Integrability

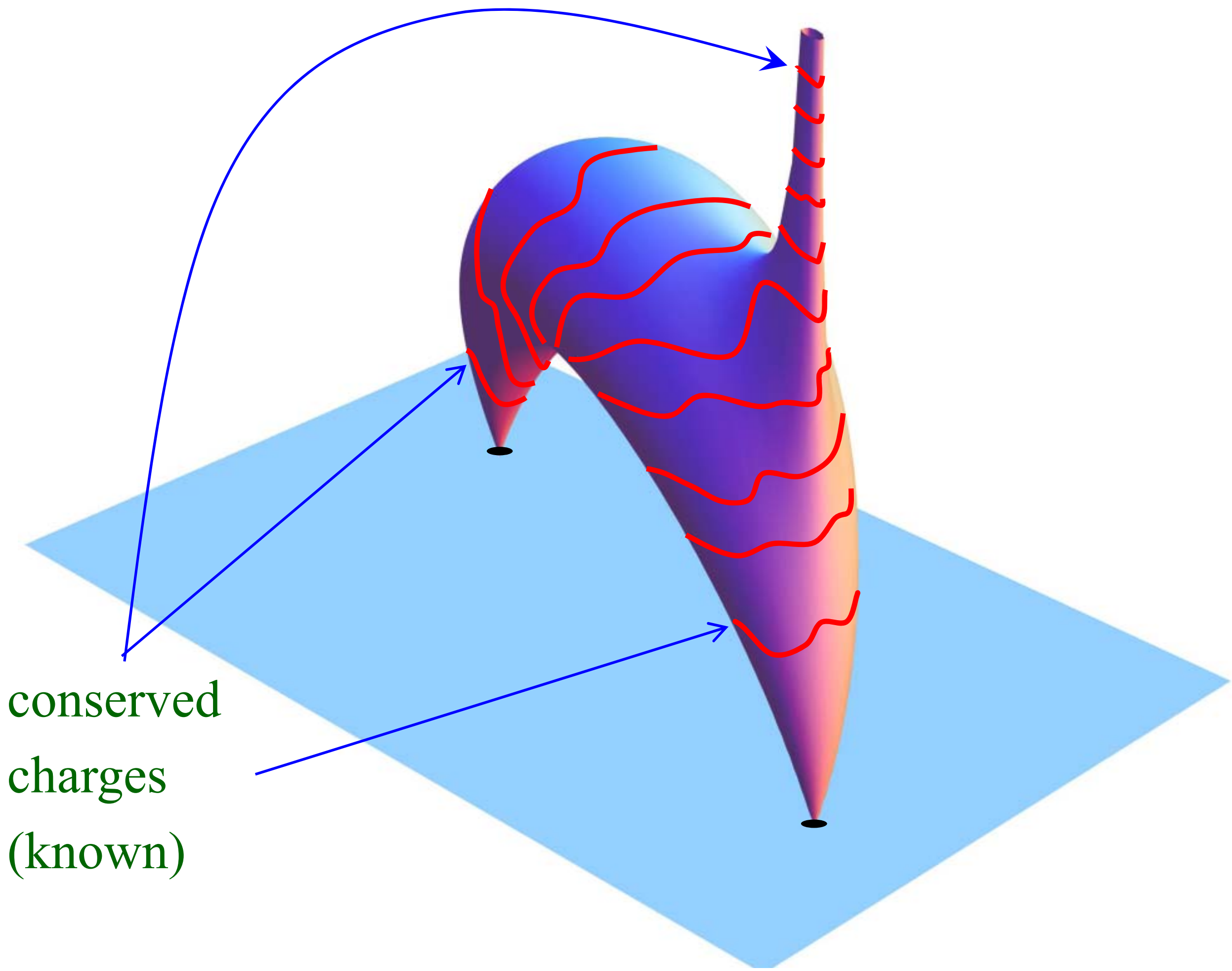
∞ number
of conservation laws



Bookeeping of
conserved charges:

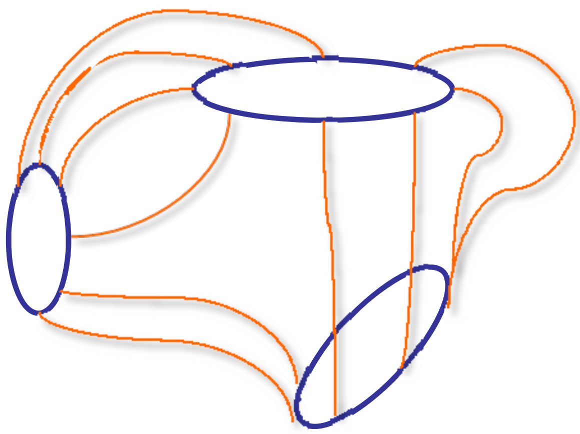


Integrability in 3-point functions?



Algebraic curves for external states + branching?

Weak coupling



…
Drukker,Plefka'09
Escobedo,Gromov,Sever,Vieira'10

- Overlap of three spin chain states
- Certain resemblance to string field theory vertex Okuyama,Tseng'04
- Can be efficiently computed using ABA Escobedo,Gromov,Sever,Vieira'10
- Still not enough to take the large-charge limit to compare to strong coupling

Questions

- Possible to compute the <LH…H> correlation functions (H – heavy semiclassical states, L – light supergravity state)
- How to calculate <HHH>?

Can give a clue to exact solution…

- How to use integrability?

Z.'10
Costa,Monteiro,Santos,Zoakos'10
Roiban,Tseytlin'10
Hernandez'10
Buchbinder,Tseytlin'10

- Vertex operators $\overset{?}{\leftrightarrow}$ Classical Solutions $\leftrightarrow$ Bethe ansatz
- Boundary conditions for generic vertex operators