Stable and Quasi-stable closed k-flux tubes in $D = 2 + 1$ $SU(N)$ Gauge Theories.

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Large-N Gauge Theories
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* With Mike Teper and Barak Bringoltz
mostly based on
arXiv:0709.0693, **arXiv:0812.0334**, **arXiv:1103.5854**
and **NEW RESULTS**

Overview

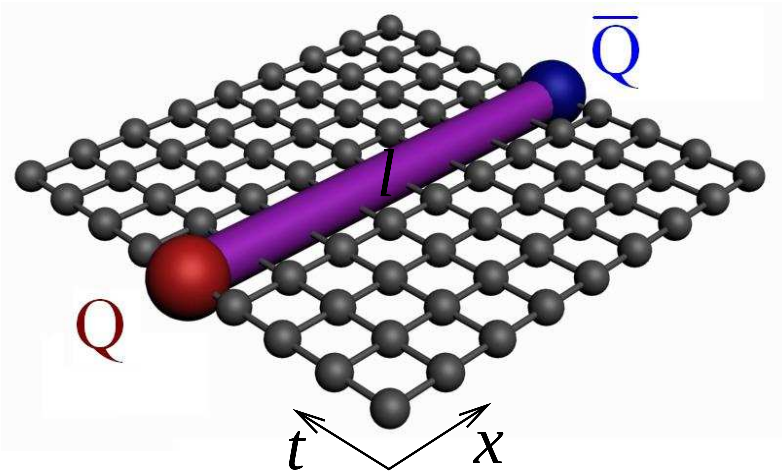
1. **General Considerations.**
2. **Open/Closed flux-tubes**
3. k**-strings.**
4. **Theoretical Expectations.**
   - **Nambu-Goto String.**
   - **Effective String Theory.**
5. **Lattice Calculation.**
6. **Quantum Numbers.**
7. **Results.**
   - $k = 1$.
   - $k = 2$.
   - $k = 3$.
8. **Conclusions.**

1. General Considerations

- What effective string theory describes the confining flux-tube?
- Open flux-tube
- **Closed flux-tube** (torelon)
- If source transforms as $\psi(x) \longrightarrow z^k \psi(x)$, $z \in Z_N \Longrightarrow k$-string.
- k-strings
  - AdS/CFT, MQCD. (Strassler, Armoni, Shiffman...)
  - Hamiltonian approach. (Karabali, Nair, ...)
  - **Lattice**. (Teper, Lucini, Bringlotz,...)
- Recently:
  - $k = 1$ spectrum in $D = 2 + 1$ close to Nambu-Goto. [arXiv:0709.0693].
  - $k = 2$ low-lying spectrum in $D = 2 + 1$ close to Nambu-Goto. [arXiv:0812.0334].
  - $k = 1$ spectrum in $D = 3 + 1$ mostly close to Nambu-Goto. [arXiv:1007.4720].
  - $k = 1$ New on $D = 2 + 1$. [arXiv:1103.5854].
- Today:
  - $k = 1$ spectrum of fundamental and two higher irreducible representations.
  - $k = 2$ spectrum of Antisymmetric and Symmetric irreducible representations
  - $k = 3$ spectrum of Antisymmetric, Mixed and Symmetric irreducible representations.

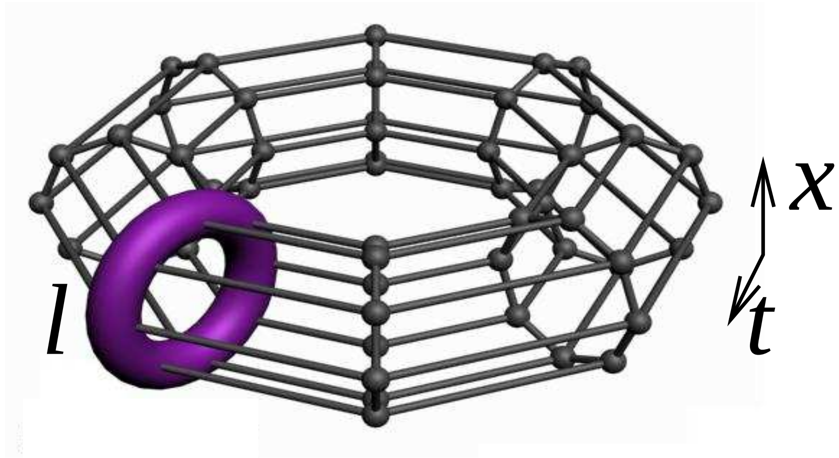
2. Open/Closed flux-tubes

Open flux-tube



Periodic B.C

Closed flux-tube



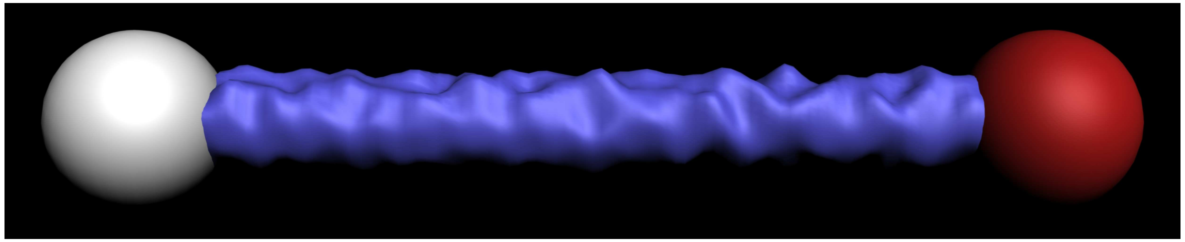
$$\Phi(l,t) = \psi^{\dagger}(0,t)U(0,l;t)\psi(l,t)$$

$$\Phi(l,t) = \mathrm{Tr}U(l;t)$$

Left with a closed tube of flux winding around the torus!

3. k-strings: General

- Confinement in 3-d $SU(N)$ leads to a linear potential between colour charges in the fundamental representation.



- For $SU(N \geq 4)$ there is a possibility of new **stable strings** which join test charges in representations higher than the fundamental!
- We can label these by the way the test charge transforms under the center of the group: $\psi(x) \longrightarrow z^k \psi(x)$, $z \in Z_N$.
- The string has N-ality k and these new **stable flux-tubes** are called k-strings.
- k-string refers to the most energetically favorable flux-tube $\rightarrow$ **STABLE**
- Flux-tubes emanating in higher irreducible representations $\rightarrow$ possible to exist!
- **UNSTABLE** since gluons will screen the sources down to the states with lower energy.
- Such screening is suppressed as $N \rightarrow \infty$, hence, **QUASI-STABLE**.
- "Casimir conjecture" $\rightarrow$ Flux-tube string tension is proportional to the Quadratic Casimir.

3. k-strings: Operators

- A tube of fundamental flux winding once around the torus of length $aL_{||}$
- A generic operator $\Phi_{\mathcal{C}}$ expected to couple to such a closed flux-tube:

$$\text{Polyakov Loop} \equiv \Phi_{\mathcal{C}} = \text{Tr}\left\{\prod_{n_{||}=1}^{L_{||}} U_{||}(n_{||},\mathcal{C})\right\}$$

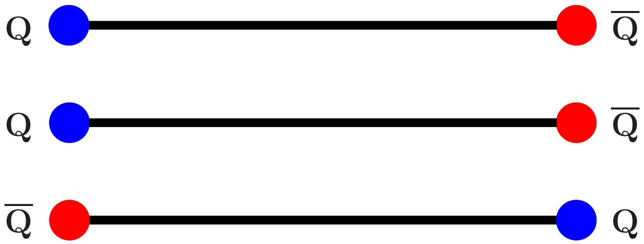
- $\Phi_{k=1} = \text{Tr}\{l_{\mathcal{C}}\}$: under the centre of the group: $l_{\mathcal{C}} \to z l_{\mathcal{C}} \equiv z^{k=1} l_{\mathcal{C}}$.
- A k-flux-tube operator will transform as $l_{\mathcal{C},k} \to z^k l_{\mathcal{C},k}$
- If k, then a generic operator would be given by:

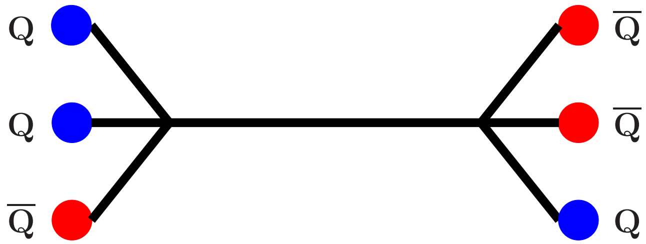
$$\Phi = \text{Tr}\{l_{\mathcal{C}}^{k+k^- - i}\}\text{Tr}\{l_{\mathcal{C}}\}^i \text{Tr}\{{l^\dagger}_{\mathcal{C}}^{k^- - j}\}\text{Tr}\{l_{\mathcal{C}}^\dagger\}^j$$

with: $k^- = 0,\ldots,k^+ - 1$, $j = 0,\ldots\ k^-$, $i = 0,\ldots,k$.

3. k-strings: Operators

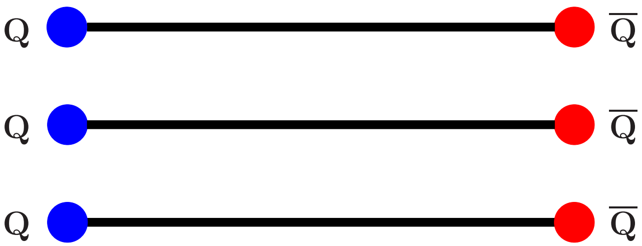
- $k = 1$ flux-tubes:

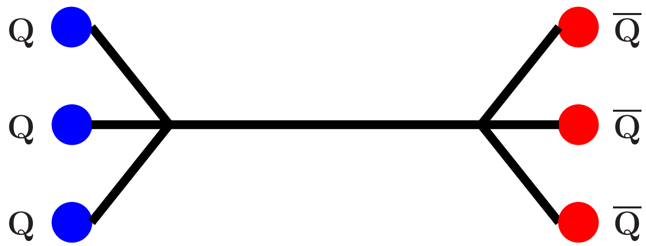




- $k = 2$ flux-tubes:

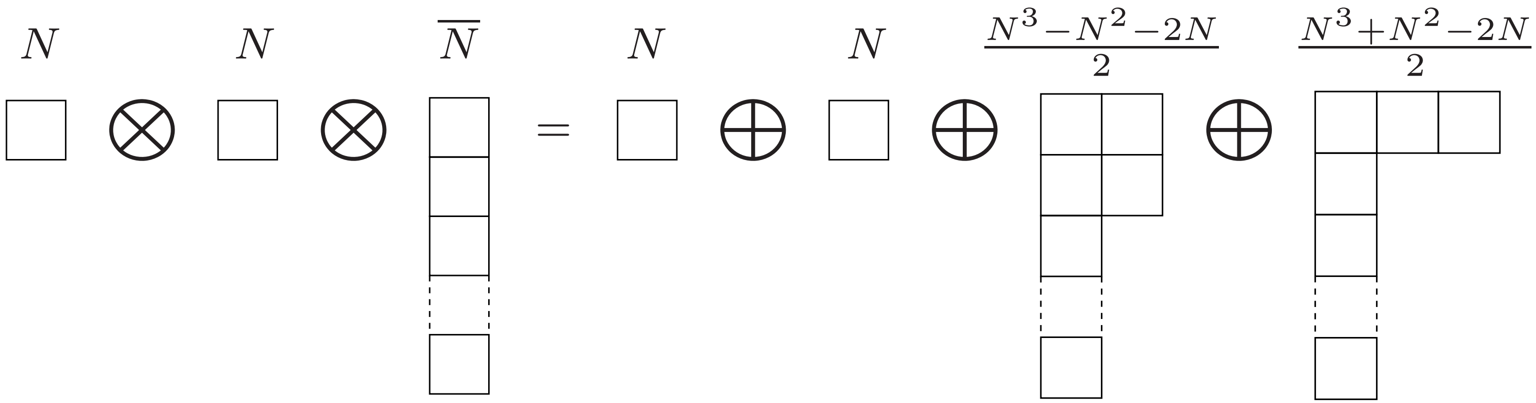
- $k = 3$ flux-tubes:





3. $k = 1$ flux-tubes: Representations and Operators

Young-Tableau Decomposition:



General Operators:

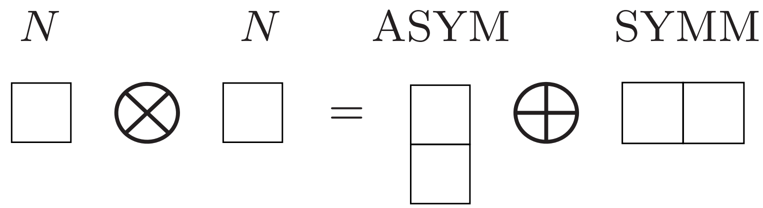
- $\Phi_1 = \mathrm{Tr}[U]$
- $\Phi_2 = \mathrm{Tr}[U^2]\mathrm{Tr}[U^\dagger]$
- $\Phi_3 = \mathrm{Tr}[U]\mathrm{Tr}[U]\mathrm{Tr}[U^\dagger]$

Projecting onto irreducible representations:

- $\Phi_f = \mathrm{Tr}[U]$
- $\Phi_{\boxplus} = \frac{1}{2}\left[\{\mathrm{Tr}[U]\}^2 - \mathrm{Tr}[U^2]\right]\mathrm{Tr}[U^\dagger] - \mathrm{Tr}[U]$
- $\Phi_{\sqsubset\!\!\square} = \frac{1}{2}\left[\{\mathrm{Tr}[U]\}^2 + \mathrm{Tr}[U^2]\right]\mathrm{Tr}[U^\dagger] - \mathrm{Tr}[U]$

3. $k = 2$ flux-tubes: Representations and Operators

Young-Tableau Decomposition:



General Operators:

- $\Phi_1 = \text{Tr}[U]\text{Tr}[U]$
- $\Phi_2 = \text{Tr}[U^2]$

Projecting onto Irreducible Representations:

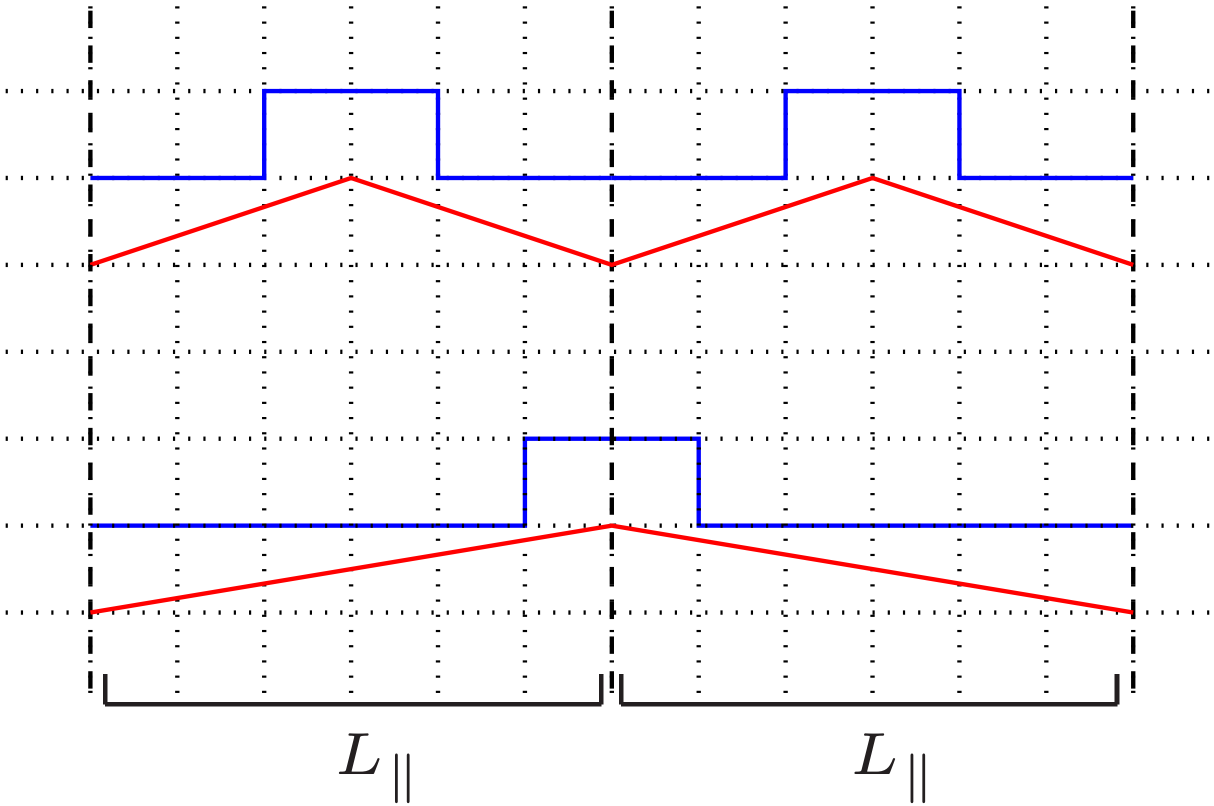
- $\Phi_{\text{AS}} = \frac{1}{2}\left[\{\text{Tr}[U]\}^2 - \text{Tr}[U^2]\right]$
- $\Phi_{\text{SY}} = \frac{1}{2}\left[\{\text{Tr}[U]\}^2 + \text{Tr}[U^2]\right]$

However!

A double winding fundamental flux-tube is a subgroup of $k = 2$ closed flux-tubes.

3. $k = 2$ flux-tubes: Representations and Operators

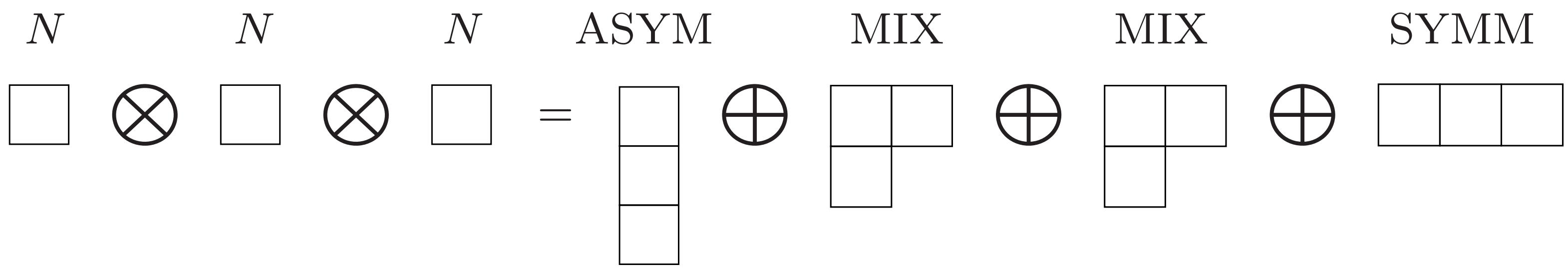
One can also think of $w = 2$ states: These would be tubes of **fundamental flux** winding twice around the torus.



As shown, extra $k = 2$ states coupled to these operators do exist! $\rightarrow$ [arXiv:0812.0334].

3. $k = 3$ flux-tubes: Representations and Operators

Young-Tableau Decomposition:



General Operators:

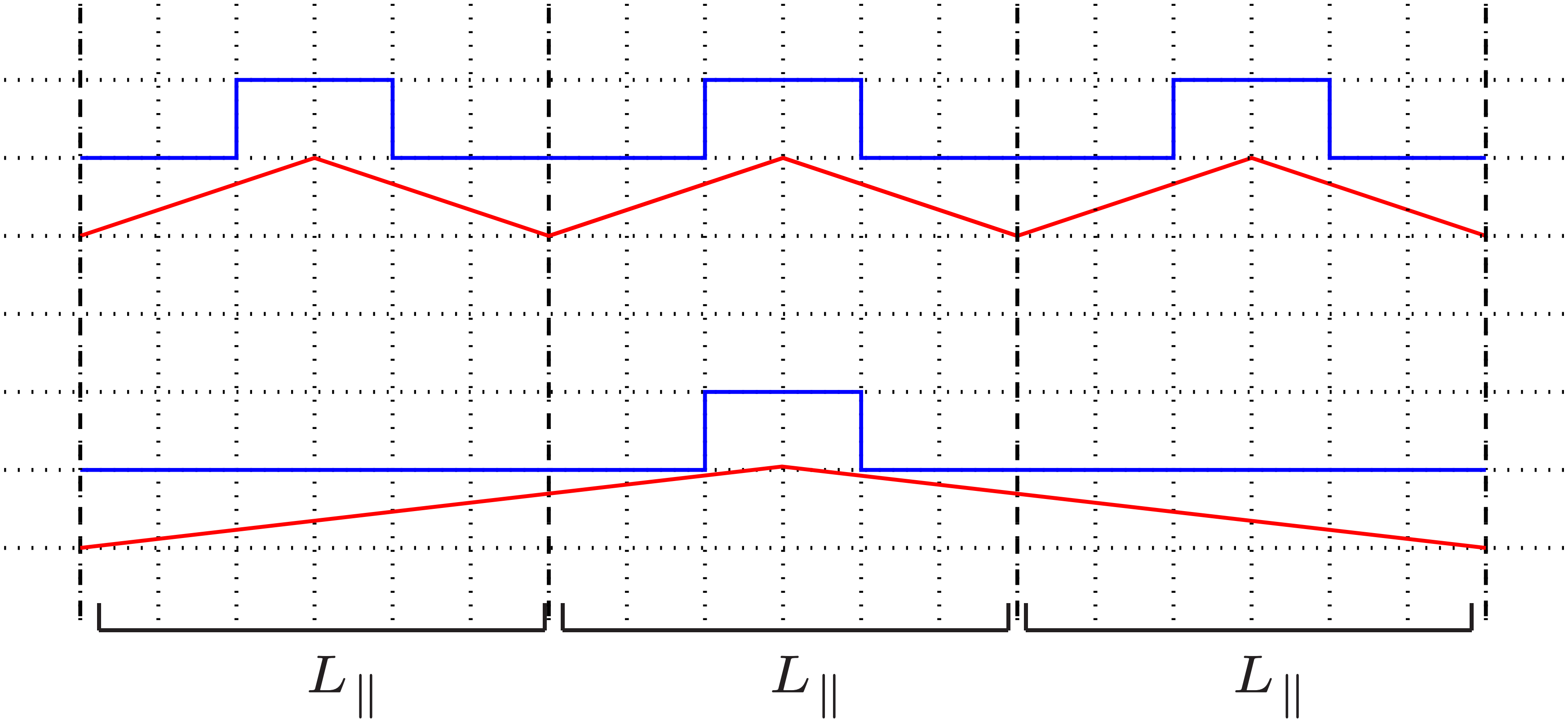
- $\Phi_1 = \mathrm{Tr}[U]\mathrm{Tr}[U]\mathrm{Tr}[U]$
- $\Phi_2 = \mathrm{Tr}[U]\mathrm{Tr}[U^2]$
- $\Phi_3 = \mathrm{Tr}[U^3]$

Projecting onto the irreducible representations:

- $\Phi_{\mathrm{AS}} = \frac{1}{6}\left[\{\mathrm{Tr}[U]\}^3 - 3\mathrm{Tr}[U]\left\{\mathrm{Tr}[U^2]\right\} + 2\mathrm{Tr}[U^3]\right]$
- $\Phi_{\mathrm{MX}} = \frac{1}{3}\left[\{\mathrm{Tr}[U]\}^3 - \mathrm{Tr}[U^3]\right]$
- $\Phi_{\mathrm{SY}} = \frac{1}{6}\left[\{\mathrm{Tr}[U]\}^3 + 3\mathrm{Tr}[U]\left\{\mathrm{Tr}[U^2]\right\} + 2\mathrm{Tr}[U^3]\right]$

3. $k = 3$ flux-tubes: Representations and Operators

One can also think of $w = 3$ states....



However, these states (if exist) would be **heavy**, **hard to identify** and **computationally and memorywise expensive**!!

NO!!

4. Theoretical Expectations: Nambu-Goto String

- The spectrum of a closed bosonic string compactified around a torus is:

$$E^2_{N_L,N_R,q,w} = (\sigma l w)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2 + p_\perp^2 .$$

- The spectrum is described by:

  1. The winding number w (w=1, 2),

  2. The winding momentum $p_{||} = 2\pi q/l$ with $q = 0, \pm 1, \pm 2, \ldots$.

  3. The transverse momentum $p_\perp$ ($p_\perp = 0$),

  4. $N_L = \sum_{k>0} n_L(k)k$ and $N_R = \sum_{k'>0} n_R(k')k'$

  5. Level-matching constrain: $N_L - N_R = qw$.

- How do we construct the string states:

$$(\alpha^{i_1}_{-k_1})^{n_L(k_1)} \ldots (\alpha^{i_{m_L}}_{-k_{m_L}})^{n_R(k_{m_L})} (\bar{\alpha}^{i'_1}_{-k'_1})^{n_R(k'_1)} \ldots (\bar{\alpha}^{i'_{m_R}}_{-k'_{m_R}})^{n_R(k'_{m_R})} |0\rangle$$

($i = 1, \ldots, D-2$)

- Example: $\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1}|0\rangle$
  - ⋆ $N_L = 3$
  - ⋆ $N_R = 1$
  - ⋆ $q = 2$

4. Theoretical Expectations: Nambu-Goto String

- The Nambu-Goto energy for $w = 1$ and $q = 0$, in dimensionless units is written as:

$$\frac{E_n(l)}{\sqrt{\sigma}} = l\sqrt{\sigma}\left(1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{1}{24}\right)\right)^{\frac{1}{2}} \quad \text{with} \quad n = \frac{N_L + N_R}{2}$$

- The above expression can be expanded in $1/l\sqrt{\sigma}$ for:

$$l\sqrt{\sigma} > l_c^{N.G}\sqrt{\sigma} = \left\{8\pi\left(n - \frac{1}{24}\right)\right\}^{\frac{1}{2}}$$

- The expansion is written as:

$$\frac{E_n\,(l)}{\sqrt{\sigma}} \overset{l\to\infty}{=} l\sqrt{\sigma} + \frac{c_1^{N.G}}{l\sqrt{\sigma}} + \frac{c_2^{N.G}}{(l\sqrt{\sigma})^3} + \frac{c_3^{N.G}}{(l\sqrt{\sigma})^5} + \frac{c_4^{N.G}}{(l\sqrt{\sigma})^7} + \cdots + \mathcal{O}\left(\frac{1}{(l\sqrt{\sigma})^{\infty}}\right)$$

- The ground state $n = 0$ becomes tachyonic for $\sigma l^2 < \pi/3$.
- In the real world the large-N deconfining transition occurs for $\sigma l^2 > \pi/3$!!

4. Theoretical Expectations: Effective String Theory.

The energy (mass) of a closed flux-tube is expected to be described as:

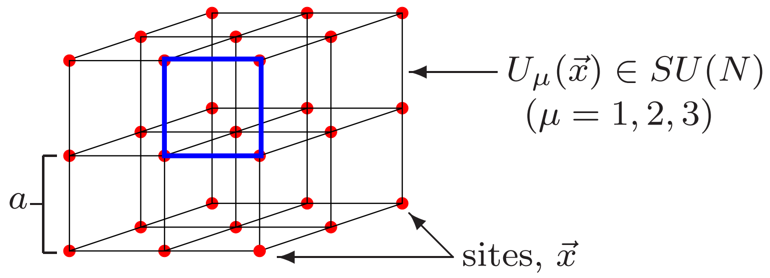
$$
\begin{aligned}
E_n \stackrel{l\to\infty}{=} \quad & \sigma l && \text{linear confinement} \\
+ \quad & \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right) && \text{Lüscher 1980, Polchinski\&Strominger 1991} \\
- \quad & \frac{8\pi^2}{\sigma l^3}\left(n - \frac{D-2}{24}\right)^2 && \text{Lüscher\&Weisz 2004, Drummond 2004} \\
+ \quad & \frac{32\pi^3}{\sigma^2 l^5}\left(n - \frac{D-2}{24}\right)^3 && \text{Aharony\&Karzbrun 2009}
\end{aligned}
$$

Relation to Nambu-Goto:

$$
\frac{E_n(l)}{\sqrt{\sigma}} \stackrel{l\to\infty}{=} l\sqrt{\sigma} + \frac{c_1^{N.G}}{l\sqrt{\sigma}} + \frac{c_2^{N.G}}{(l\sqrt{\sigma})^3} + \frac{c_3^{N.G}}{(l\sqrt{\sigma})^5} + \mathcal{O}\left(\frac{1}{(l\sqrt{\sigma})^7}\right)
$$

5. Lattice Calculation: Lattice Setup

- Define the gauge theory on a $D = 3$ discretized periodic Euclidean space-time with $L_{\|} \times L_{\perp} \times L_T$ sites.



- We use the standard Wilson Action:

$$
\begin{aligned}
S_L &= \beta \sum_p \{1 - \frac{1}{N_c} \mathrm{ReTr} U_p\} \\
\beta &= \frac{2N_c}{ag^2} (D = 2+1) \\
\lambda &= g^2 N \\
\beta &= \frac{2N_c^2}{a\lambda}
\end{aligned}
$$

- Energies can be calculated using the correlation functions of specific operators:

$$
\begin{aligned}
C(t) &= \langle \Phi^\dagger(t)\Phi(0)\rangle = \langle \Phi^\dagger(0) e^{-Ht} \Phi(0)\rangle \\
&= |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t} + \sum_{n=1} |\langle n|\Phi(0)|vac\rangle|^2 e^{-E_n t} \stackrel{t\to\infty}{\longrightarrow} |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t}
\end{aligned}
$$

5. Lattice Calculation: Variational Technique

- Construct a large basis of Operators $\Phi_i : i = 1, 2, \ldots$ described by the right quantum numbers

- Calculate the correlation function (Matrix) $C_{ij}(t) = \langle \Phi_i^{\dagger}(t) \Phi_j(0) \rangle$

- Diagonalise the matrix $C^{-1}(t = 0)C(t = a)$

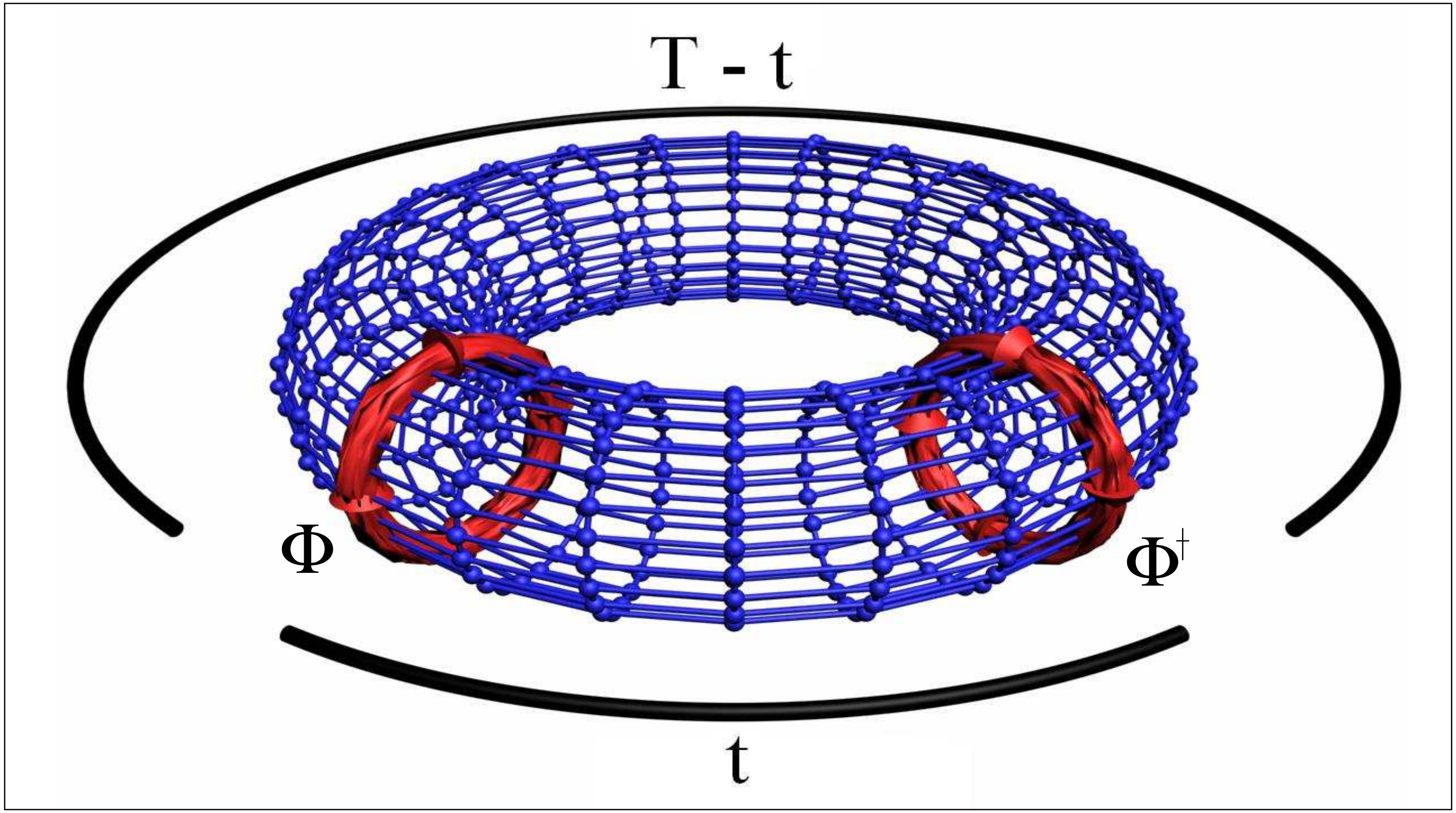
- Extract the eigenvectors

- Extract the correlator for each state ($\sim \mathrm{e}^{-E_n t}$)

- By fitting the results, we extract the mass (energy) for each state.

5. Lattice Calculation: Correlation Function

Pictorialisation of the Correlation Function



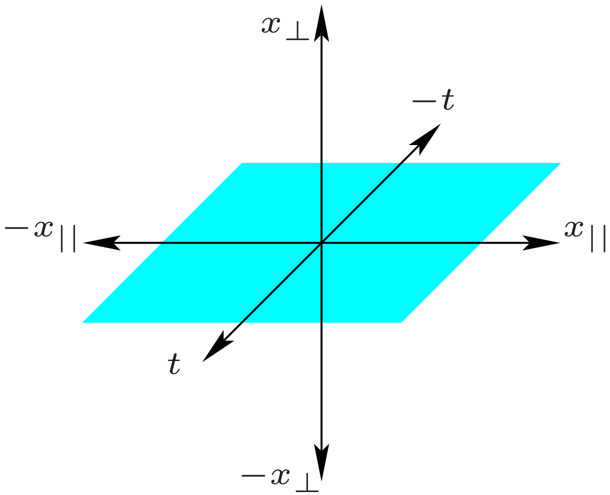
5. Lattice Calculation: Several Computations

Since 2005 a systematic attempt to investigate the closed-flux-tube spectrum.

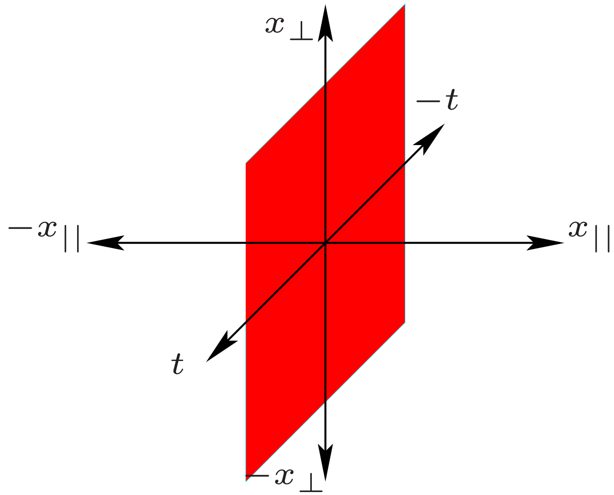
Gauge Group	Inverse coupling β	Lattice spacing $a\sqrt{\sigma}$	N-alities
$SU(3)$	21.0	0.17392(11)	$k = 1$
$SU(3)$	40.0	0.08712(10)	$k = 1$
$SU(4)$	50.0	0.13084(21)	$k = 1, 2$
$SU(5)$	80.0	0.12976(11)	$k = 1, 2$
$SU(6)$	90.0	0.17184(12)	$k = 1$
$SU(6)$	171.0	0.08582(5)	$k = 1, 2, 3$

6. Quantum Numbers: Parity

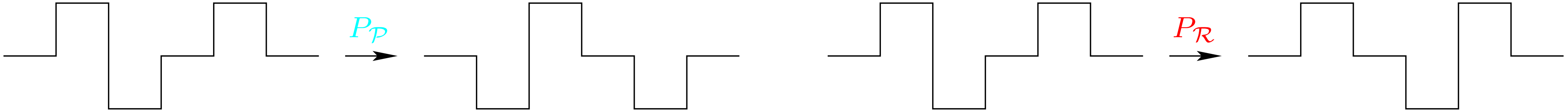
$\mathcal{P}$-Parity reflection plane



$\mathcal{R}$-Parity reflection plane



Transformation of an operator under the Parity:



6. Quantum Numbers: Parity

$P_{\mathcal{P}}$ Parity:

- Under $P_{\mathcal{P}}$ parity $(x_{||}, x_{\perp}) \rightarrow (x_{||}, -x_{\perp})$ and, therefore, $\alpha_{-k} \longleftrightarrow -\alpha_{-k}$ and $\bar{\alpha}_{-k} \longleftrightarrow -\bar{\alpha}_{-k}$.
- The parity of a state is given:

$$P_{\mathcal{P}} = (-1)^{number\ of\ phonons}$$

- For instance:
  - Even number of phonons, for example $\alpha_{-2}\bar{\alpha}_{-1}|0\rangle$, transforms as $P_{\mathcal{P}} = +$.
  - Odd number of phonons, for example $\alpha_{-1}|0\rangle$, transforms as $P_{\mathcal{P}} = -$.

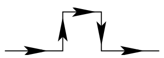
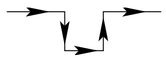
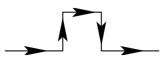
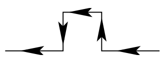
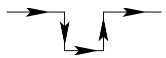
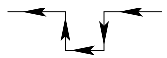
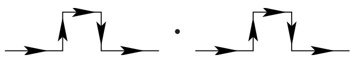
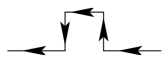
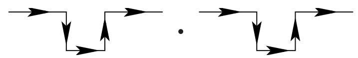
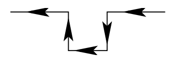
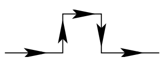
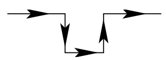
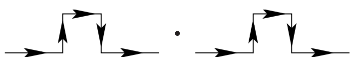
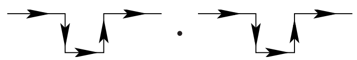
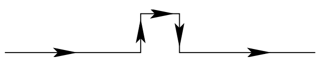
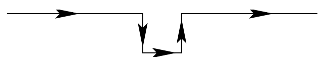
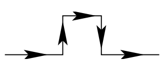
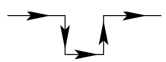
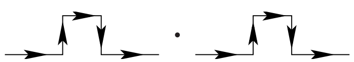
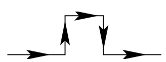
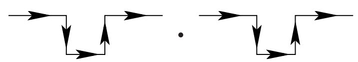
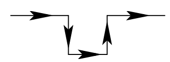
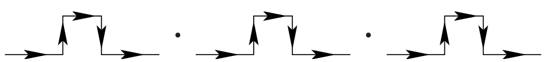
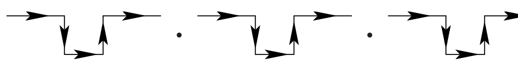
$P_{\mathcal{R}}$ Parity:

- Under $P_{\mathcal{R}}$ Parity: $\alpha_{-k} \longleftrightarrow \bar{\alpha}_{-k}$
- Only useful in the $q = 0$ sector
- The only non-null pair of states with $P_{\mathcal{R}} = \pm$ is for $P_{\mathcal{P}} = -$:

$$\{\alpha_{-2}\bar{\alpha}_{-1}\bar{\alpha}_{-1} \pm \alpha_{-1}\alpha_{-1}\bar{\alpha}_{-2}\}|0\rangle$$

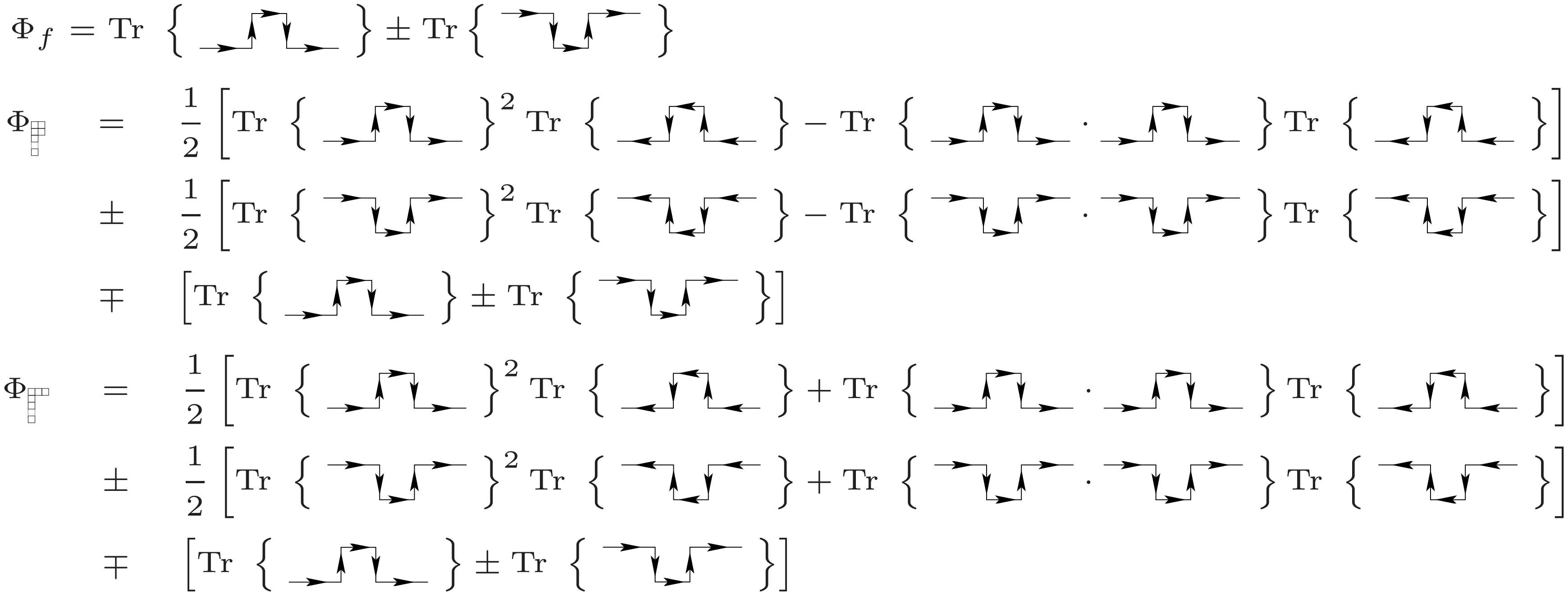
- This is quite heavy!
- In practice this Quantum Number is of minor utility.

6. Quantum Numbers: Operators

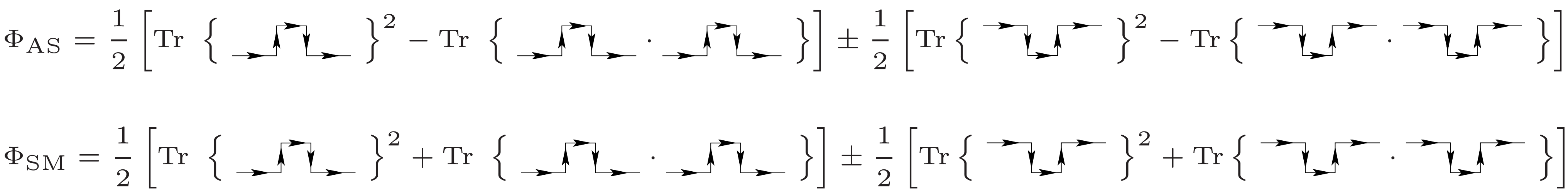
- Excited States: $p \neq 0$, $P_{\mathcal{P}} = \pm$
  - Trivial Case: Simple line Polyakov Loop $\longrightarrow$ $P_{\mathcal{P}} = +$ and $p_{||} = 0$
- Example: operators with tranverse deformations for $P_{\mathcal{P}} = \pm$ and $k = 1$:
  - $\phi_1 = \mathrm{Tr}$ {  } $\pm$ Tr {  }
  - $\phi_2 = \mathrm{Tr}$ {  }^{2} Tr {  } $\pm$ Tr {  }^{2} Tr {  }
  - $\phi_3 = \mathrm{Tr}$ {  } Tr {  } $\pm$ Tr {  } Tr {  }
- Example: operators with tranverse deformations for $P_{\mathcal{P}} = \pm$ and $k = 2$:
  - $\phi_1 = \mathrm{Tr}$ {  }^{2} $\pm$ Tr {  }^{2}
  - $\phi_2 = \mathrm{Tr}$ {  } $\pm$ Tr {  }
  - $\phi_3 = \mathrm{Tr}$ {  } $\pm$ Tr {  } (doubly wound)
- Example: operators with tranverse deformations for $P_{\mathcal{P}} = \pm$ and $k = 3$:
  - $\phi_1 = \mathrm{Tr}$ {  }^{3} $\pm$ Tr {  }^{3}
  - $\phi_2 = \mathrm{Tr}$ {  } Tr {  } $\pm$ Tr {  } Tr {  }
  - $\phi_3 = \mathrm{Tr}$ {  } $\pm$ Tr {  }

6. Quantum Numbers: Operators

- Projection onto the $k = 1$ irreducible representations:

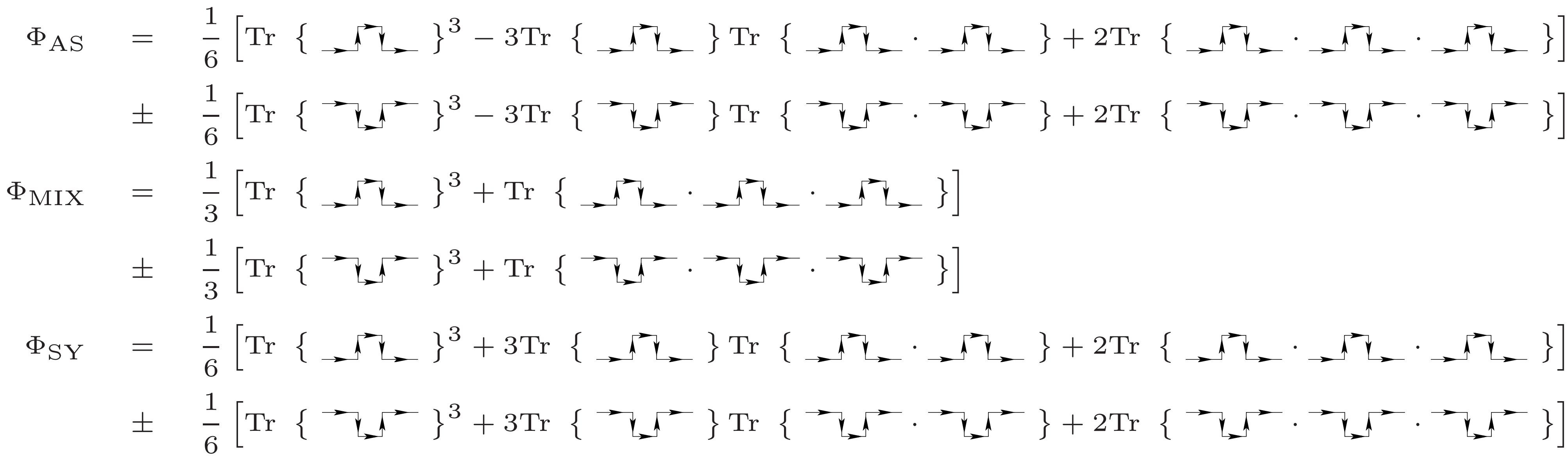


- Projection onto the $k = 2$ irreducible representation:



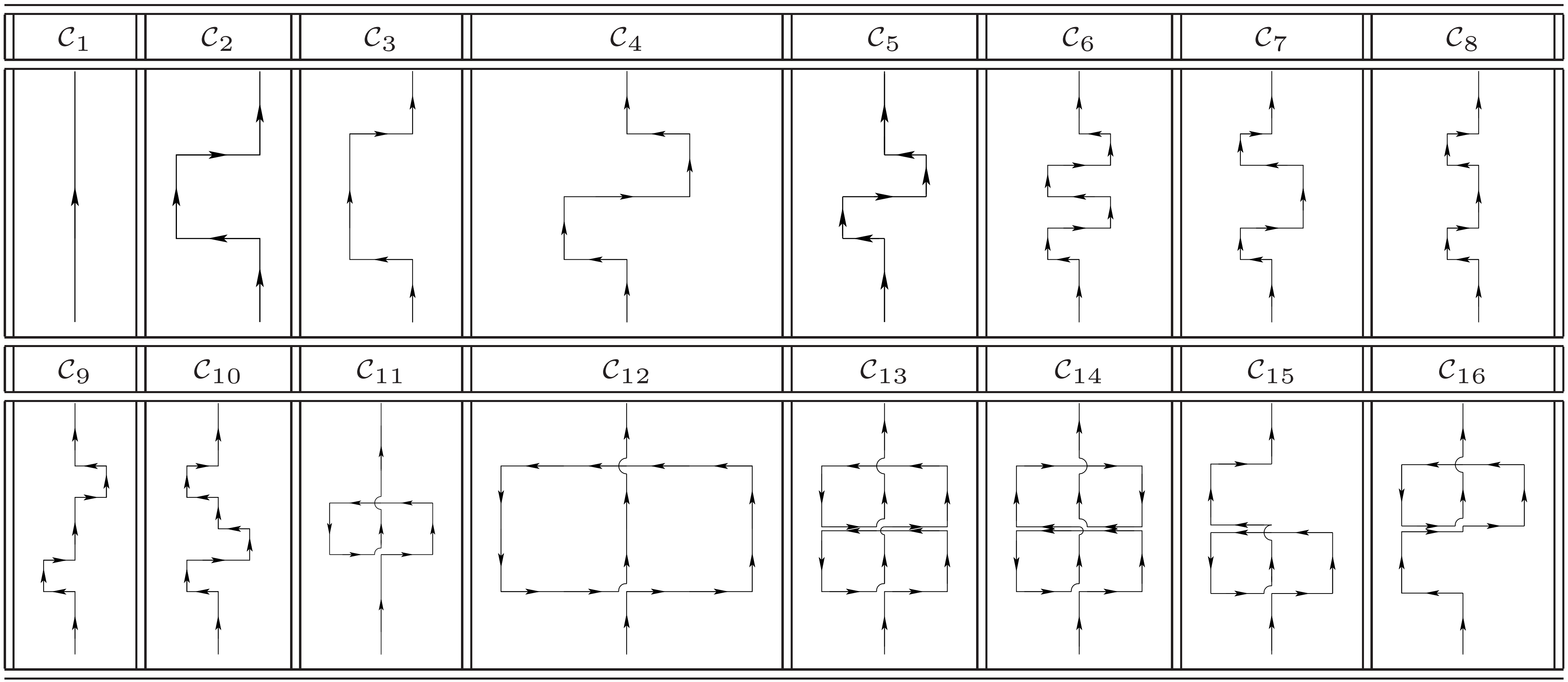
6. Quantum Numbers: Operators

- Projection onto the $k = 3$ irreducible representations:



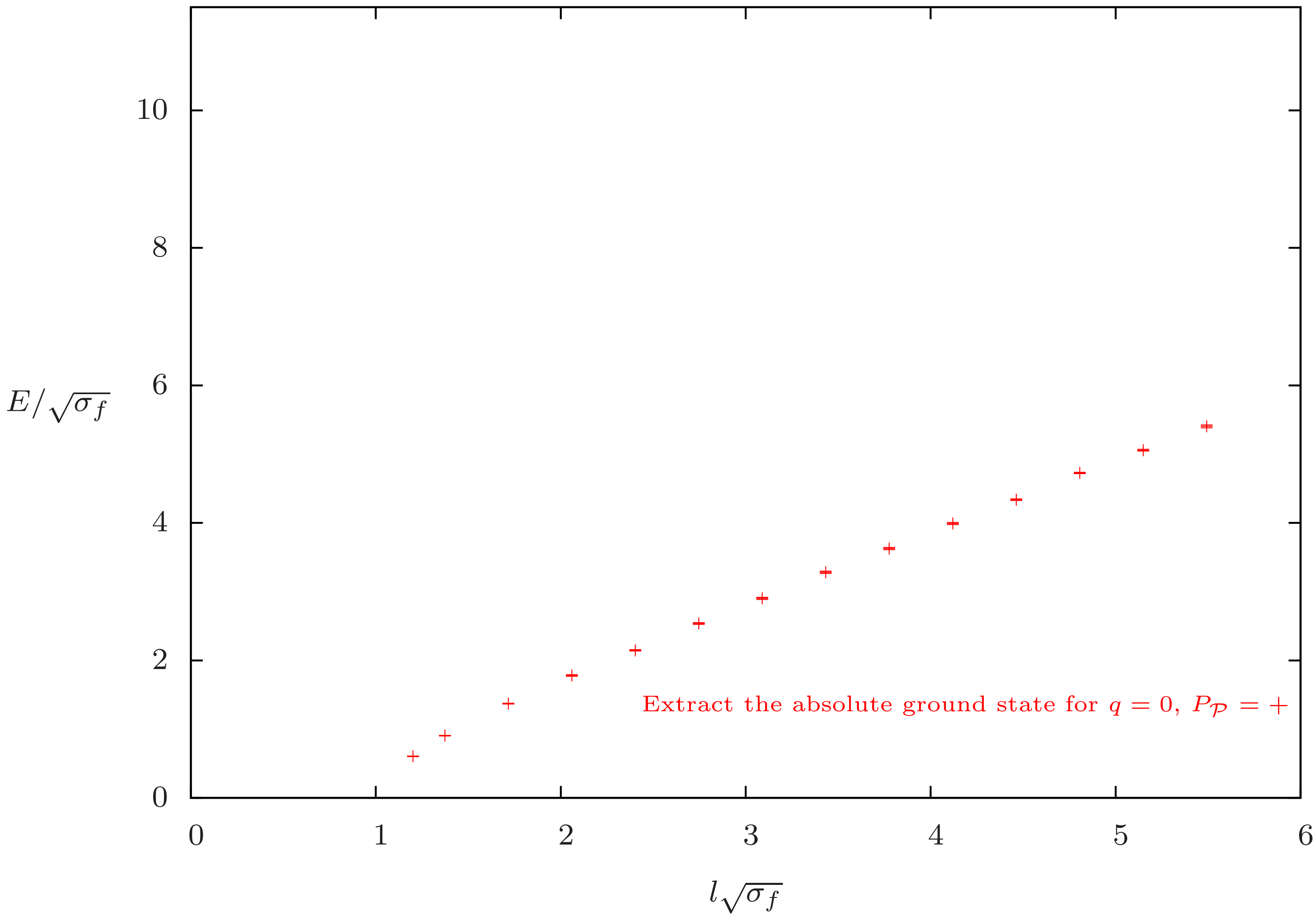
6. Quantum Numbers: Trasnverse Deformations

- We construct $\sim 200 - 300$ (general) operators for every configuration of $P_{\mathcal{P}}$, q and k.
- For each irreducible representation we have ~ 100 operators.
- The transverse deformations we use to construct the operators are the following:

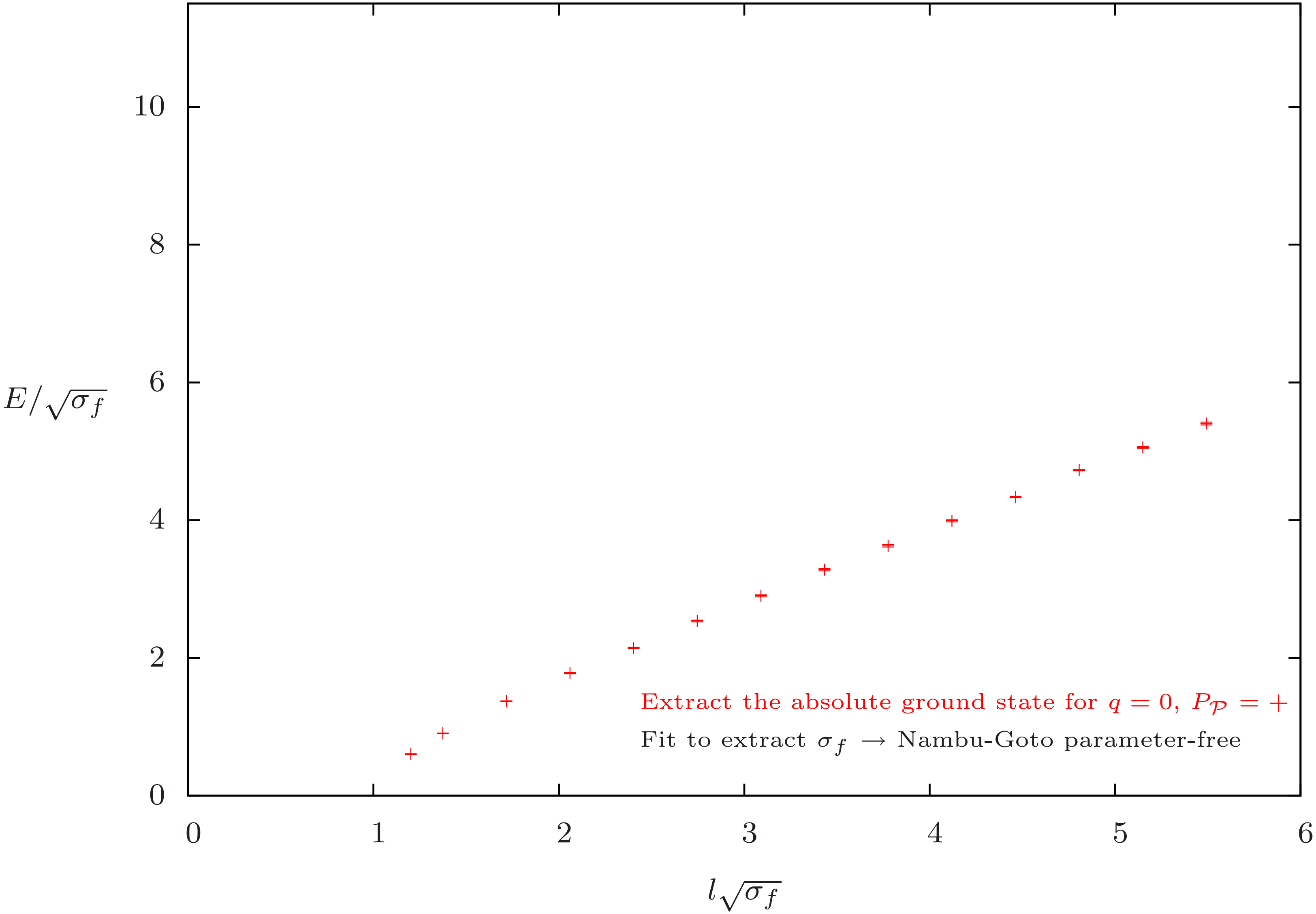


7. RESULTS FOR $k = 1$ FUNDAMENTAL REPRESENTATION

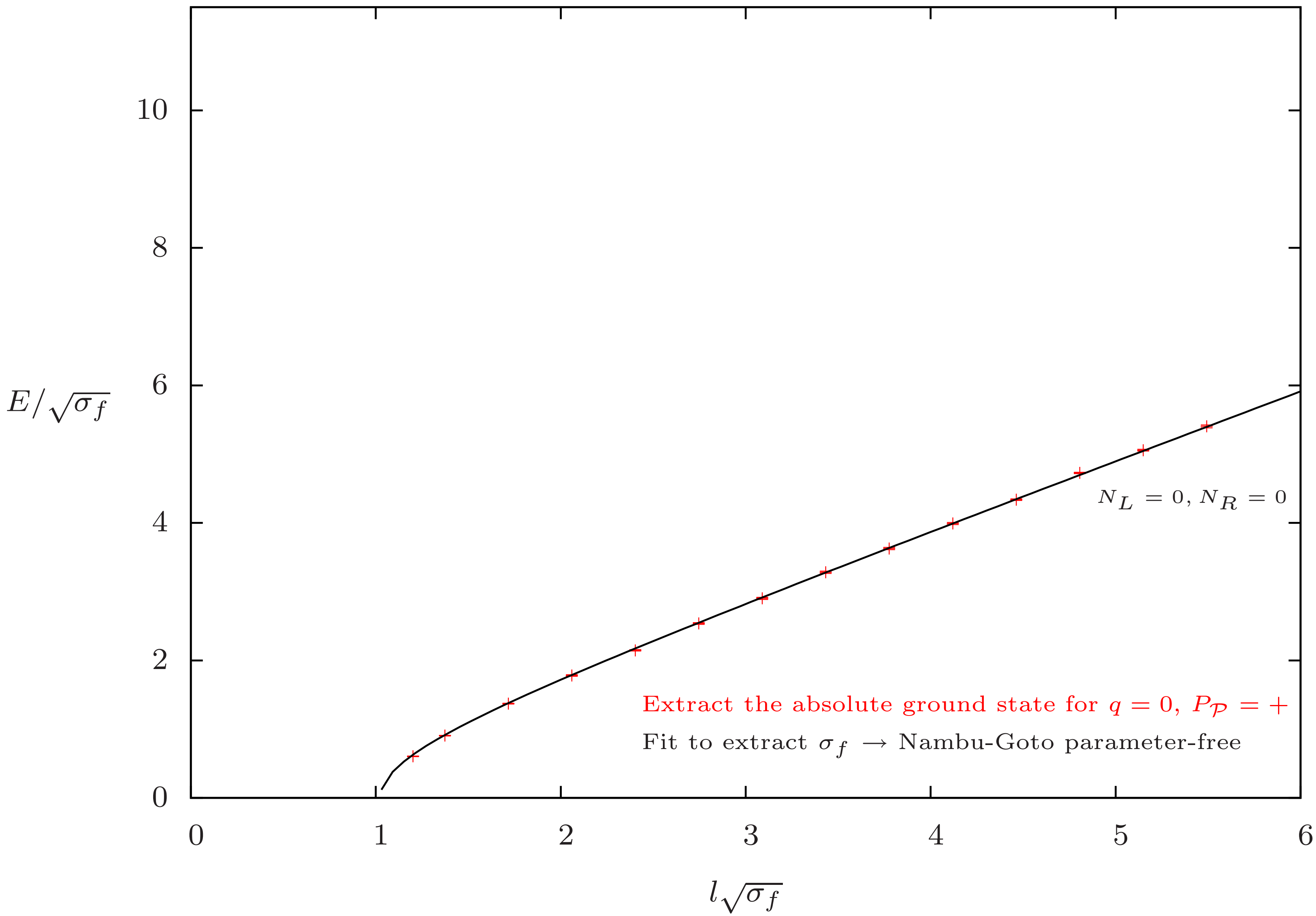
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



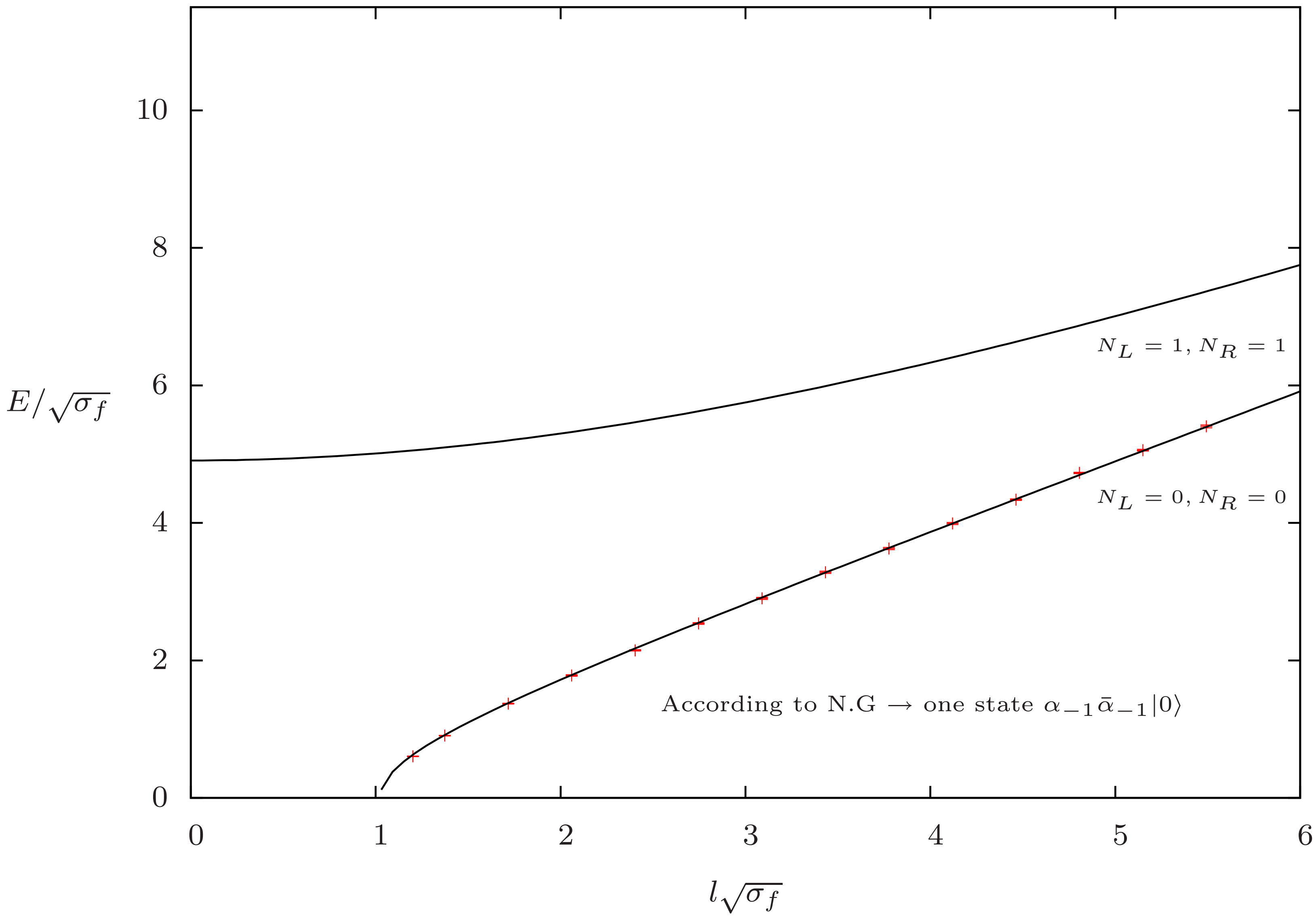
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



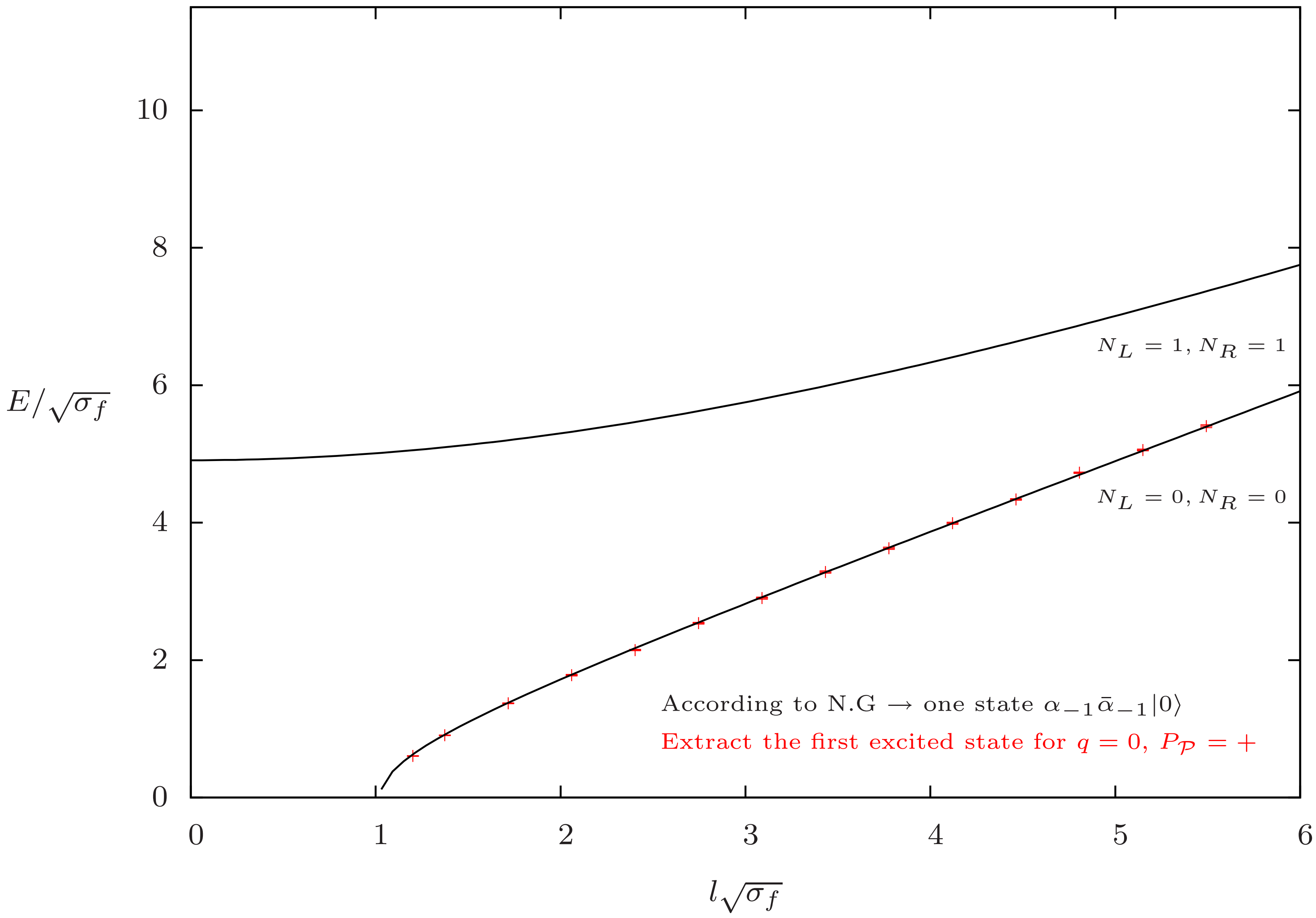
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



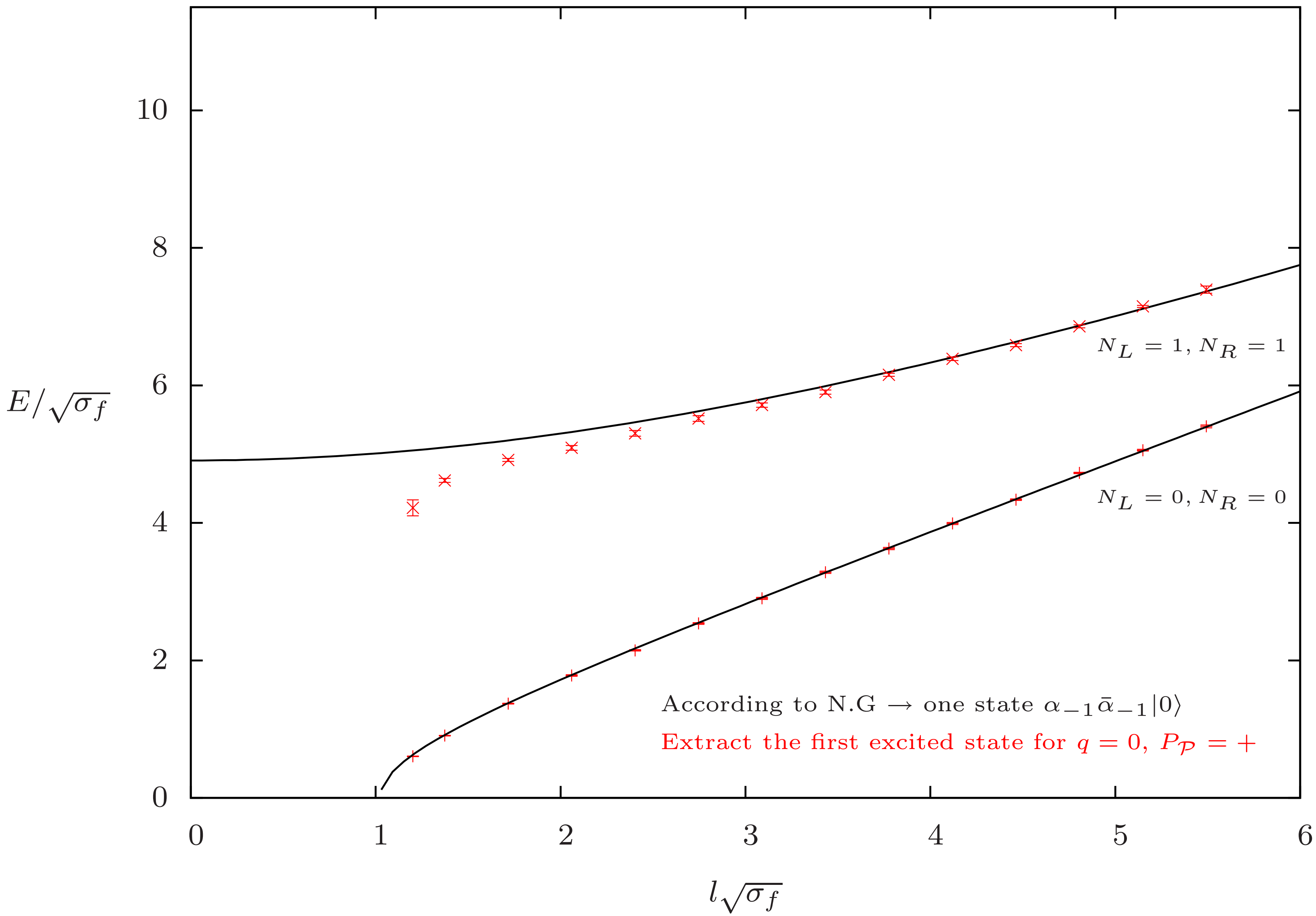
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



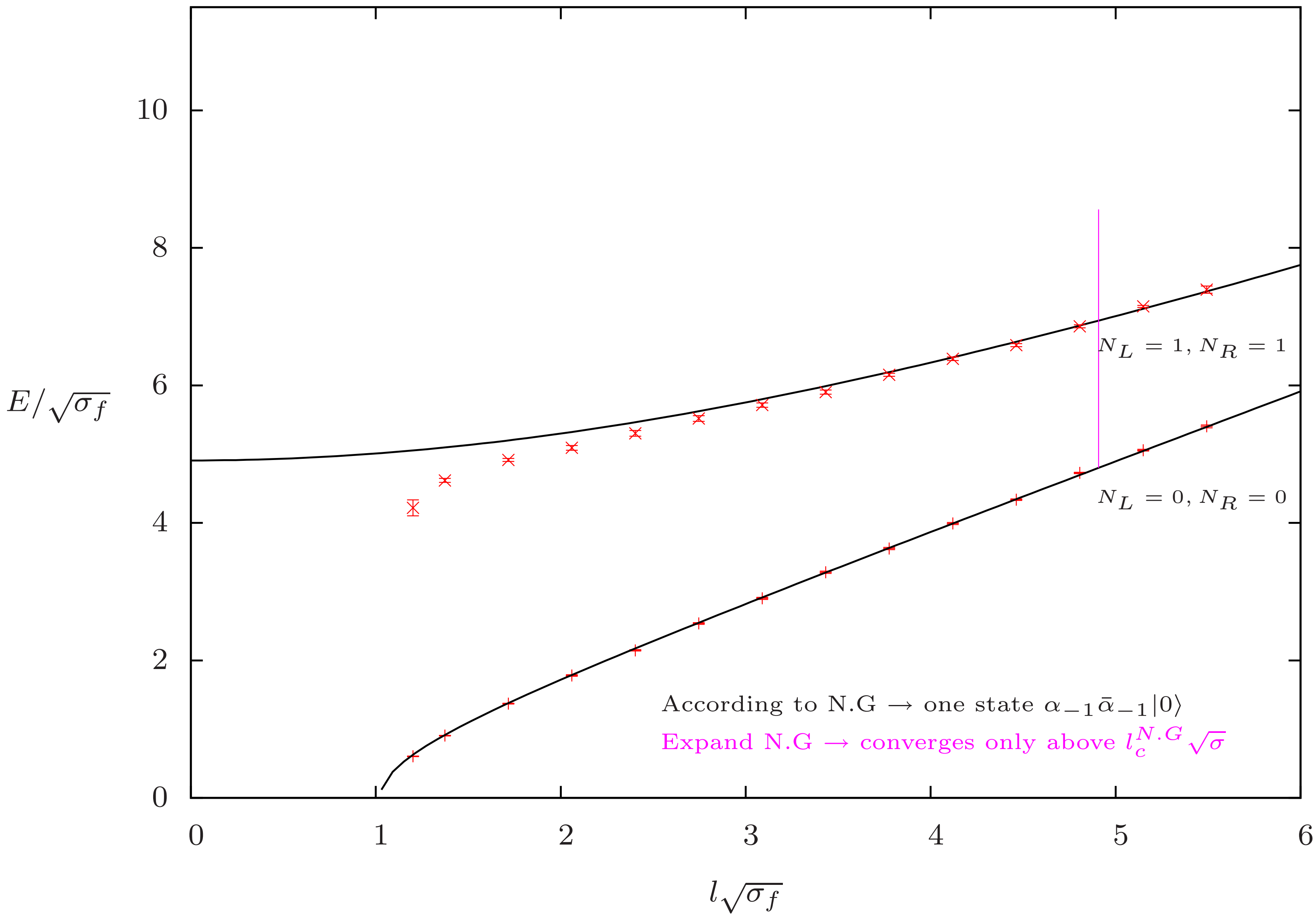
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



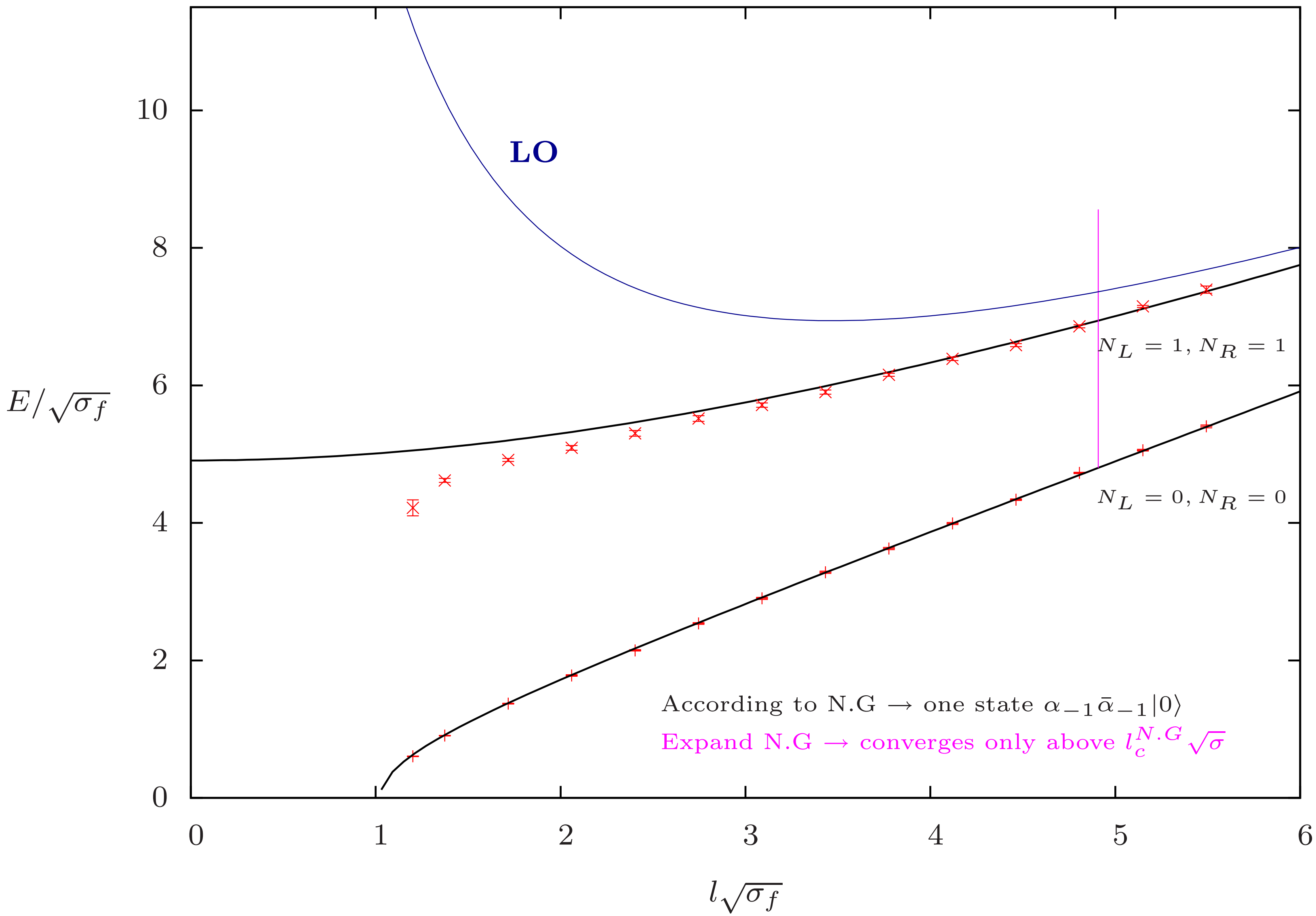
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



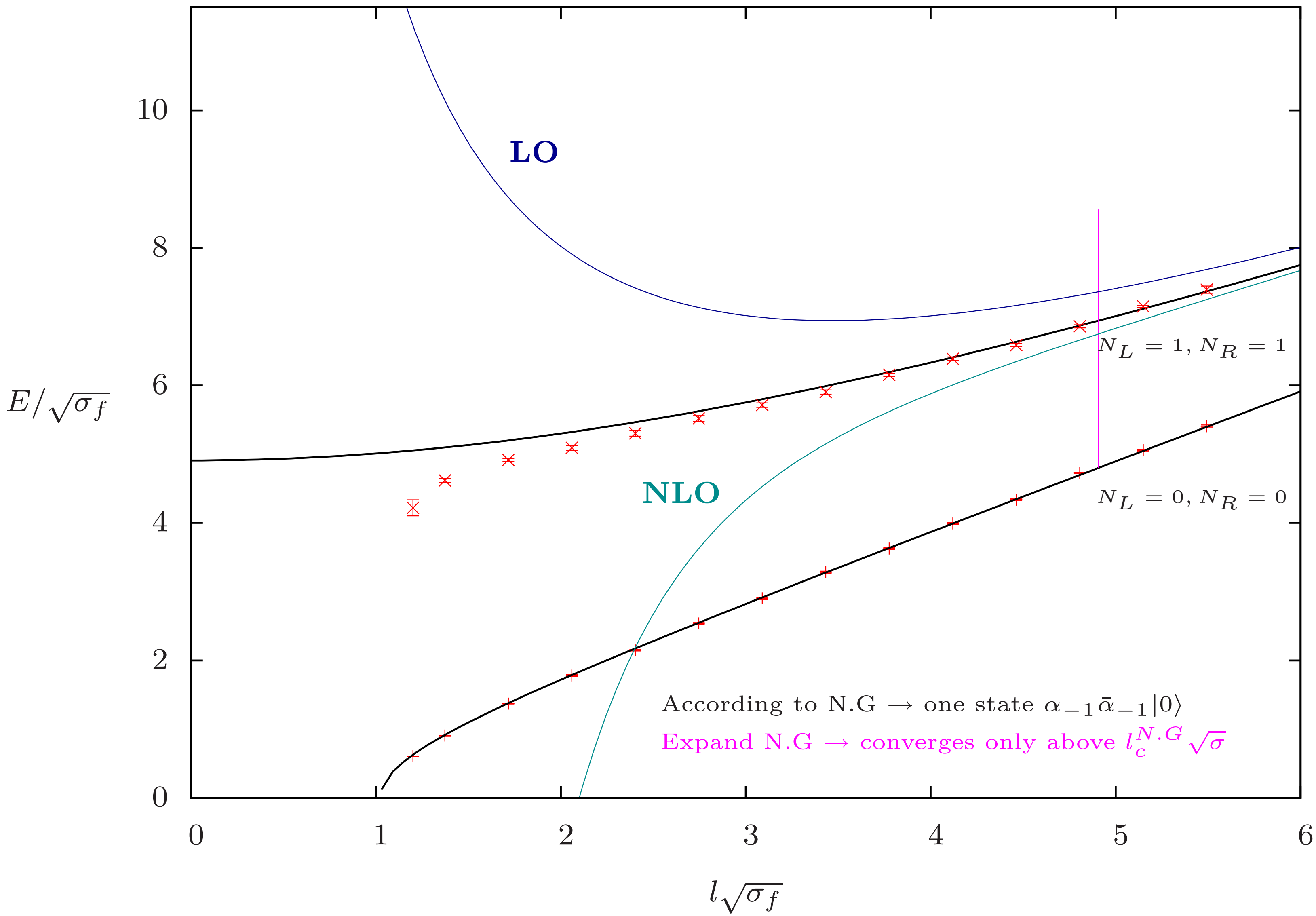
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



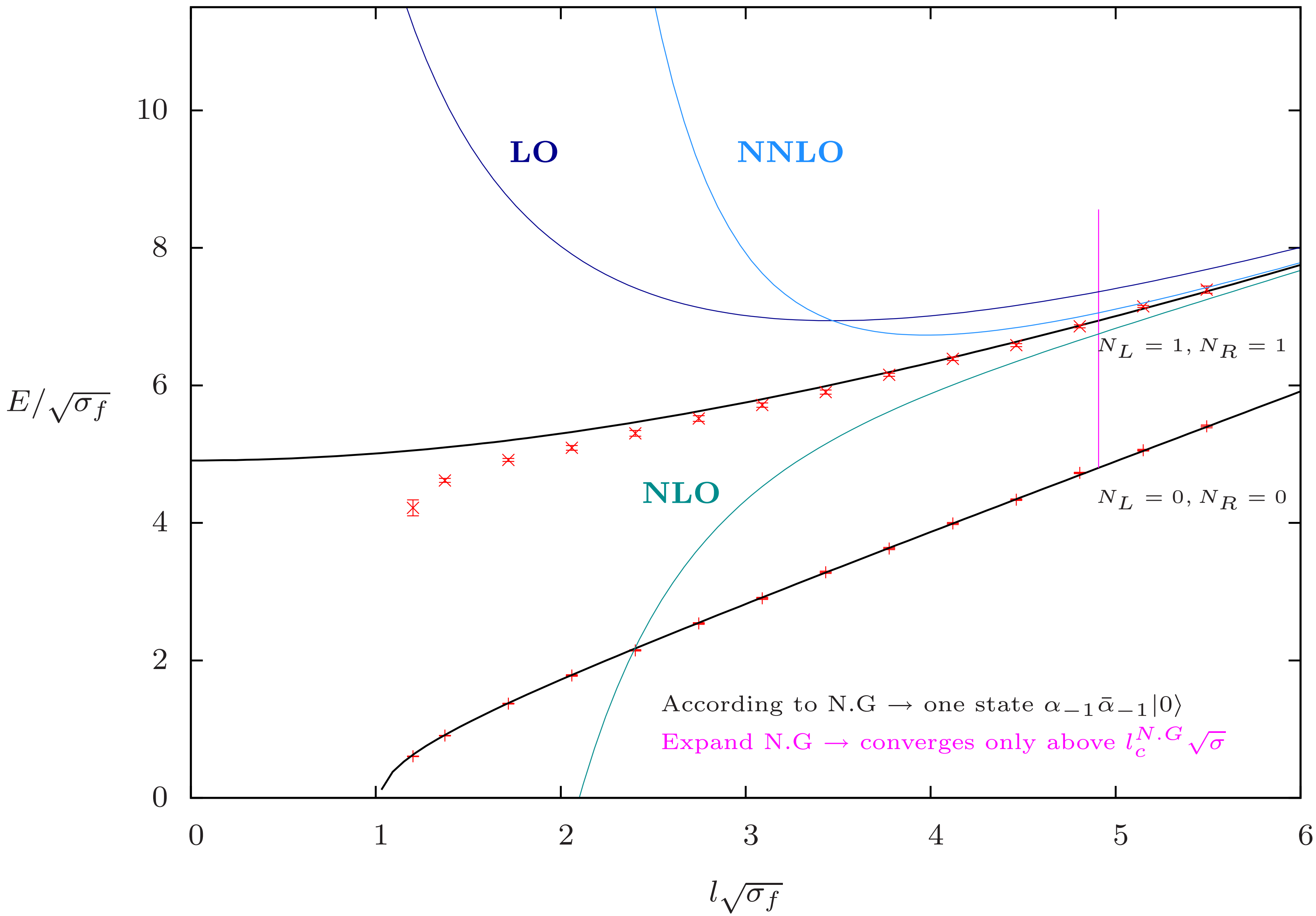
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



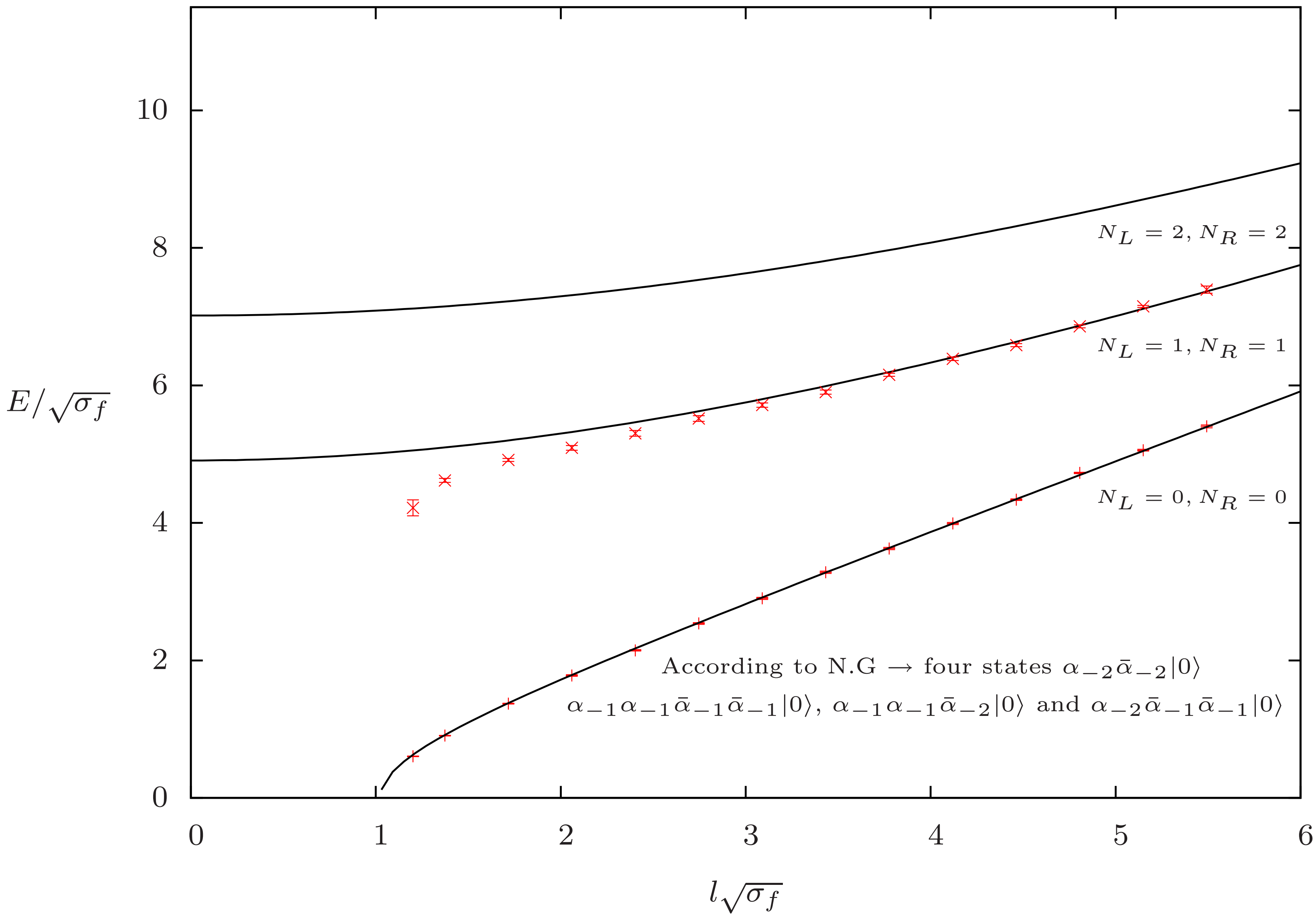
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



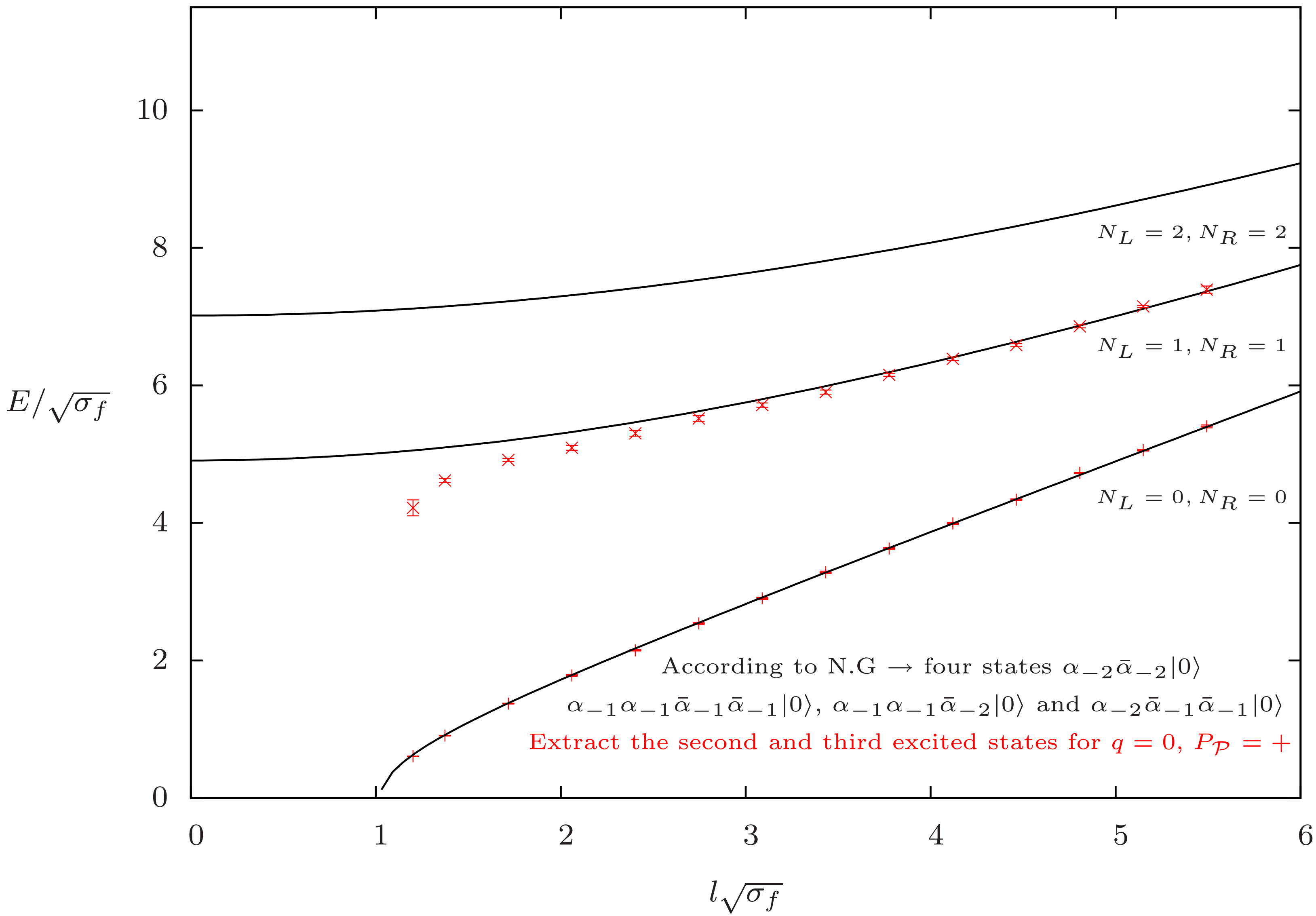
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



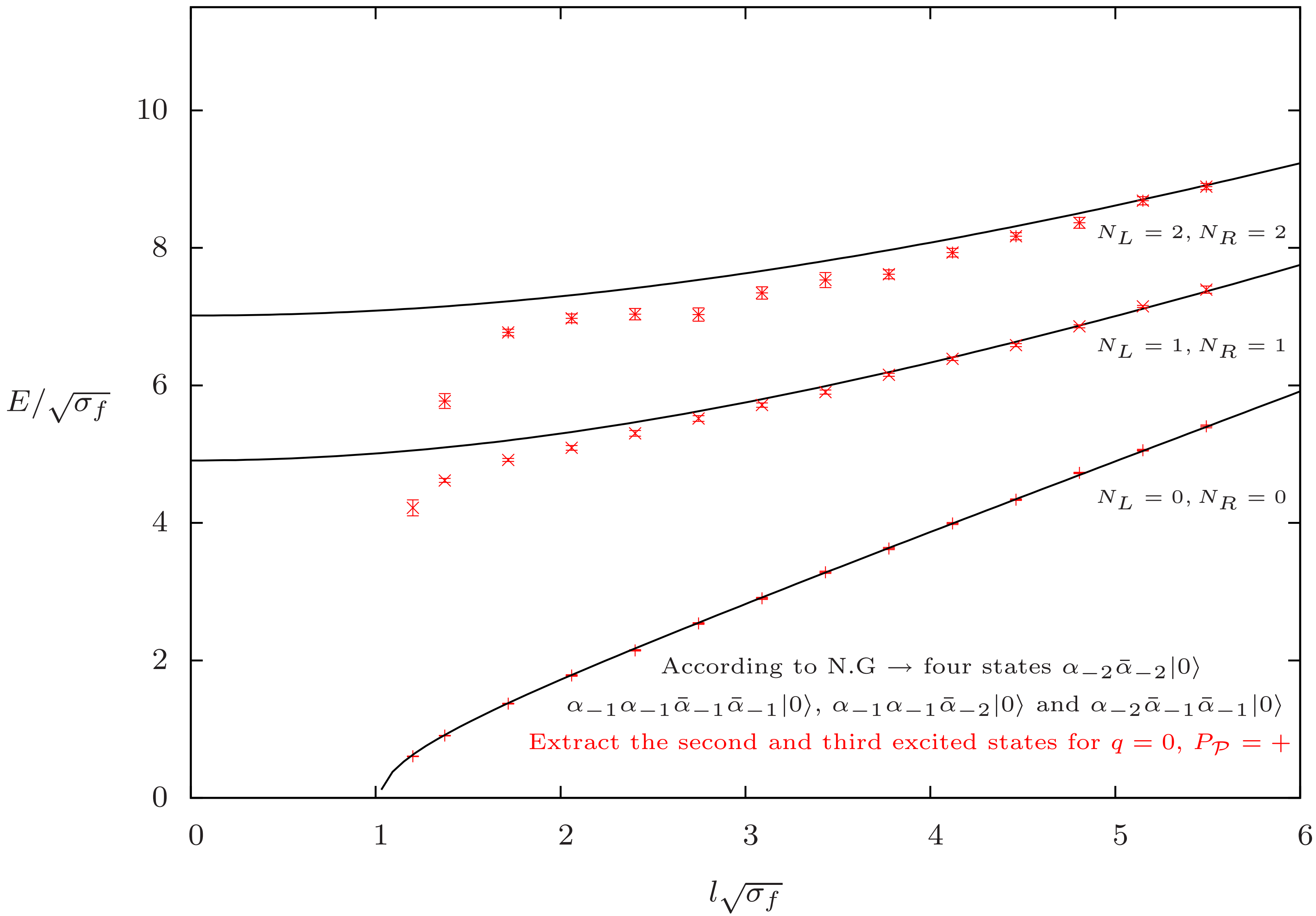
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



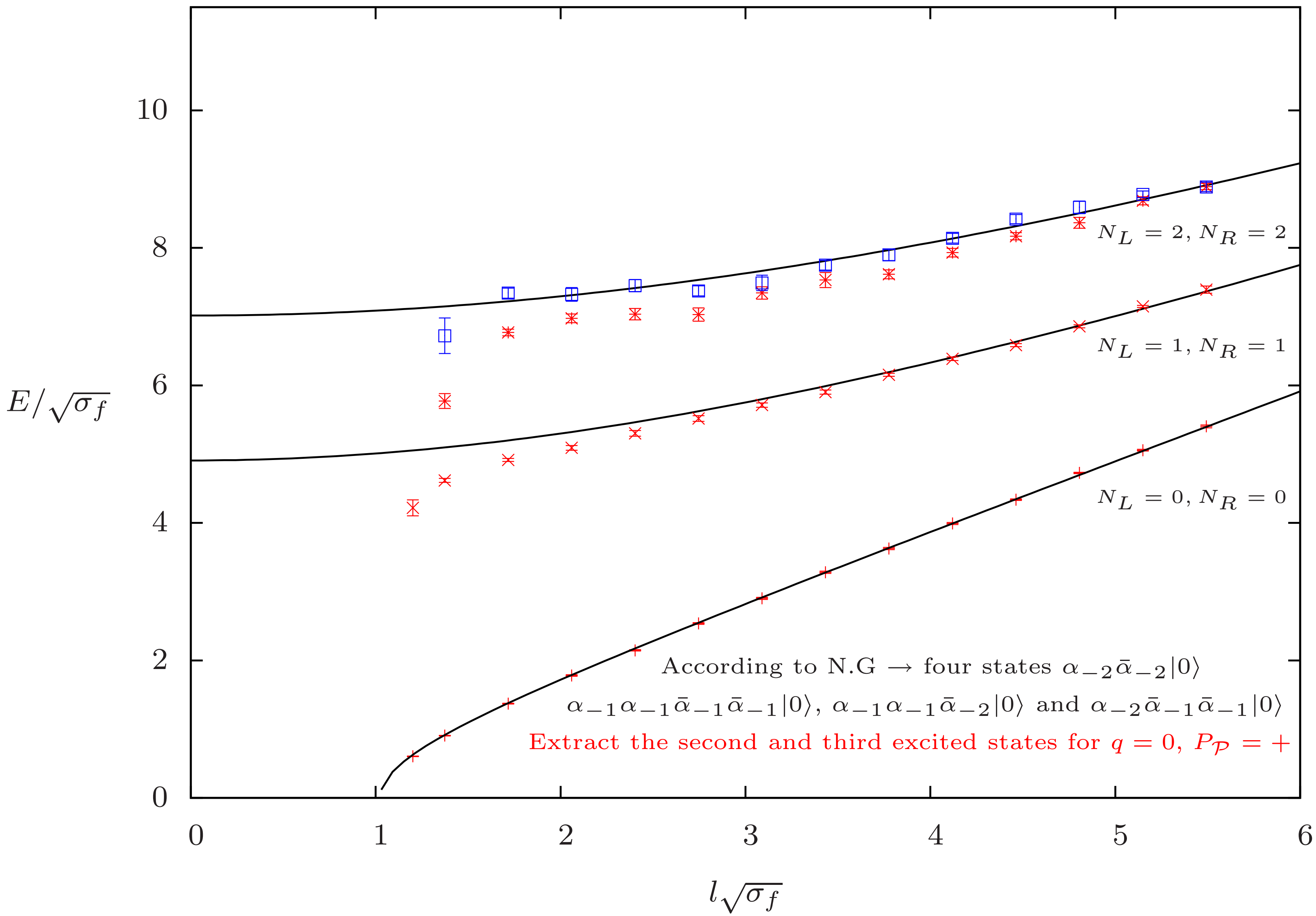
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



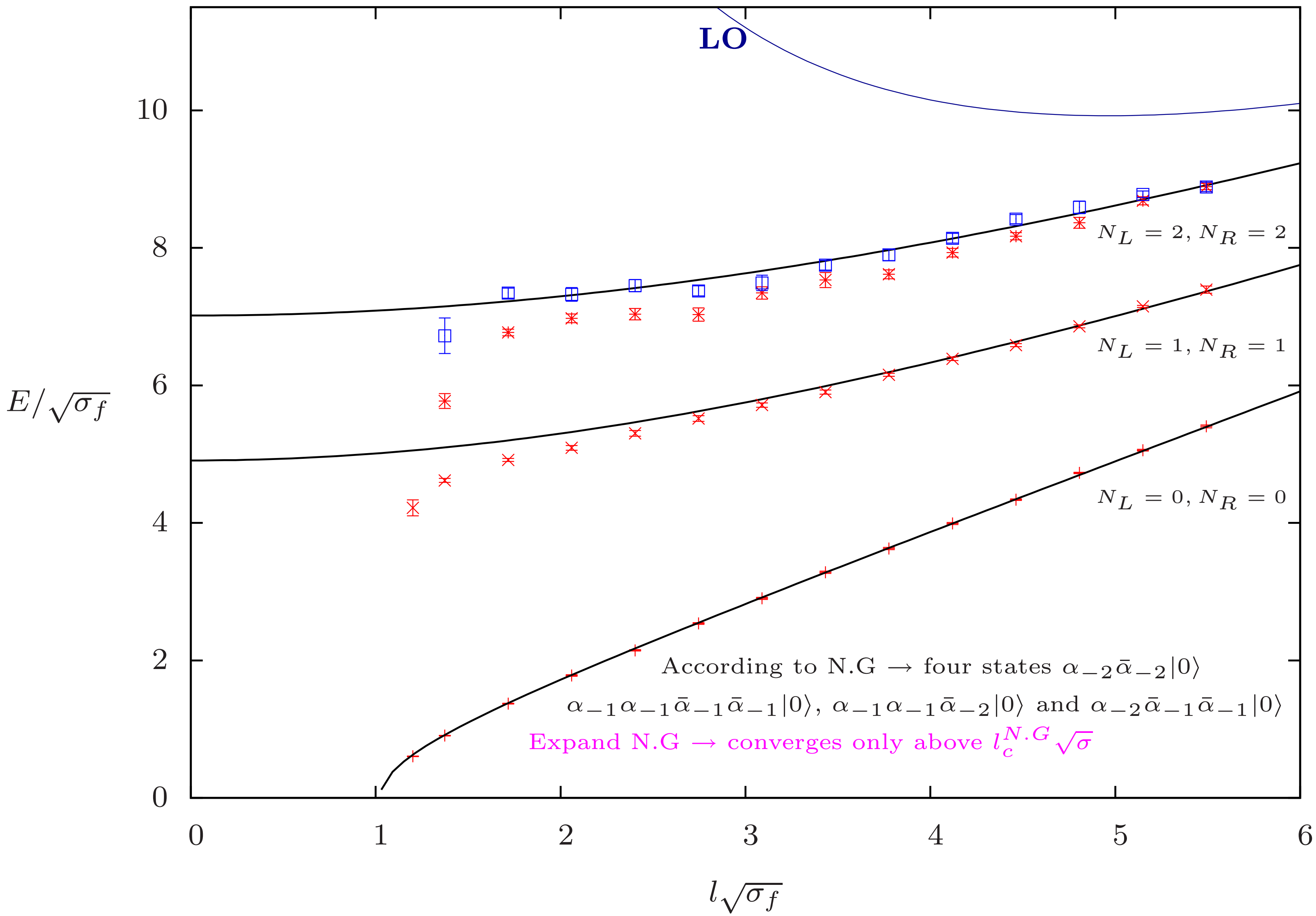
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



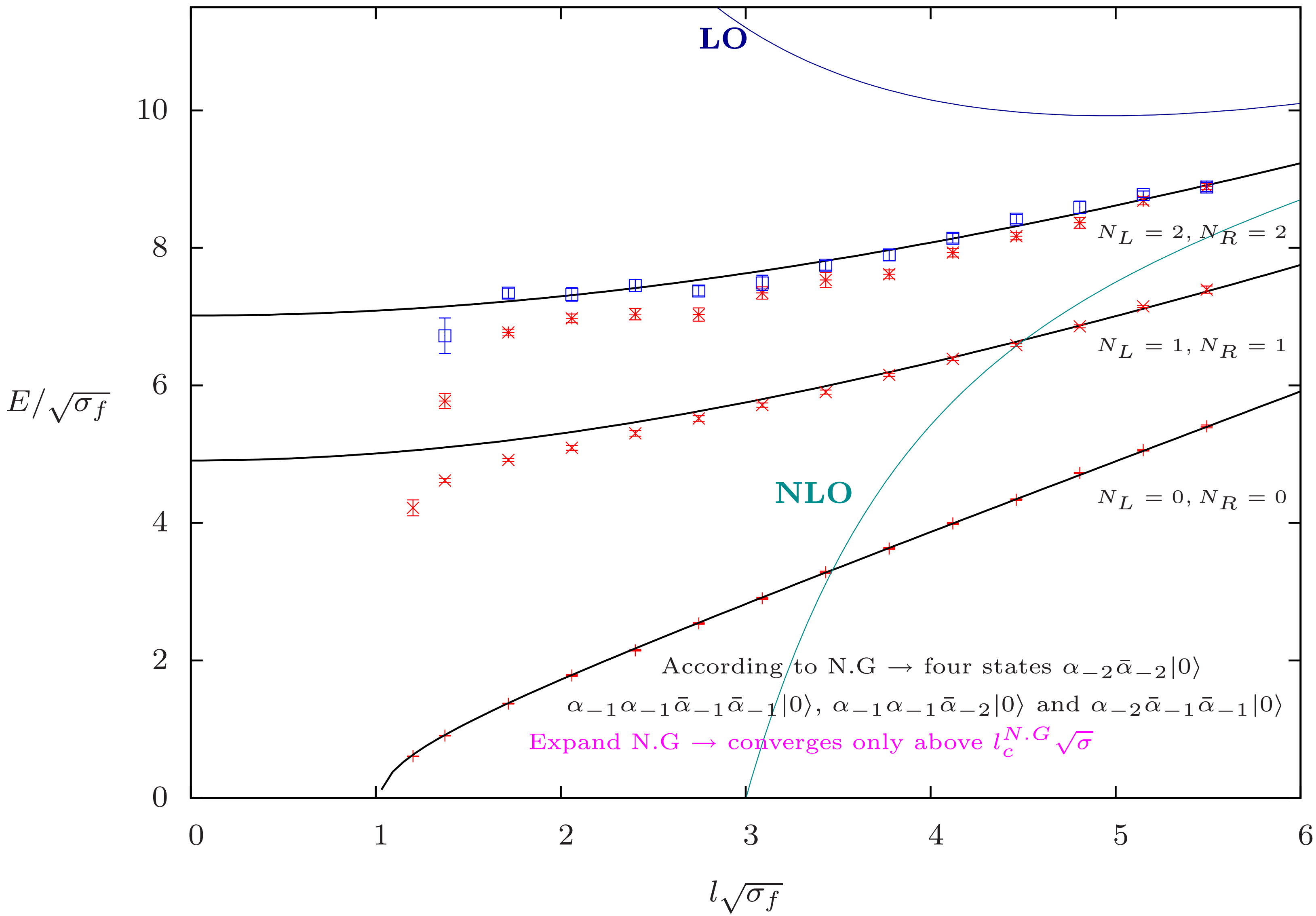
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



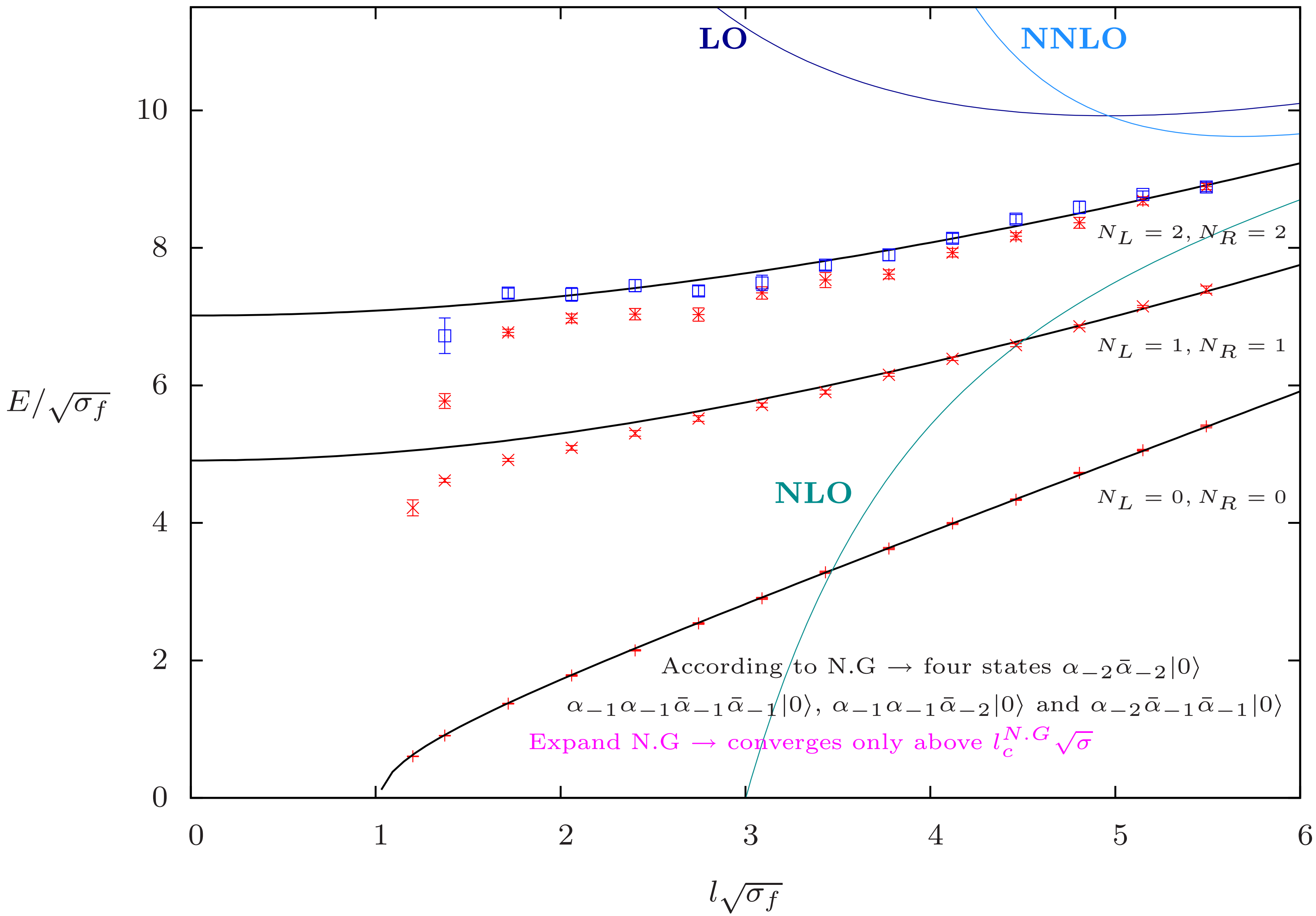
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



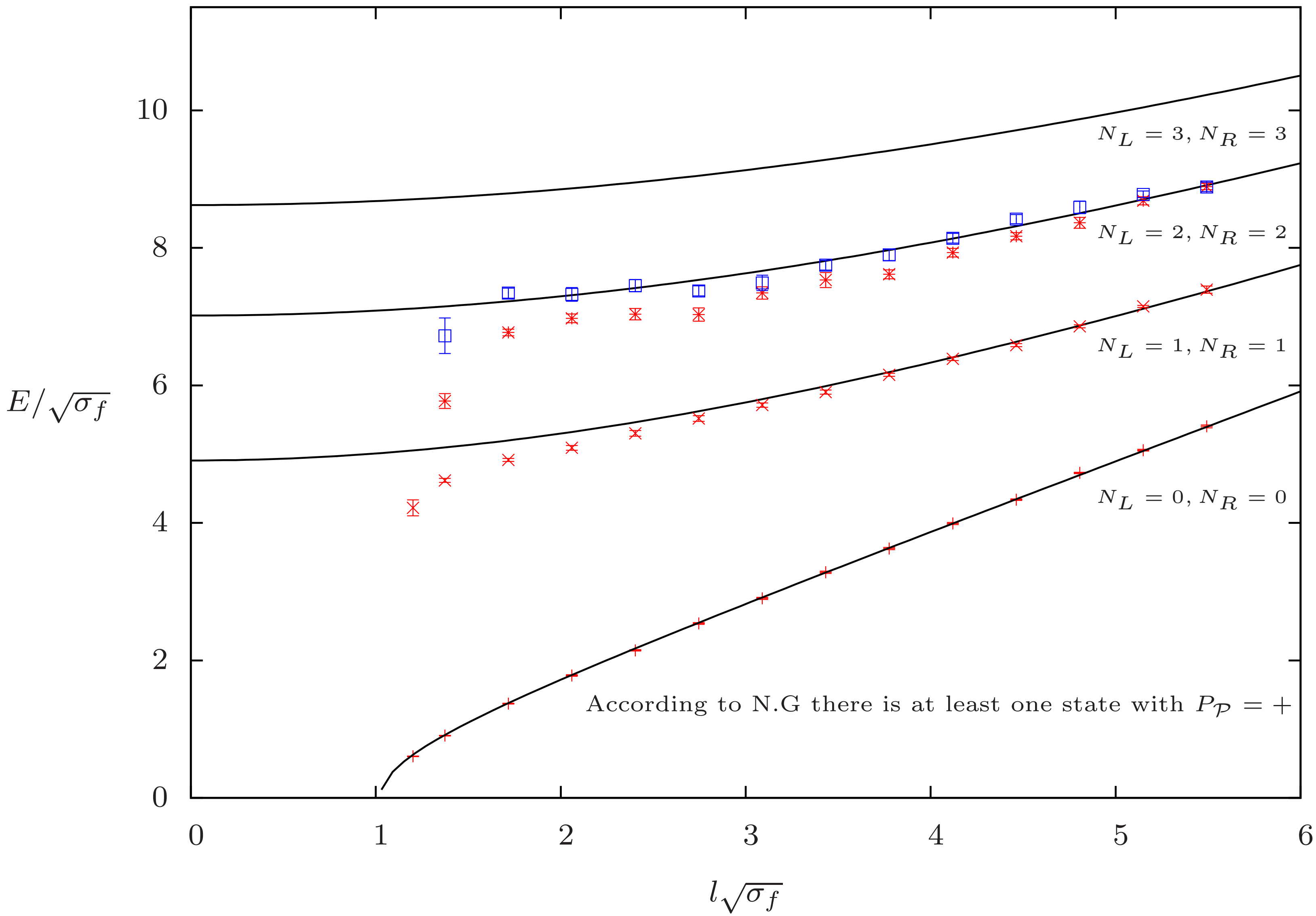
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



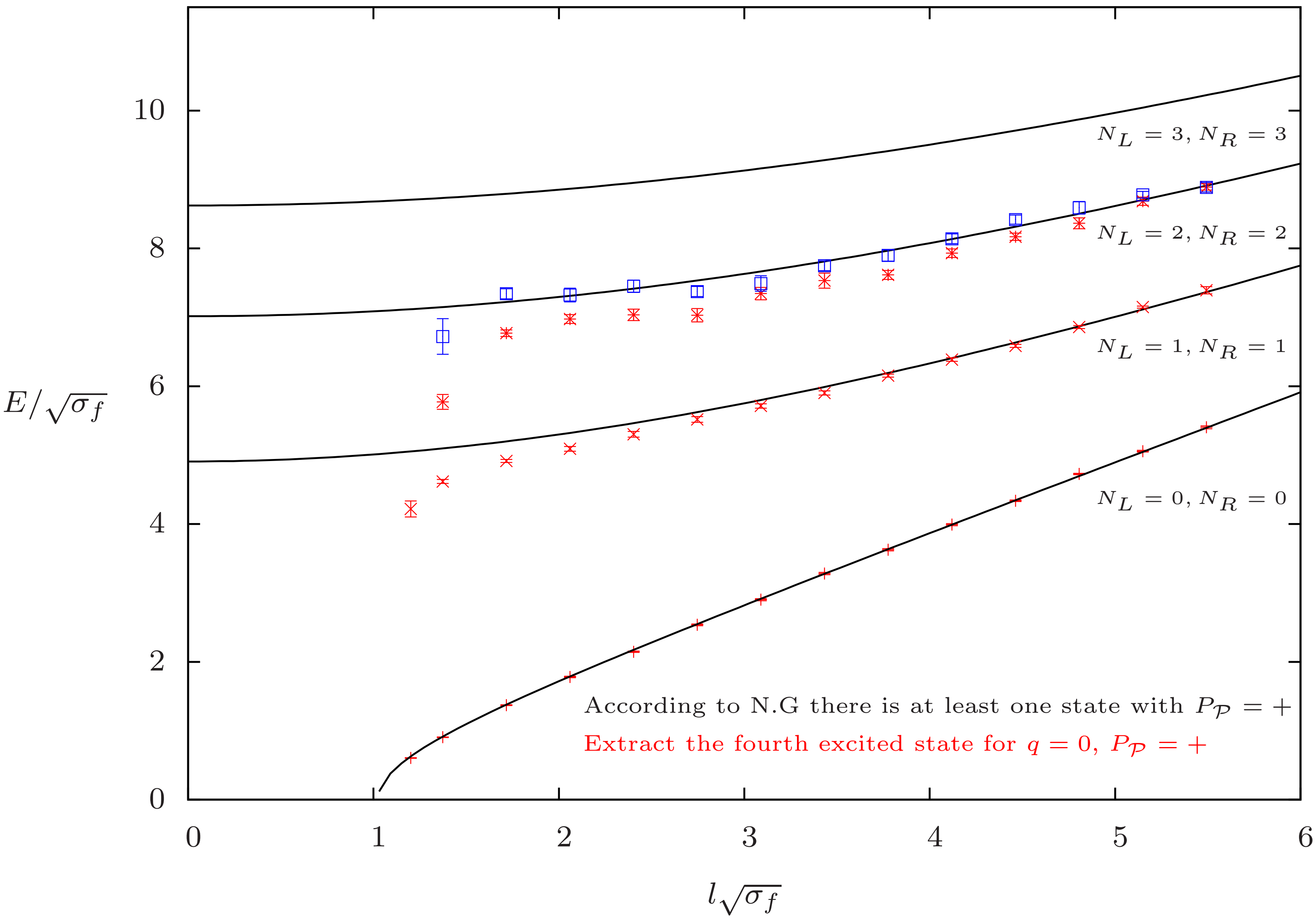
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



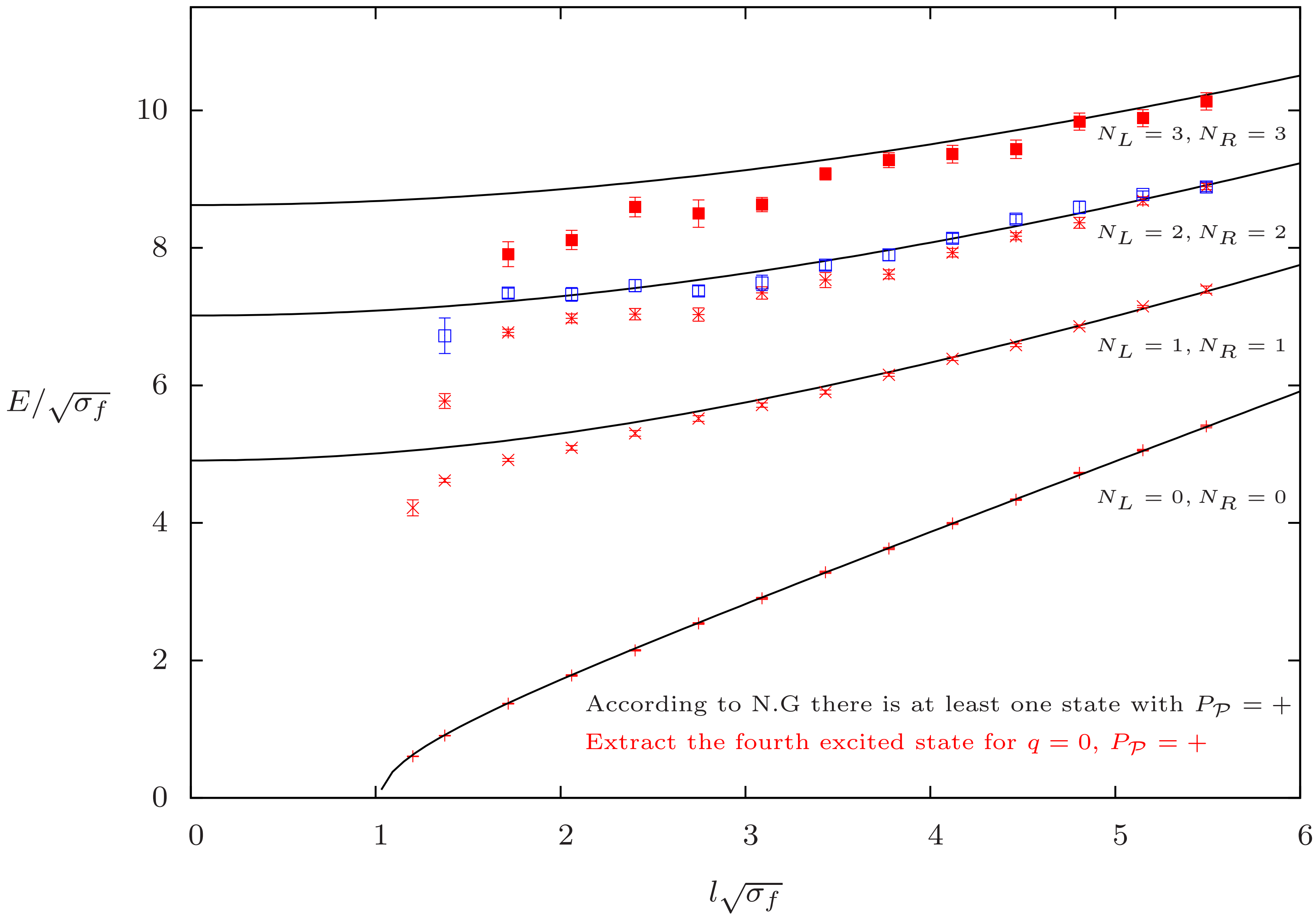
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



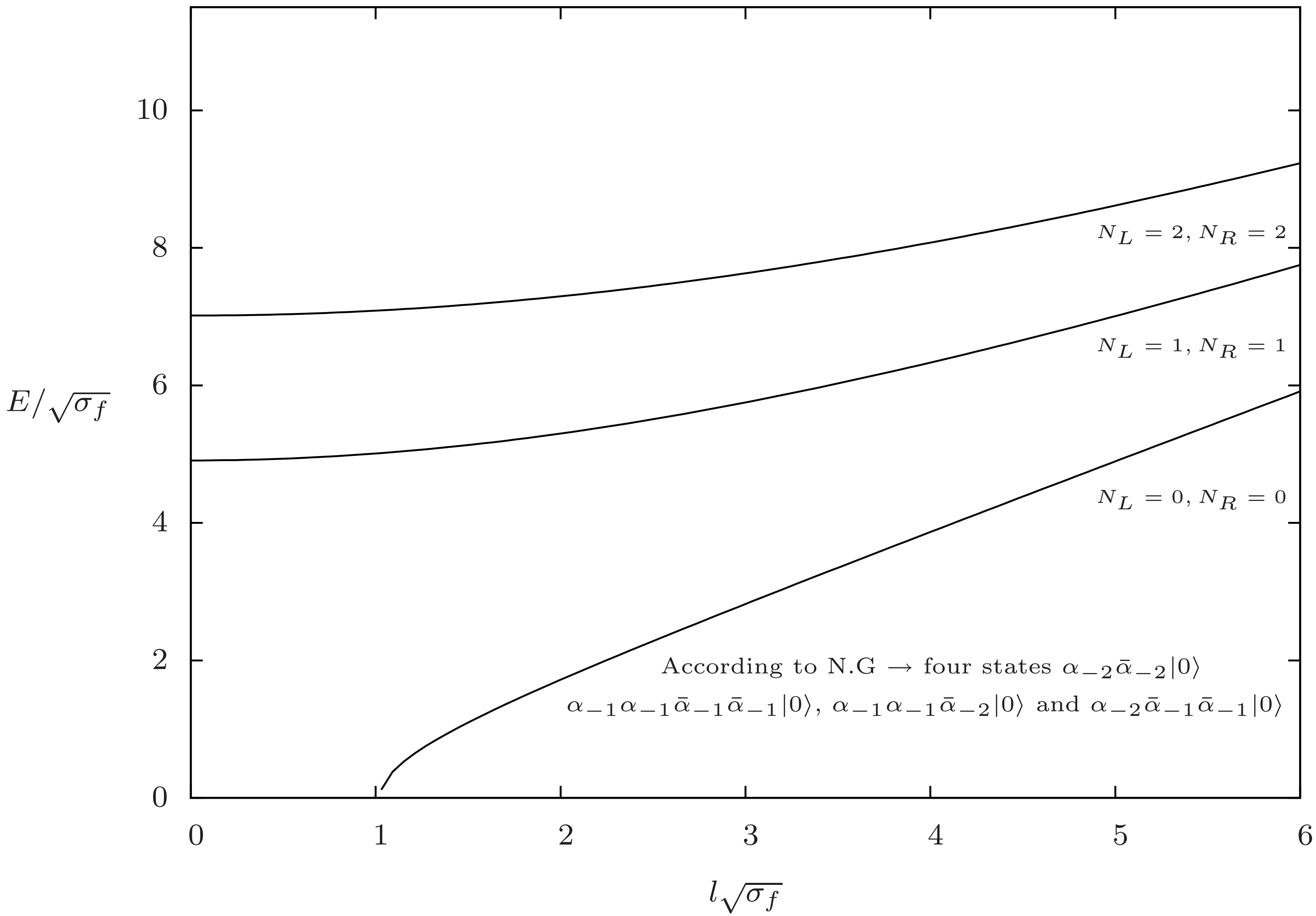
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



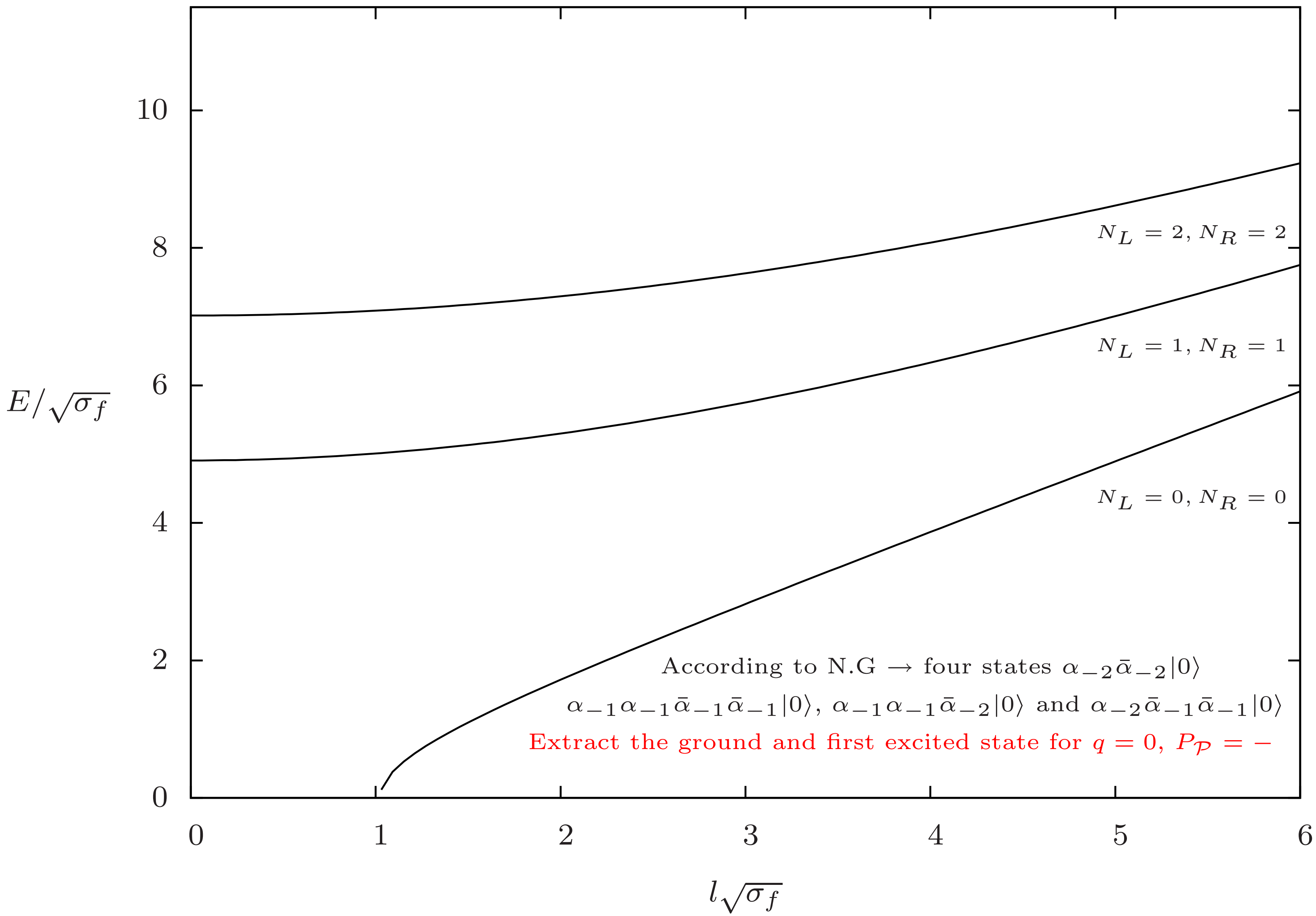
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ fundamental Representation



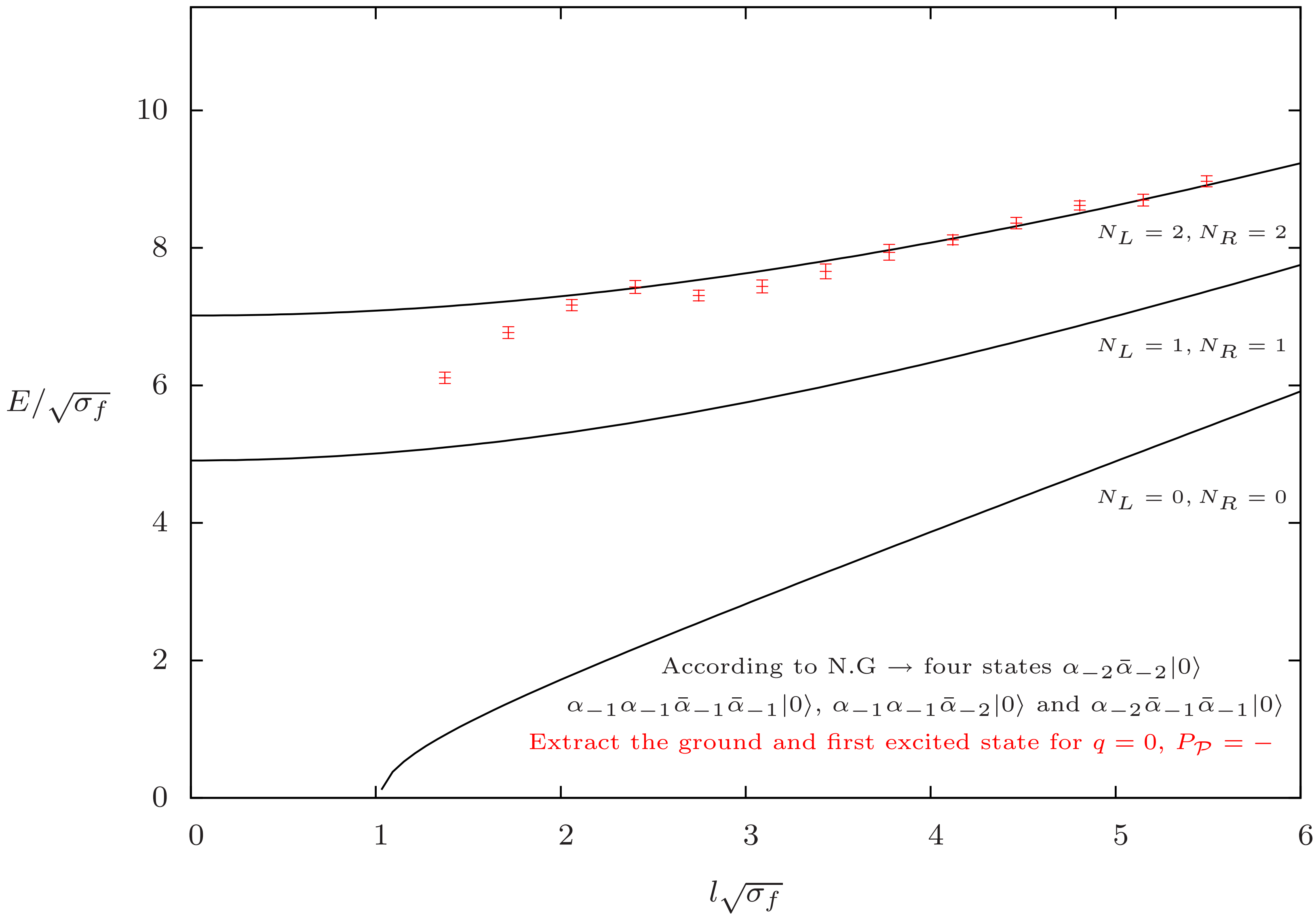
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



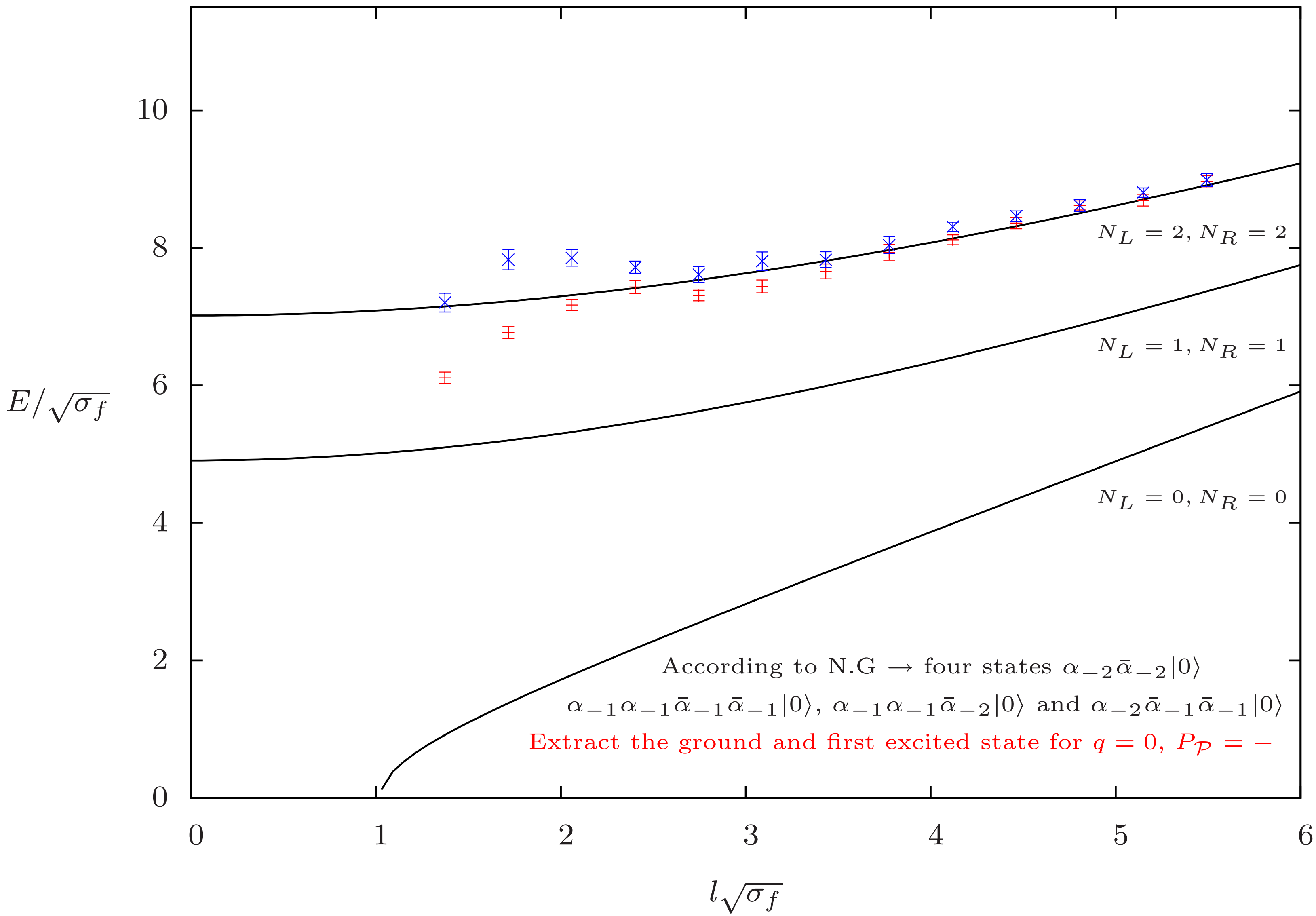
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



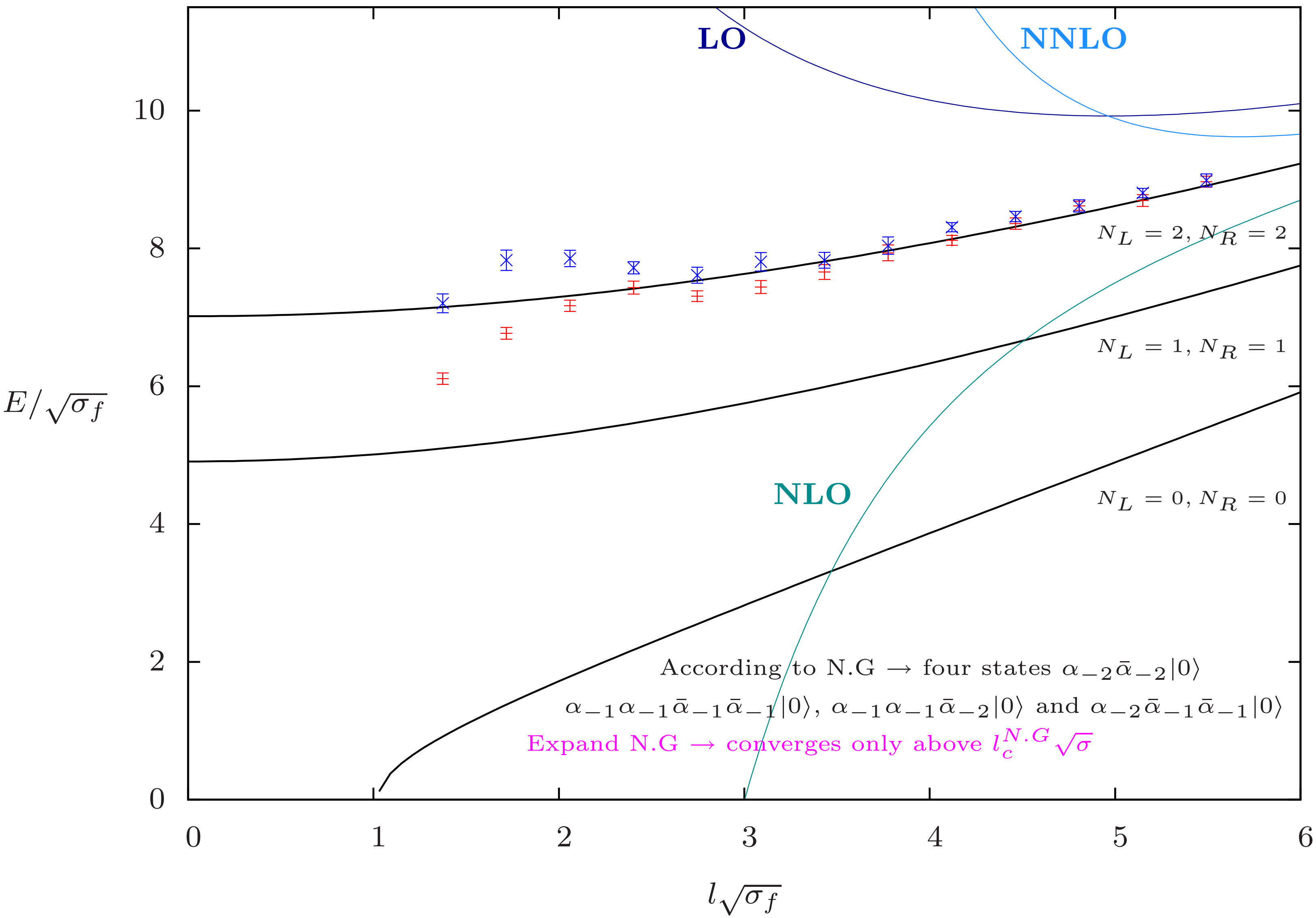
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



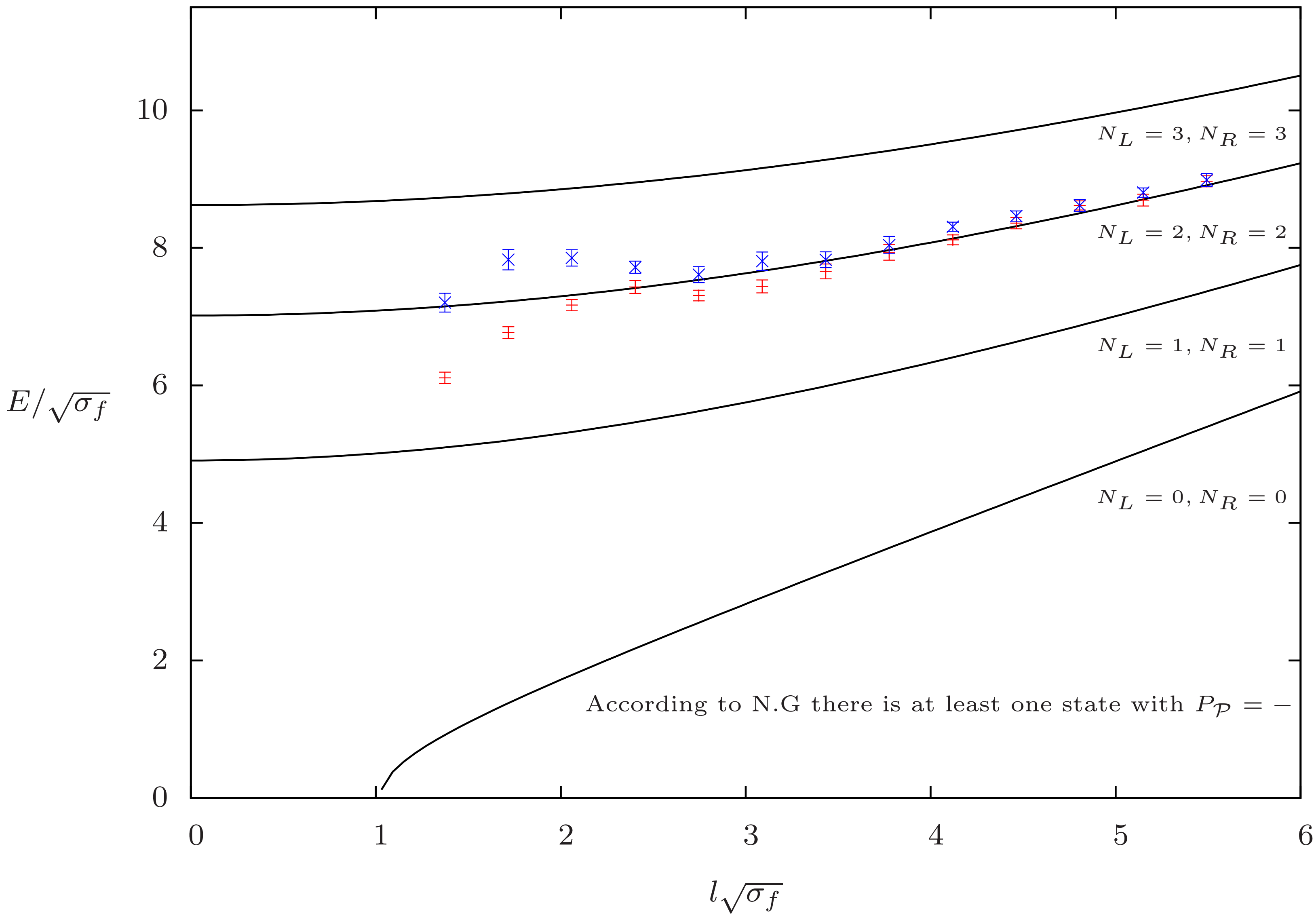
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



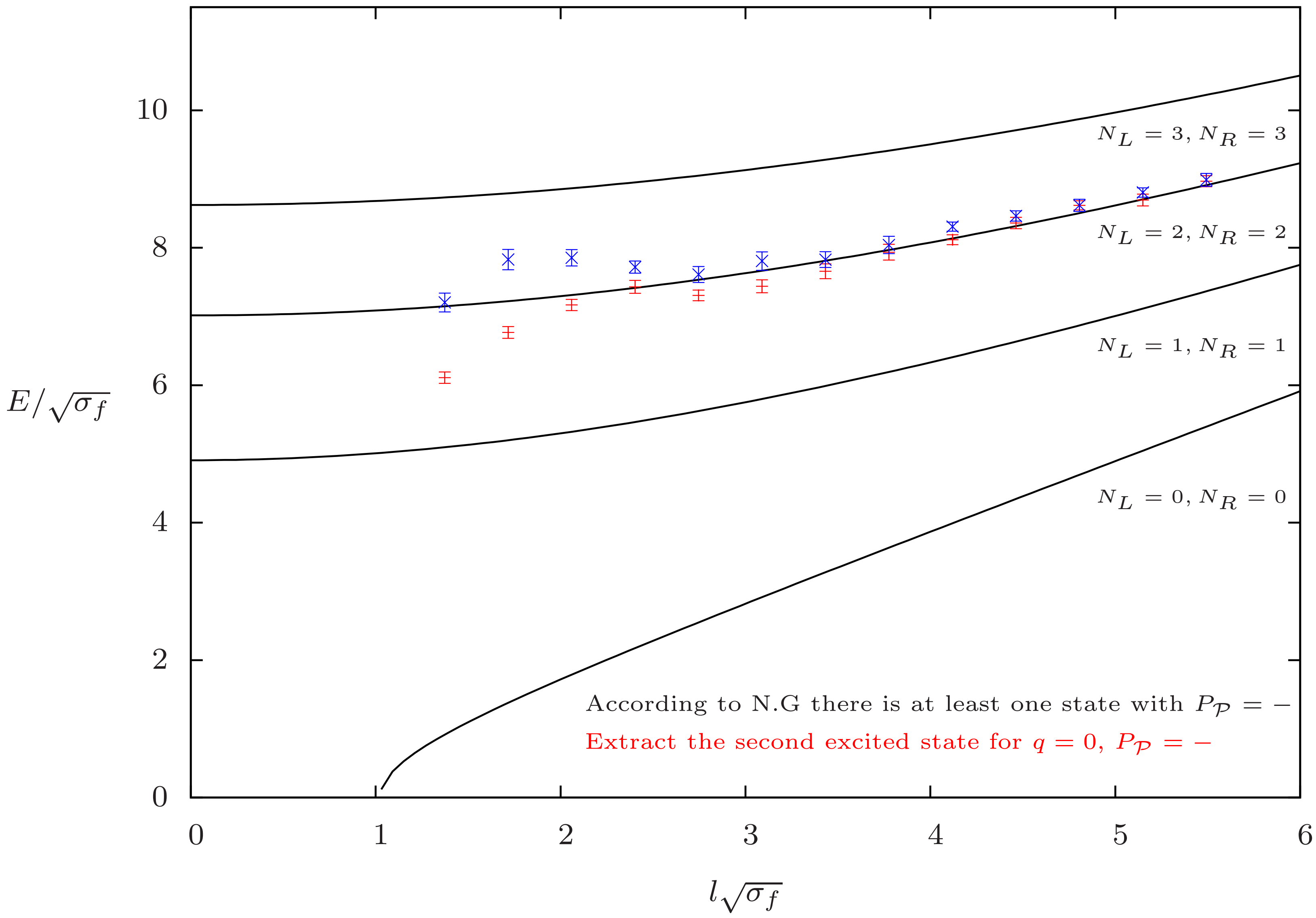
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



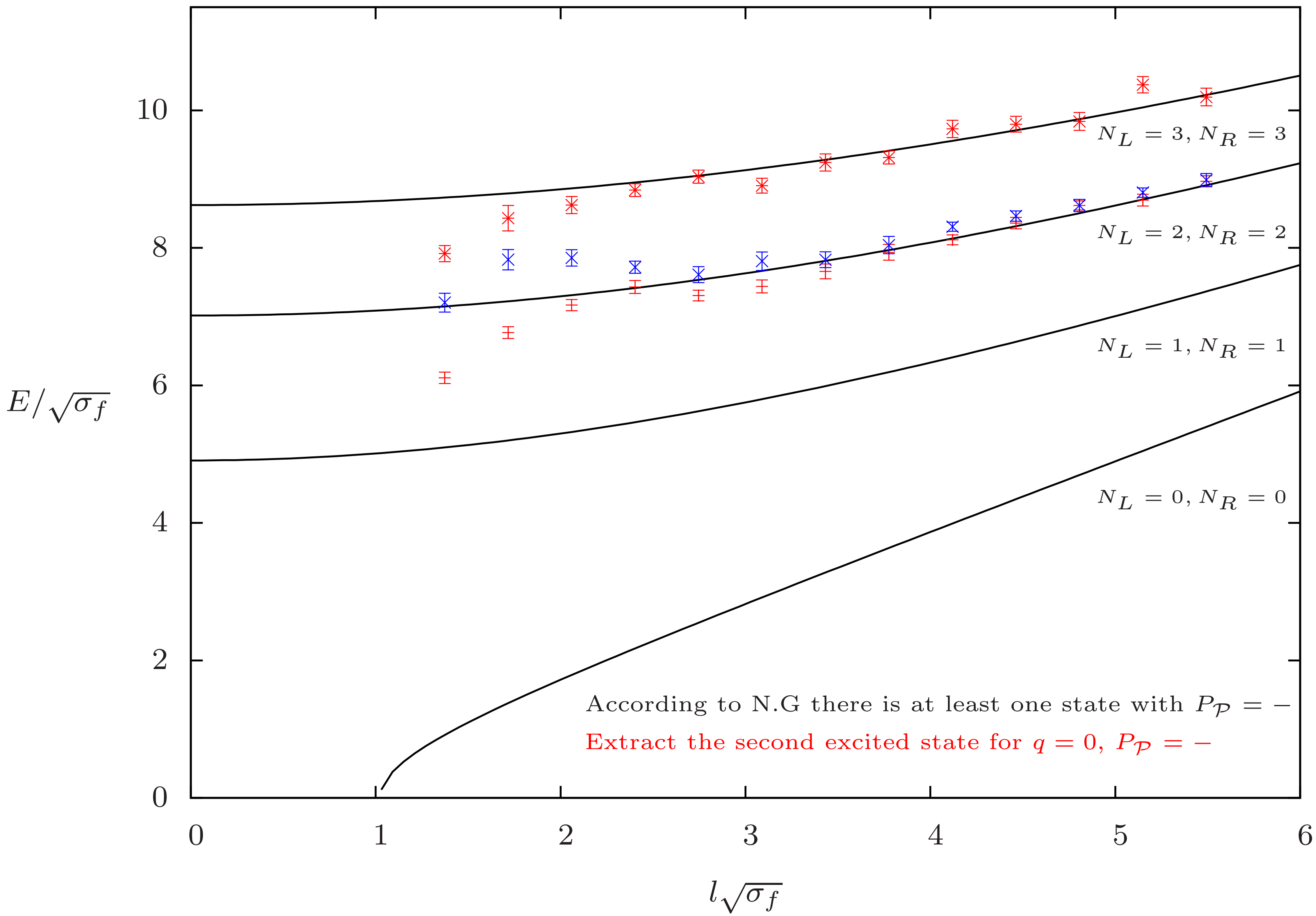
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



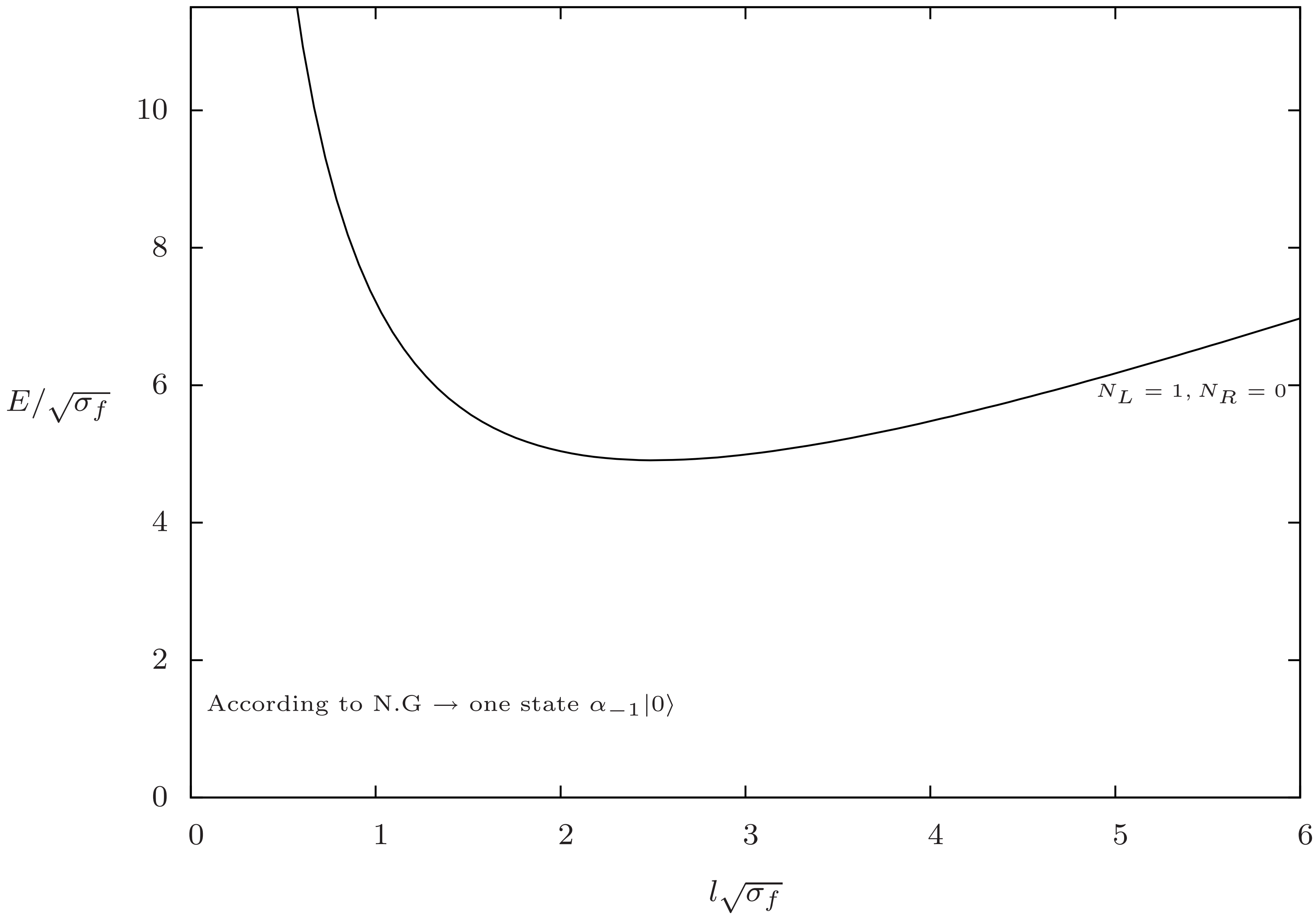
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



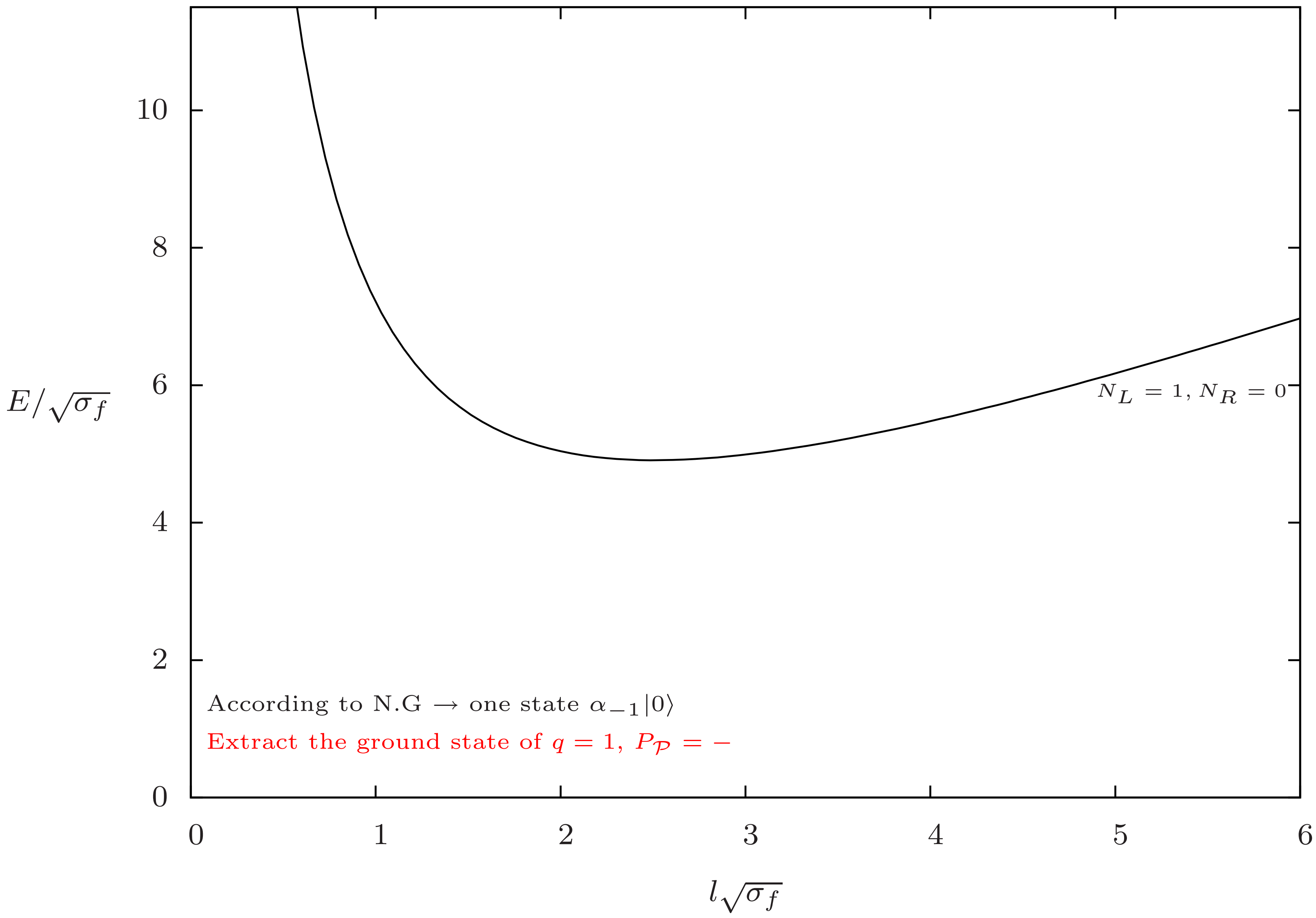
7. Results for: $q = 0$, $P_{\mathcal{P}} = -$ fundamental Representation



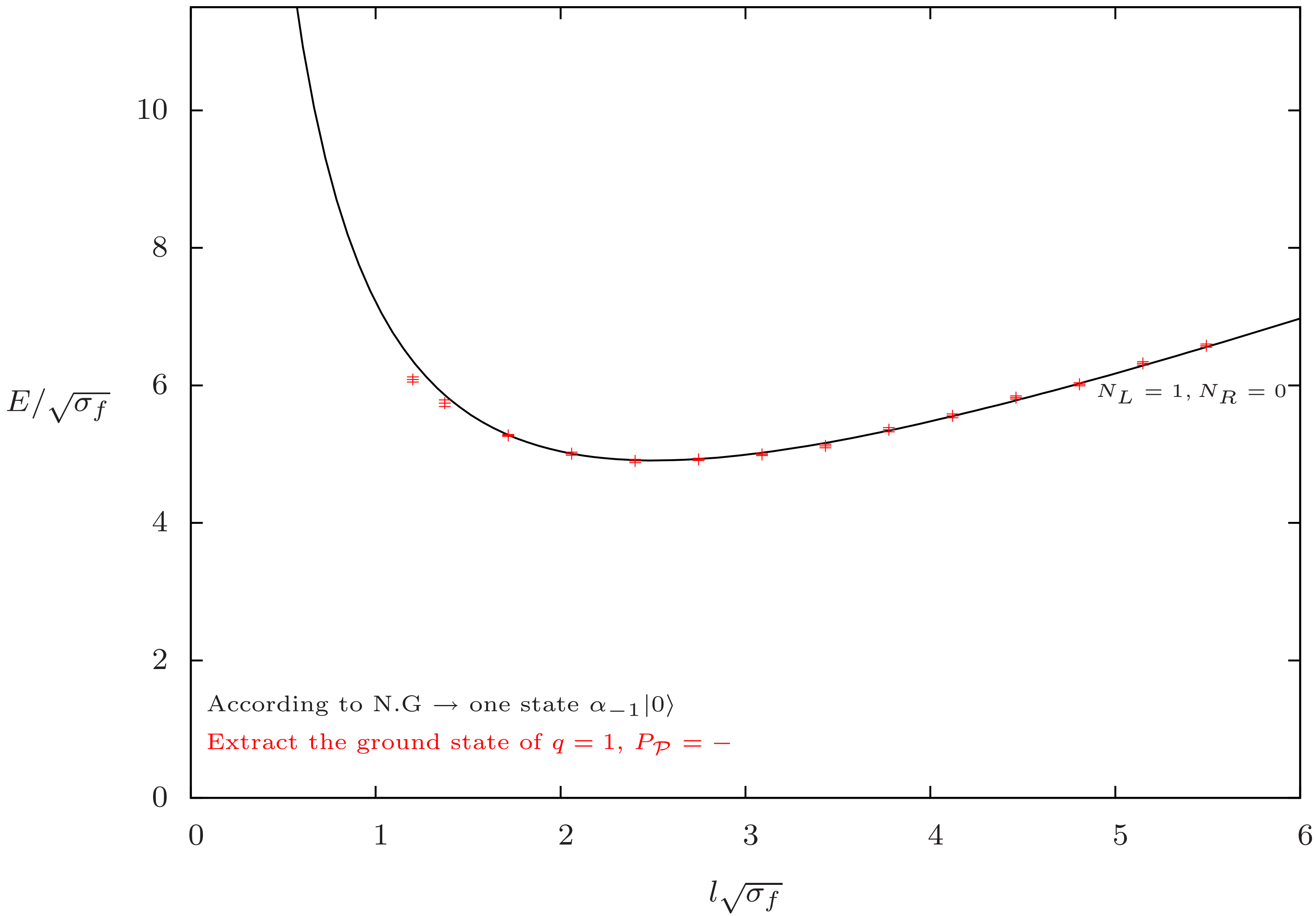
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



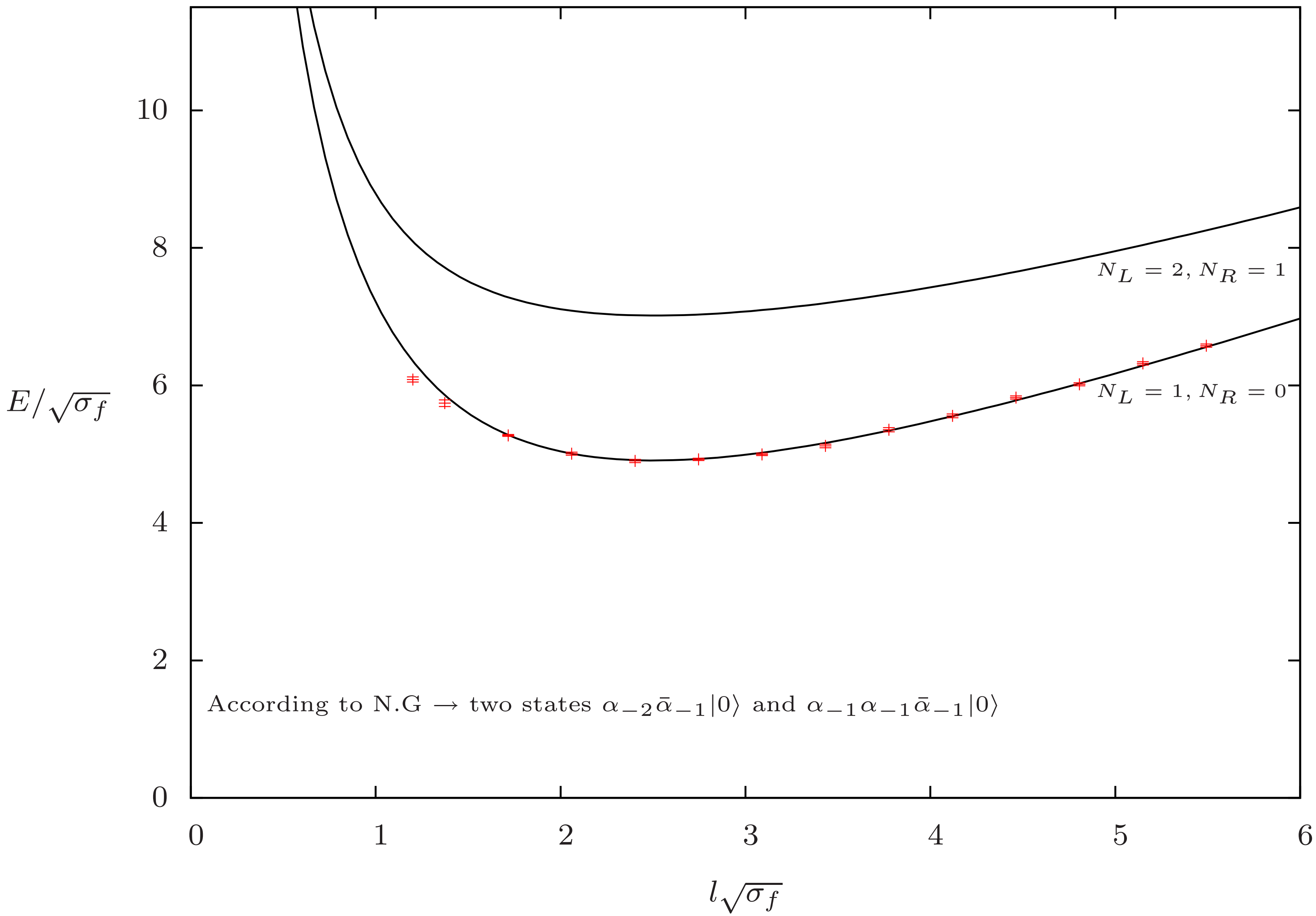
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



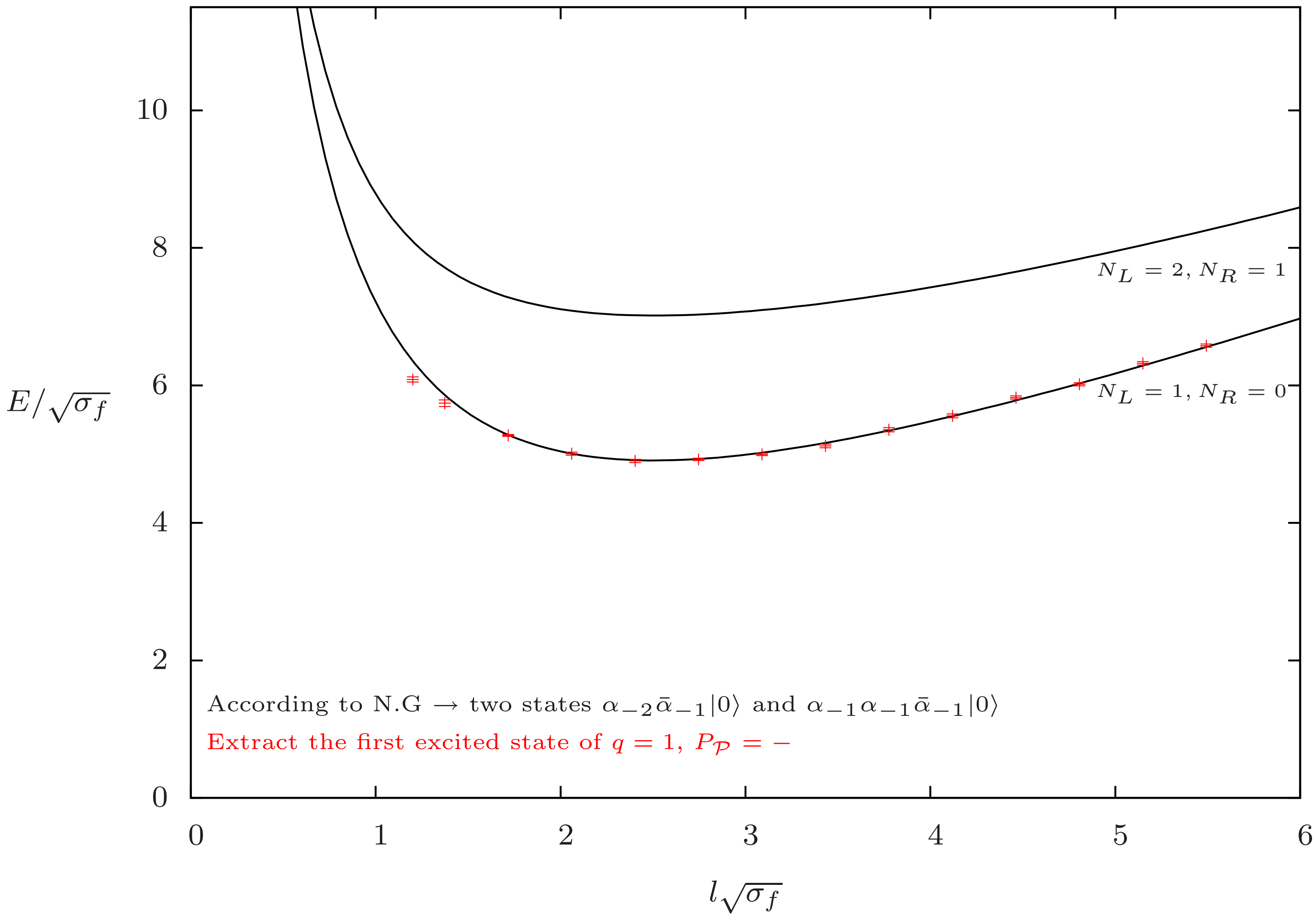
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



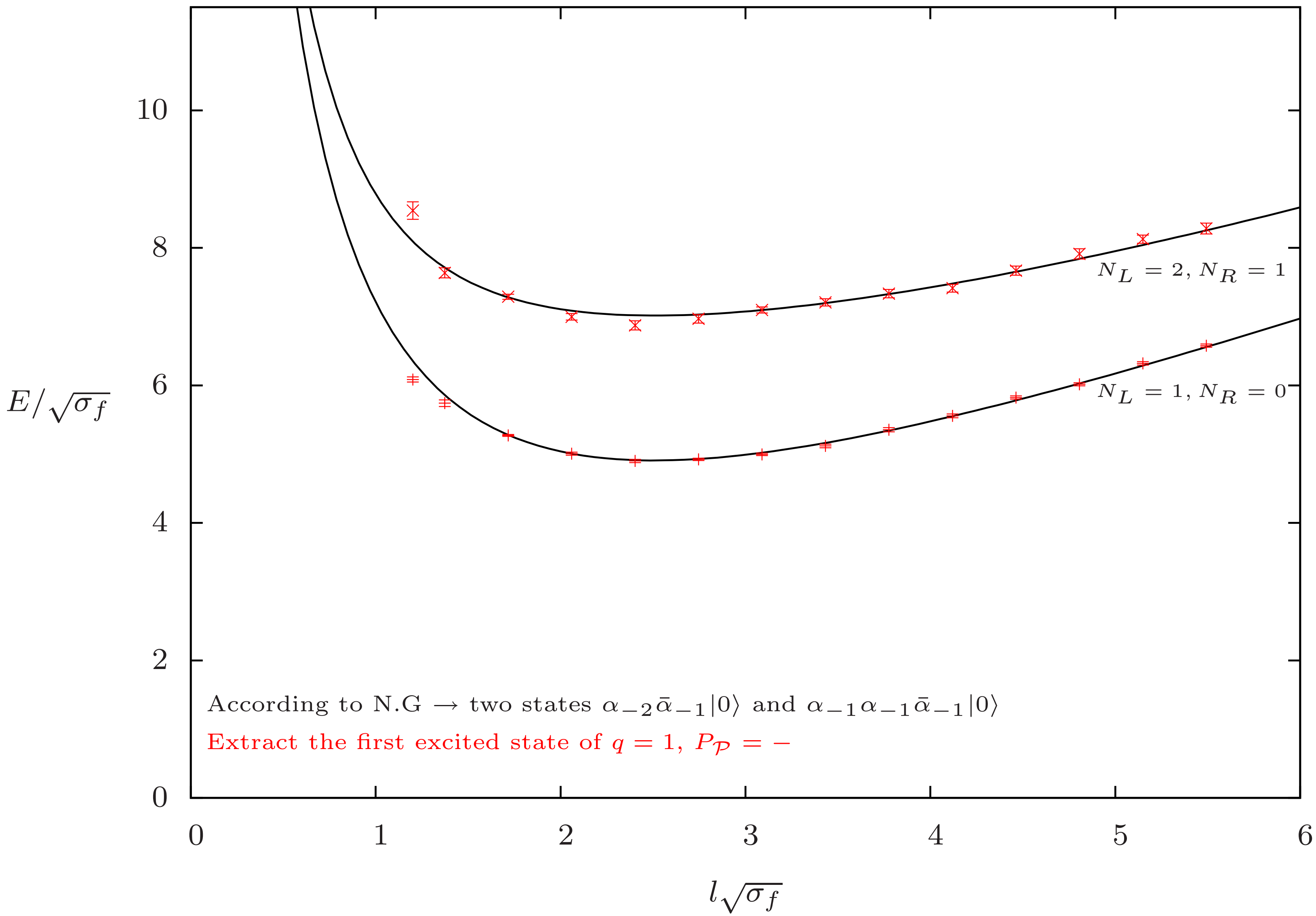
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



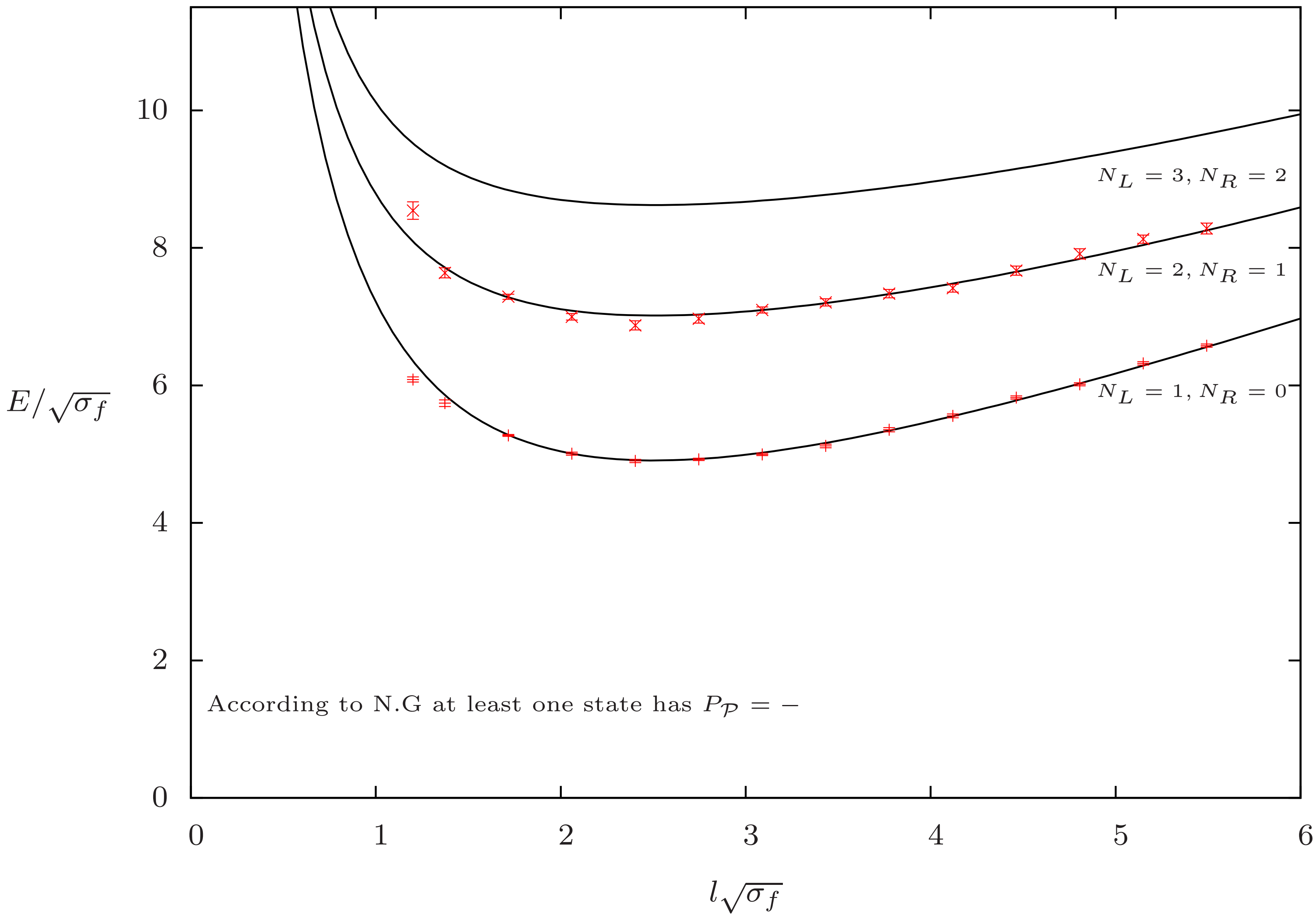
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



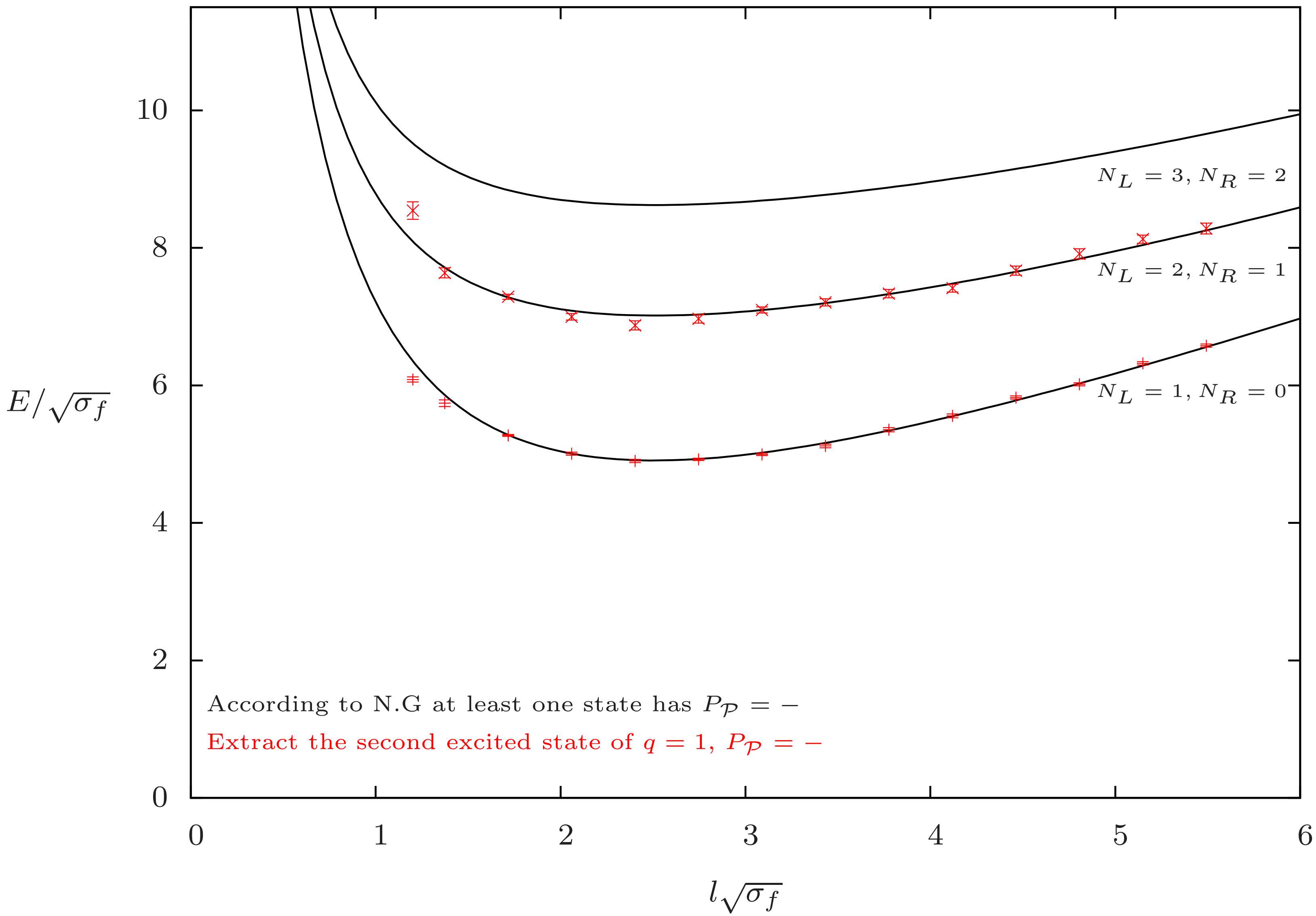
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



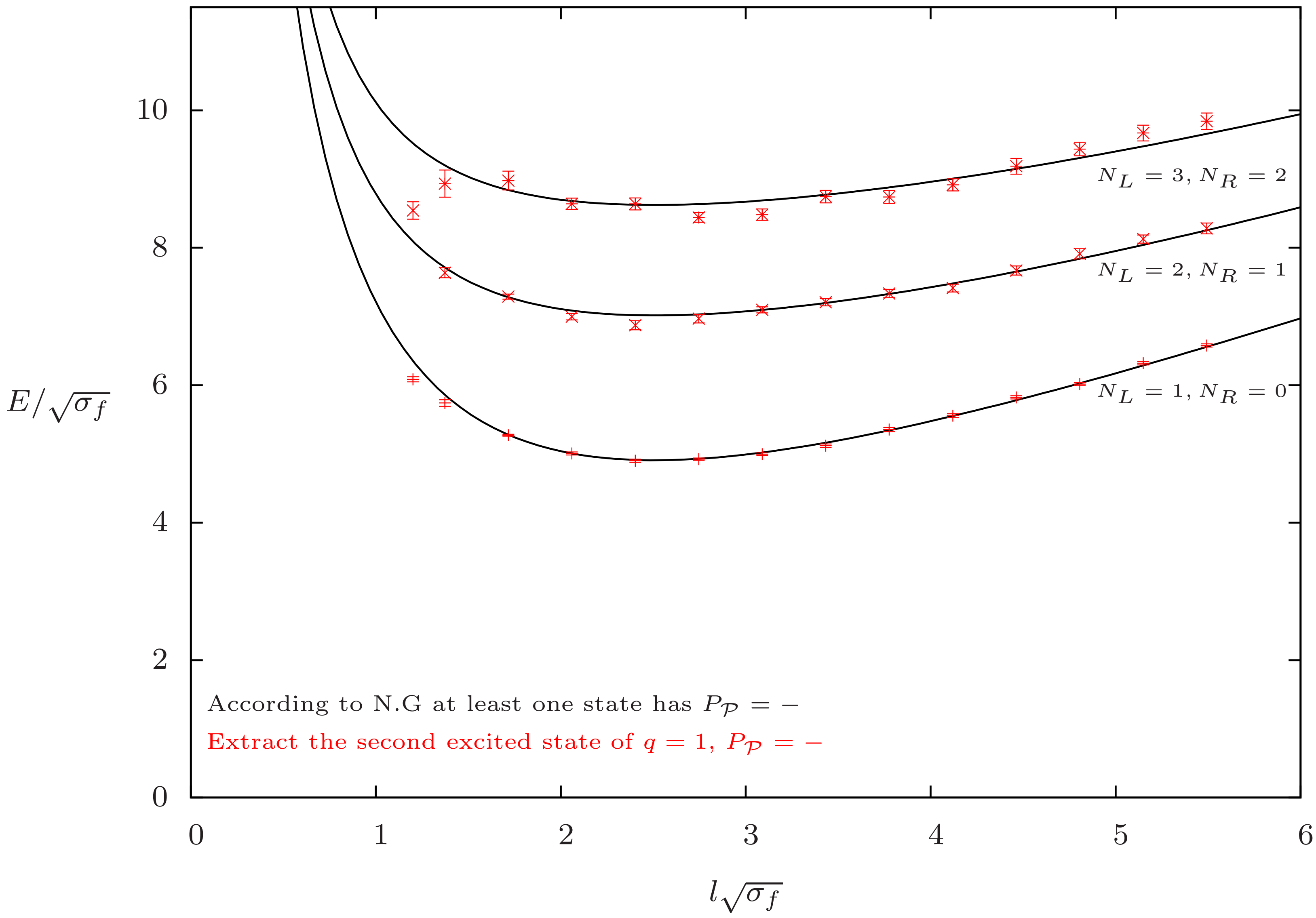
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



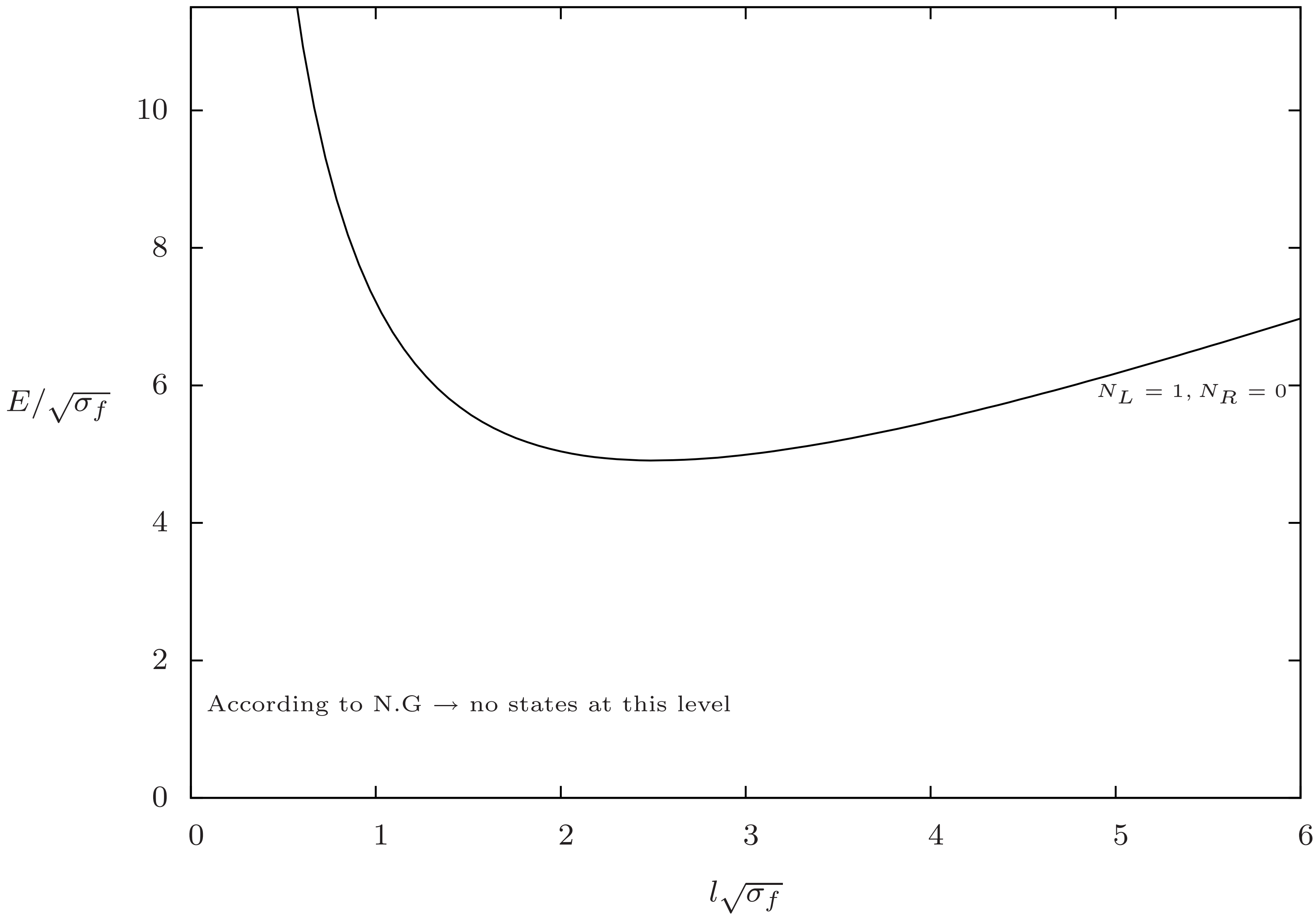
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



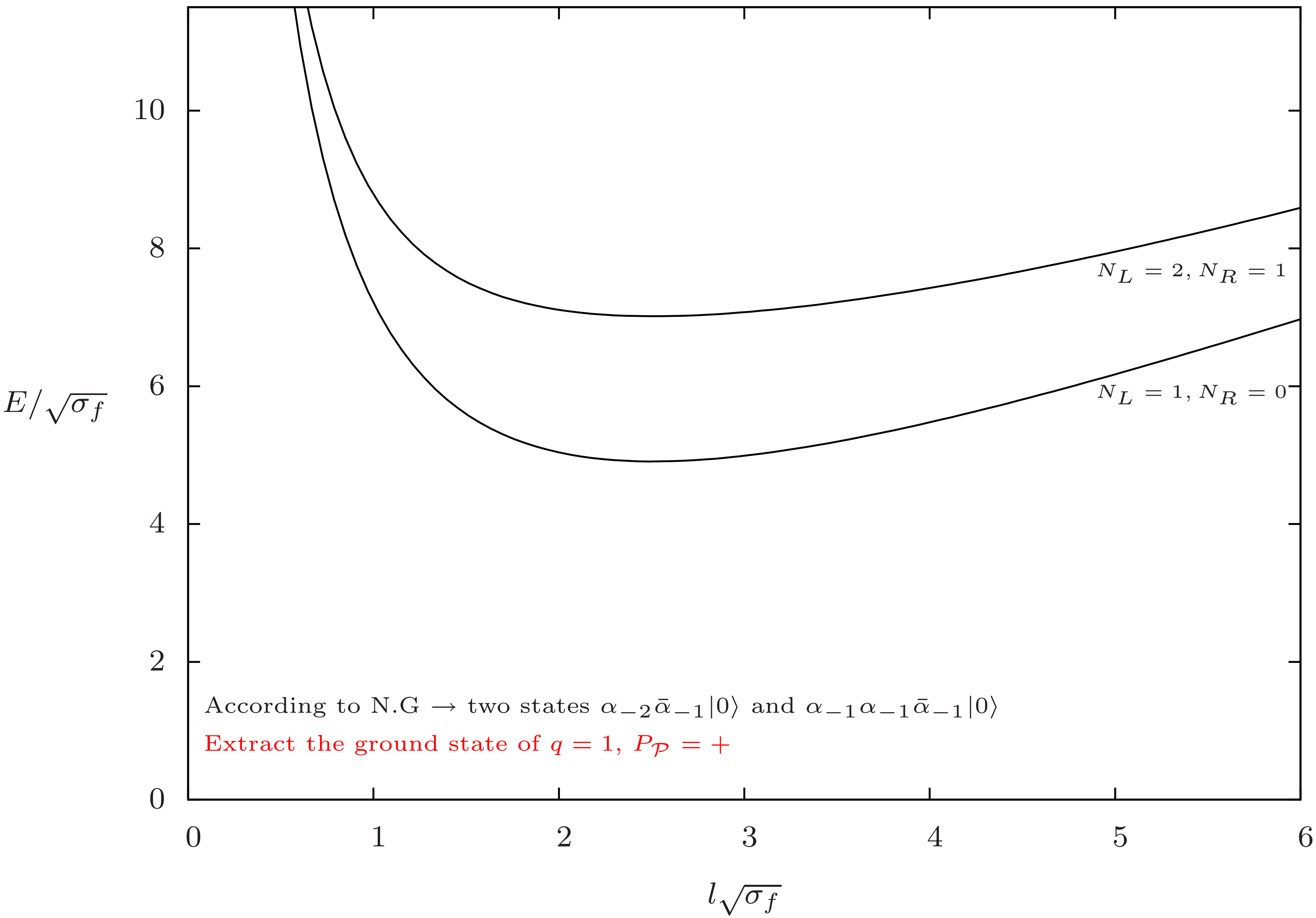
7. Results for: $q = 1$, $P_{\mathcal{P}} = -$ fundamental Representation



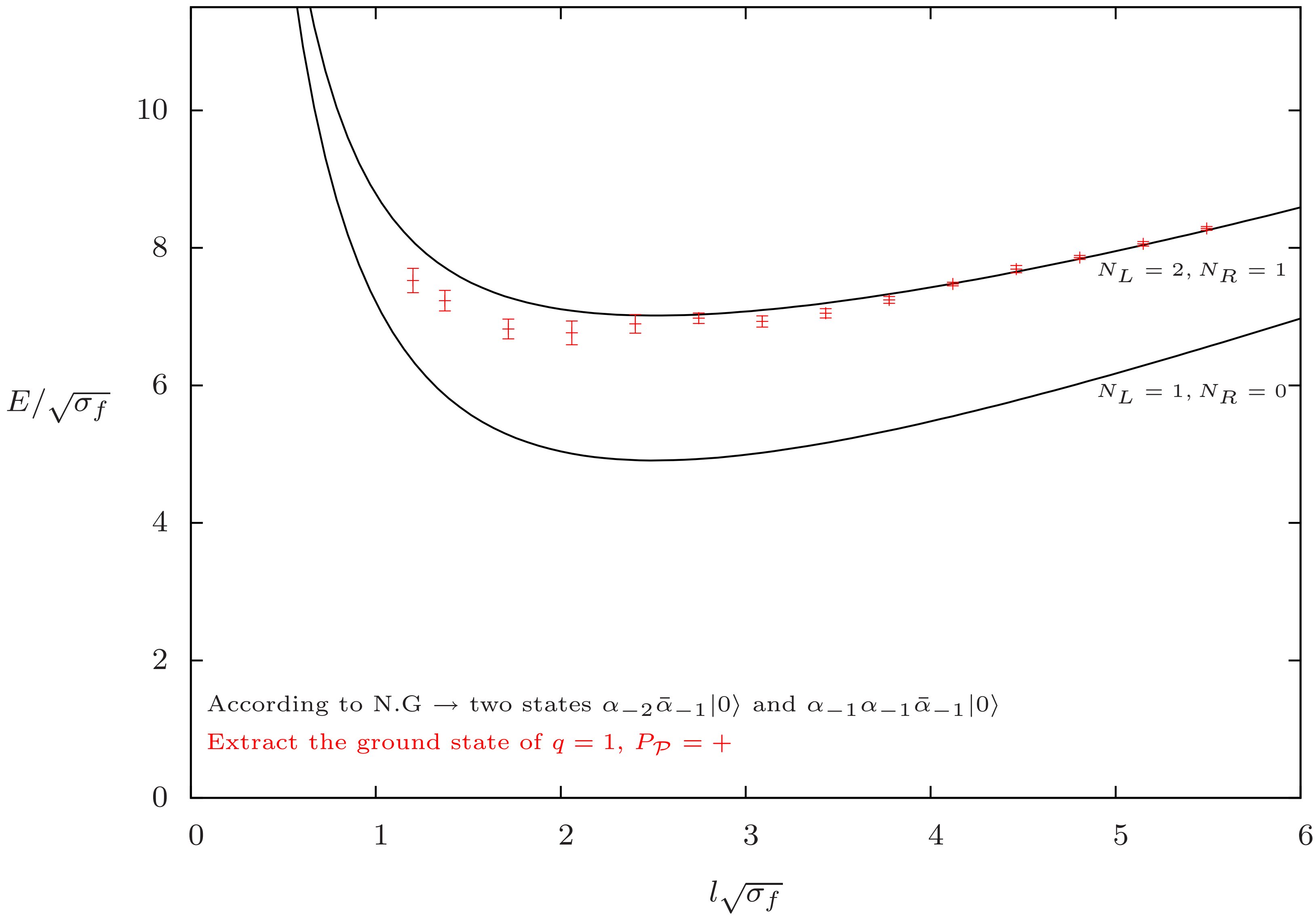
7. Results for: $q = 1$, $P_{\mathcal{P}} = +$ fundamental Representation



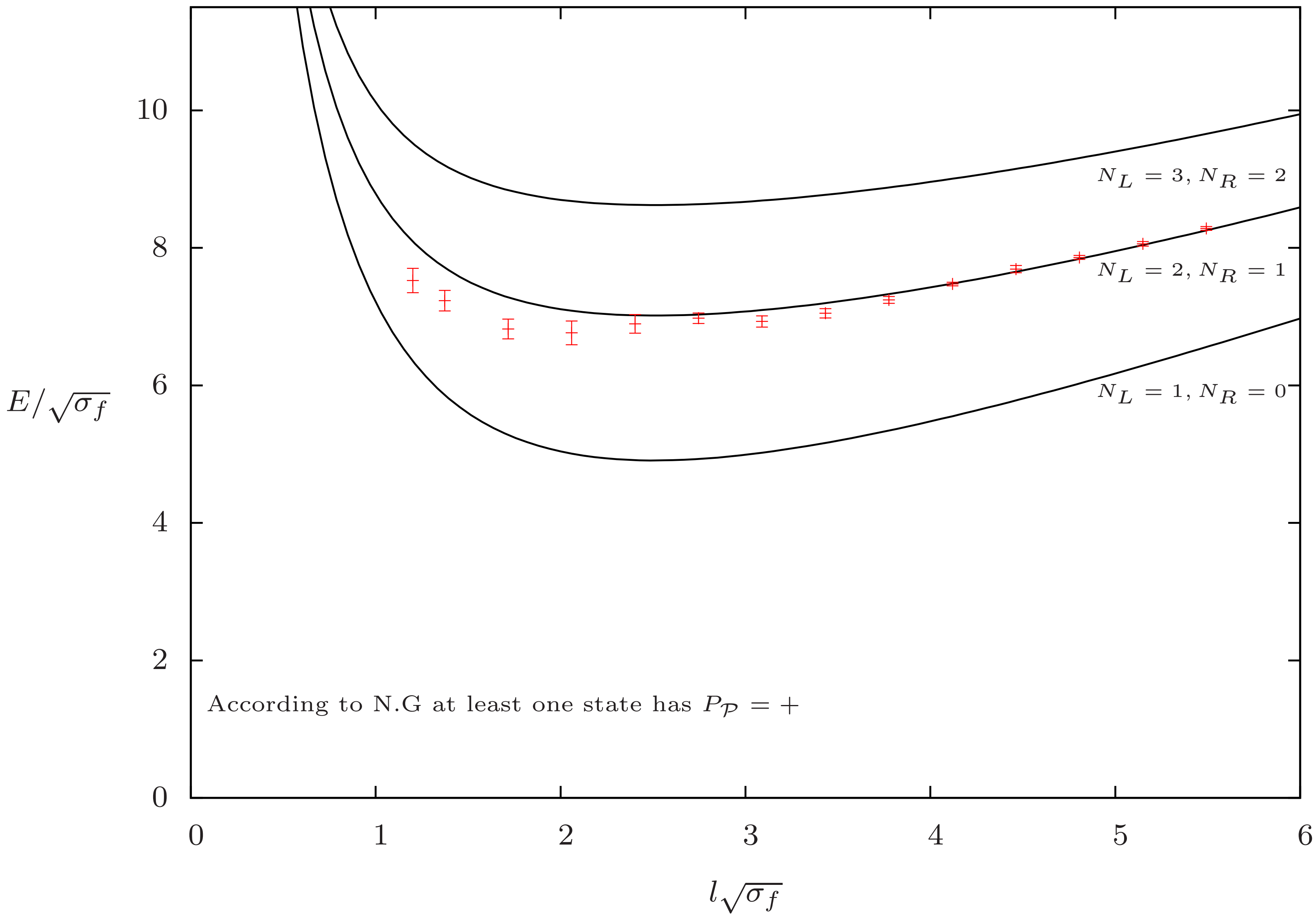
7. Results for: $q = 1$, $P_{\mathcal{P}} = +$ fundamental Representation



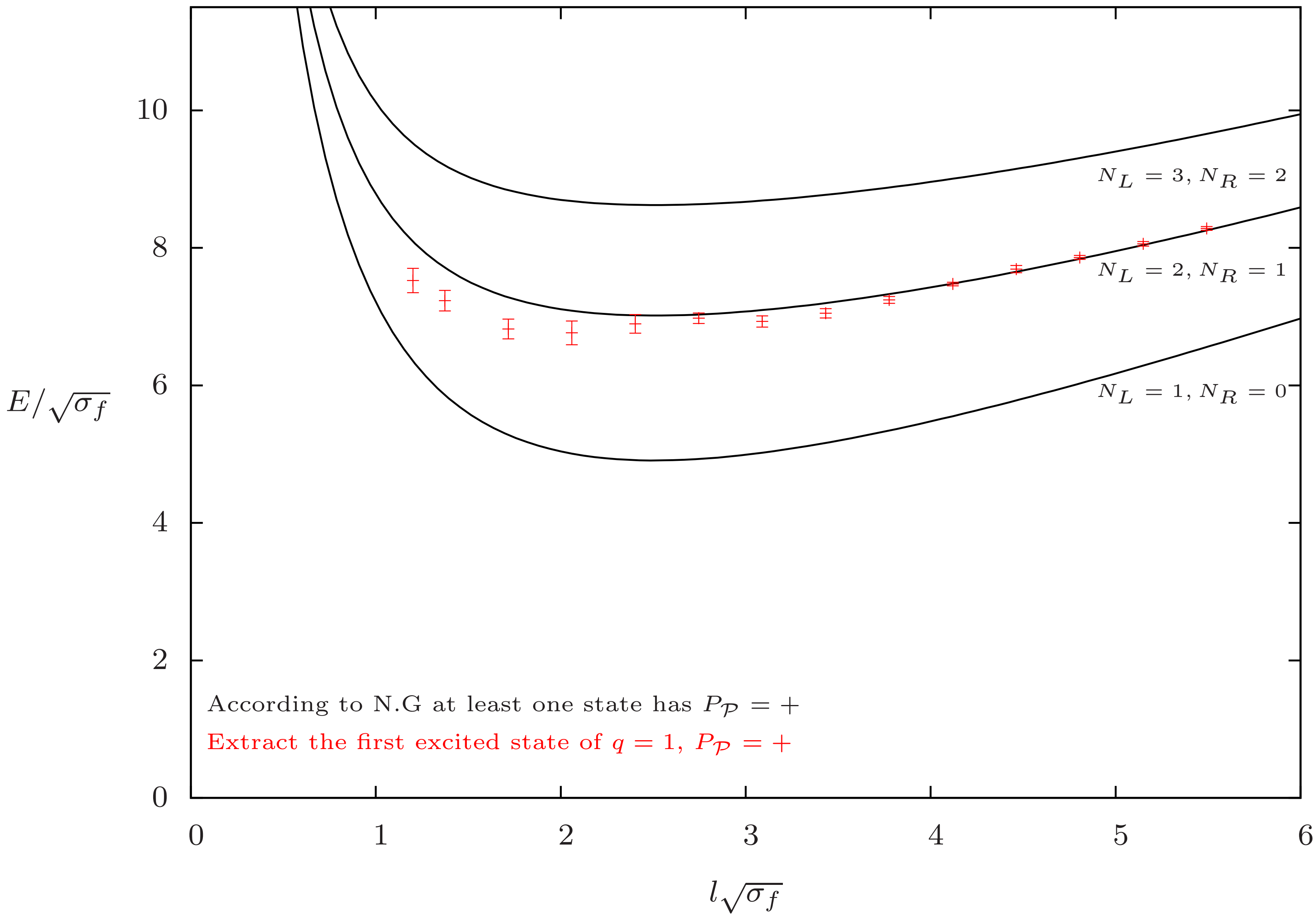
7. Results for: $q = 1$, $P_{\mathcal{P}} = +$ fundamental Representation



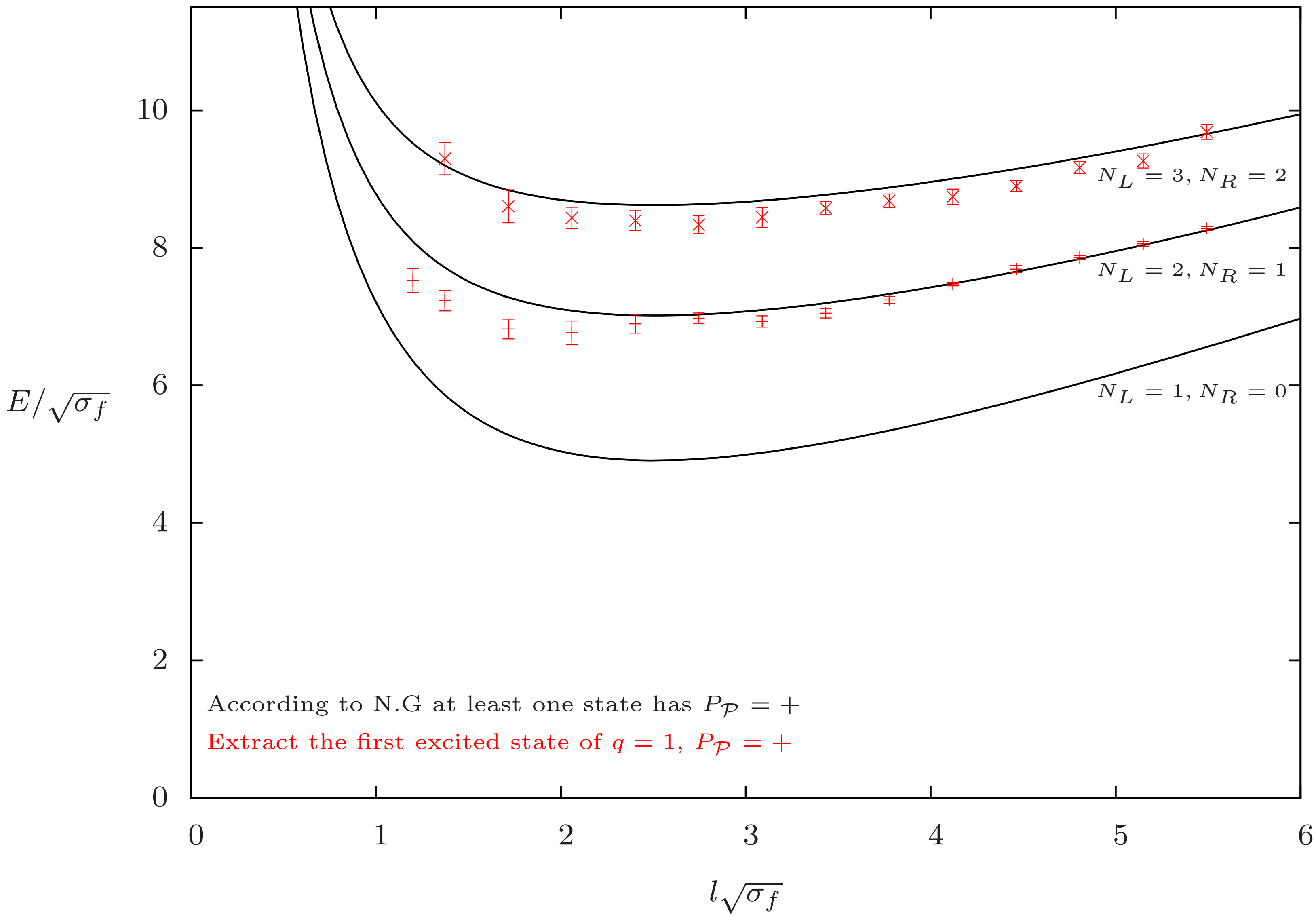
7. Results for: $q = 1$, $P_{\mathcal{P}} = +$ fundamental Representation



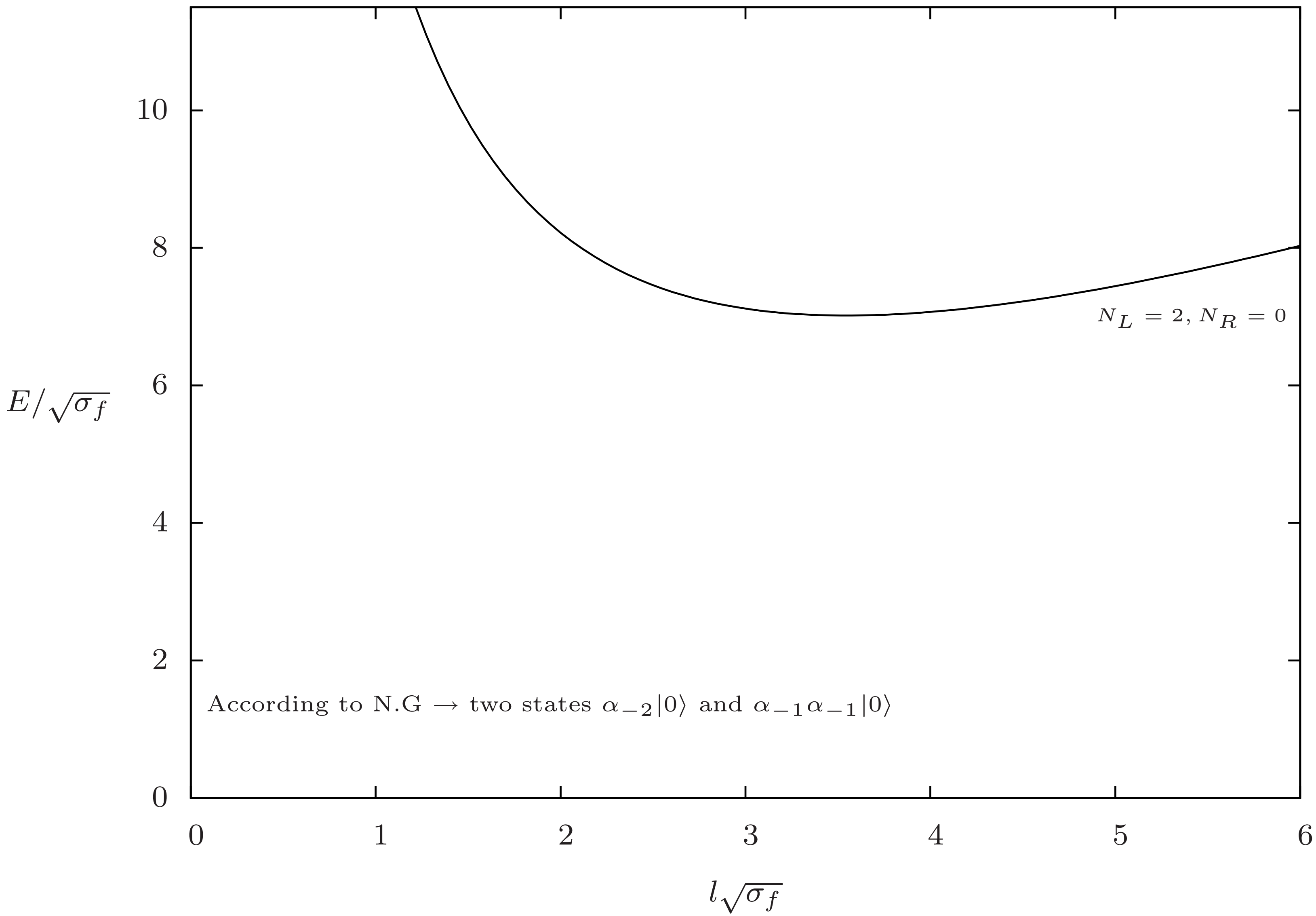
7. Results for: $q = 1$, $P_{\mathcal{P}} = +$ fundamental Representation



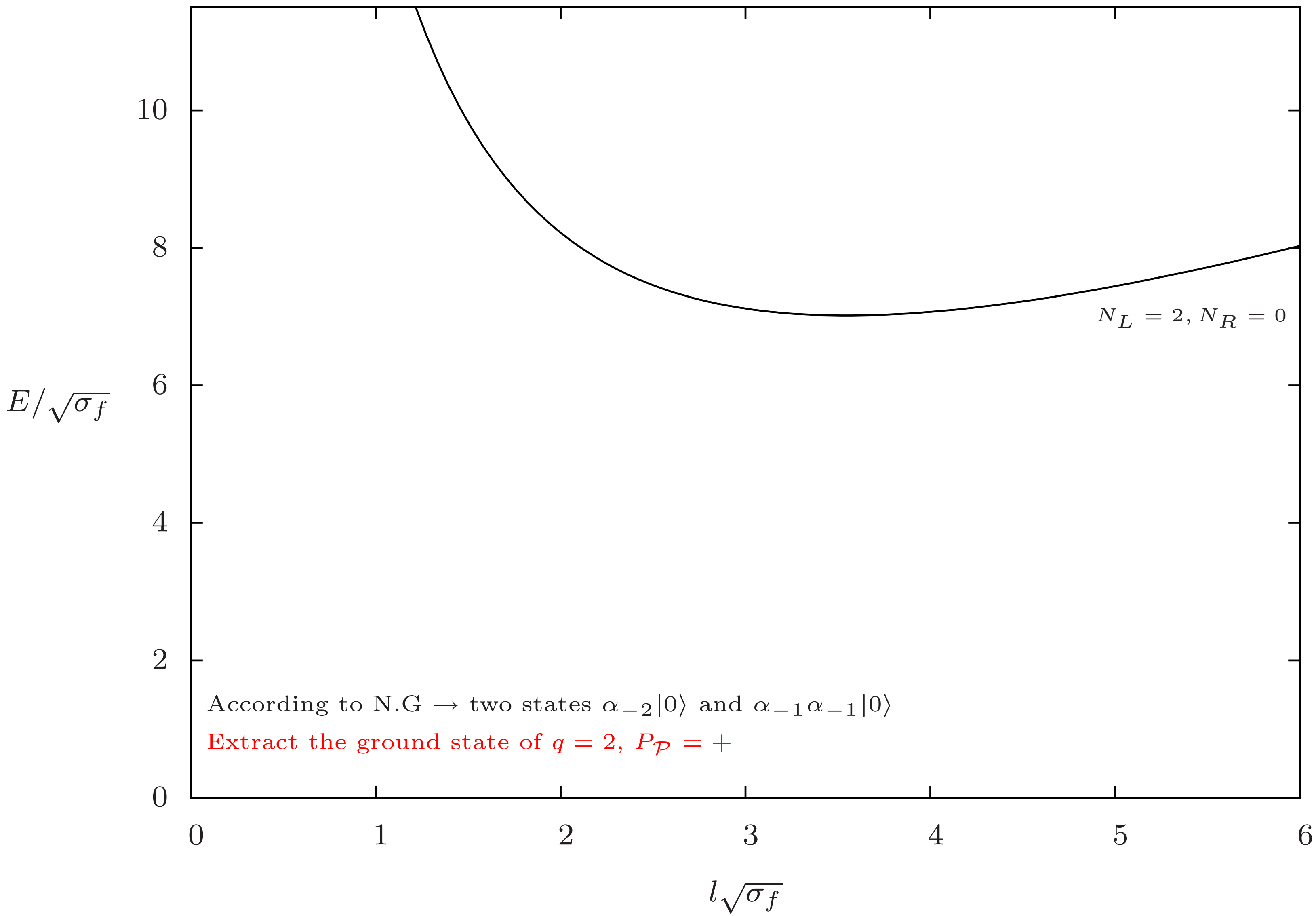
7. Results for: $q = 1$, $P_{\mathcal{P}} = +$ fundamental Representation



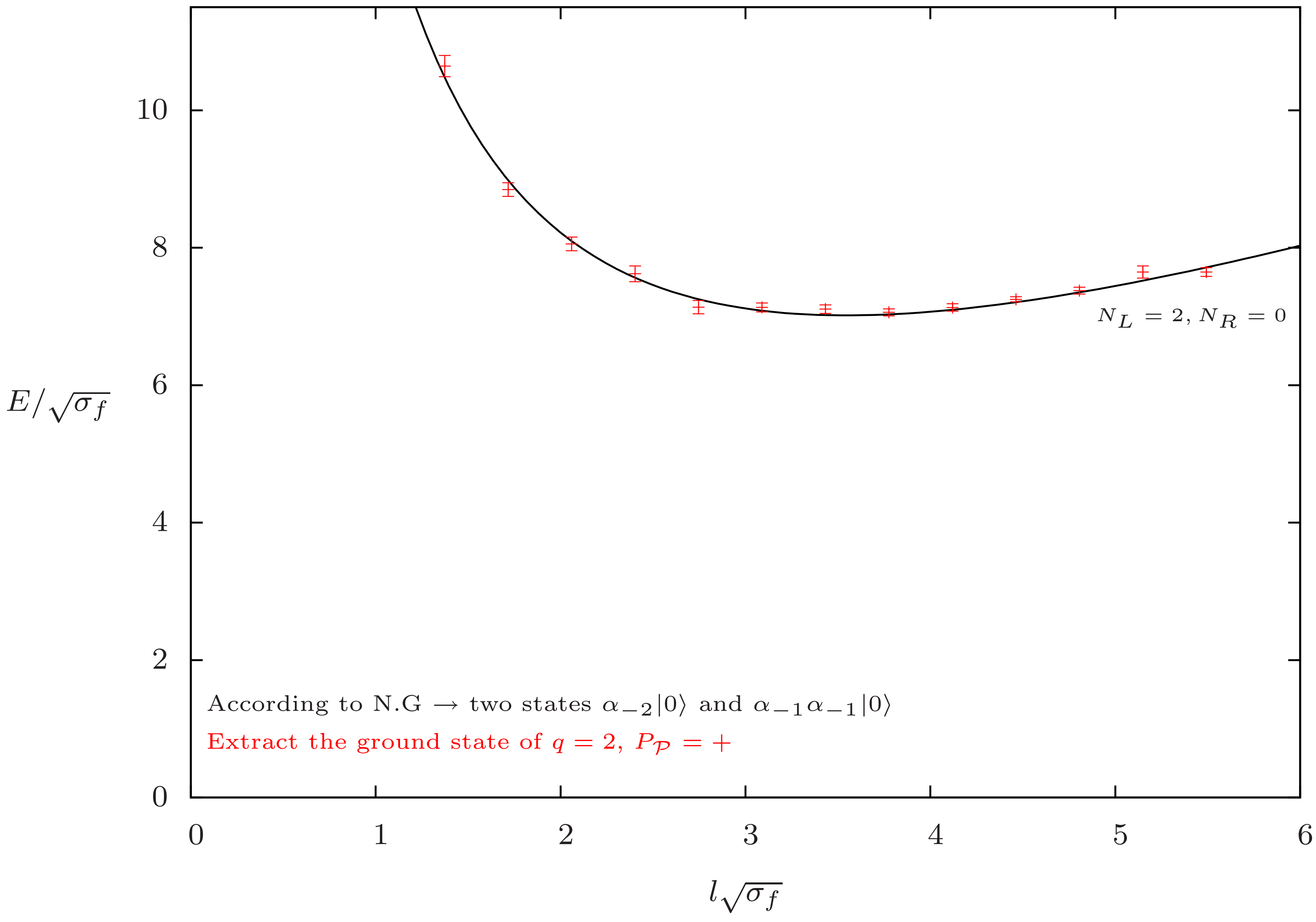
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



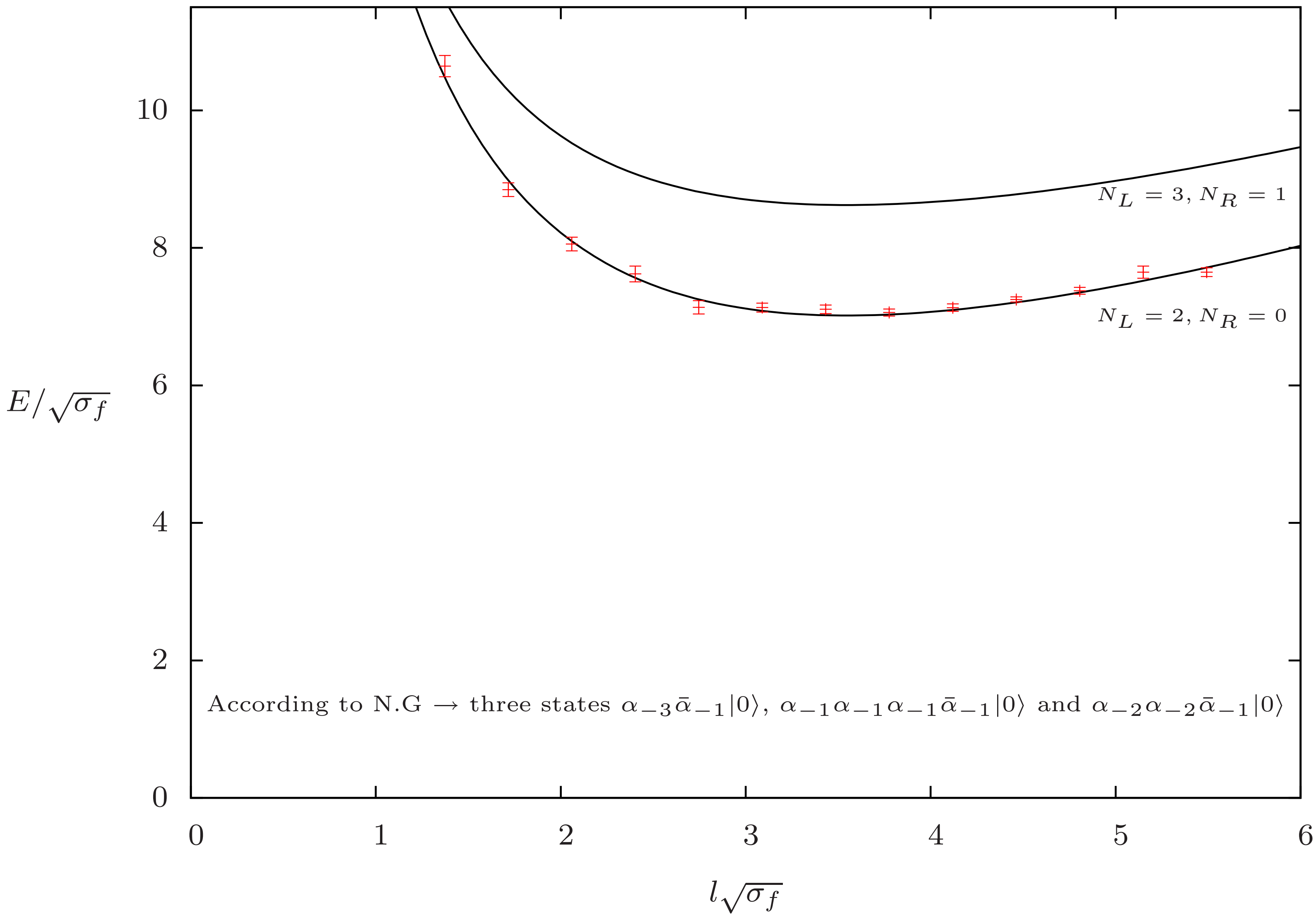
7. Results for: $q = 2,\ P_{\mathcal{P}} = +$ fundamental Representation



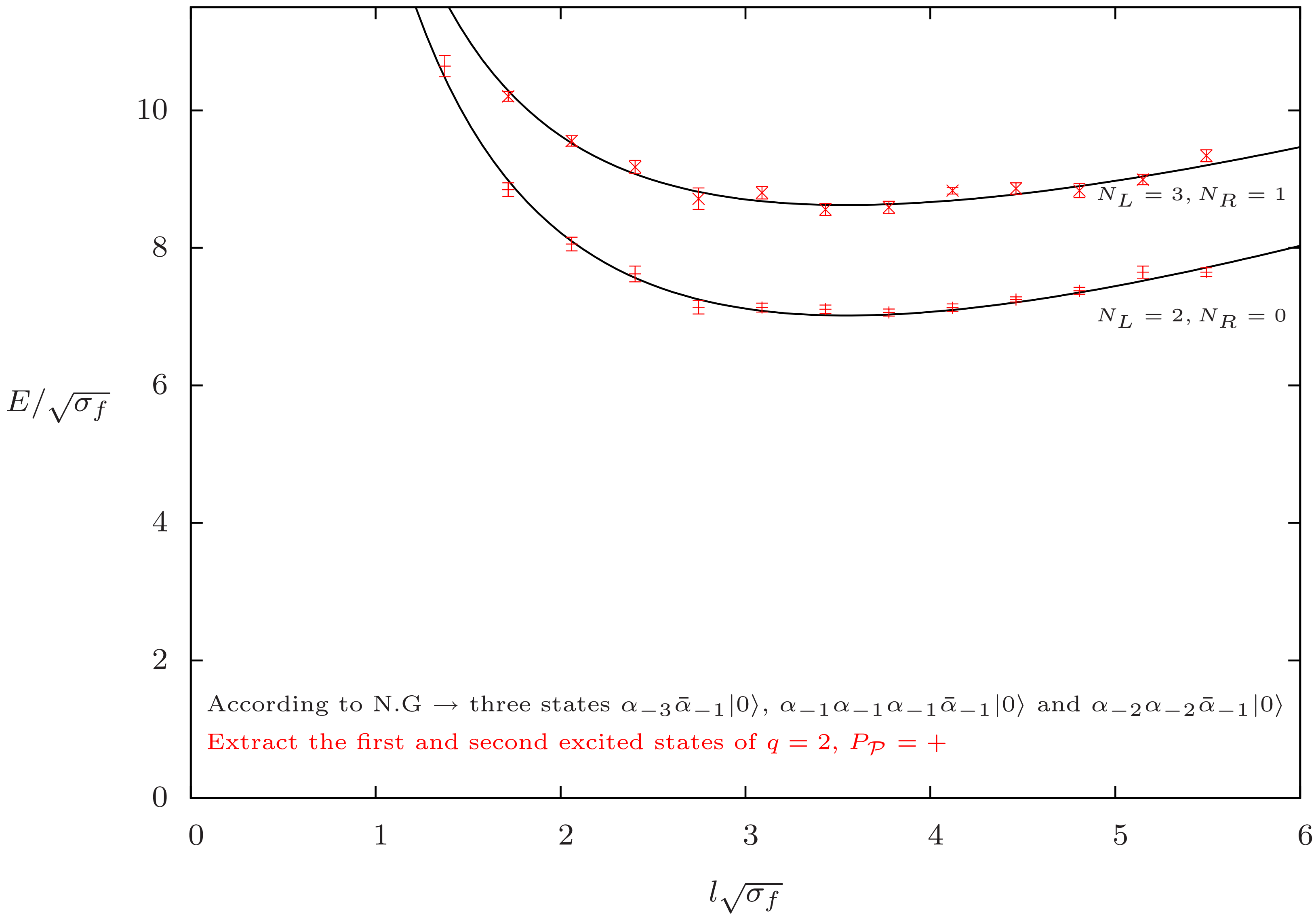
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



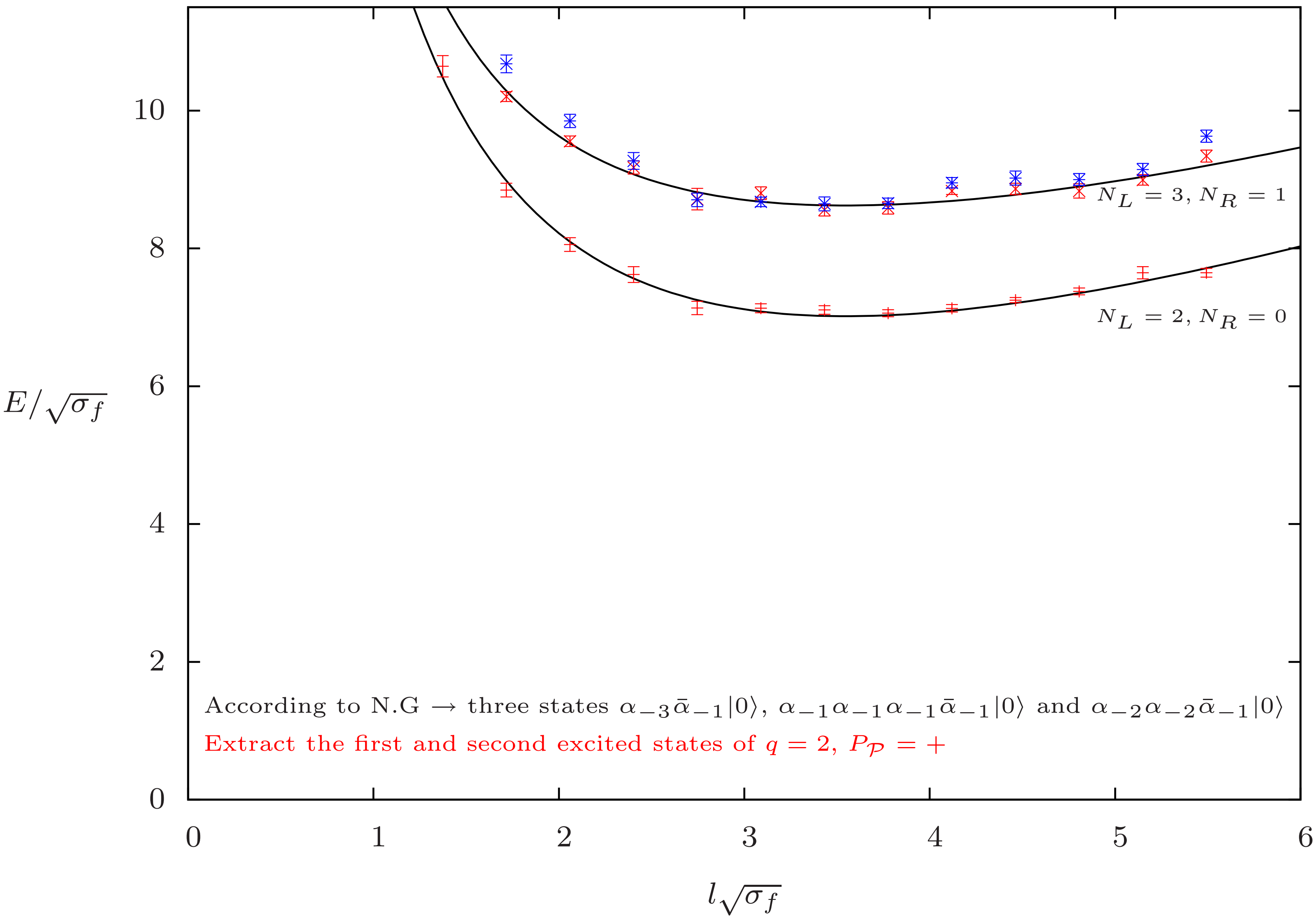
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



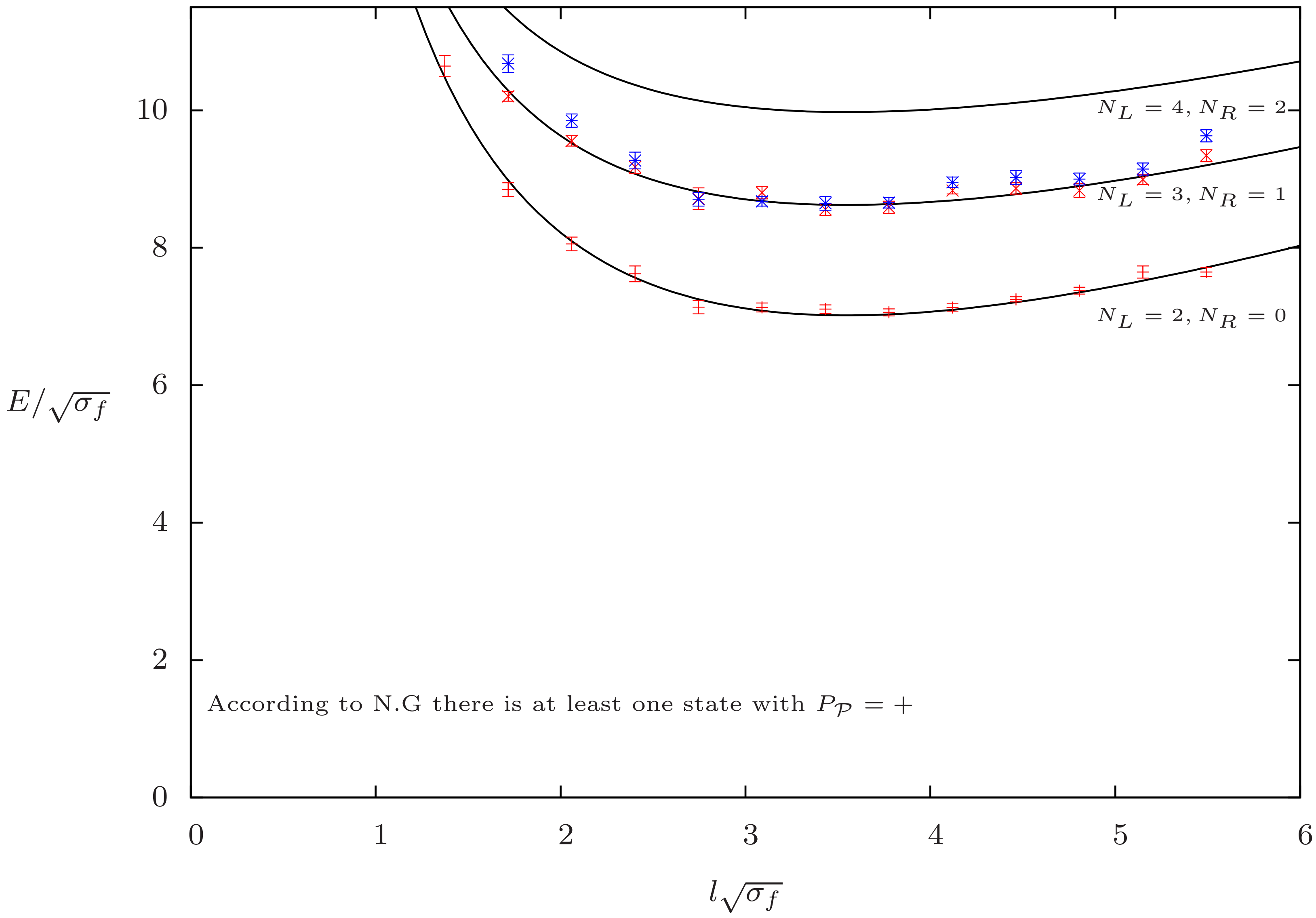
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



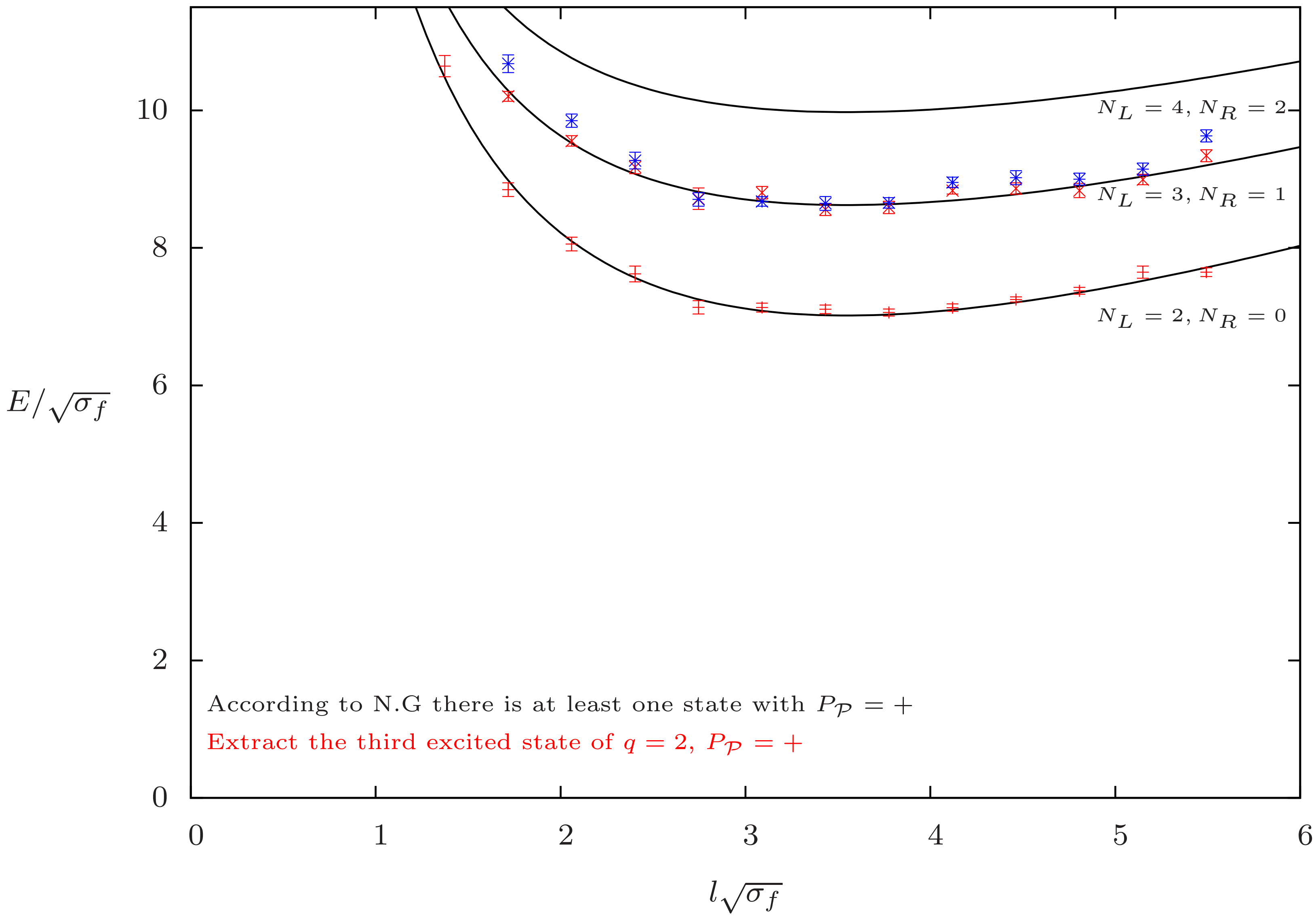
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



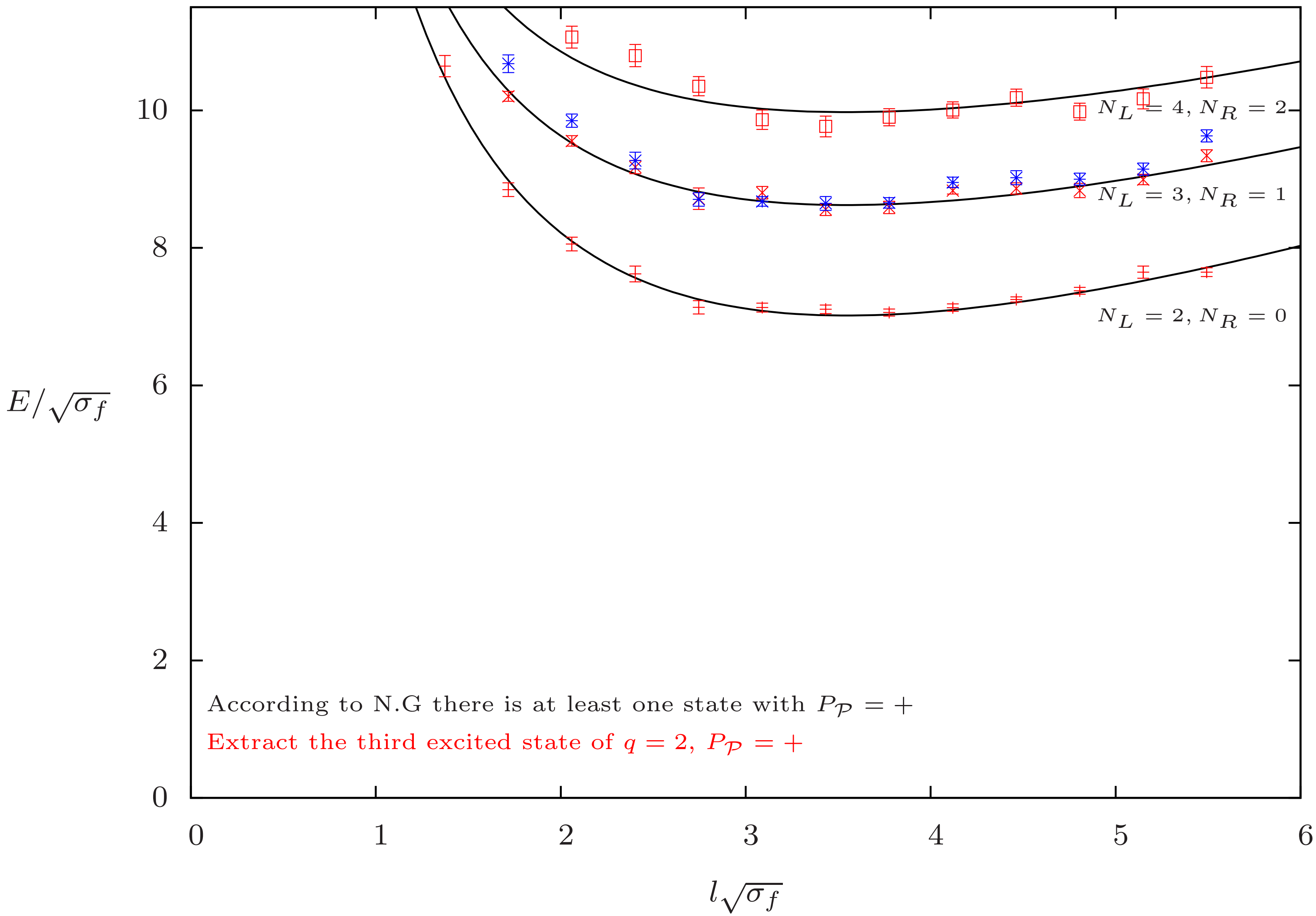
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



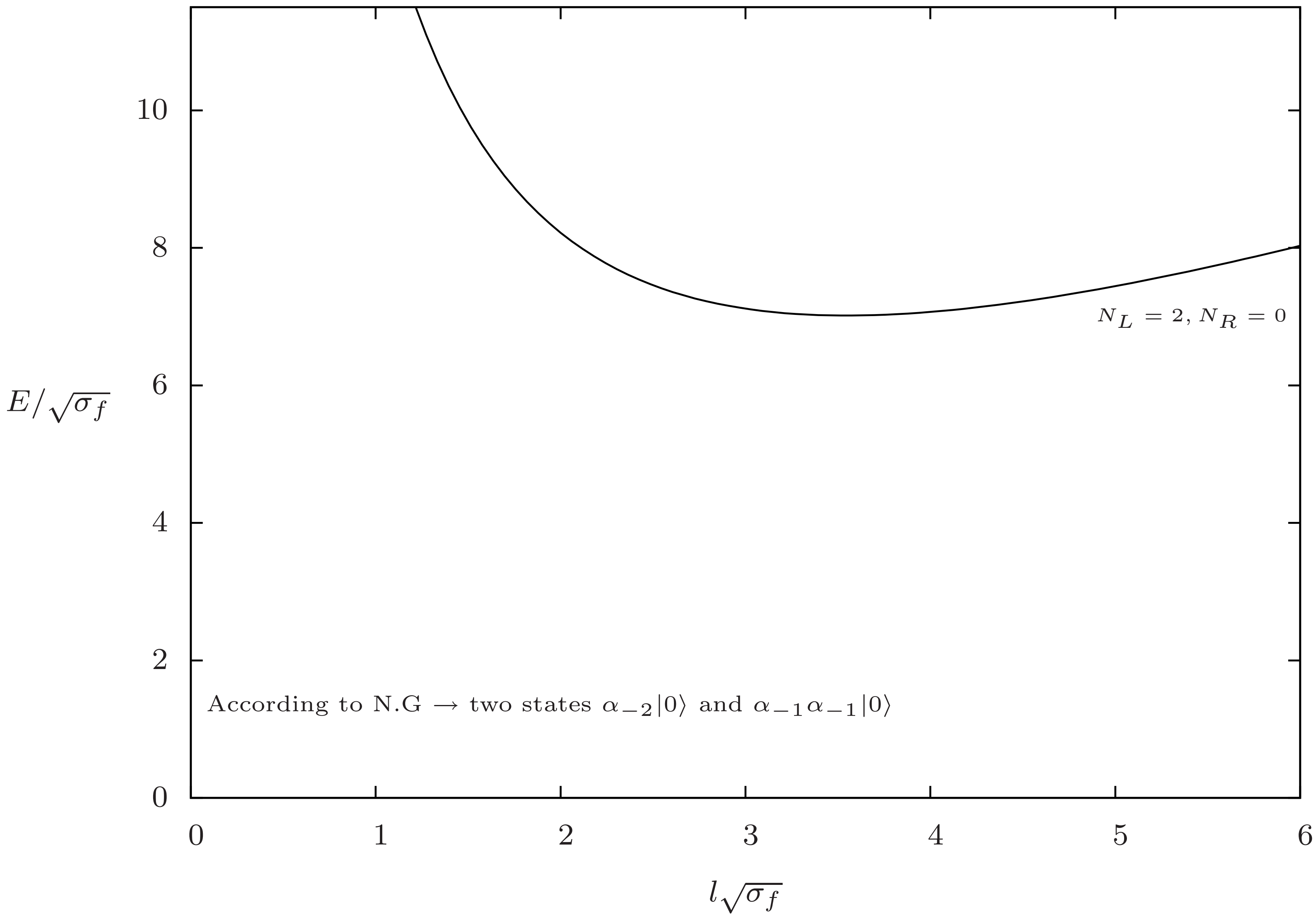
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



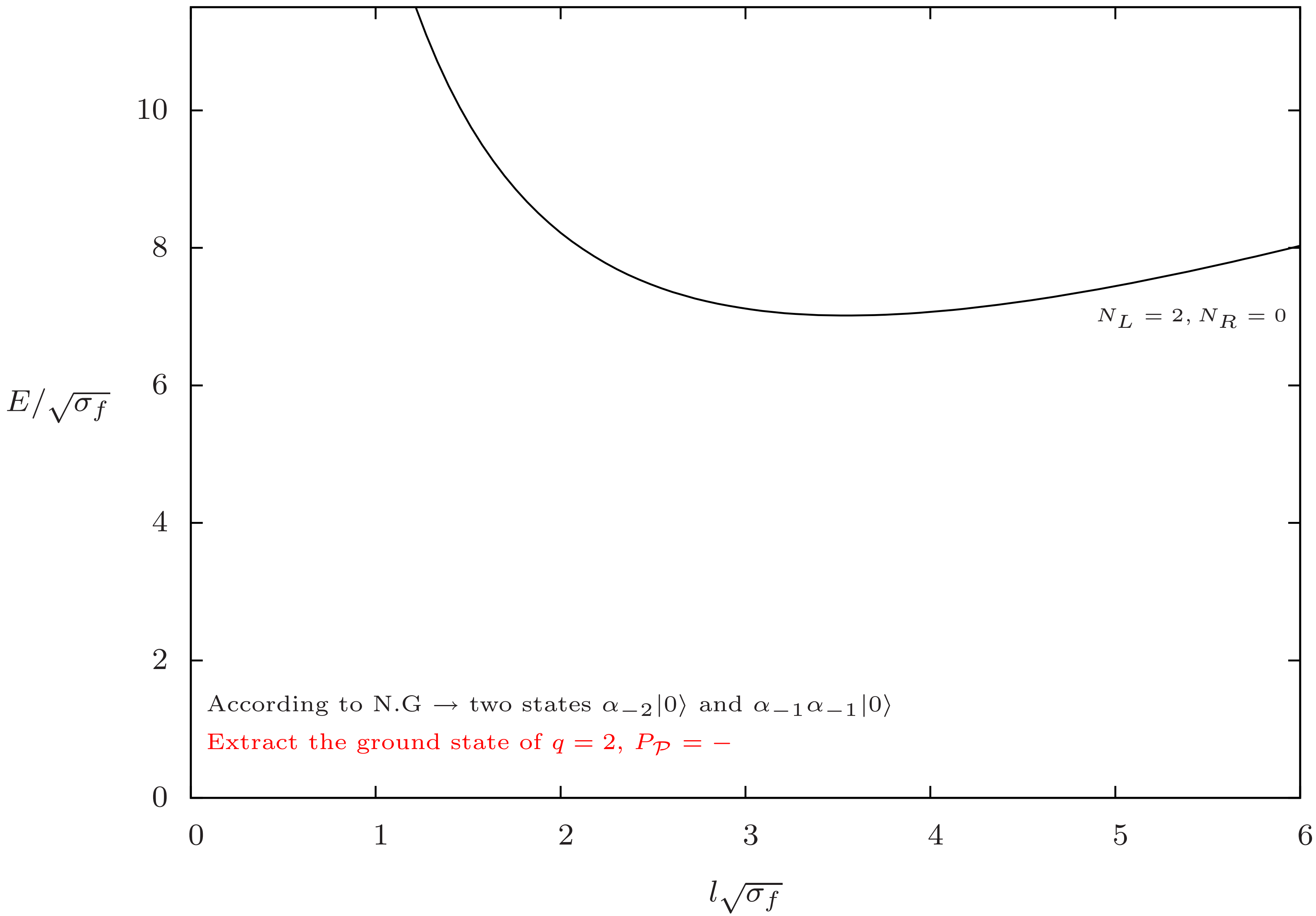
7. Results for: $q = 2$, $P_{\mathcal{P}} = +$ fundamental Representation



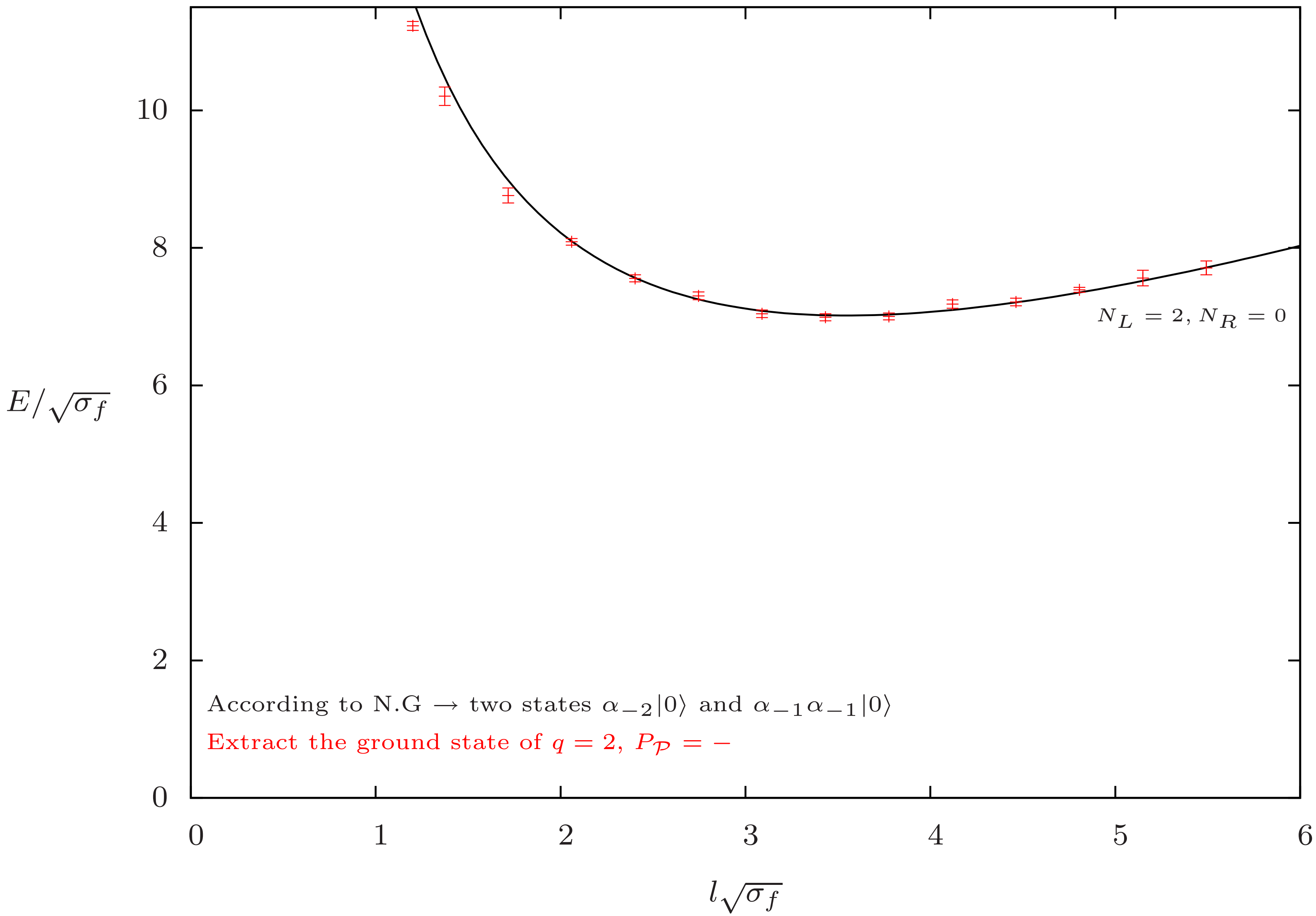
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



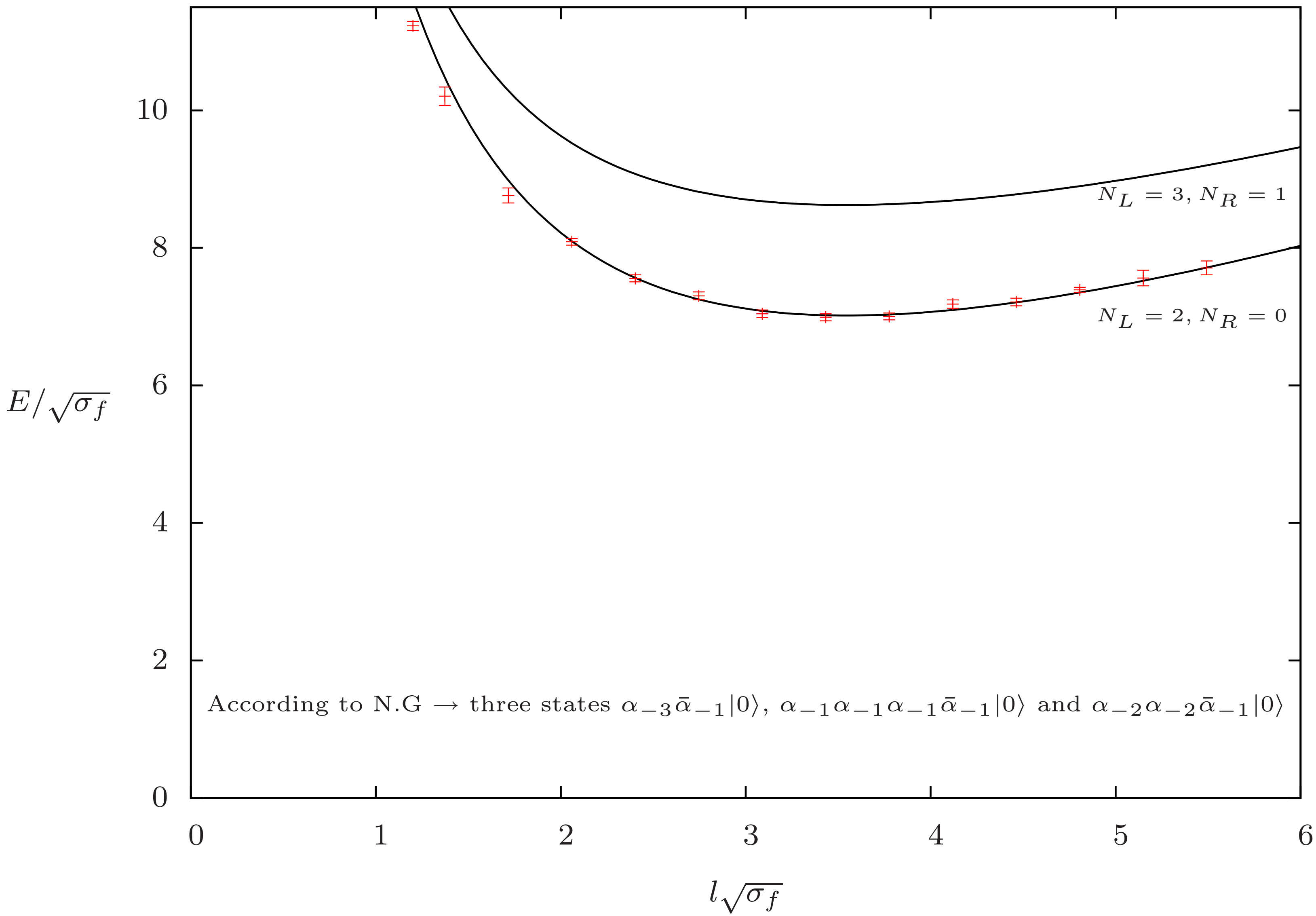
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



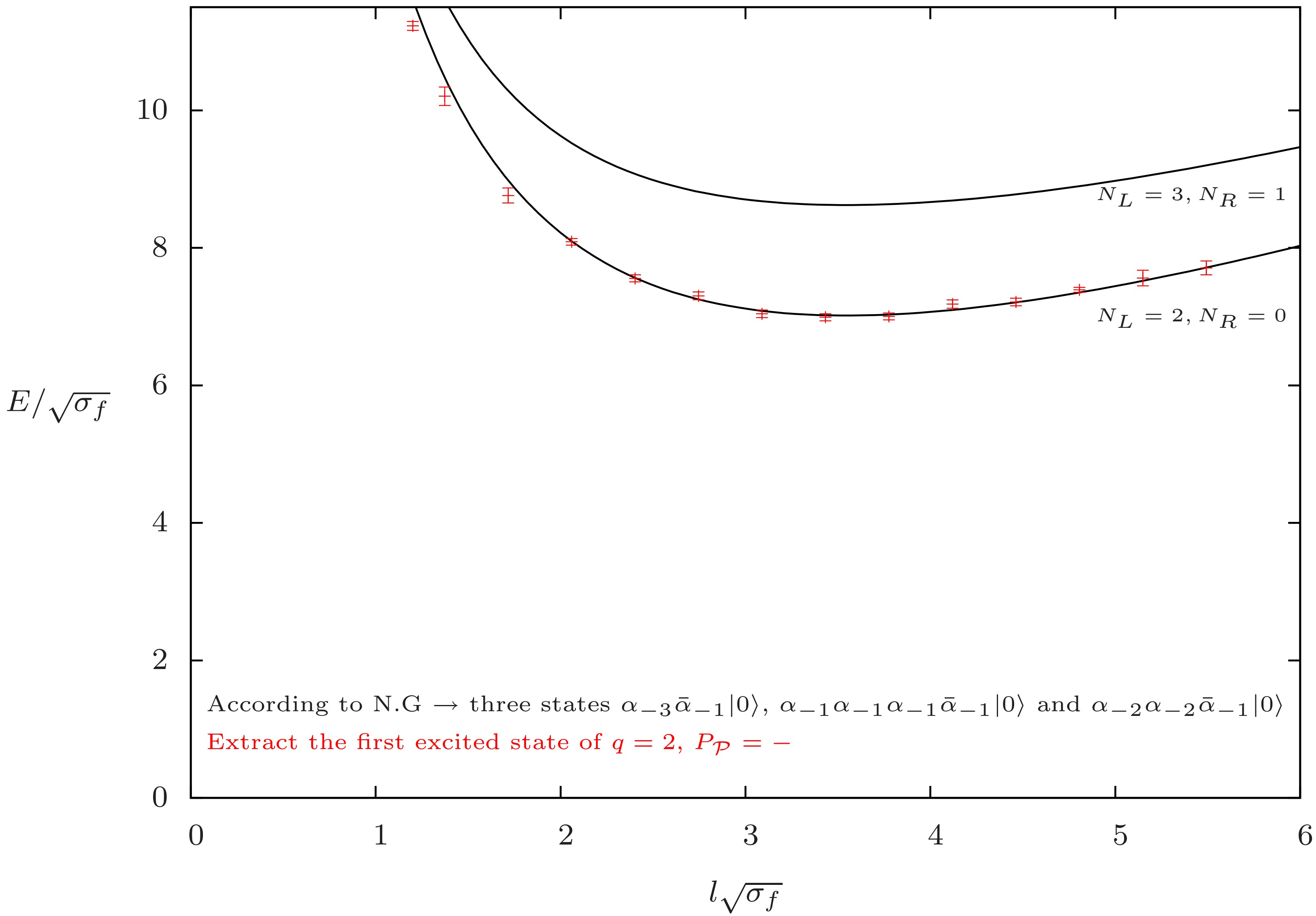
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



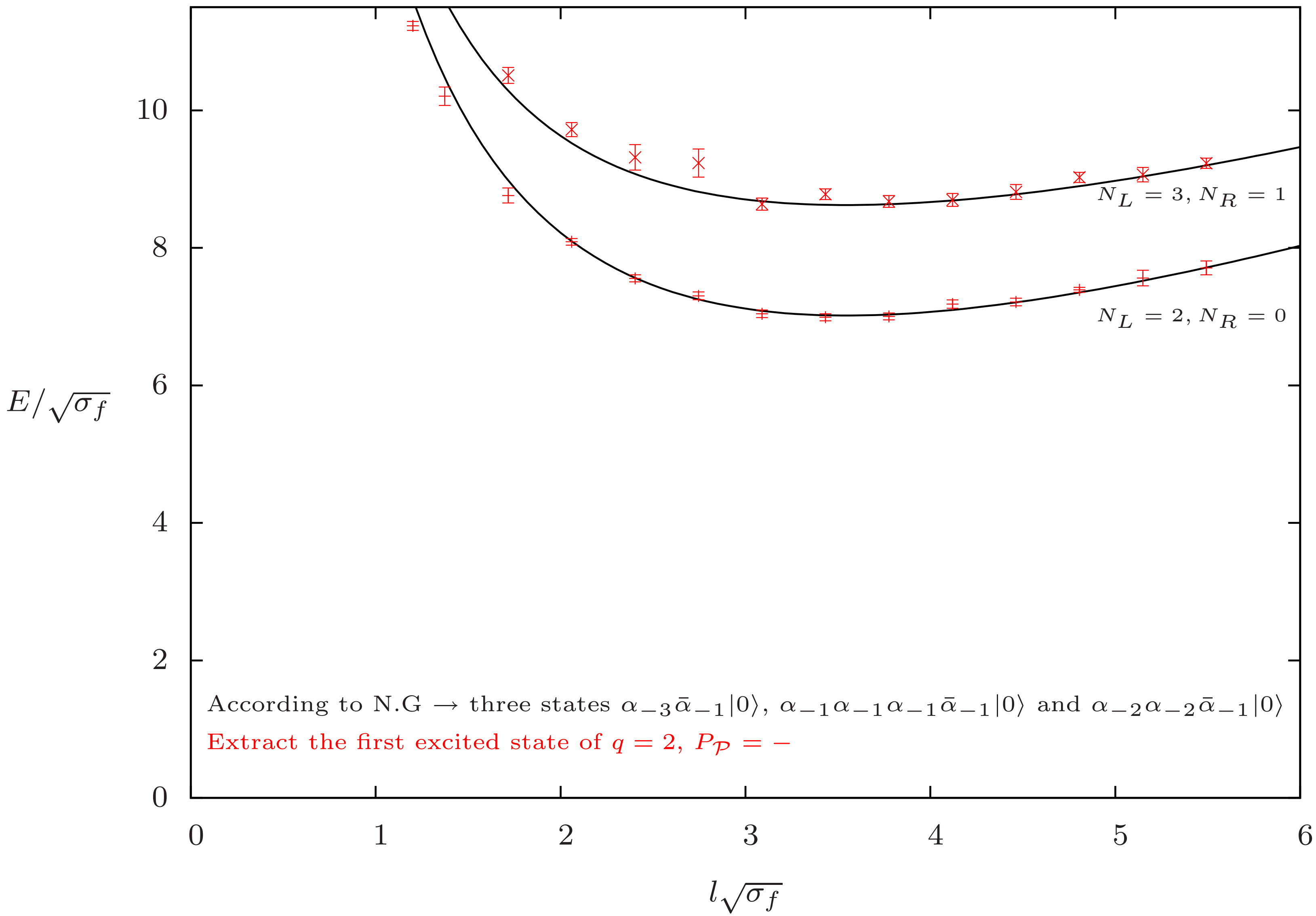
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



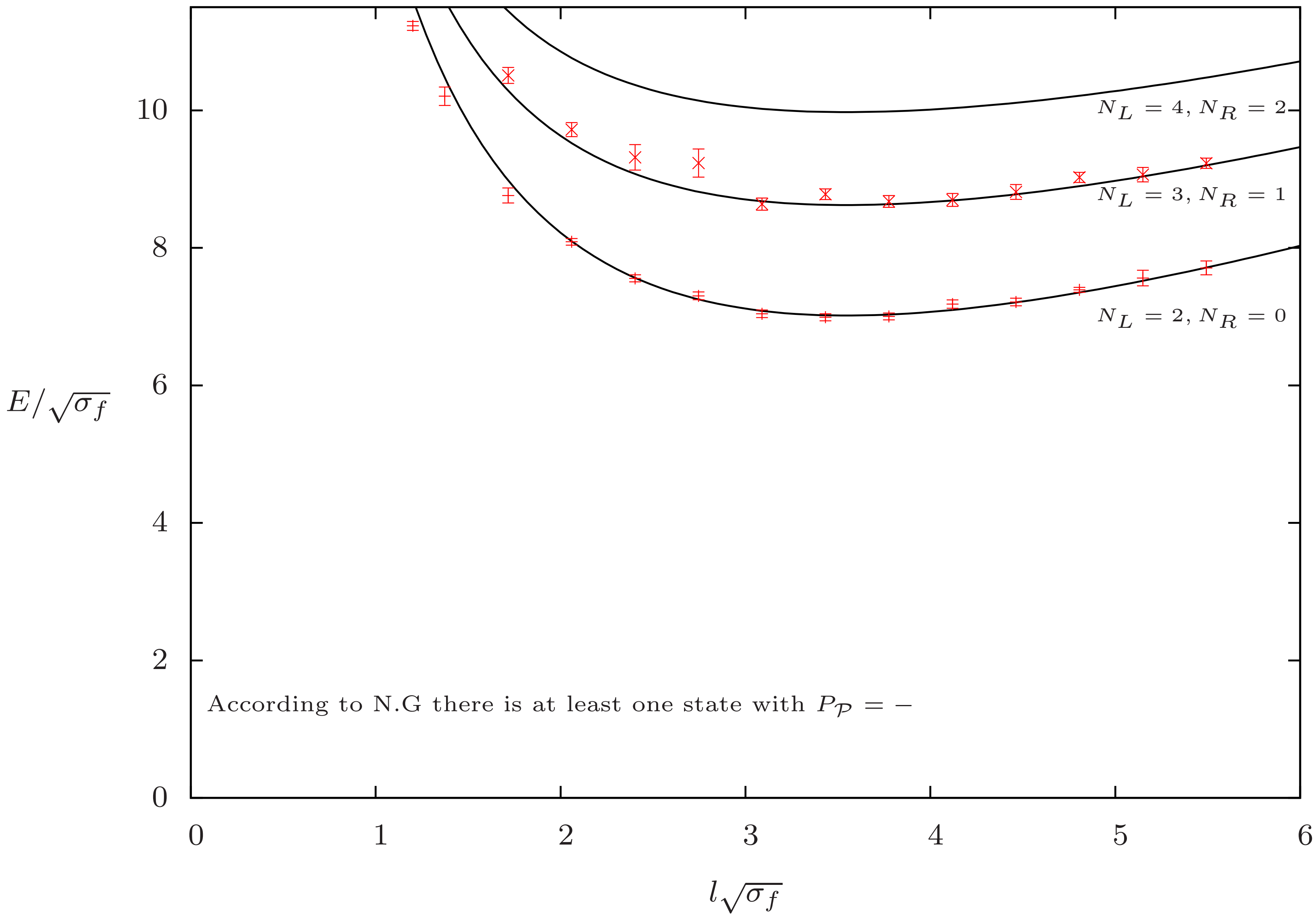
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



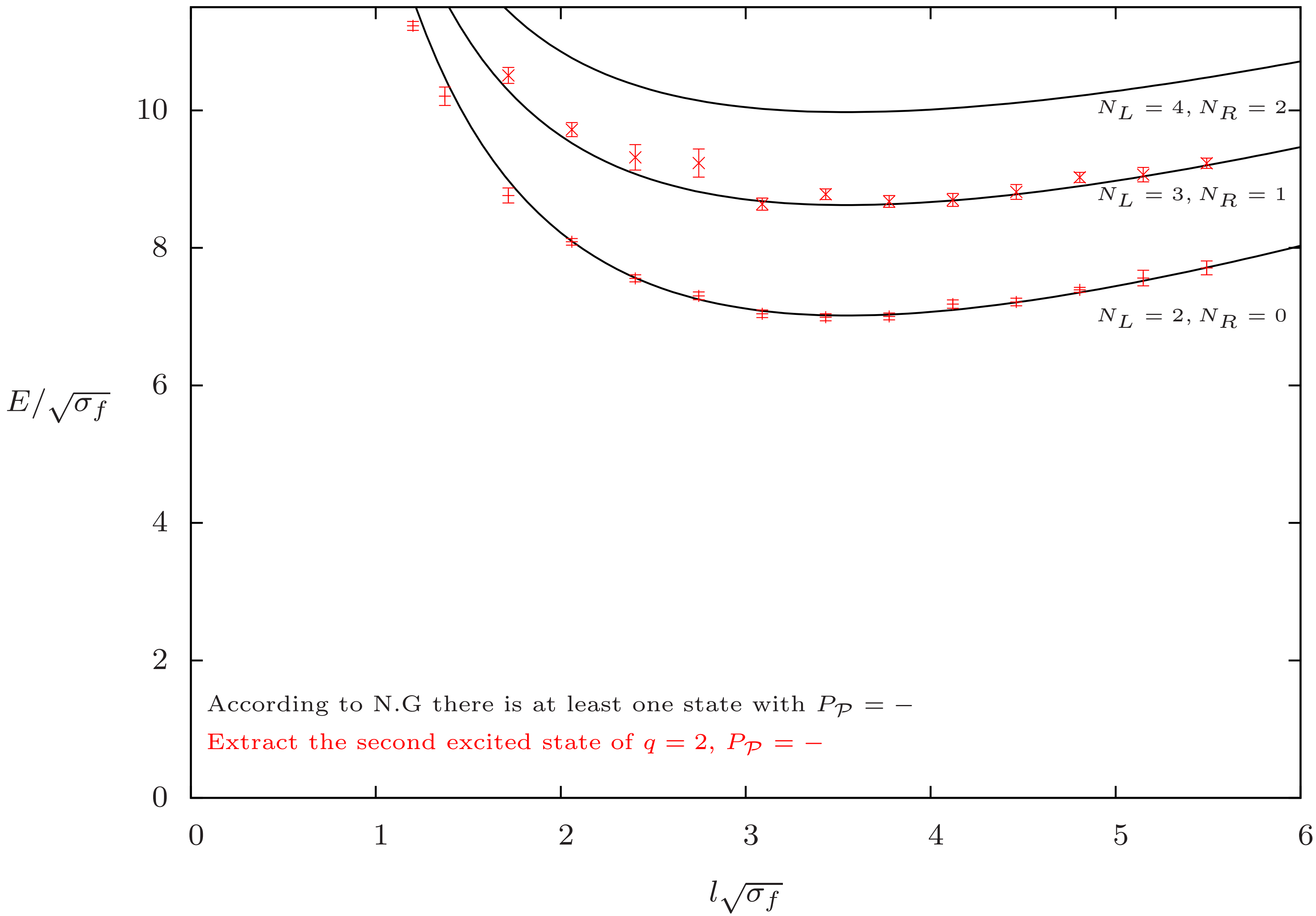
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



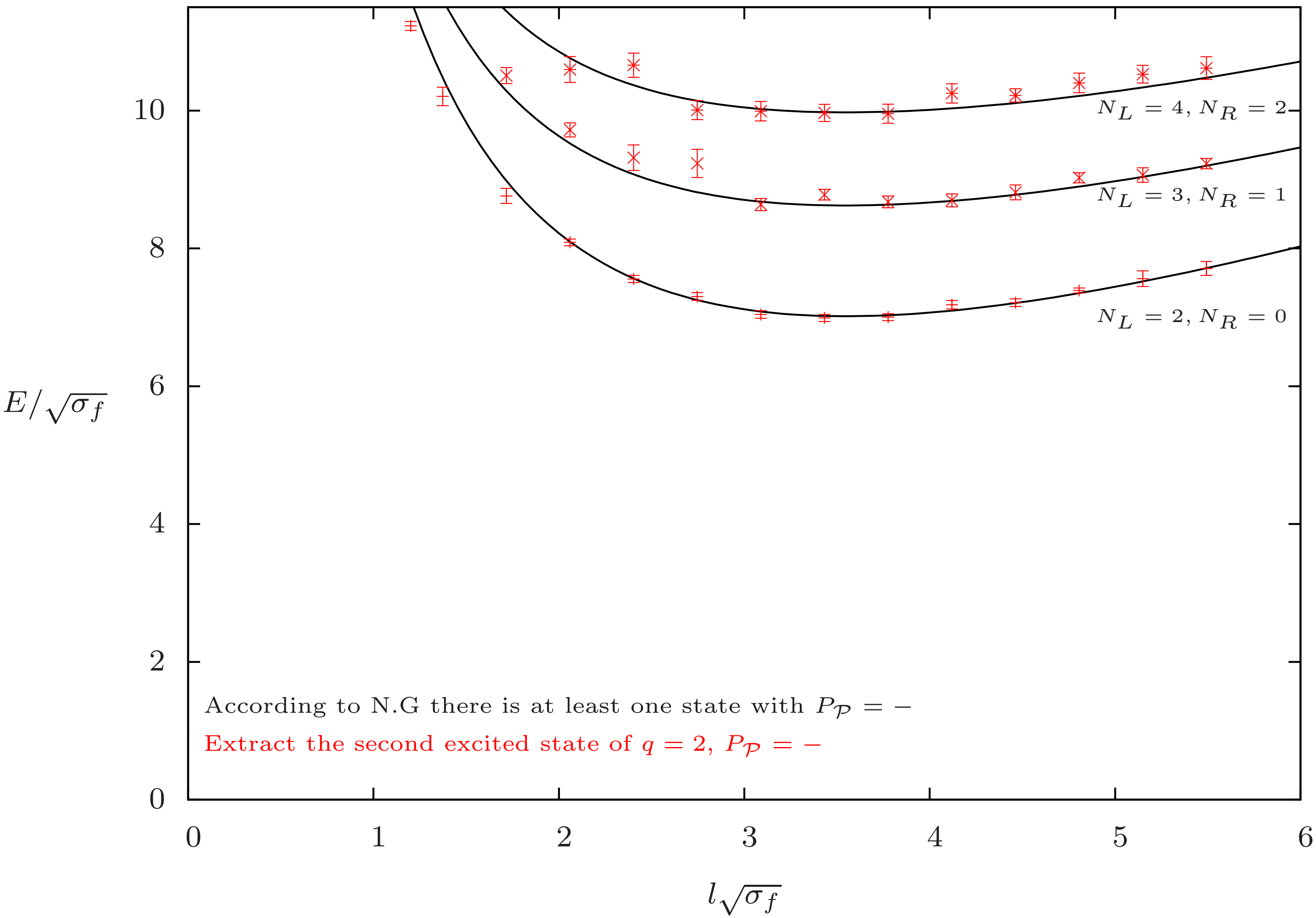
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



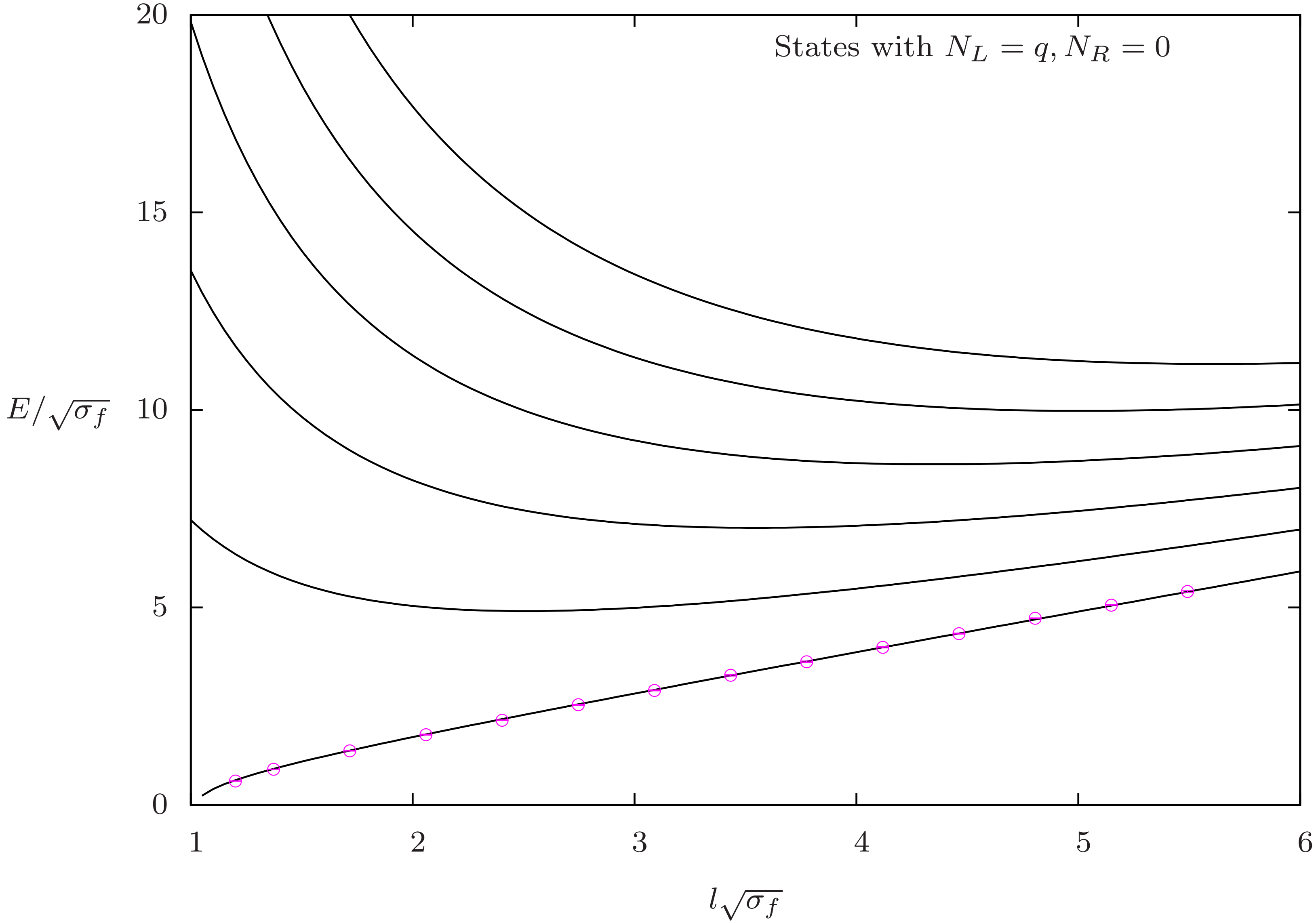
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



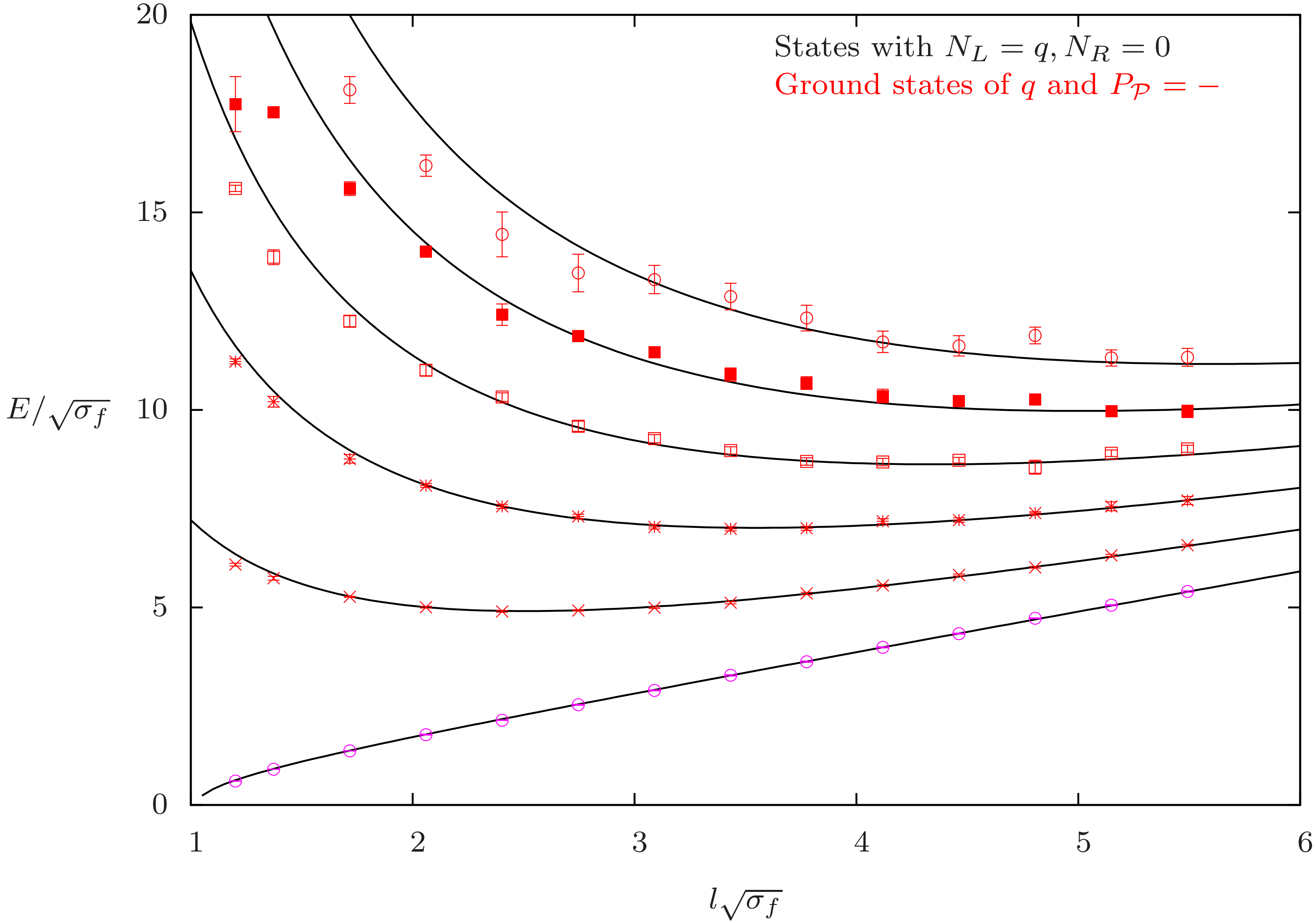
7. Results for: $q = 2$, $P_{\mathcal{P}} = -$ fundamental Representation



7. Results for: $P_{\mathcal{P}} = -$, $q = 1, 2, 3, 4, 5$ fundamental Representation

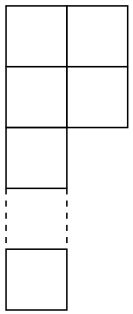


7. Results for: $P_{\mathcal{P}} = -$, $q = 1, 2, 3, 4, 5$ fundamental Representation

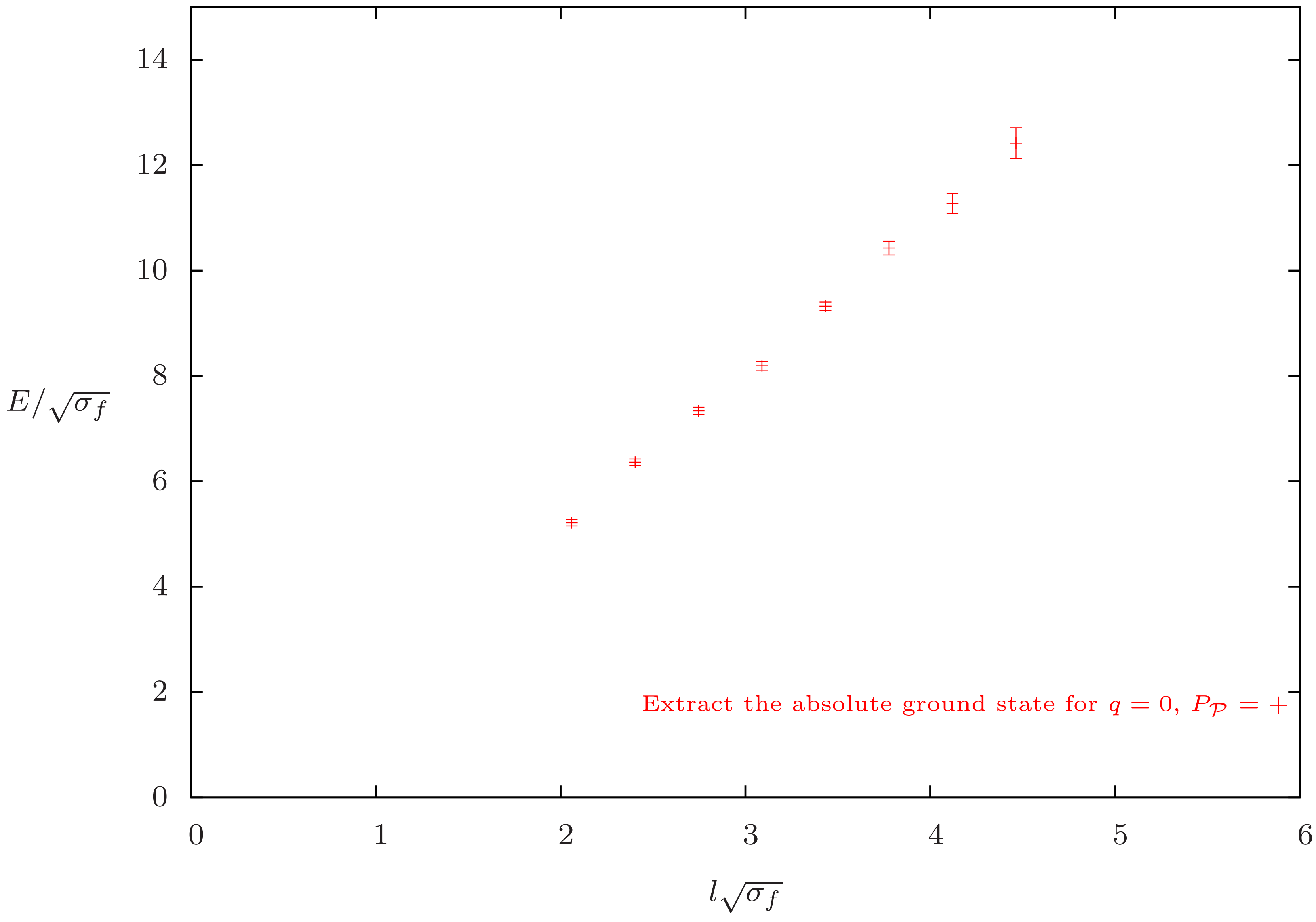


7. RESULTS FOR $k = 1$

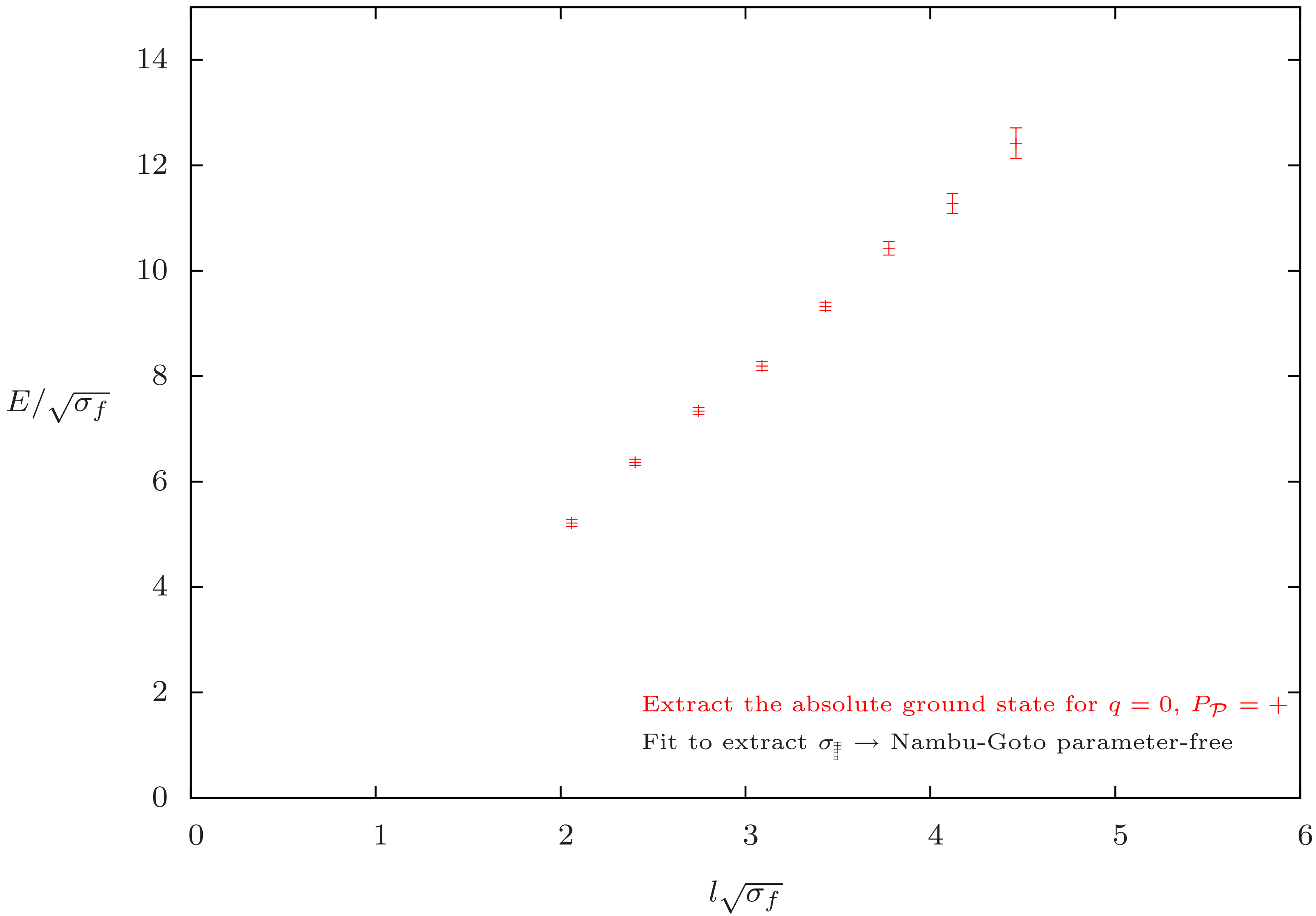
84



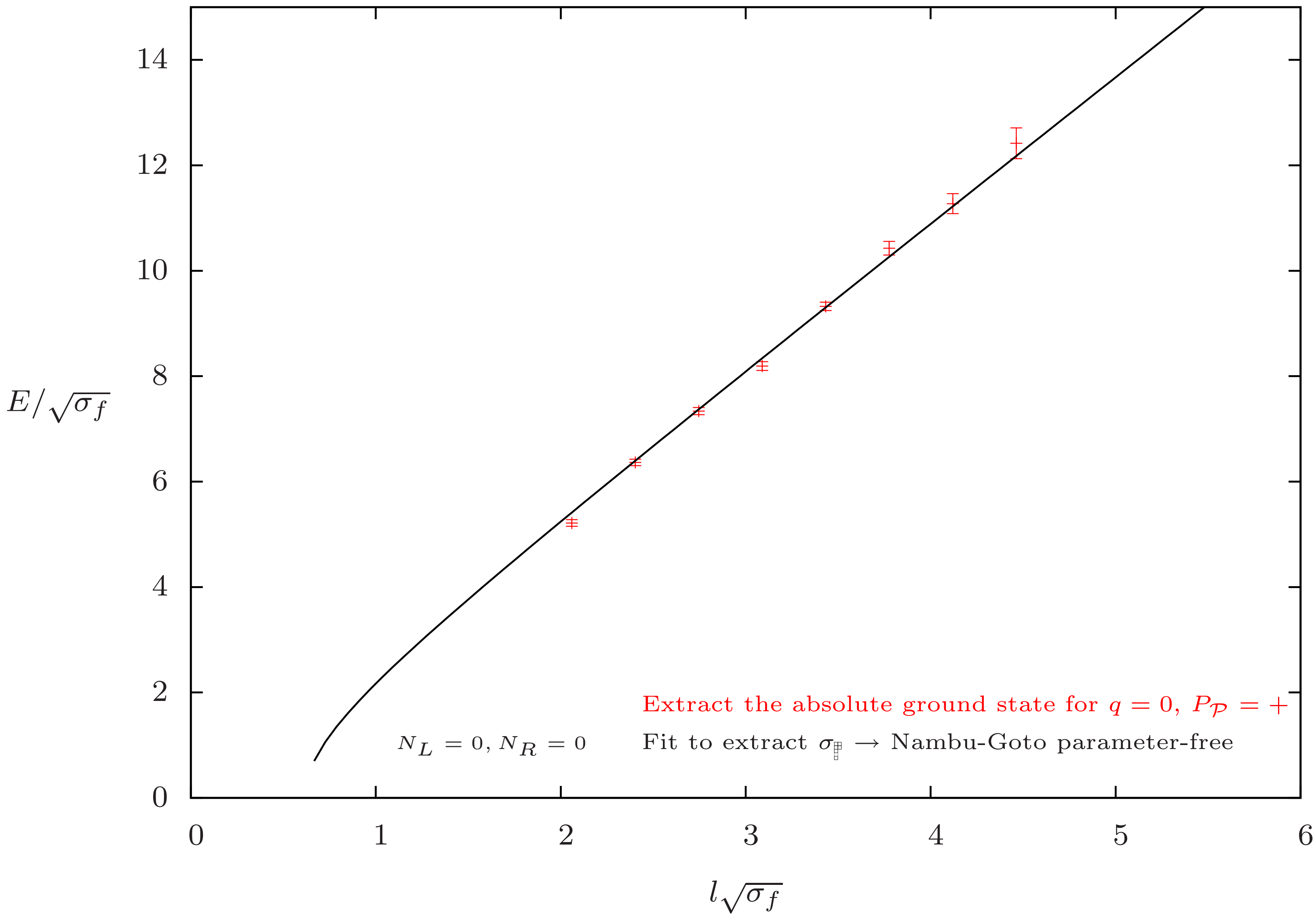
7. Results for $k = 1$, 84



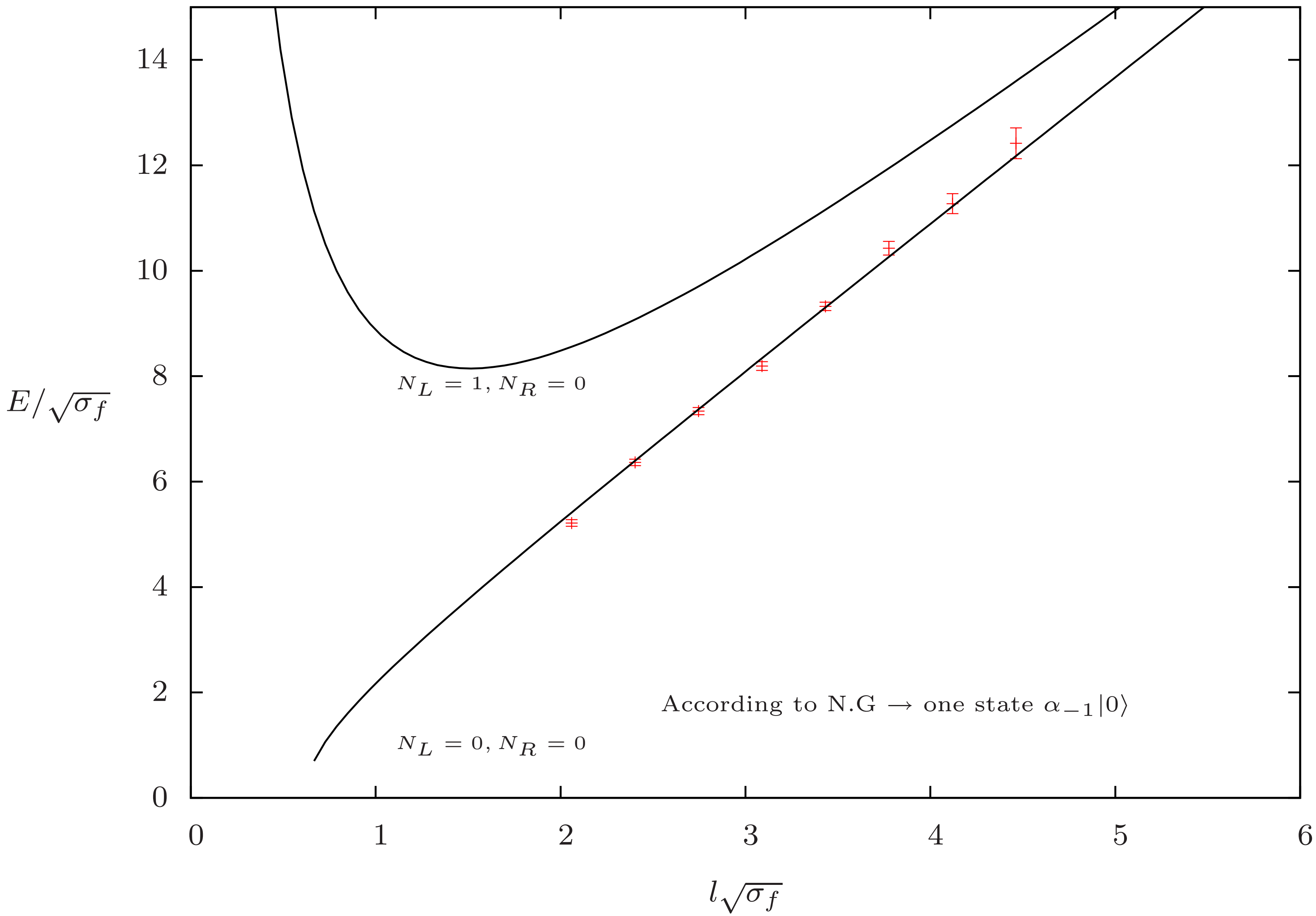
7. Results for $k = 1$, 84



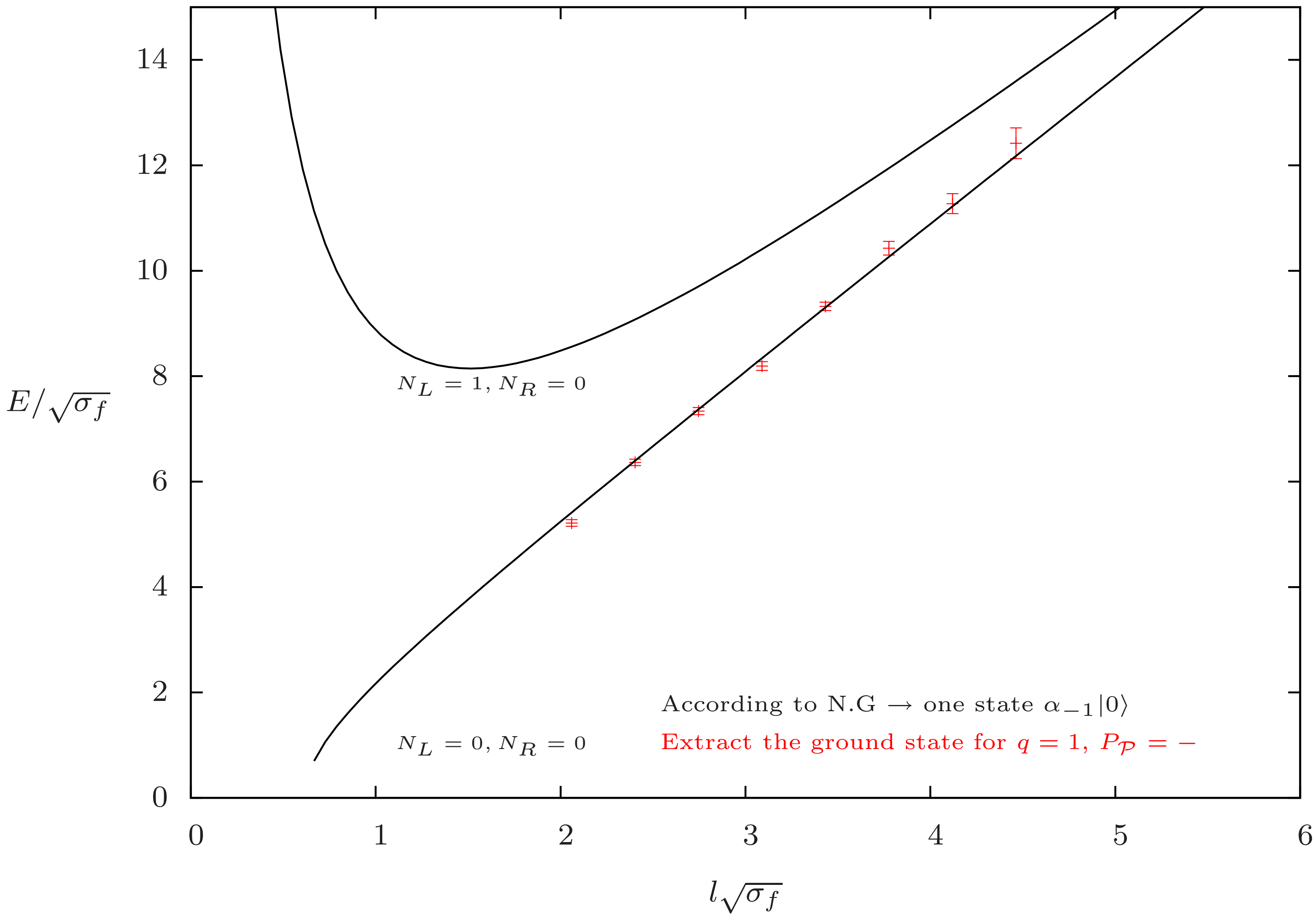
7. Results for $k = 1$, 84



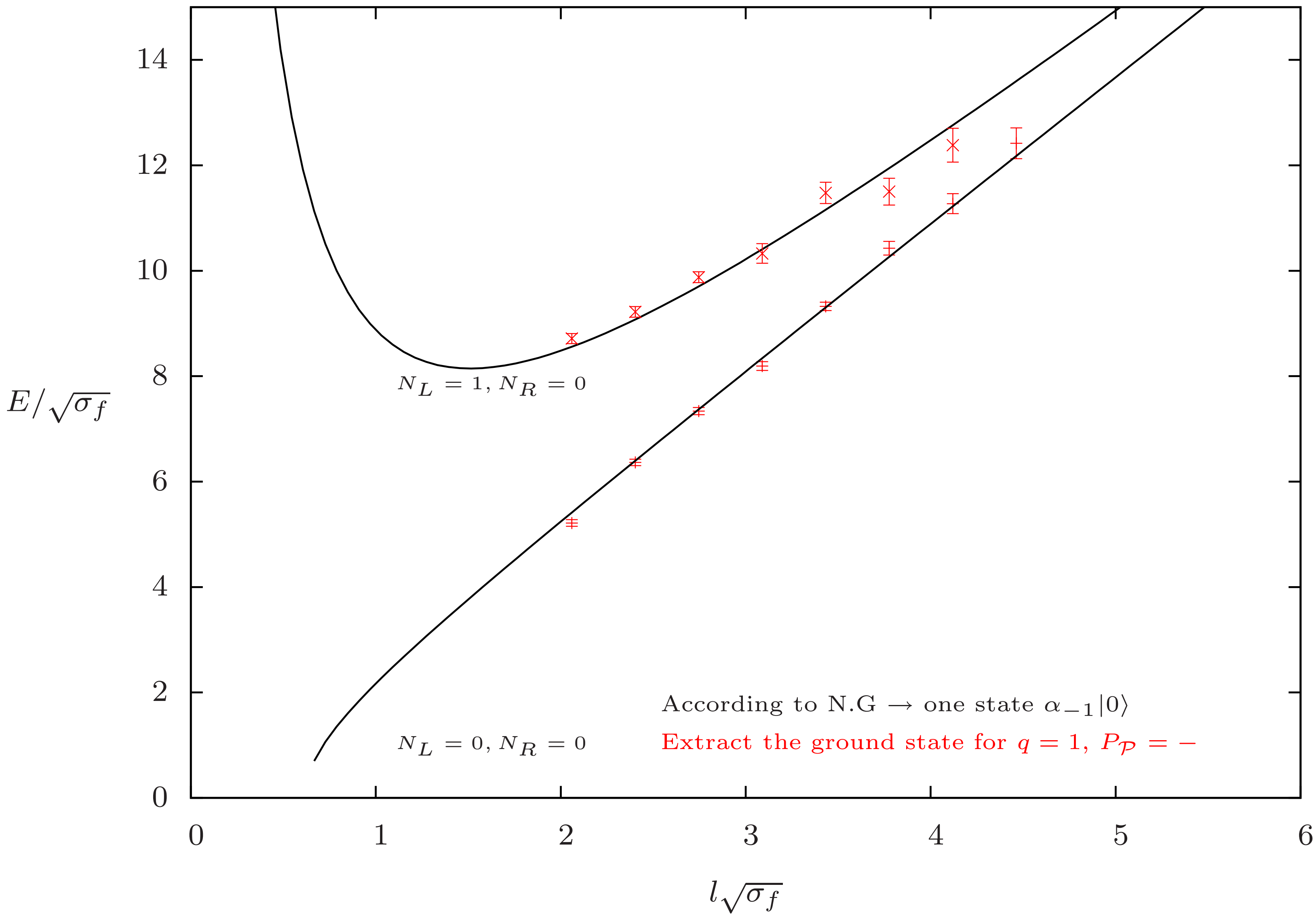
7. Results for $k = 1$, 84



7. Results for $k = 1$, 84

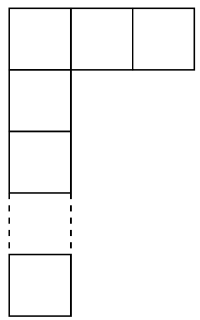


7. Results for $k = 1$, 84

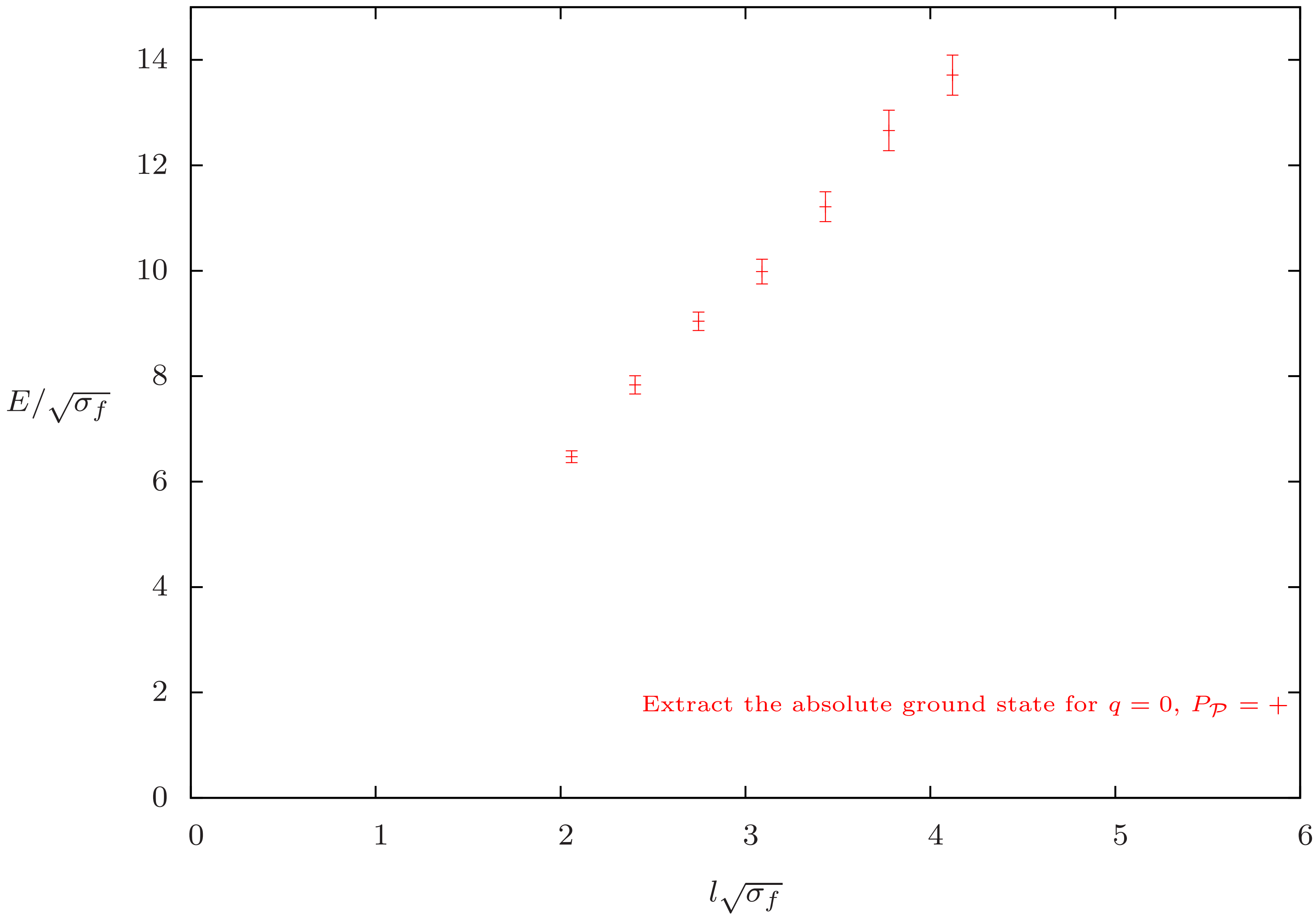


7. RESULTS FOR $k = 1$

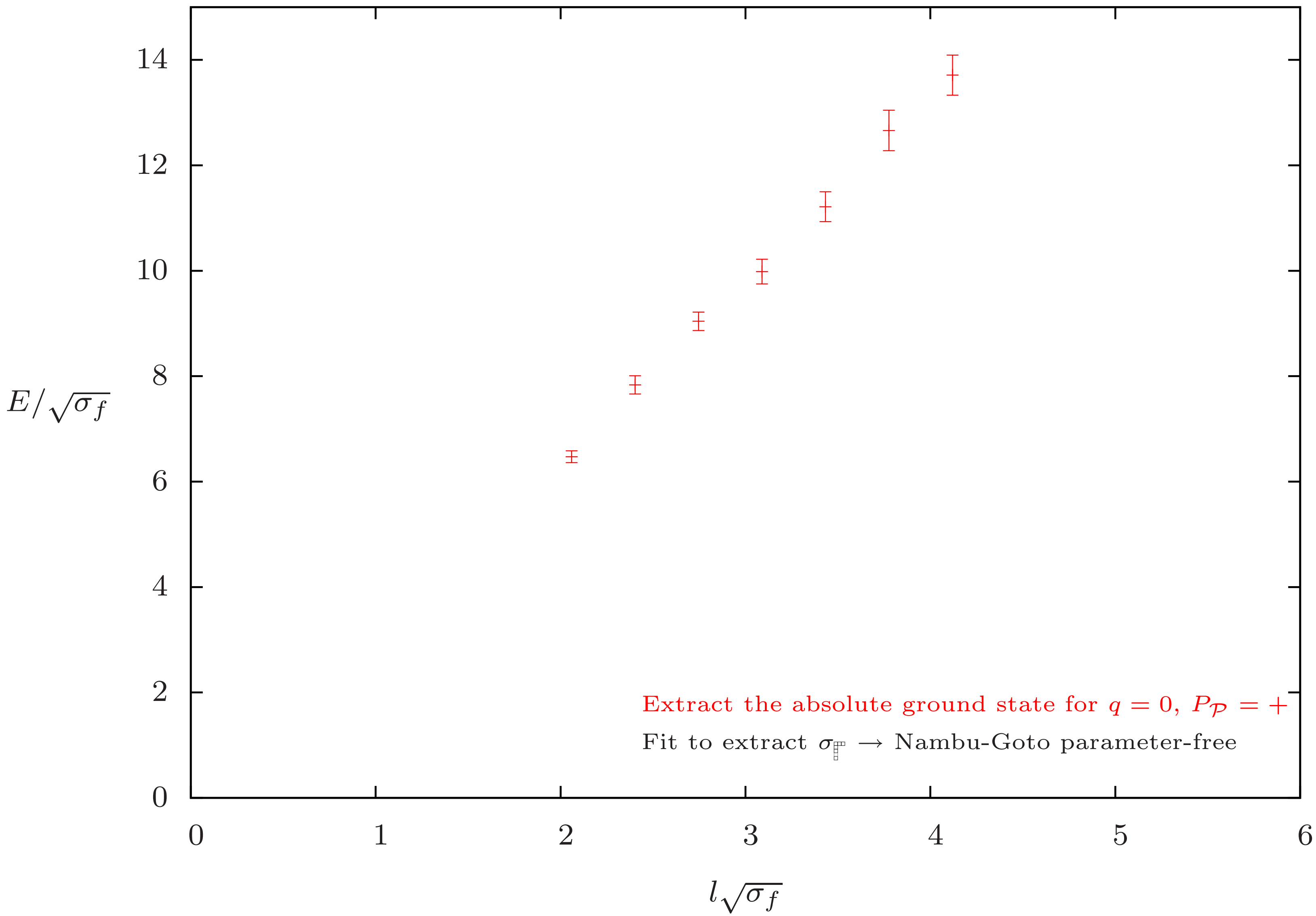
120



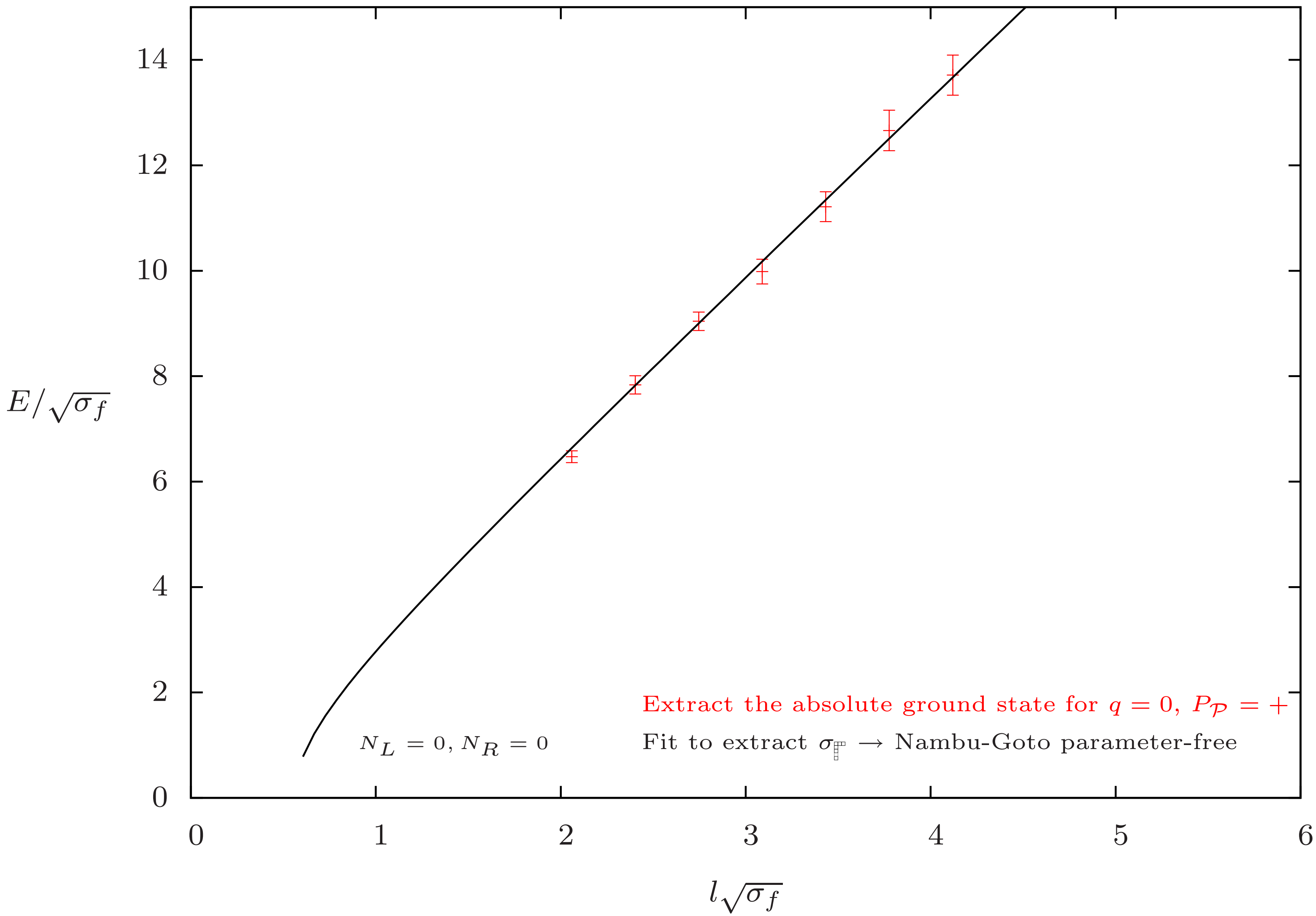
7. Results for $k = 1$, 120



7. Results for $k = 1$, 120

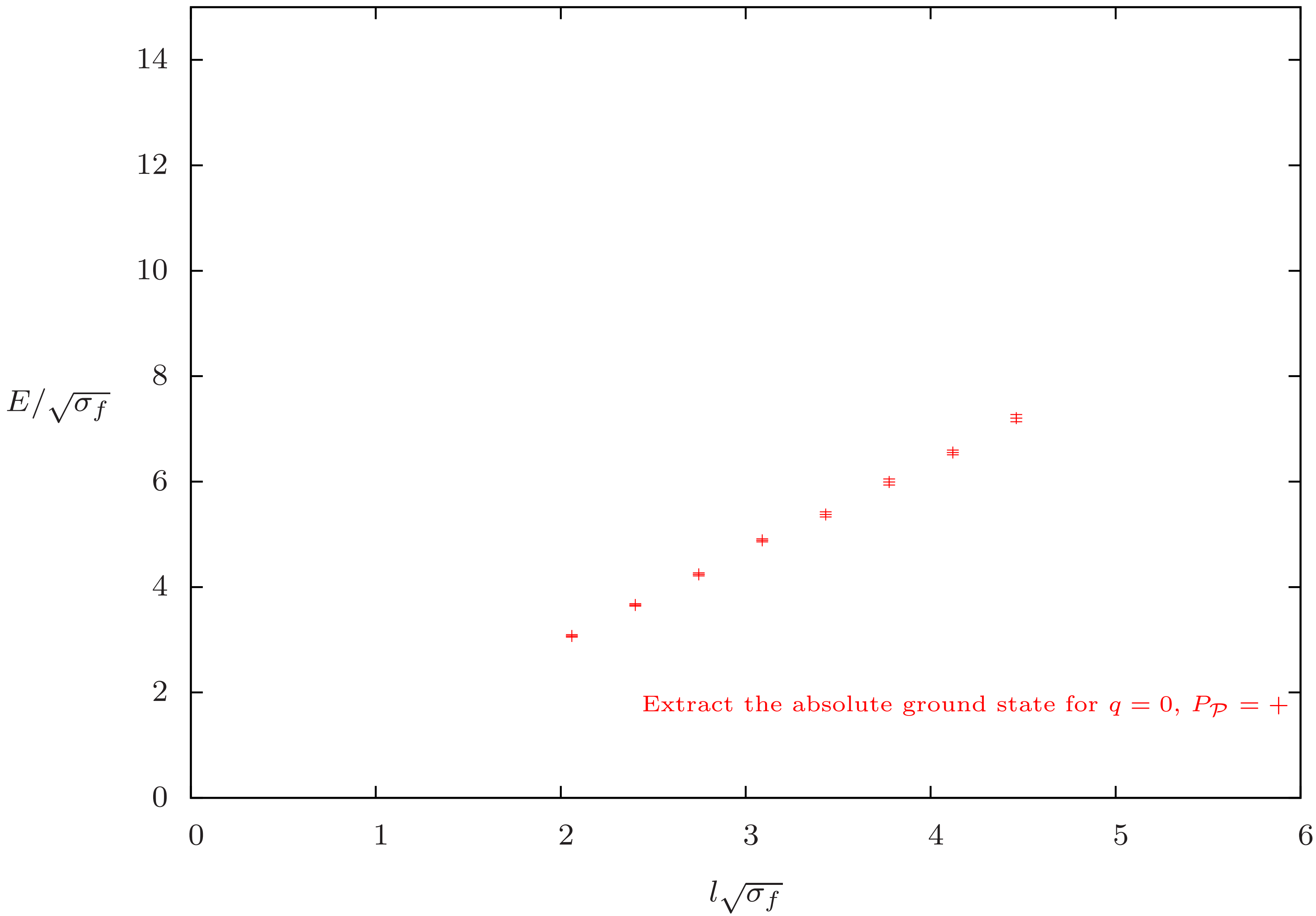


7. Results for $k = 1$, 120

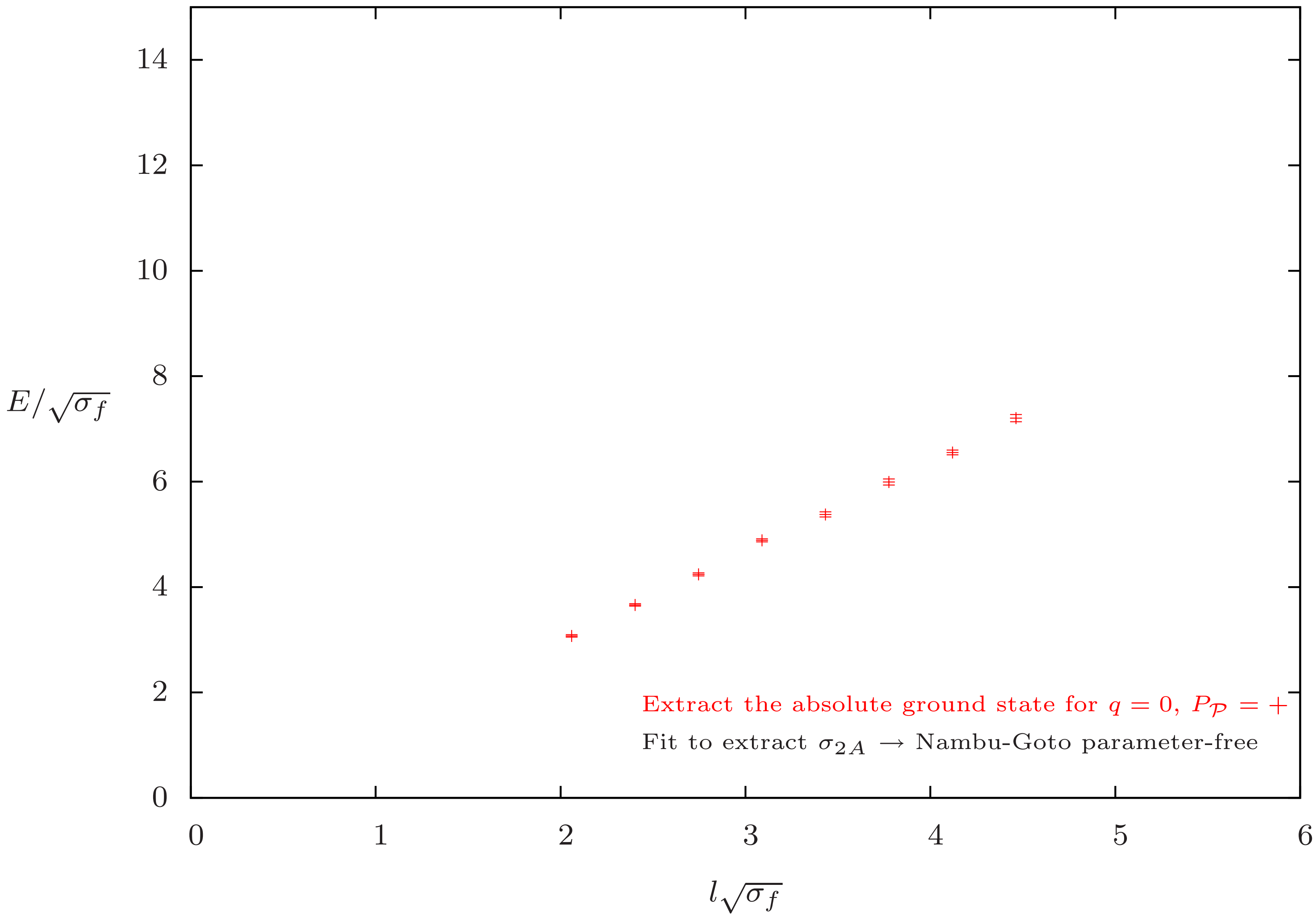


7. RESULTS FOR $k = 2$ ANTISYMMETRIC REPRESENTATION

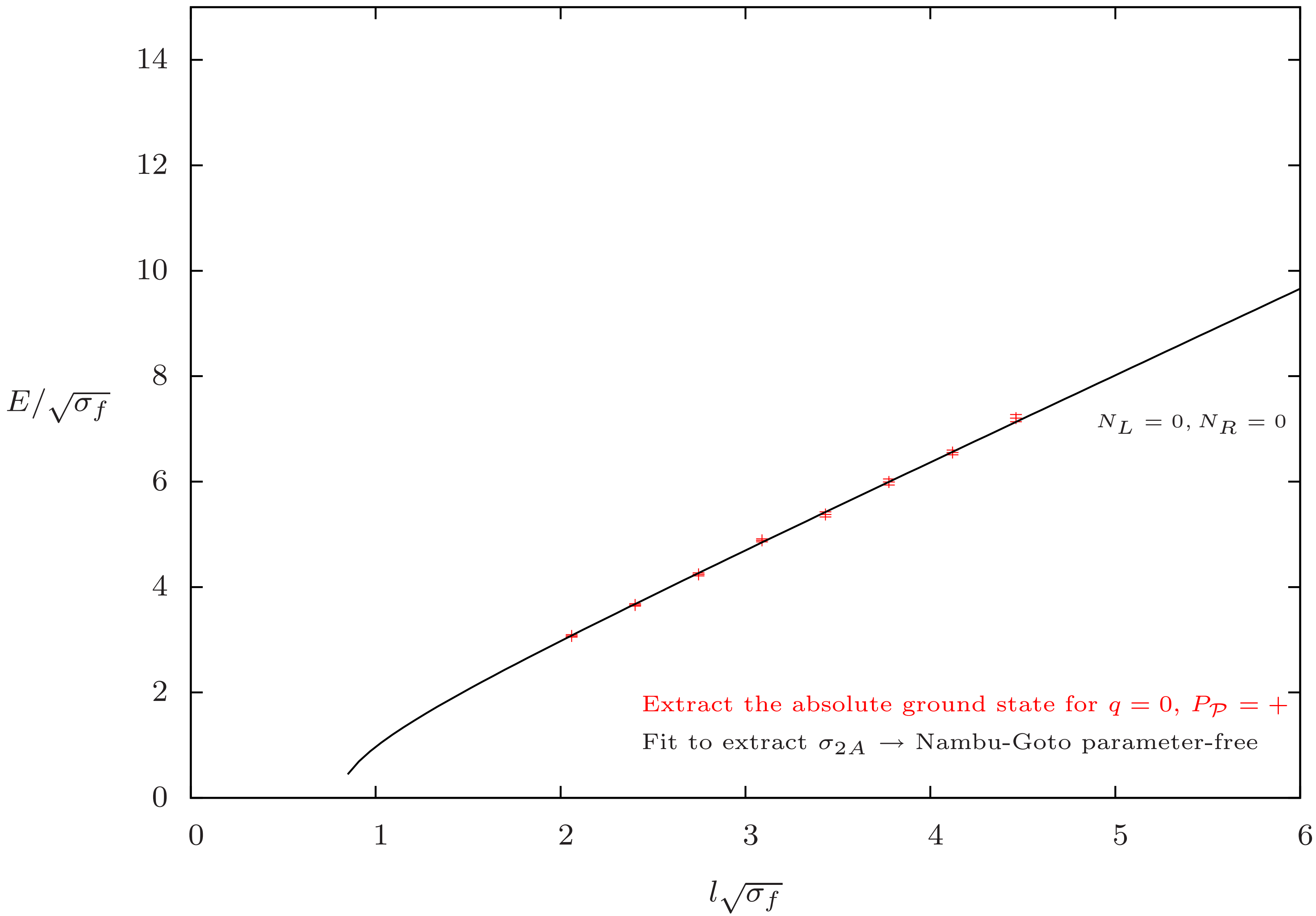
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ and $k = 2$ antisymmetric representation



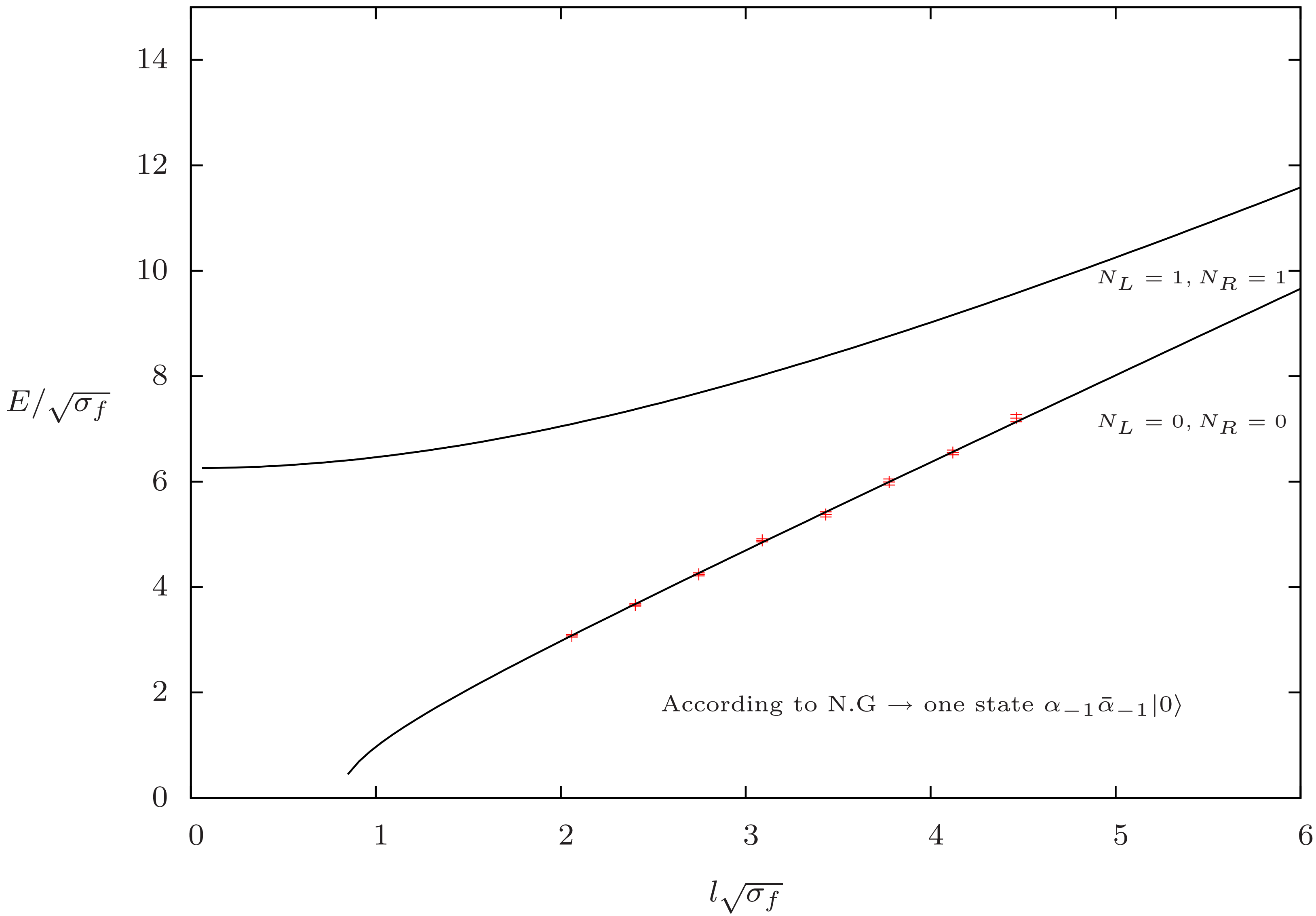
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ and $k = 2$ antisymmetric representation



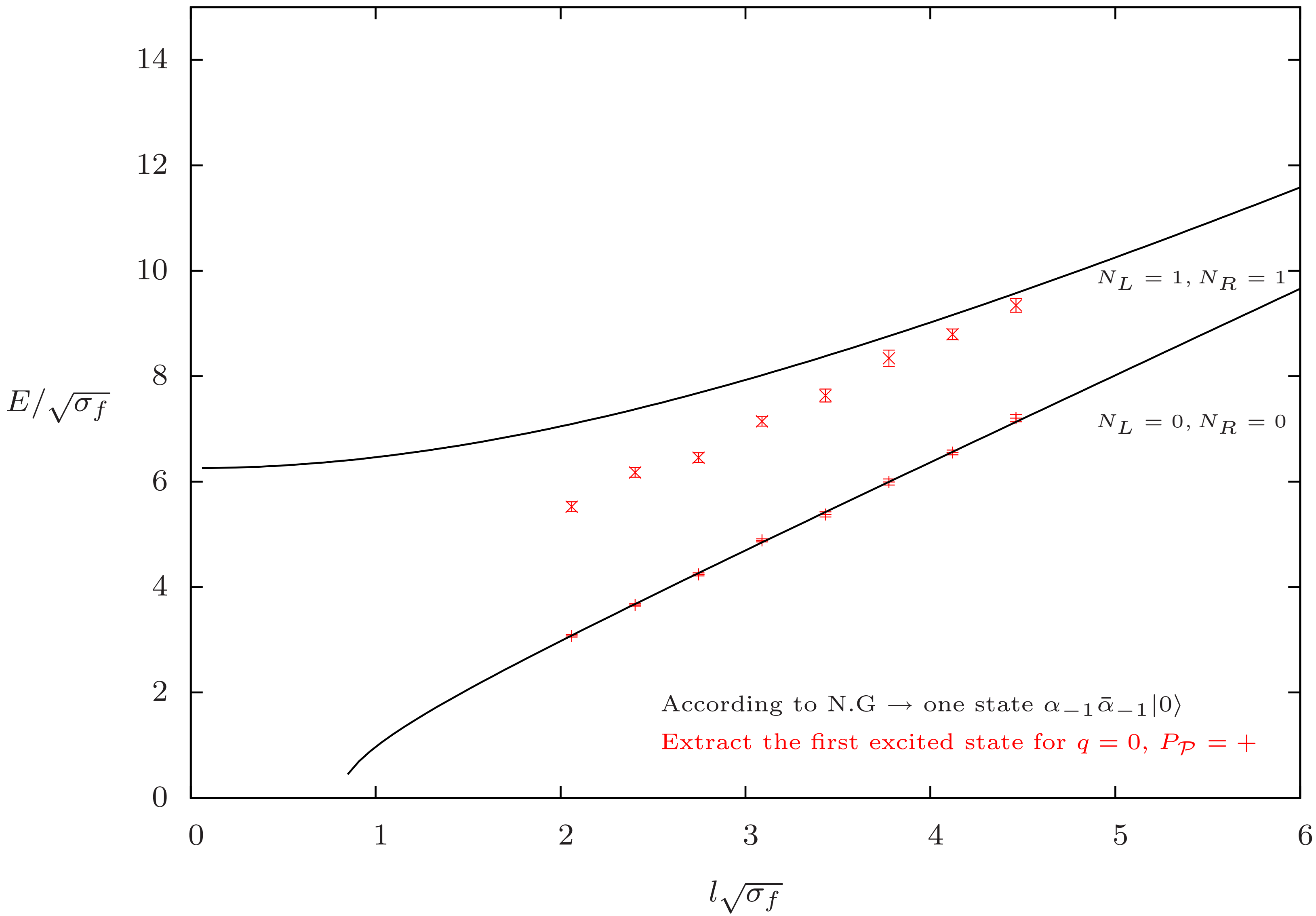
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ and $k = 2$ antisymmetric representation



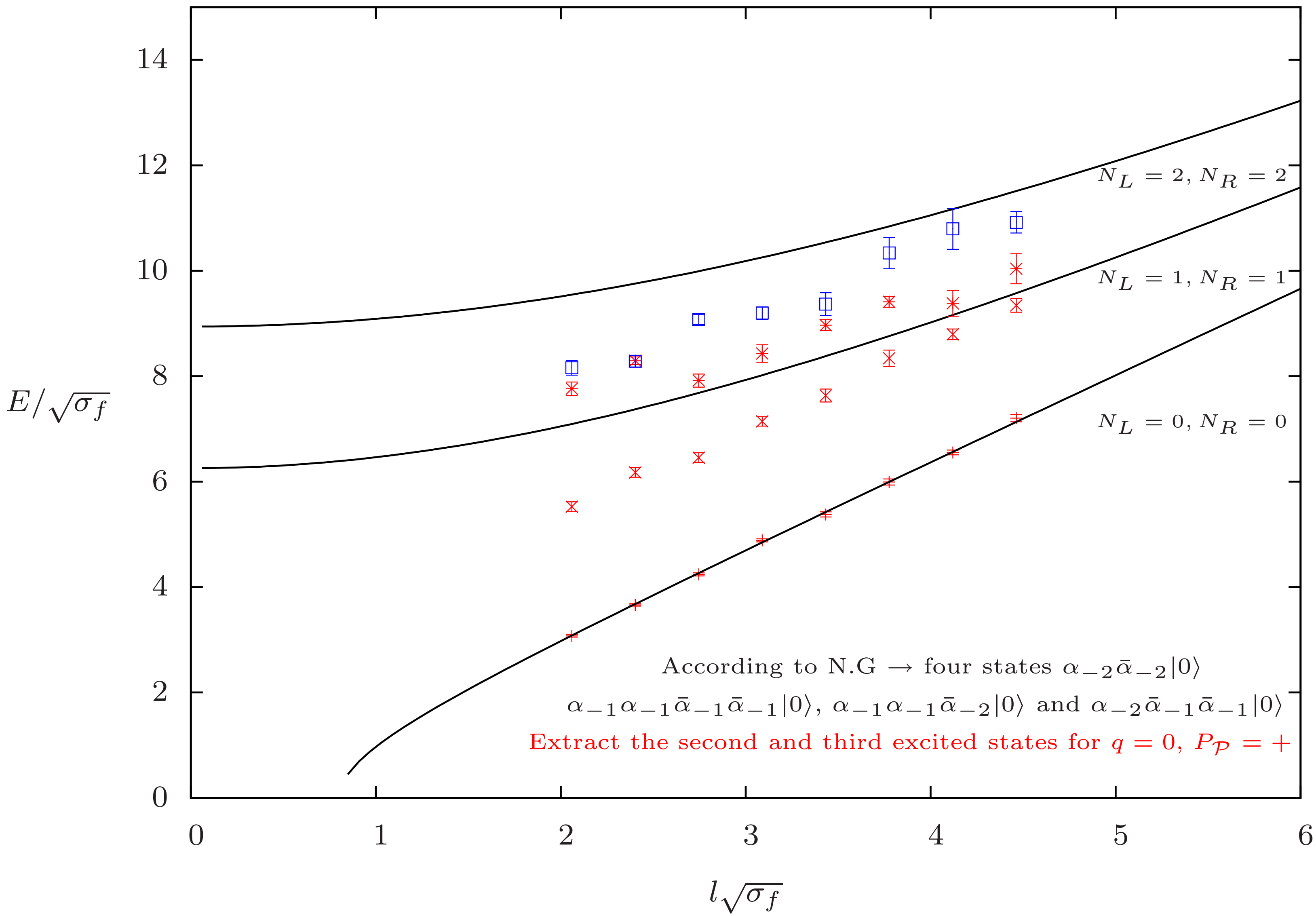
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ and $k = 2$ antisymmetric representation



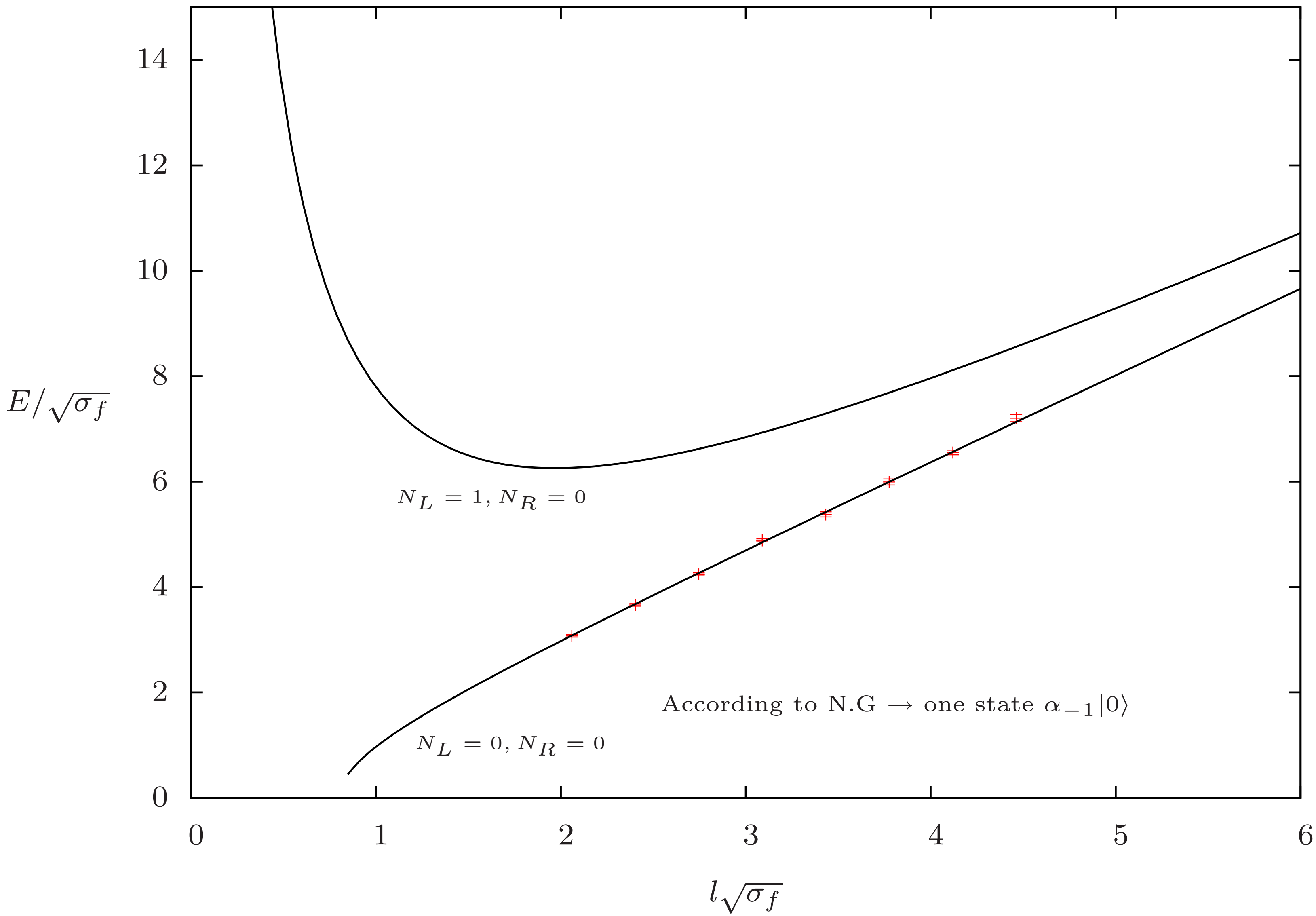
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ and $k = 2$ antisymmetric representation



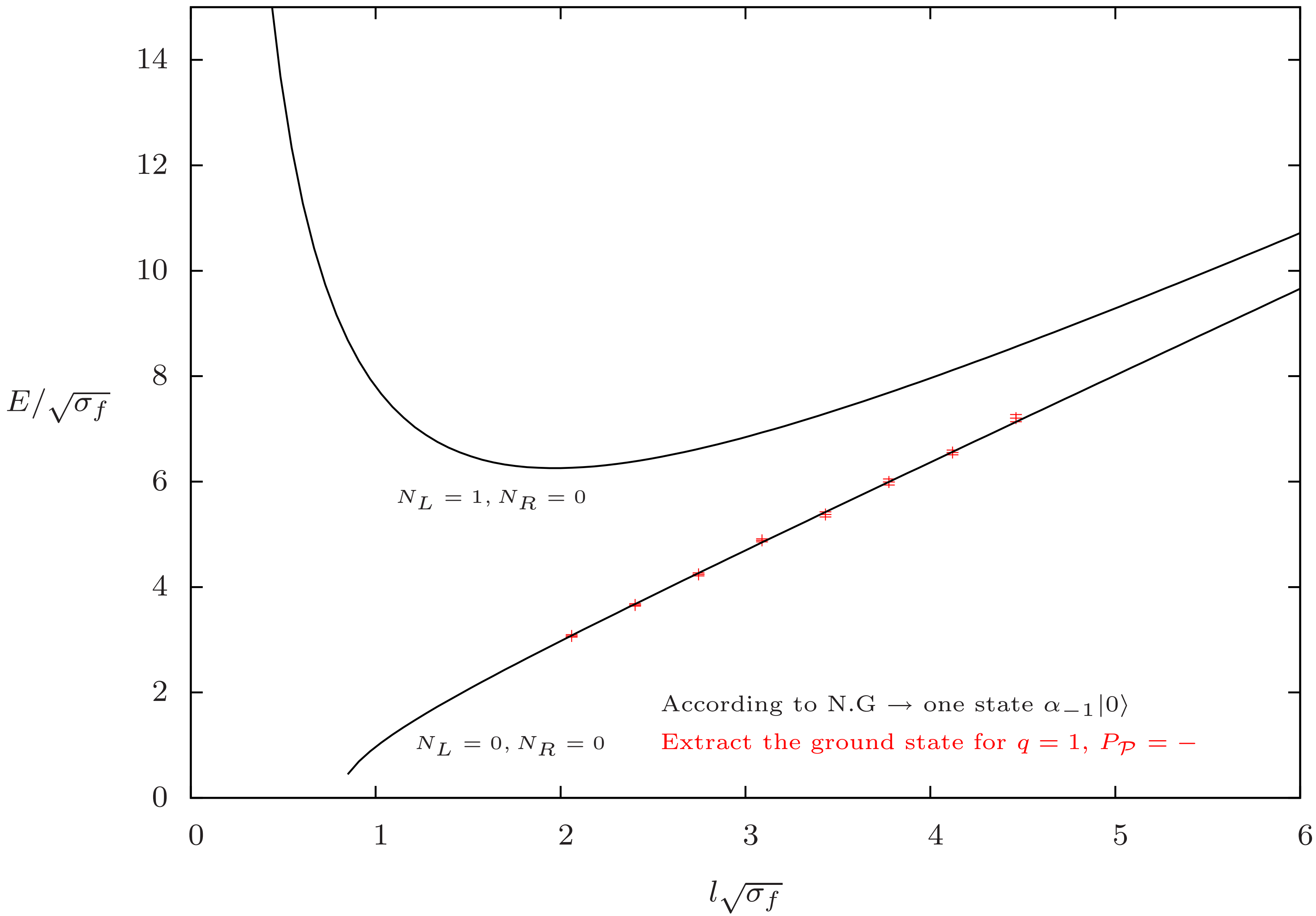
7. Results for: $q = 0$, $P_{\mathcal{P}} = +$ and $k = 2$ antisymmetric representation



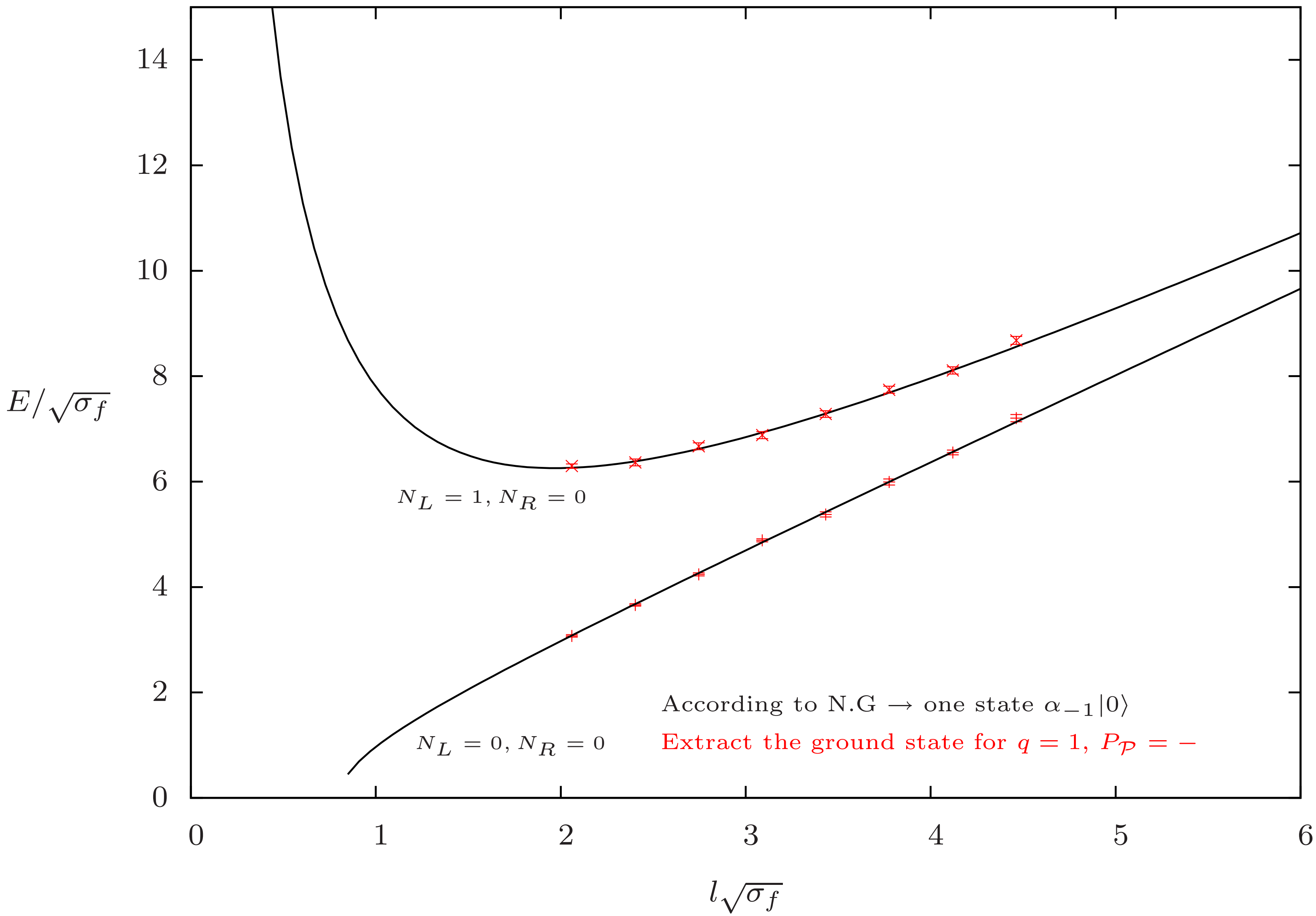
7. Results for ground states in the $k = 2$ antisymmetric representation



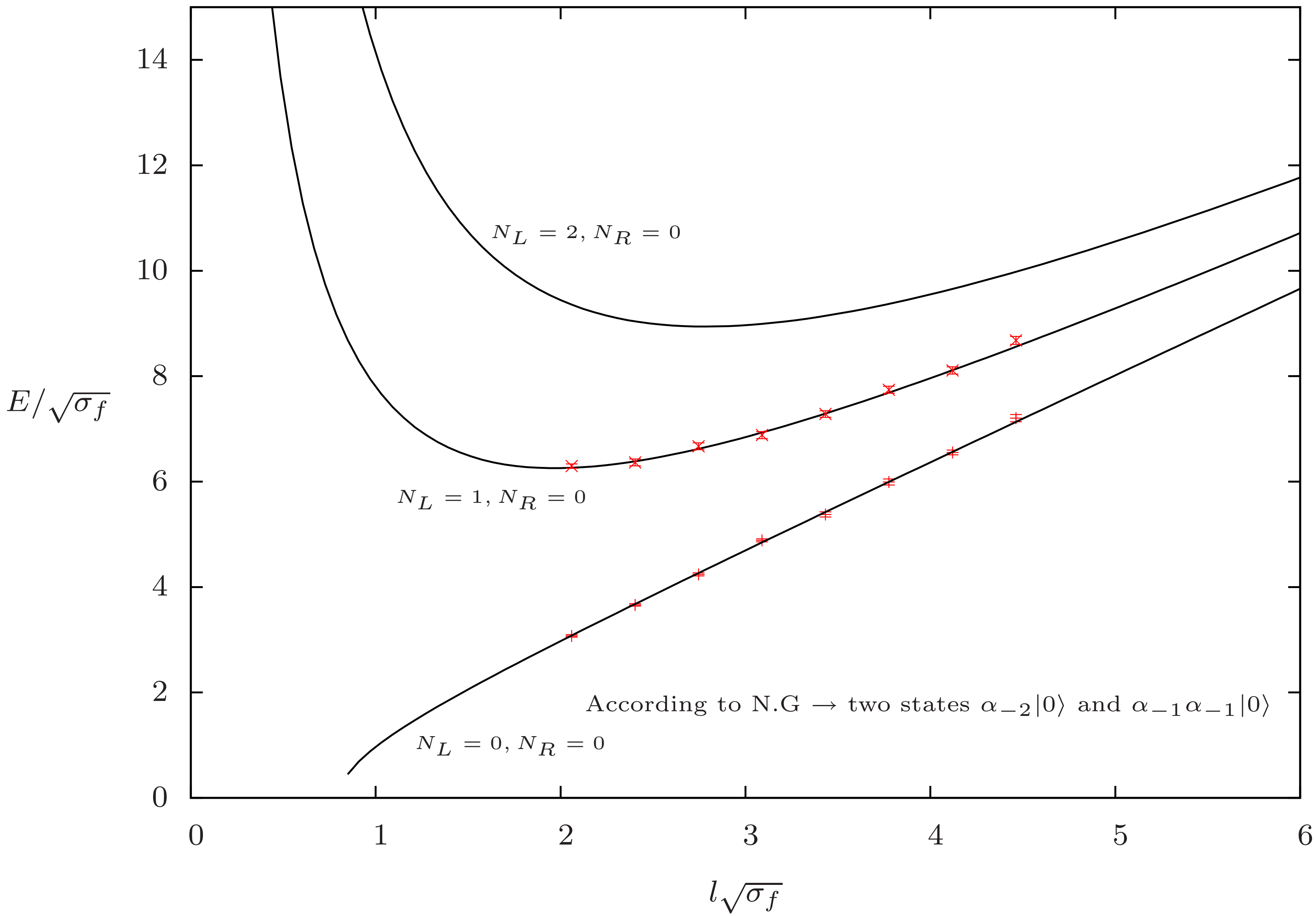
7. Results for ground states in the $k = 2$ antisymmetric representation



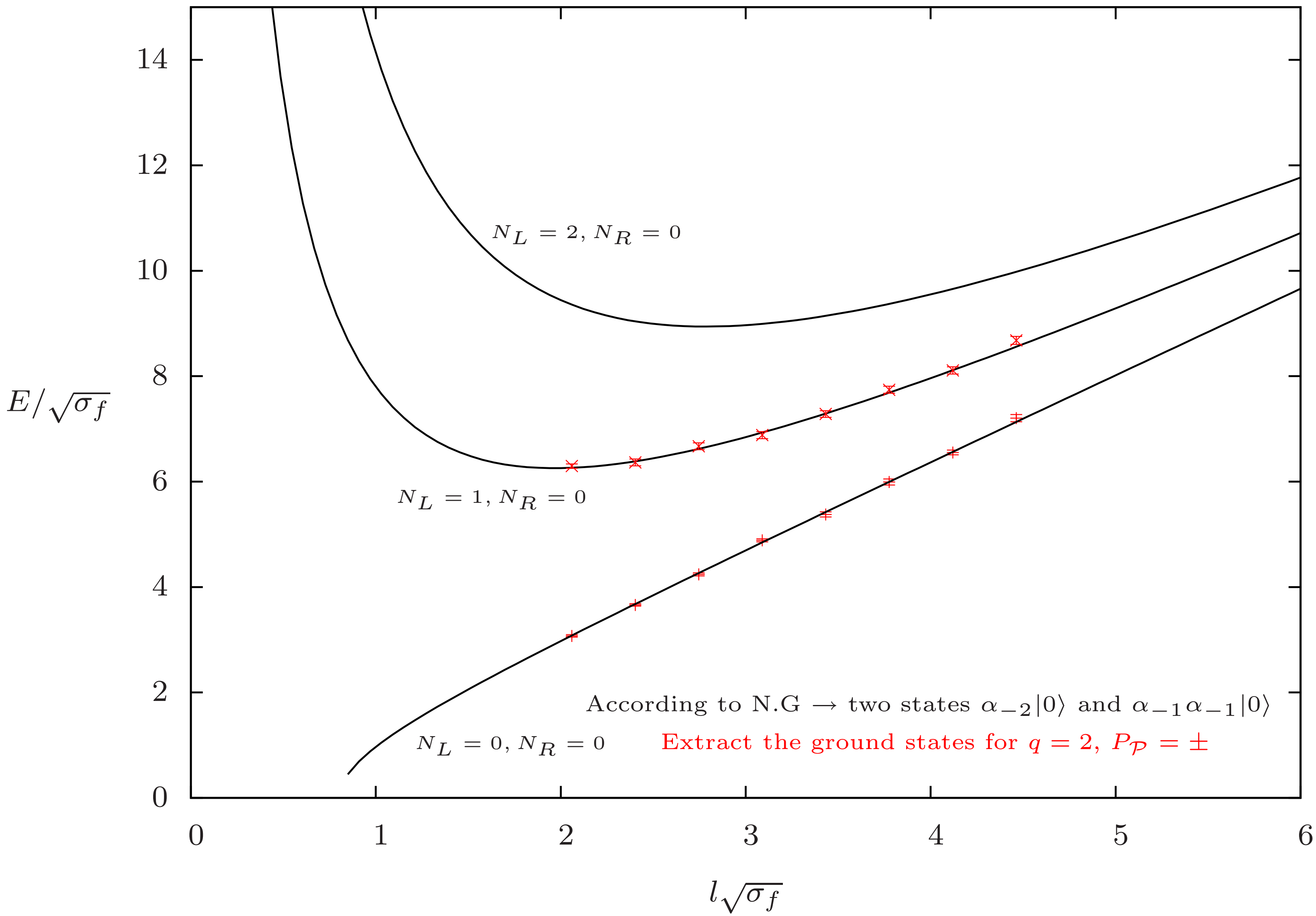
7. Results for ground states in the $k = 2$ antisymmetric representation



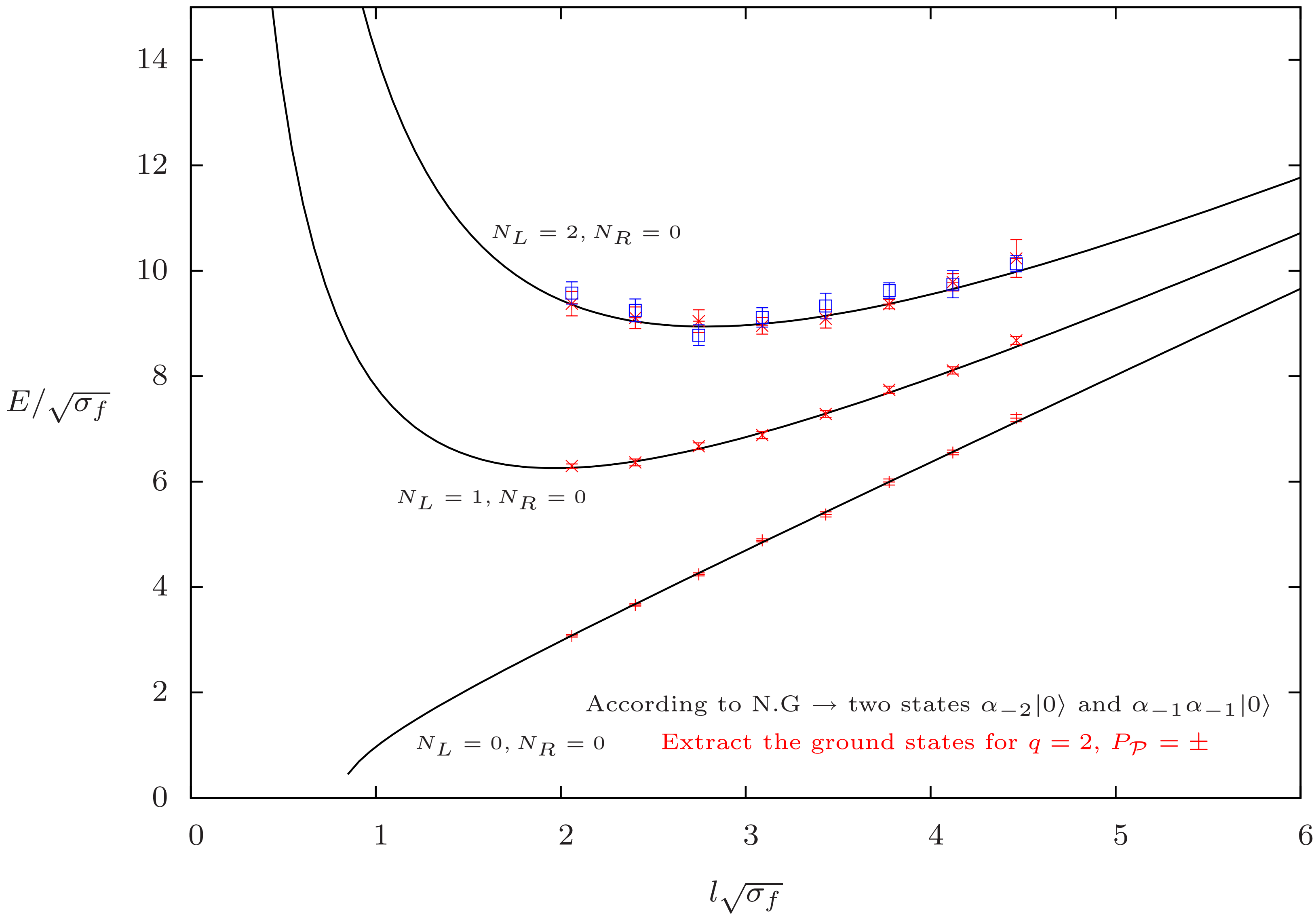
7. Results for ground states in the $k = 2$ antisymmetric representation



7. Results for ground states in the $k = 2$ antisymmetric representation

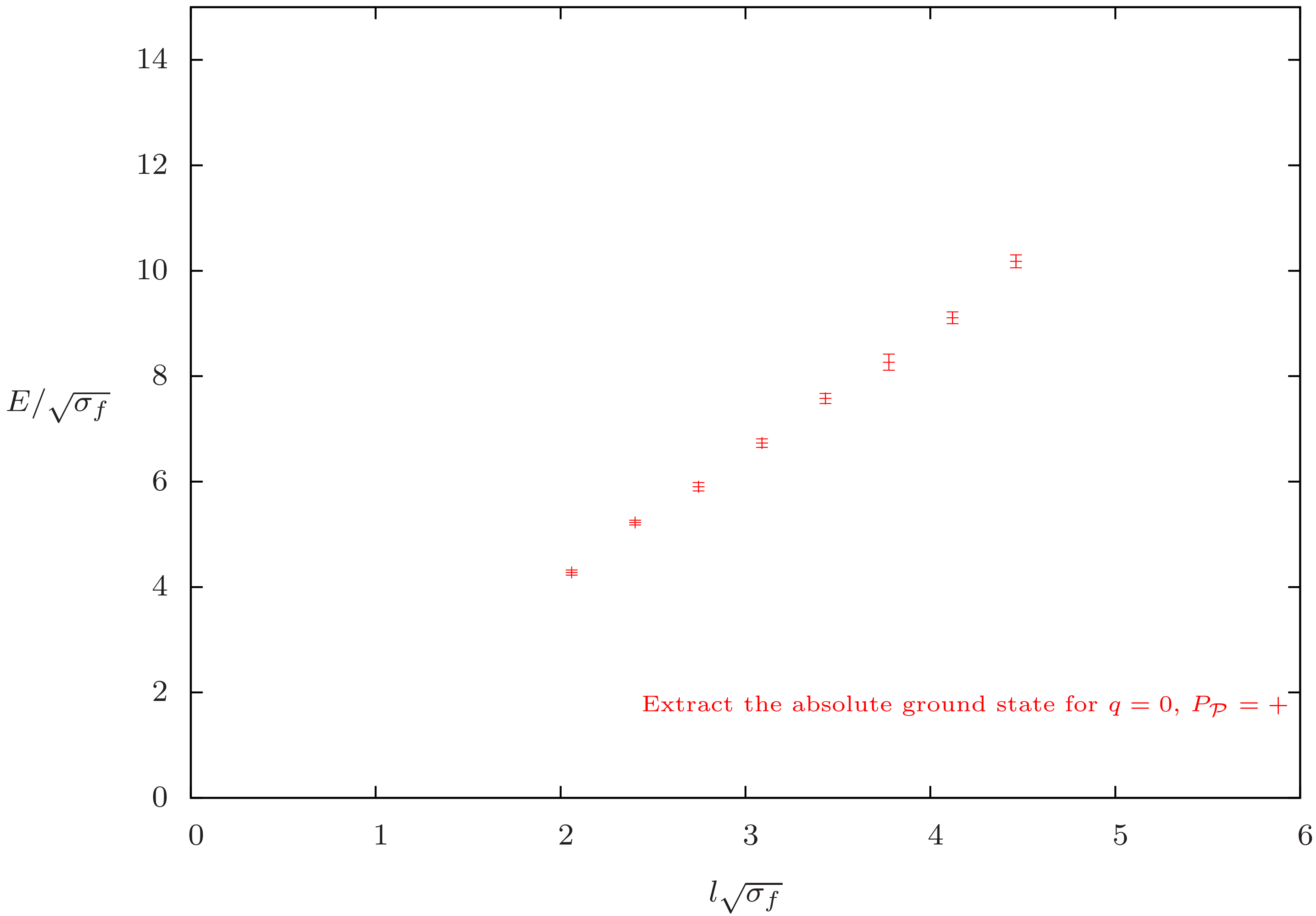


7. Results for ground states in the $k = 2$ antisymmetric representation

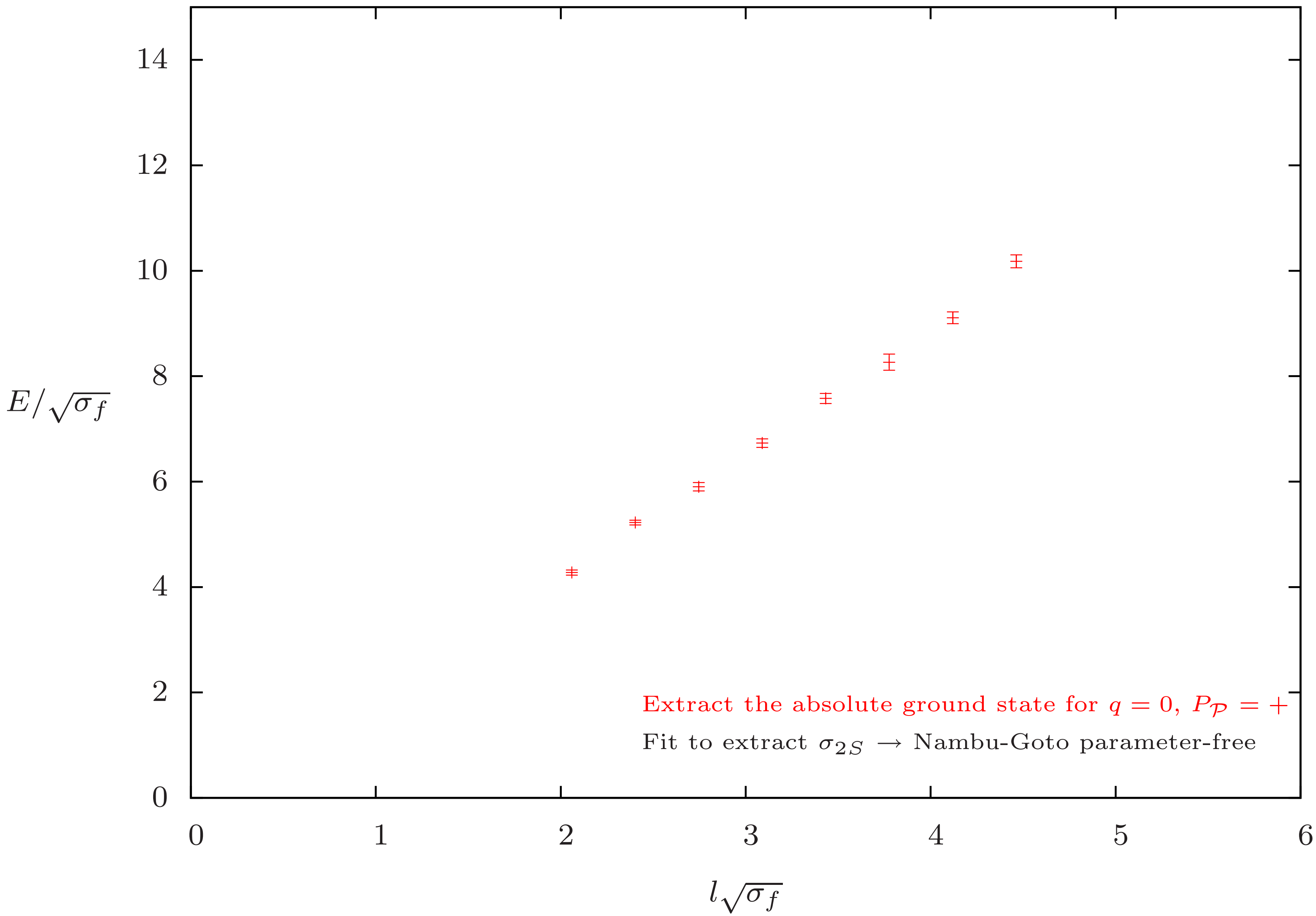


7. RESULTS FOR $k = 2$ SYMMETRIC REPRESENTATION

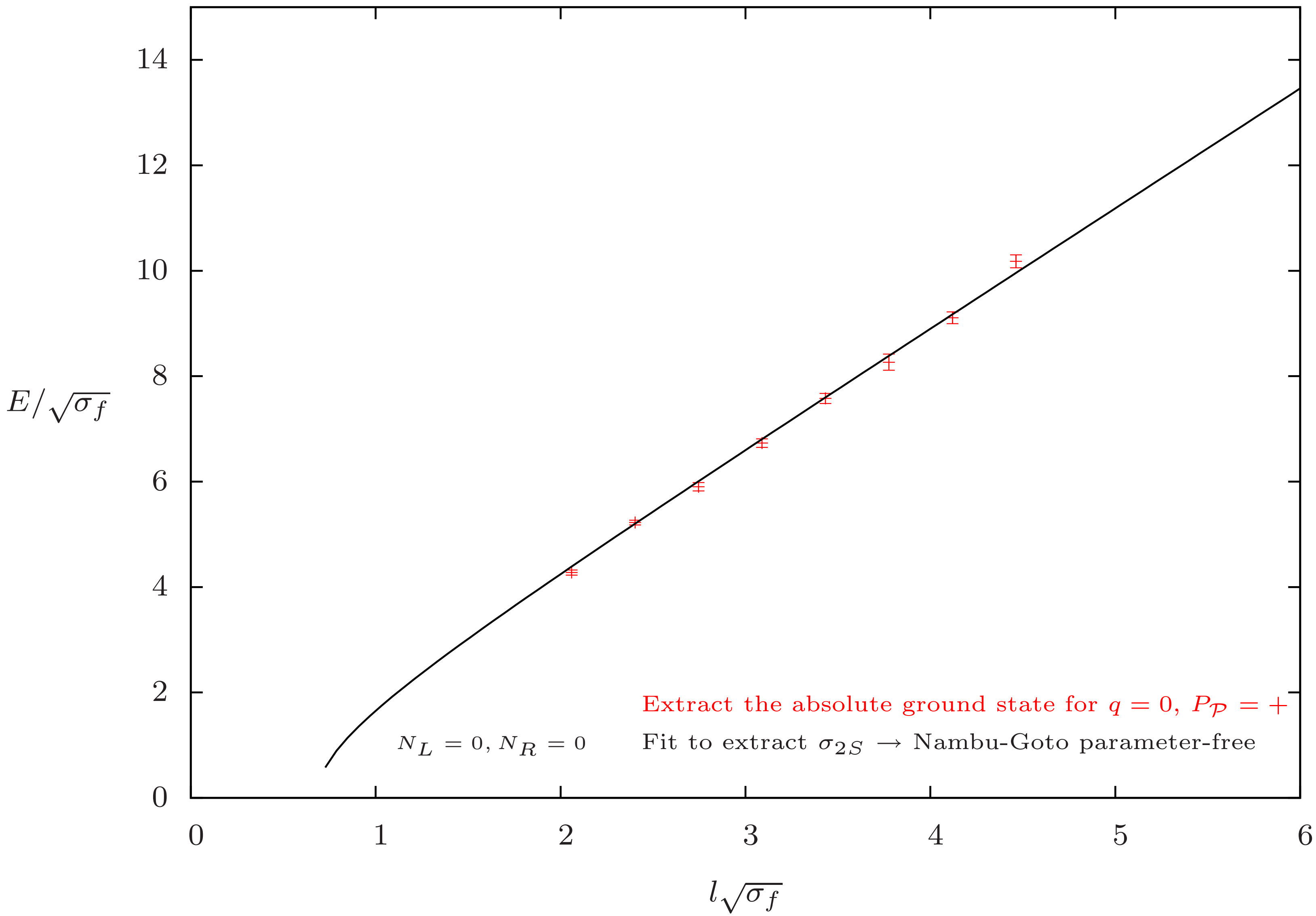
7. Results for $k = 2$ Symmetric Representation



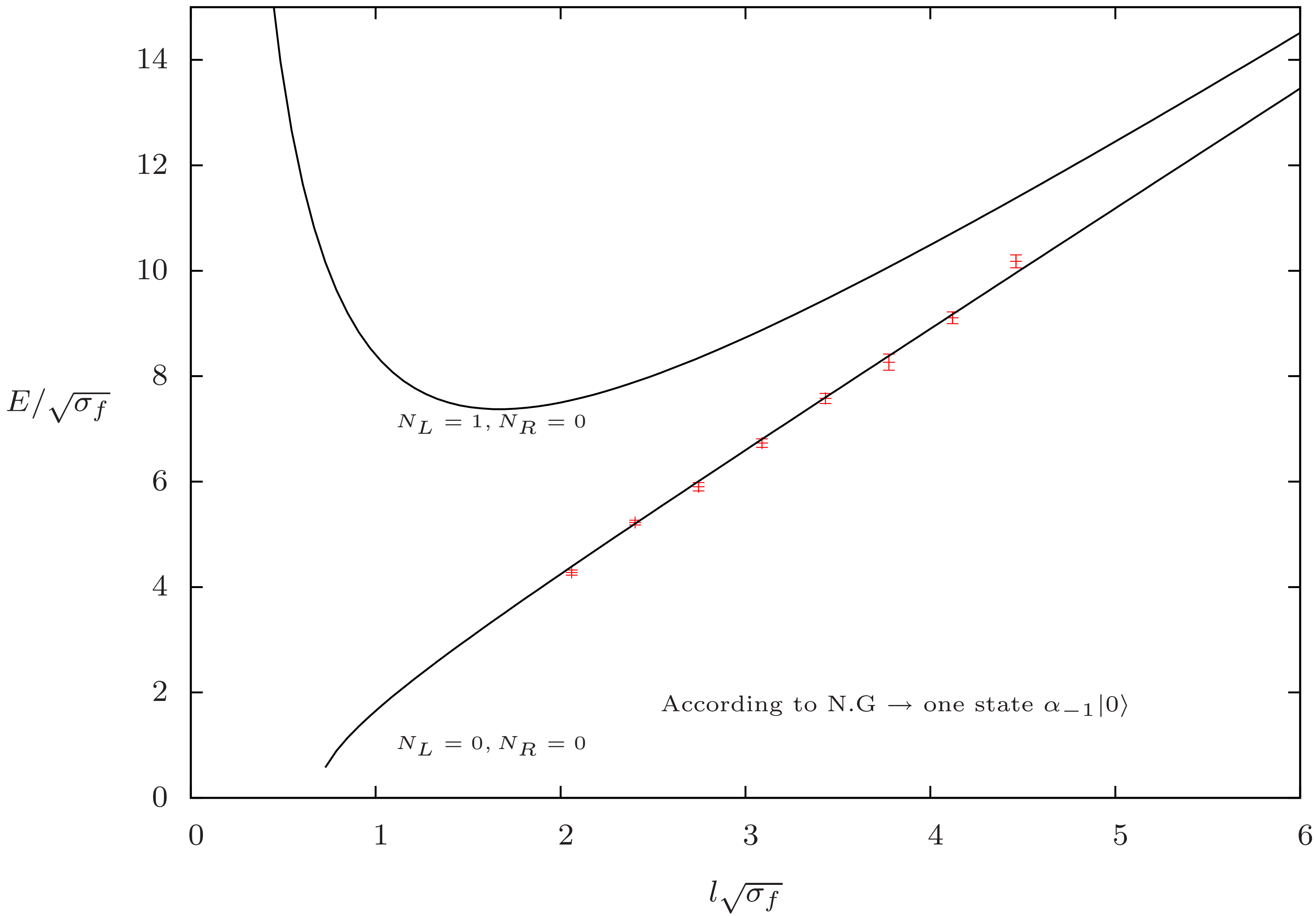
7. Results for $k = 2$ Symmetric Representation



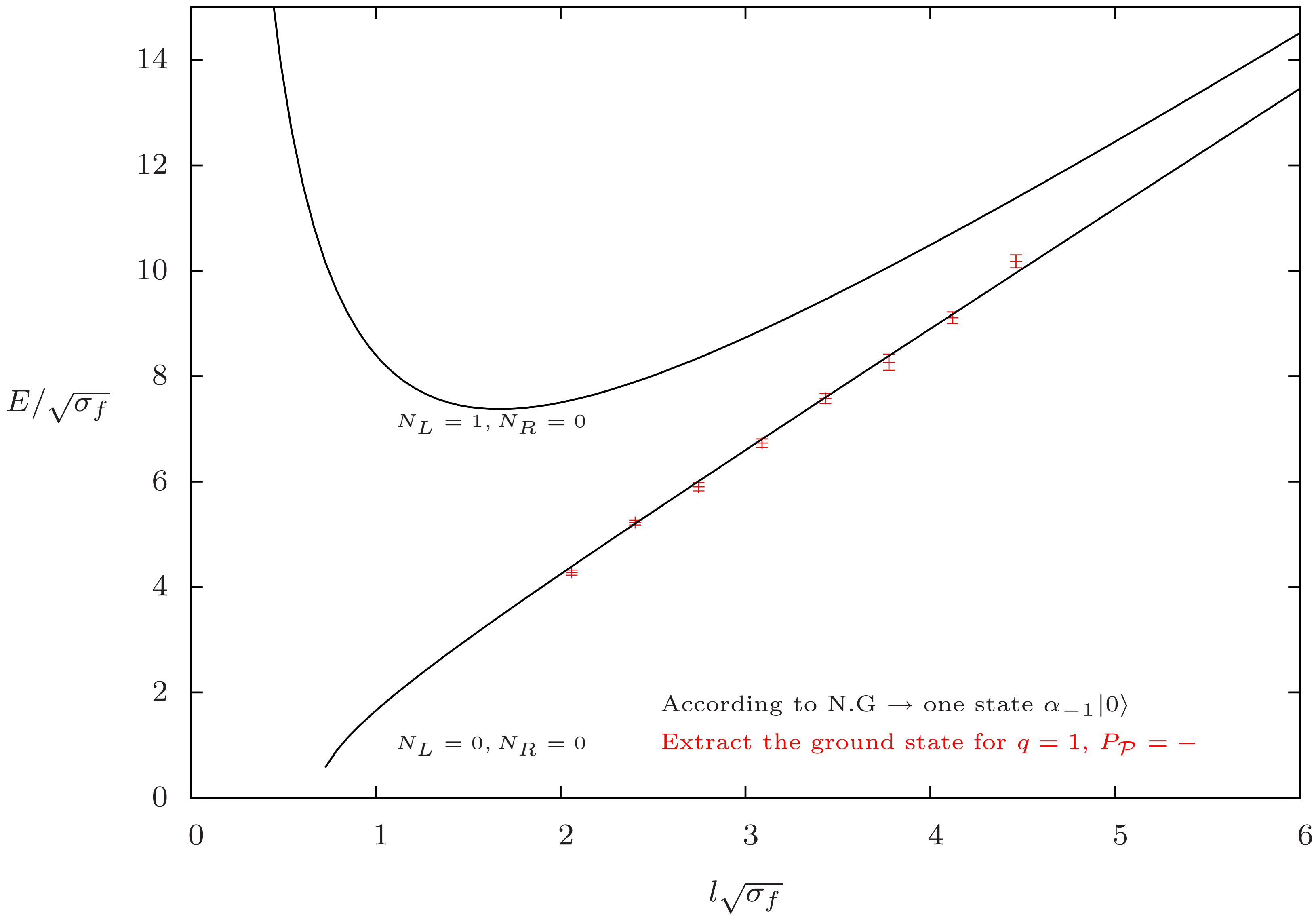
7. Results for $k = 2$ Symmetric Representation



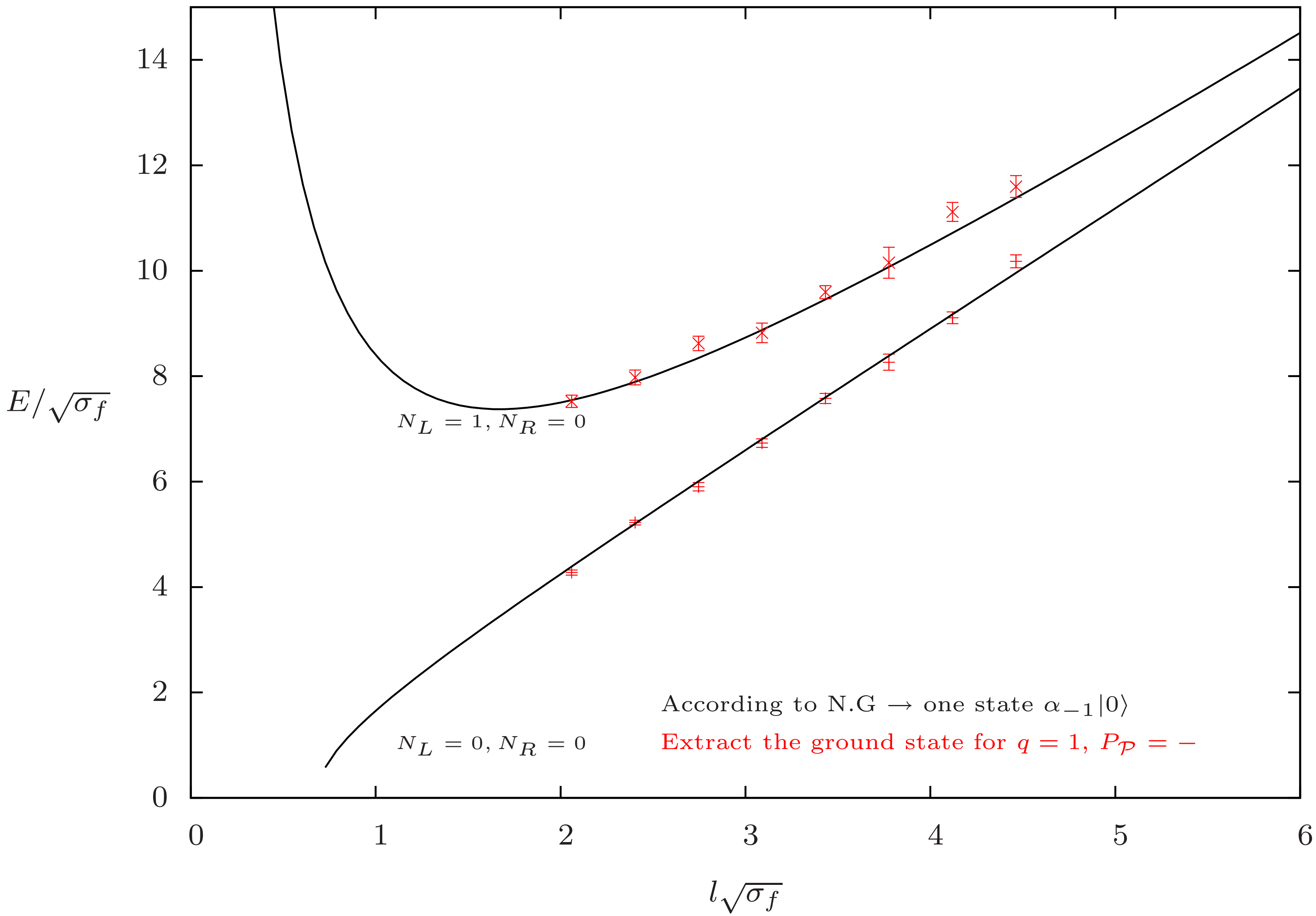
7. Results for $k = 2$ Symmetric Representation



7. Results for $k = 2$ Symmetric Representation

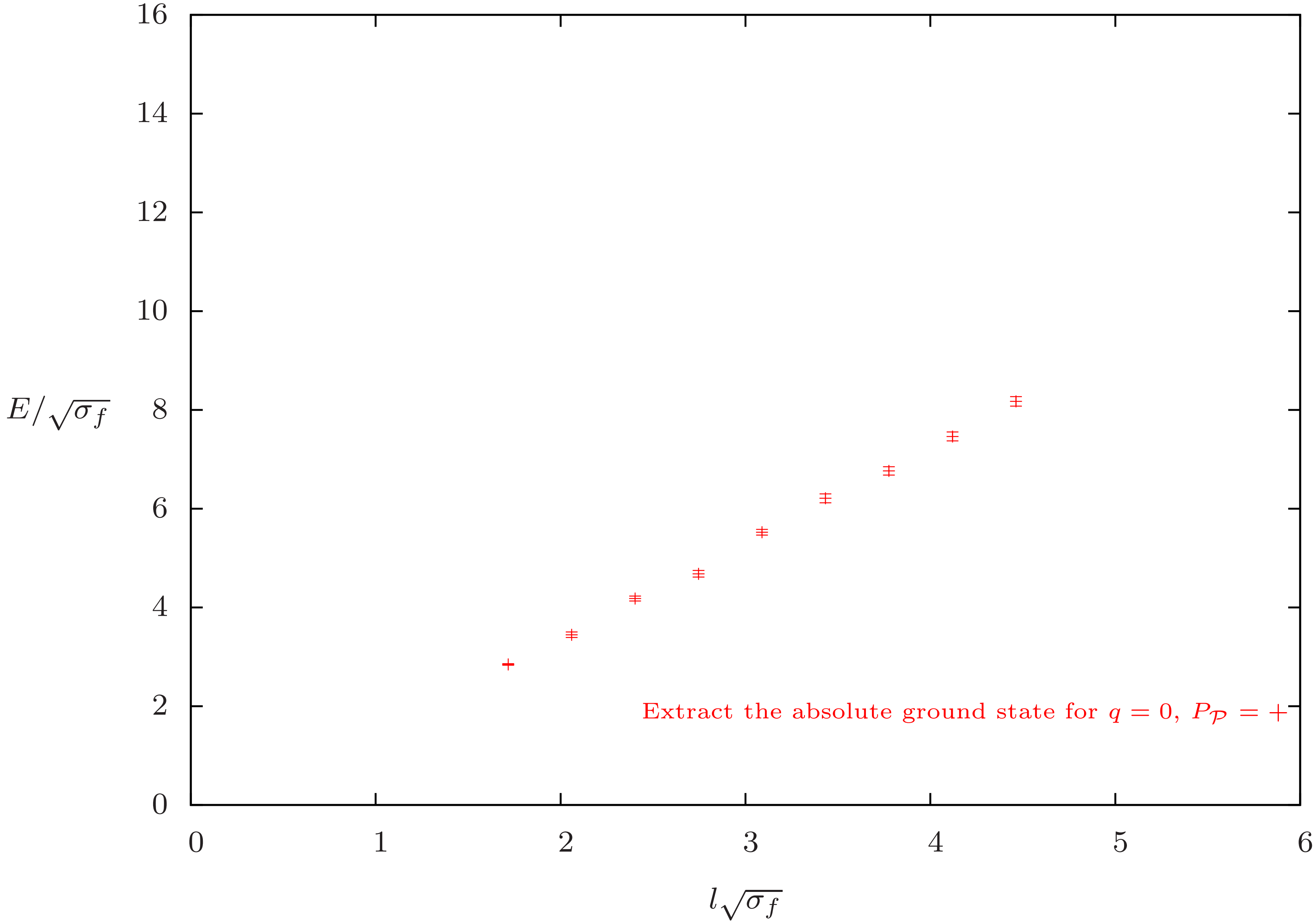


7. Results for $k = 2$ Symmetric Representation

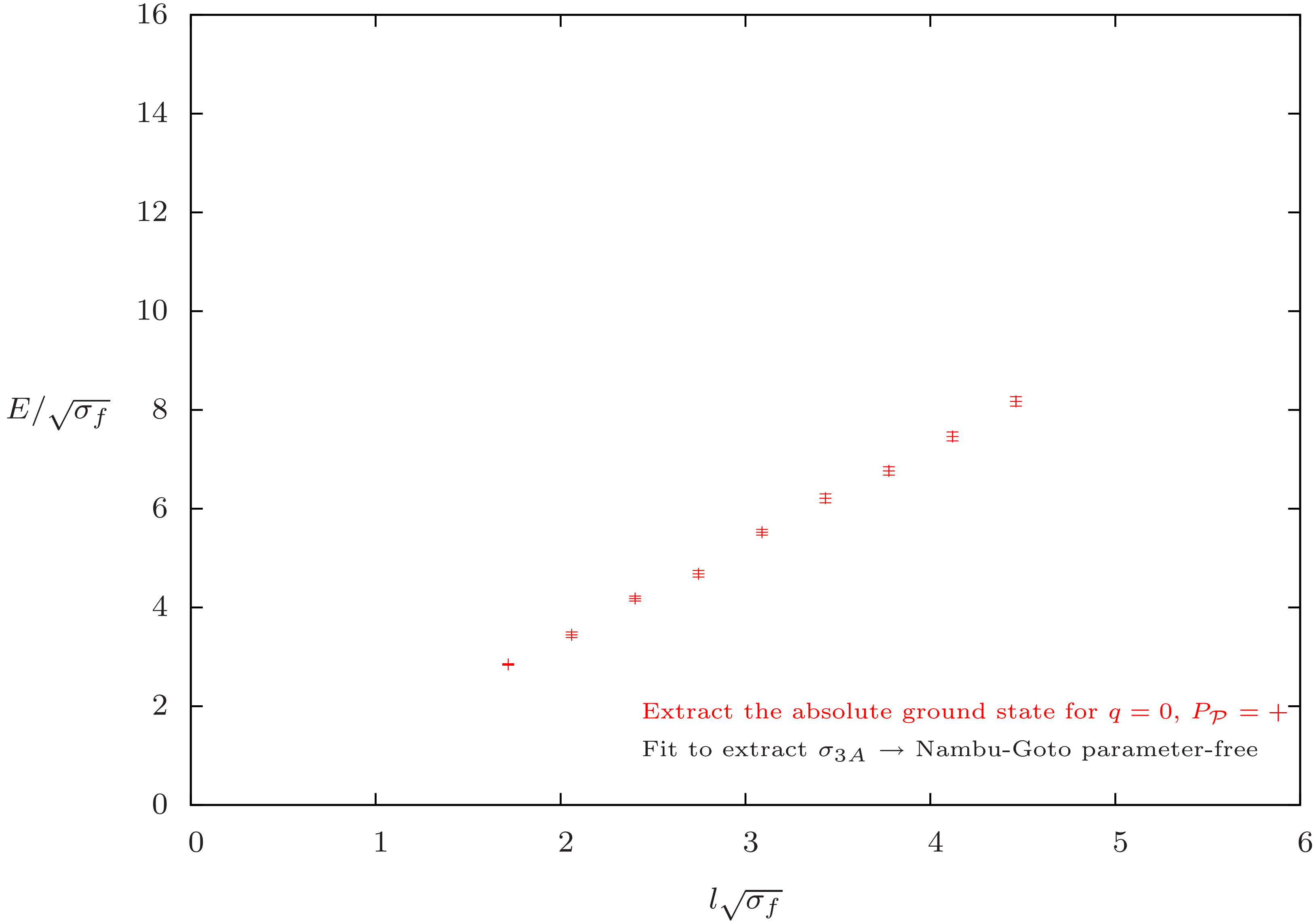


7. RESULTS FOR $k = 3$ ANTISYMMETRIC REPRESENTATION

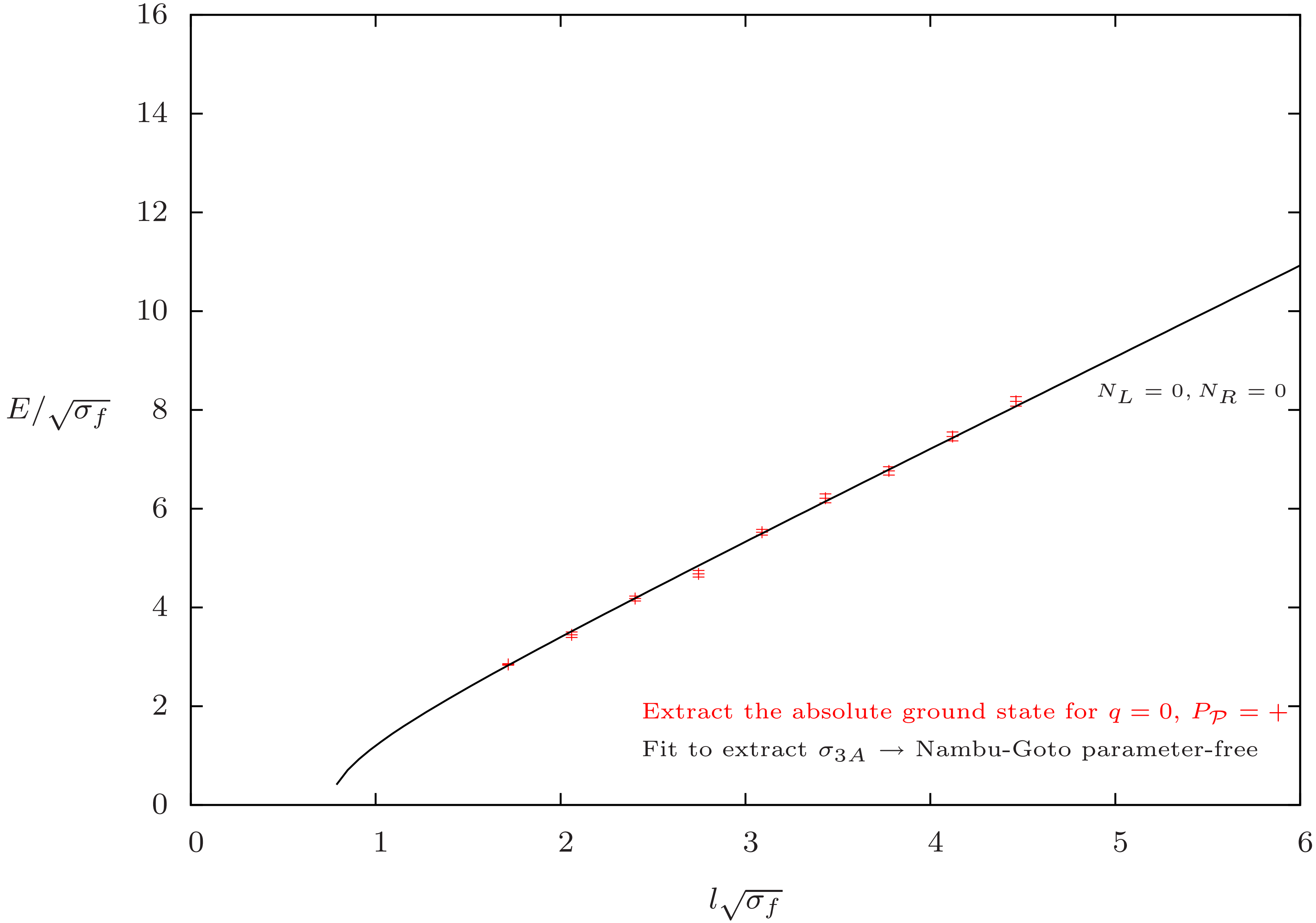
7. Results for ground states in the $k = 3$ antisymmetric representation



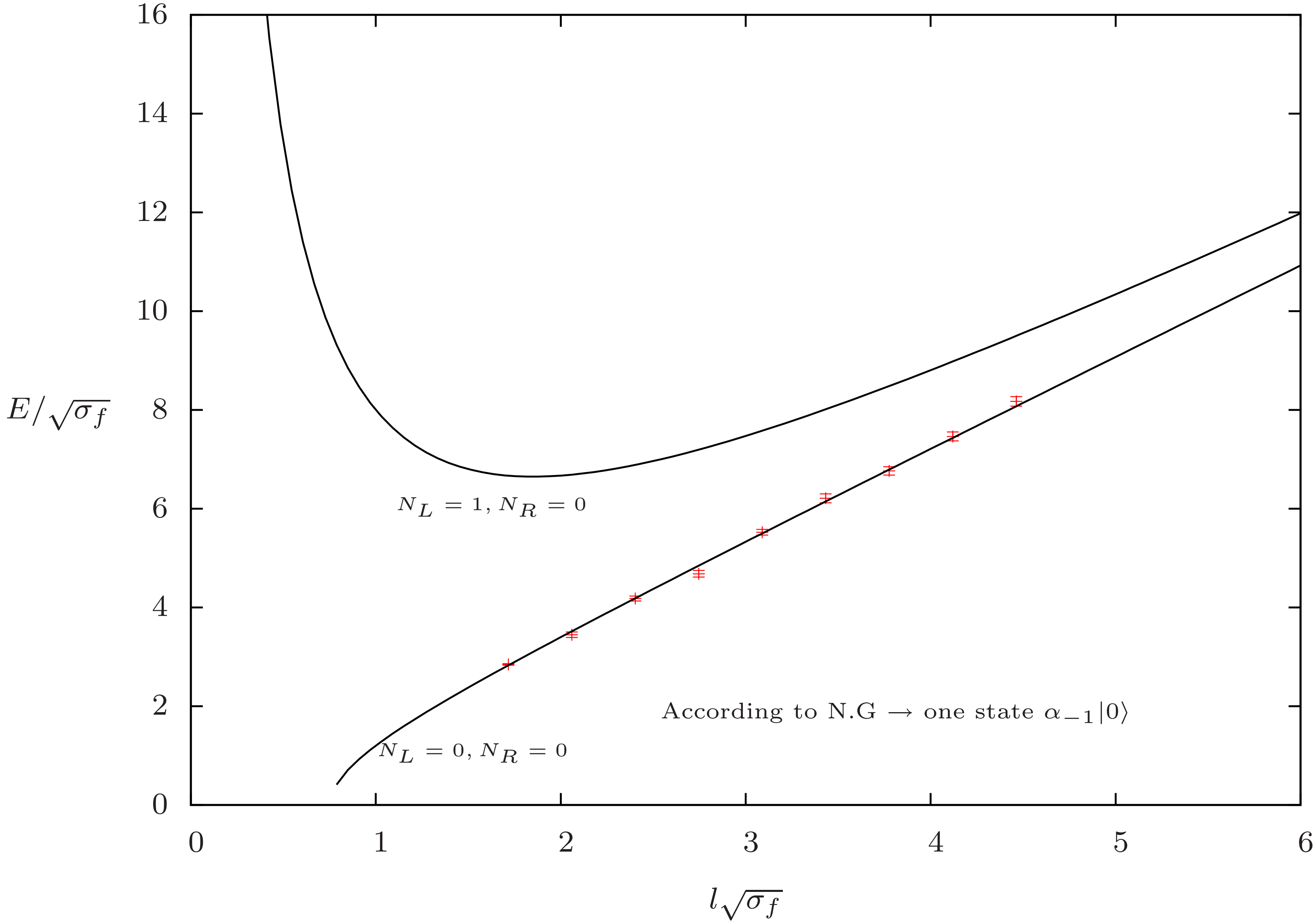
7. Results for ground states in the $k = 3$ antisymmetric representation



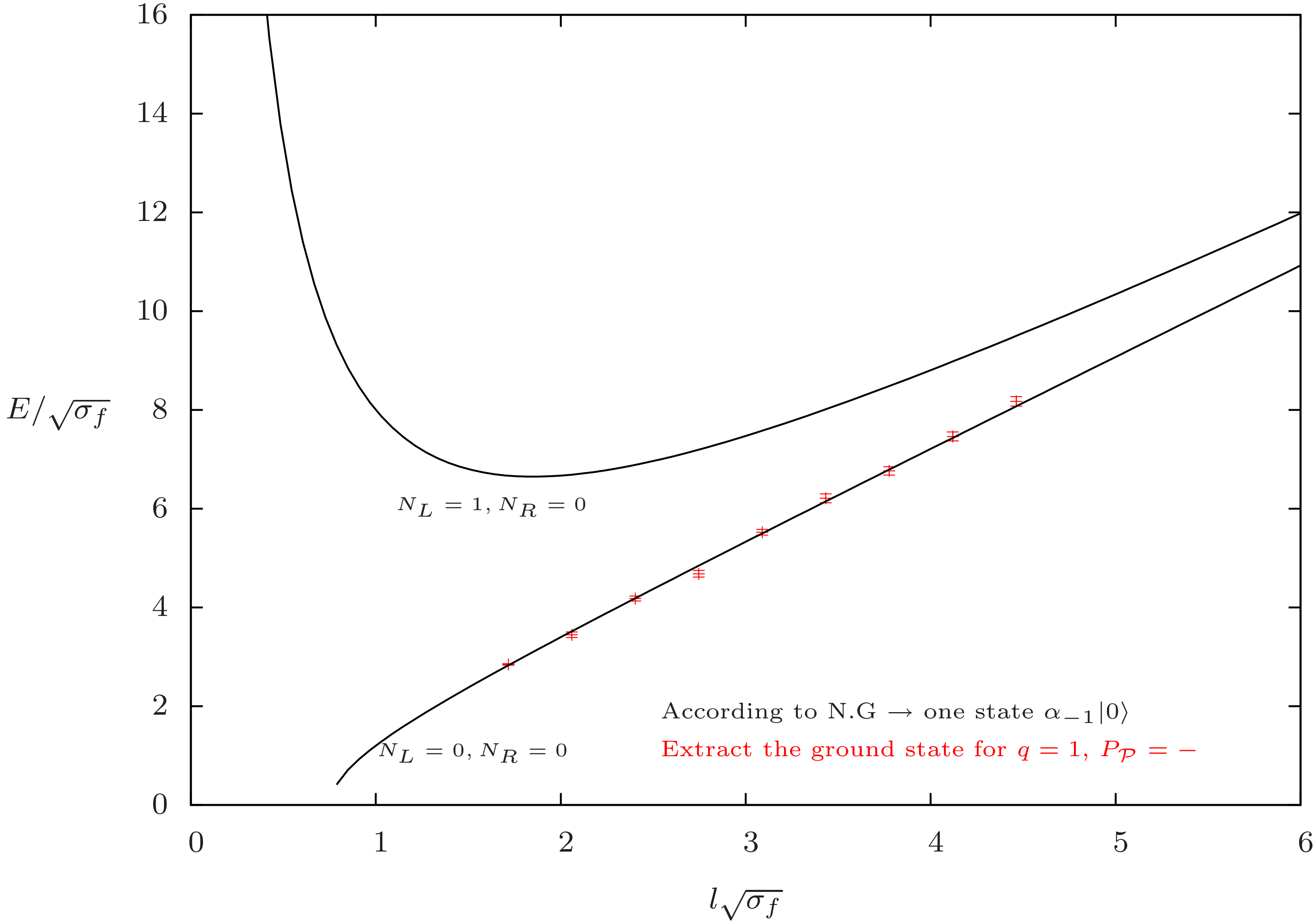
7. Results for ground states in the $k = 3$ antisymmetric representation



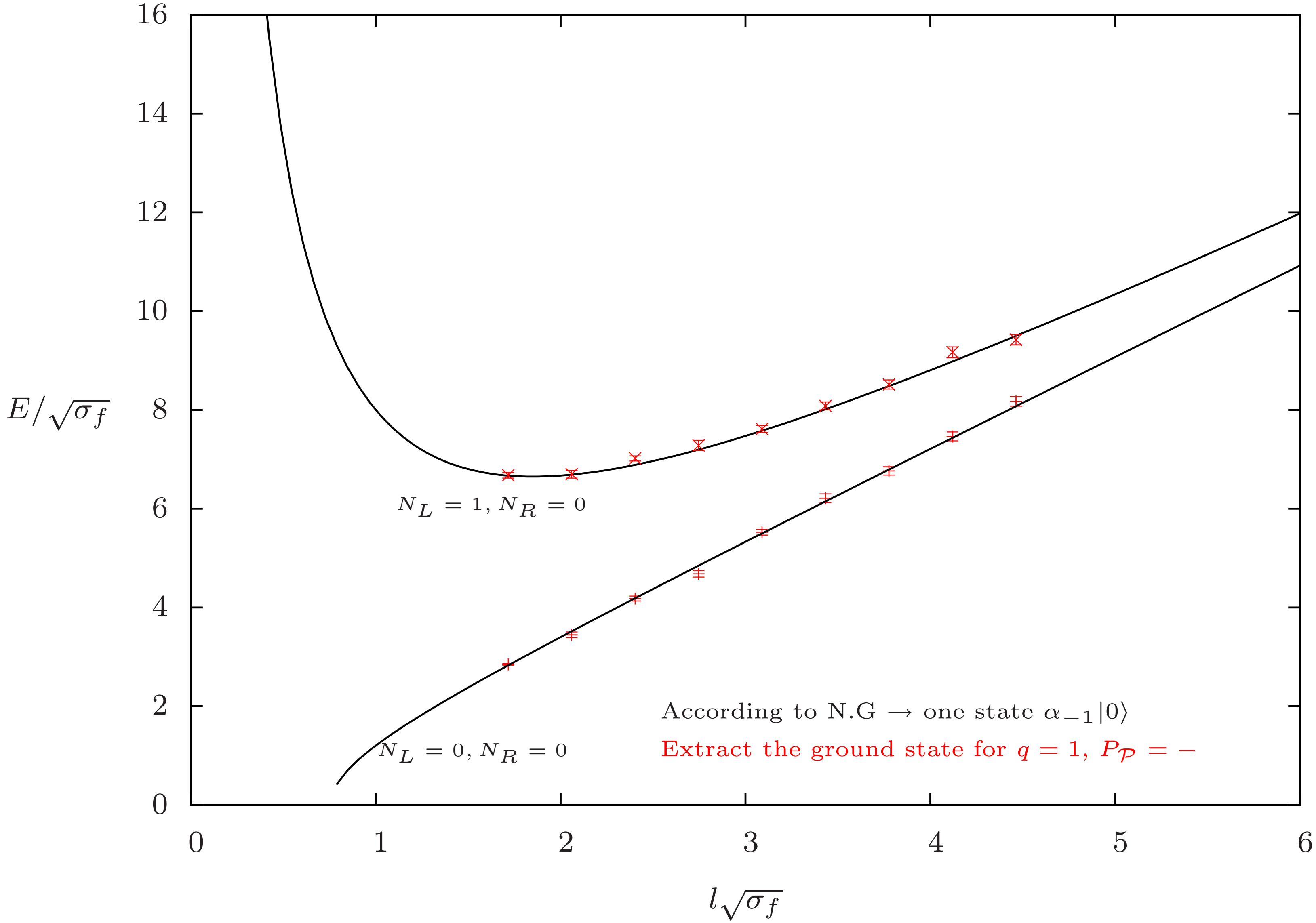
7. Results for ground states in the $k = 3$ antisymmetric representation



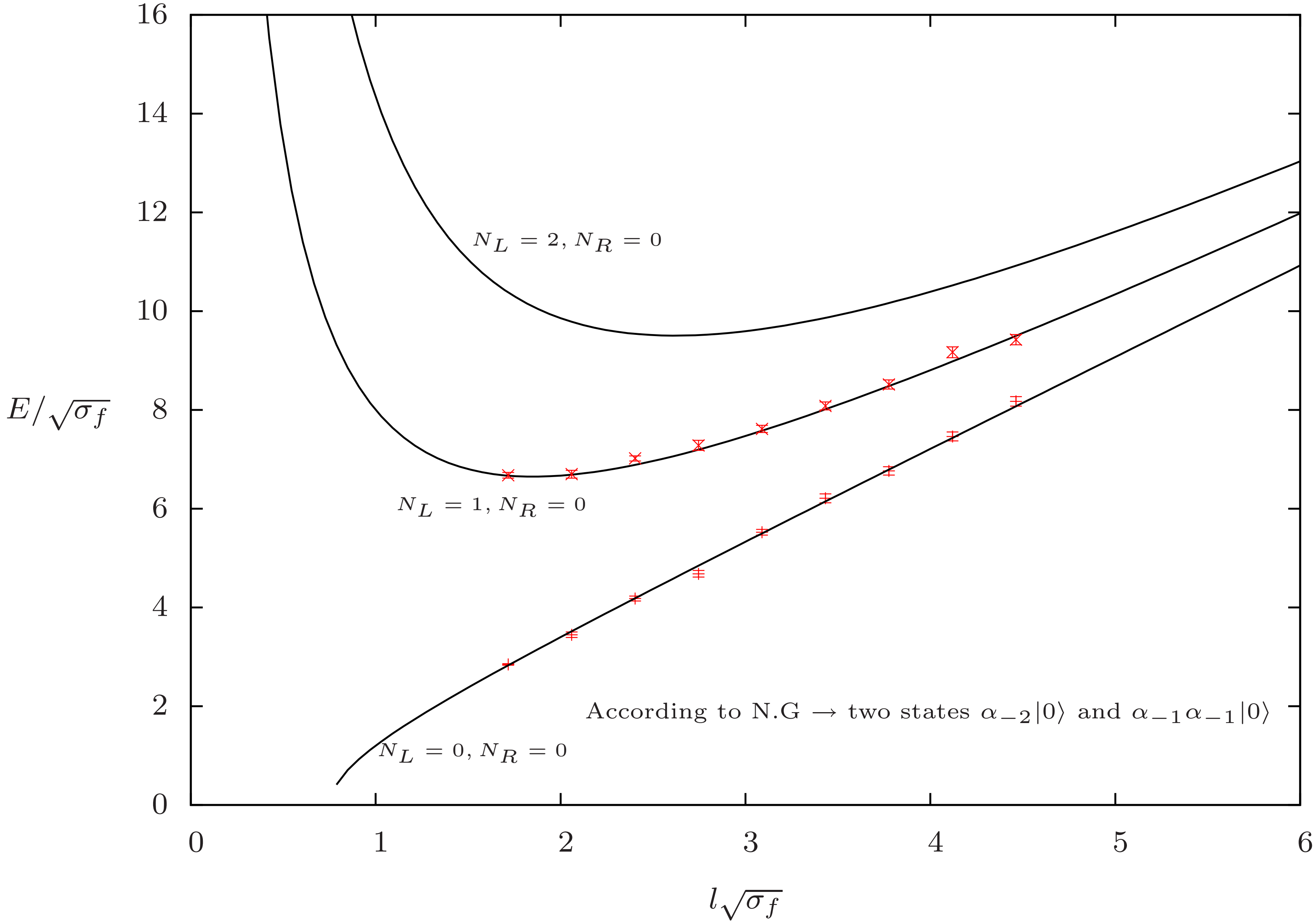
7. Results for ground states in the $k = 3$ antisymmetric representation



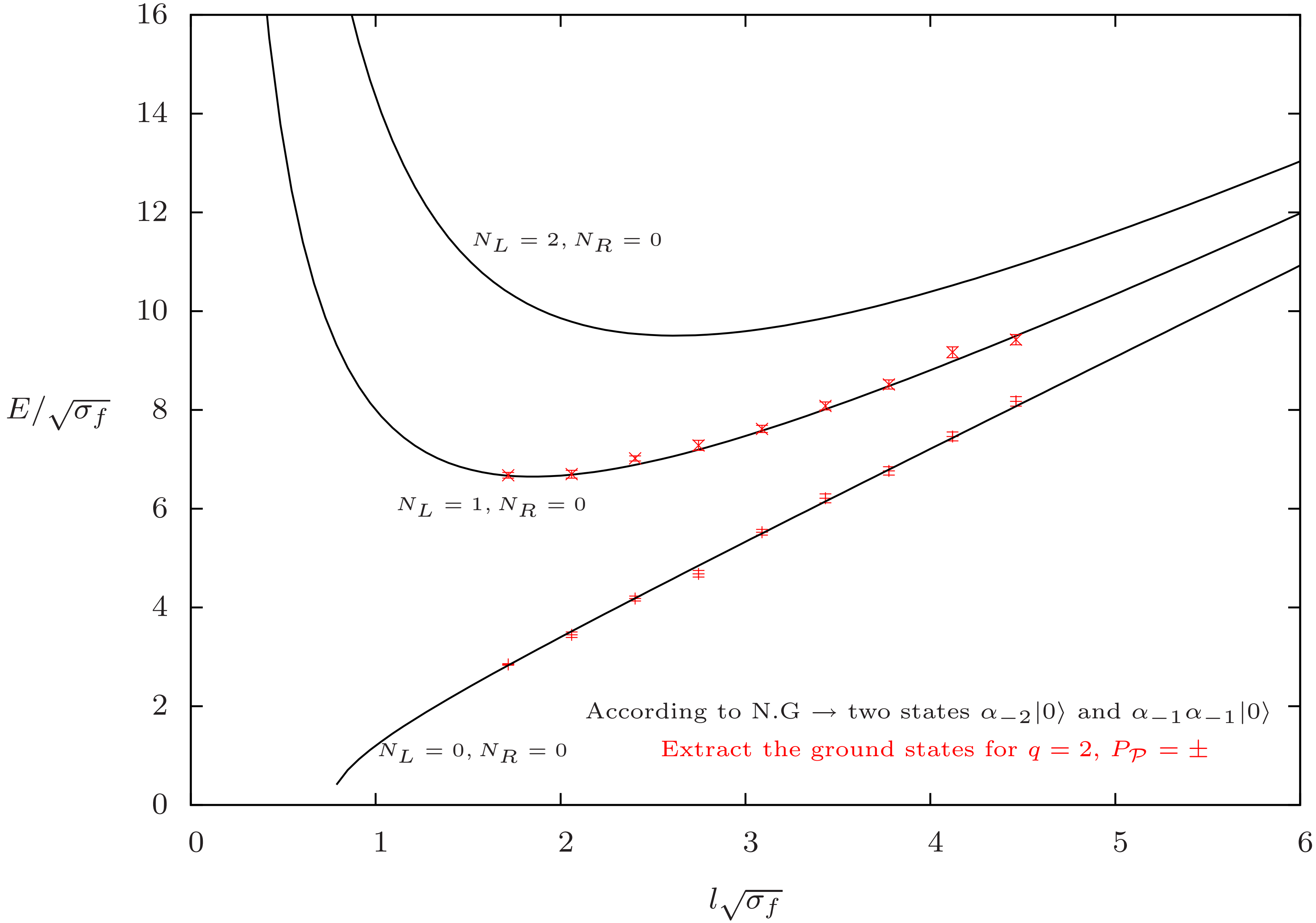
7. Results for ground states in the $k = 3$ antisymmetric representation



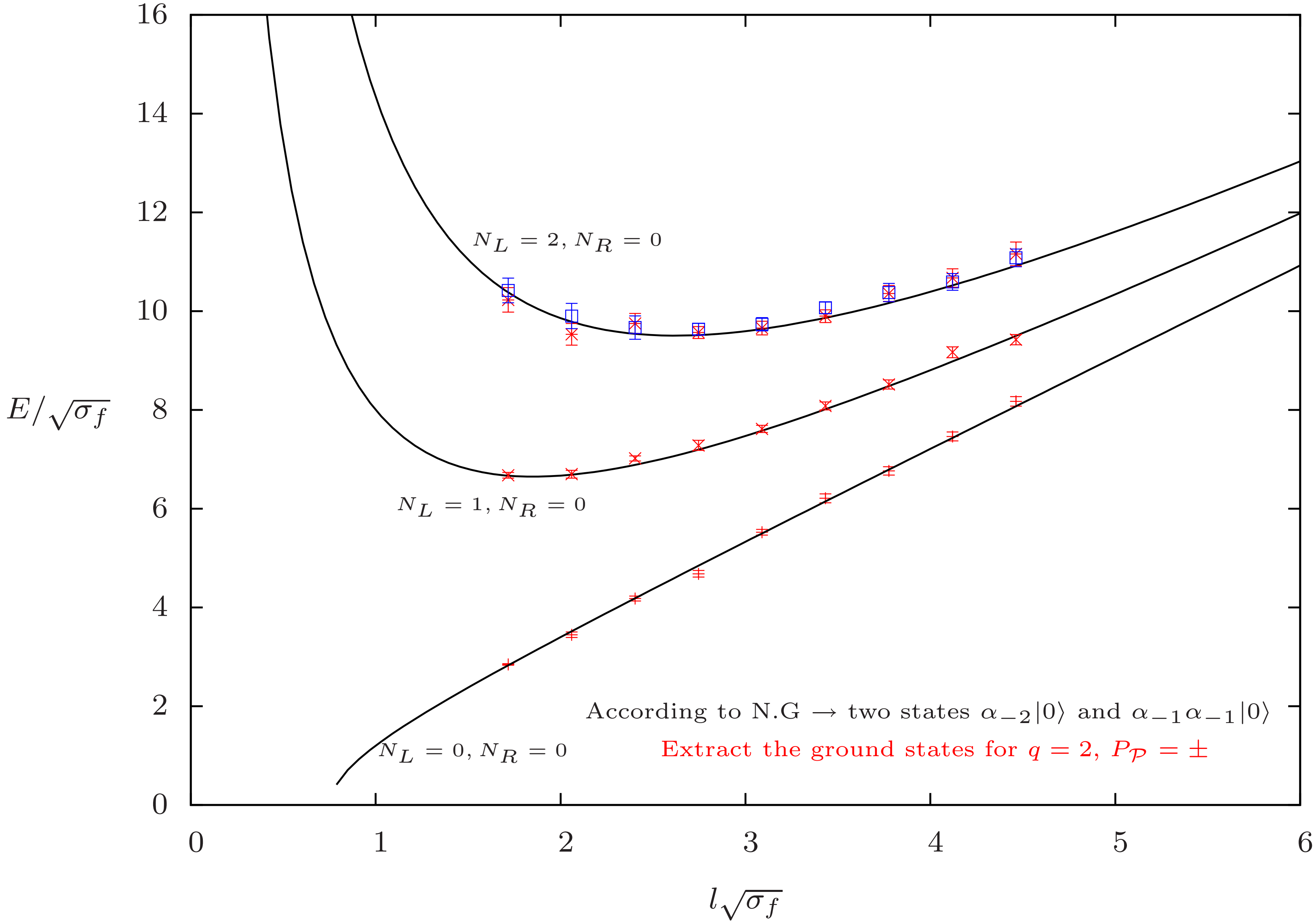
7. Results for ground states in the $k = 3$ antisymmetric representation



7. Results for ground states in the $k = 3$ antisymmetric representation

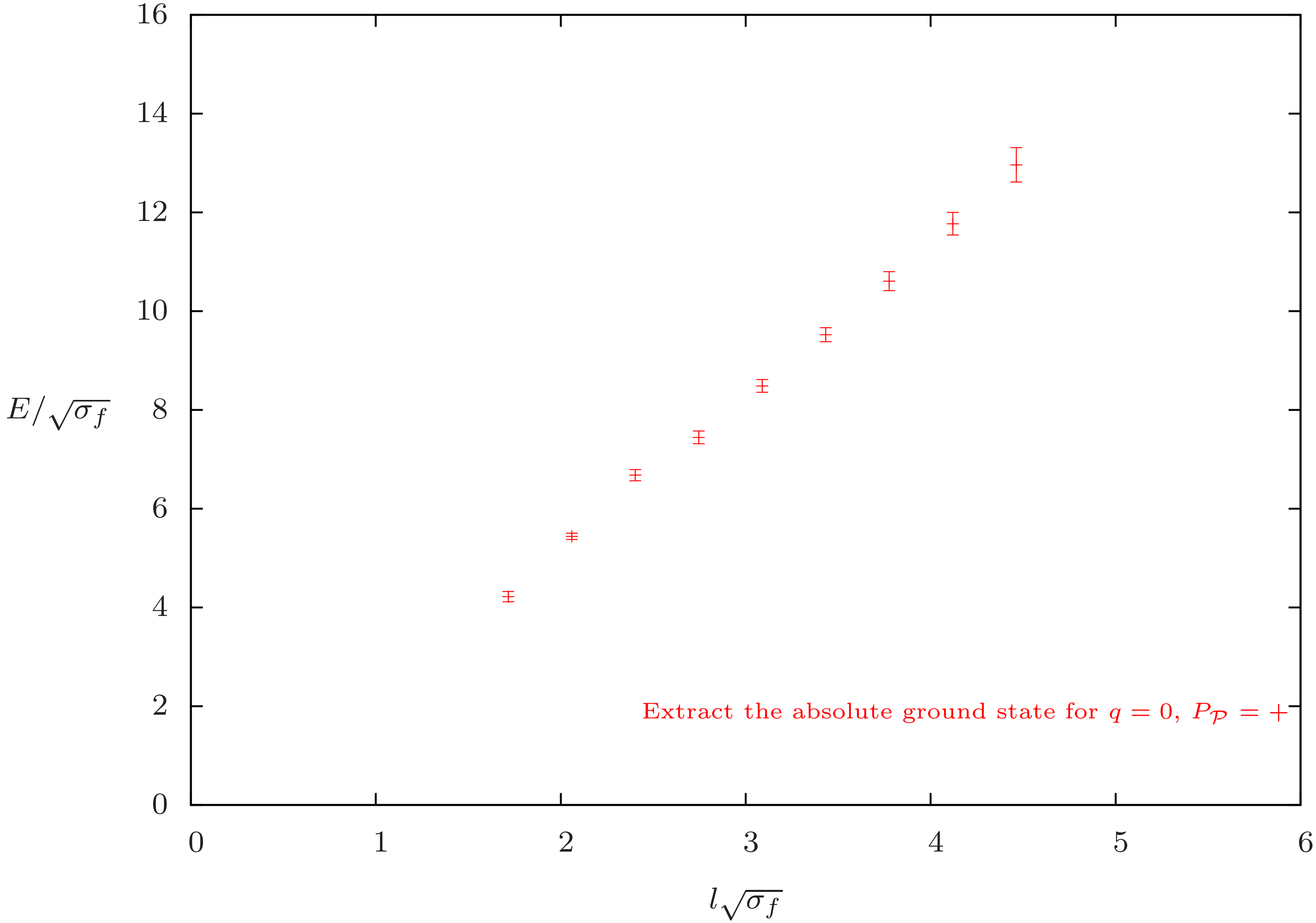


7. Results for ground states in the $k = 3$ antisymmetric representation

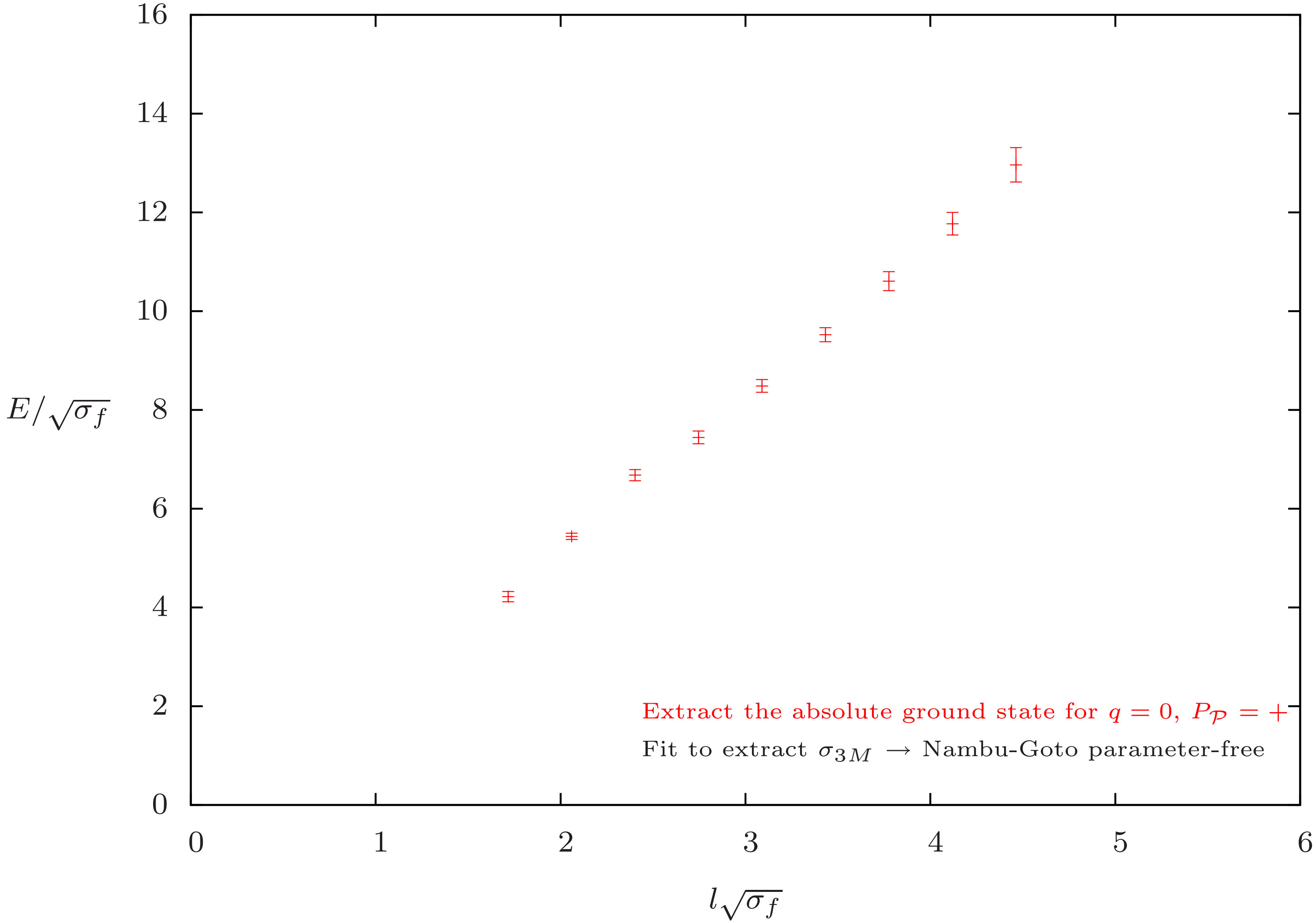


7. RESULTS FOR $k = 3$ MIXED REPRESENTATION

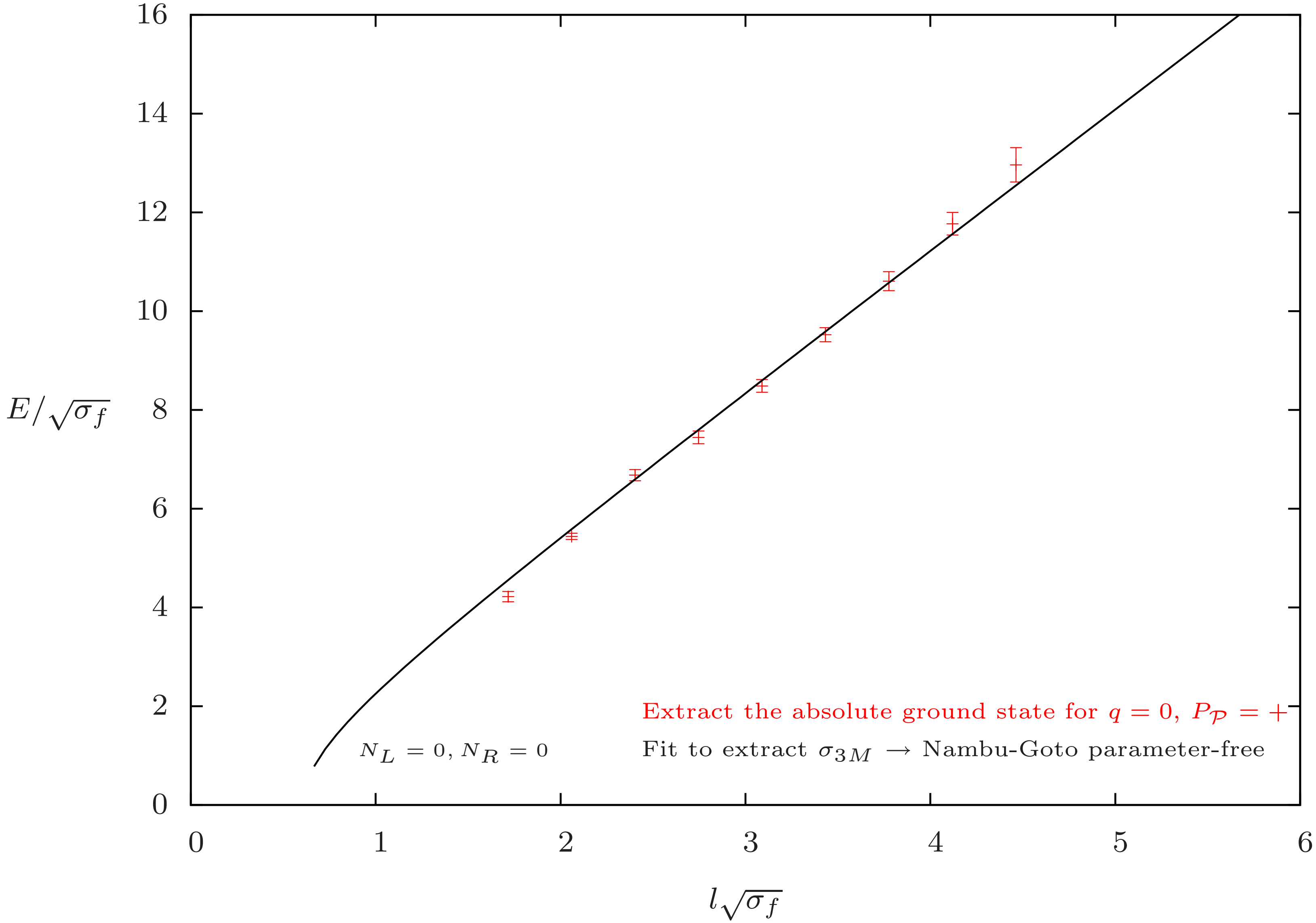
7. Results for $k = 3$ Mixed Representation



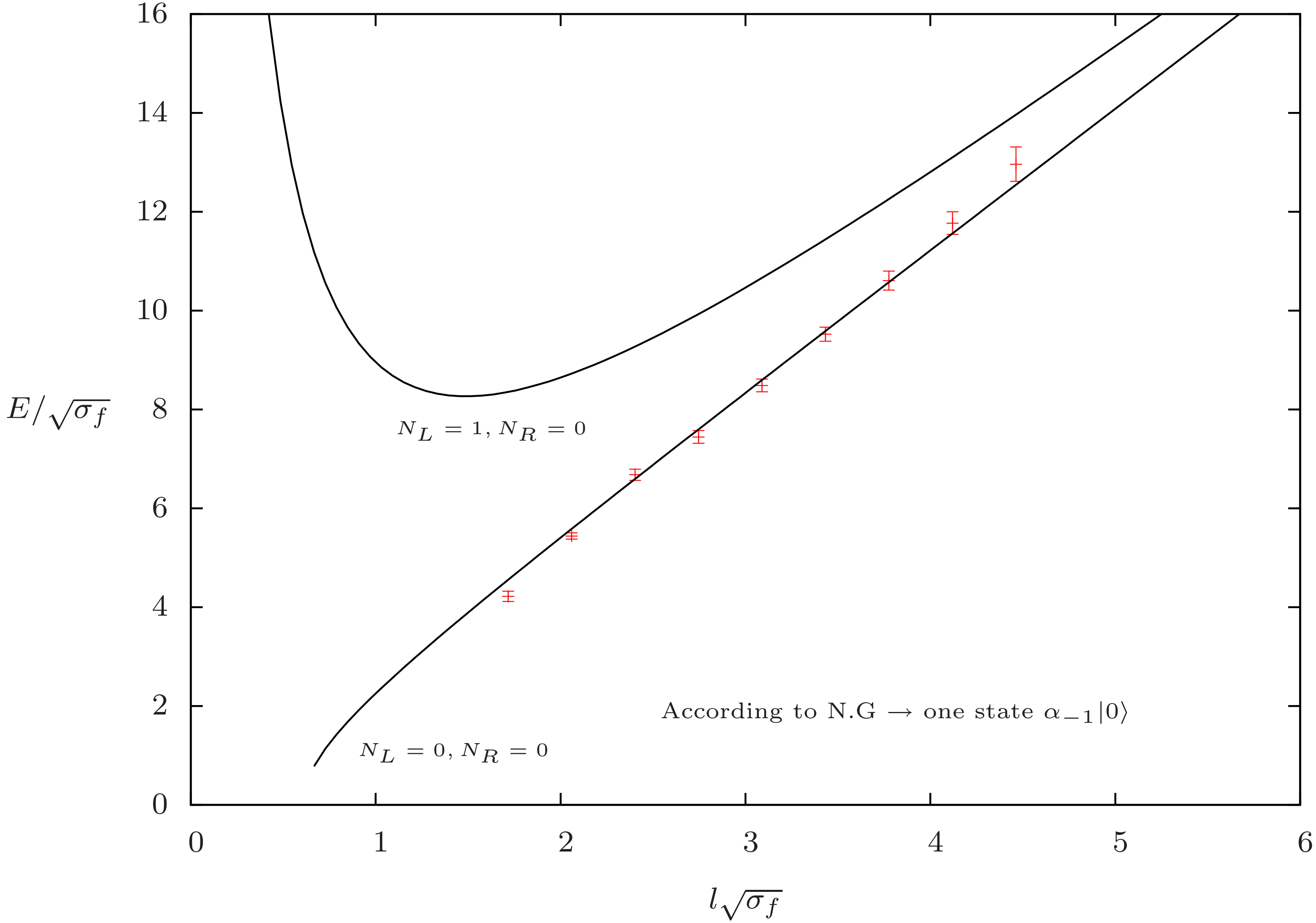
7. Results for $k = 3$ Mixed Representation



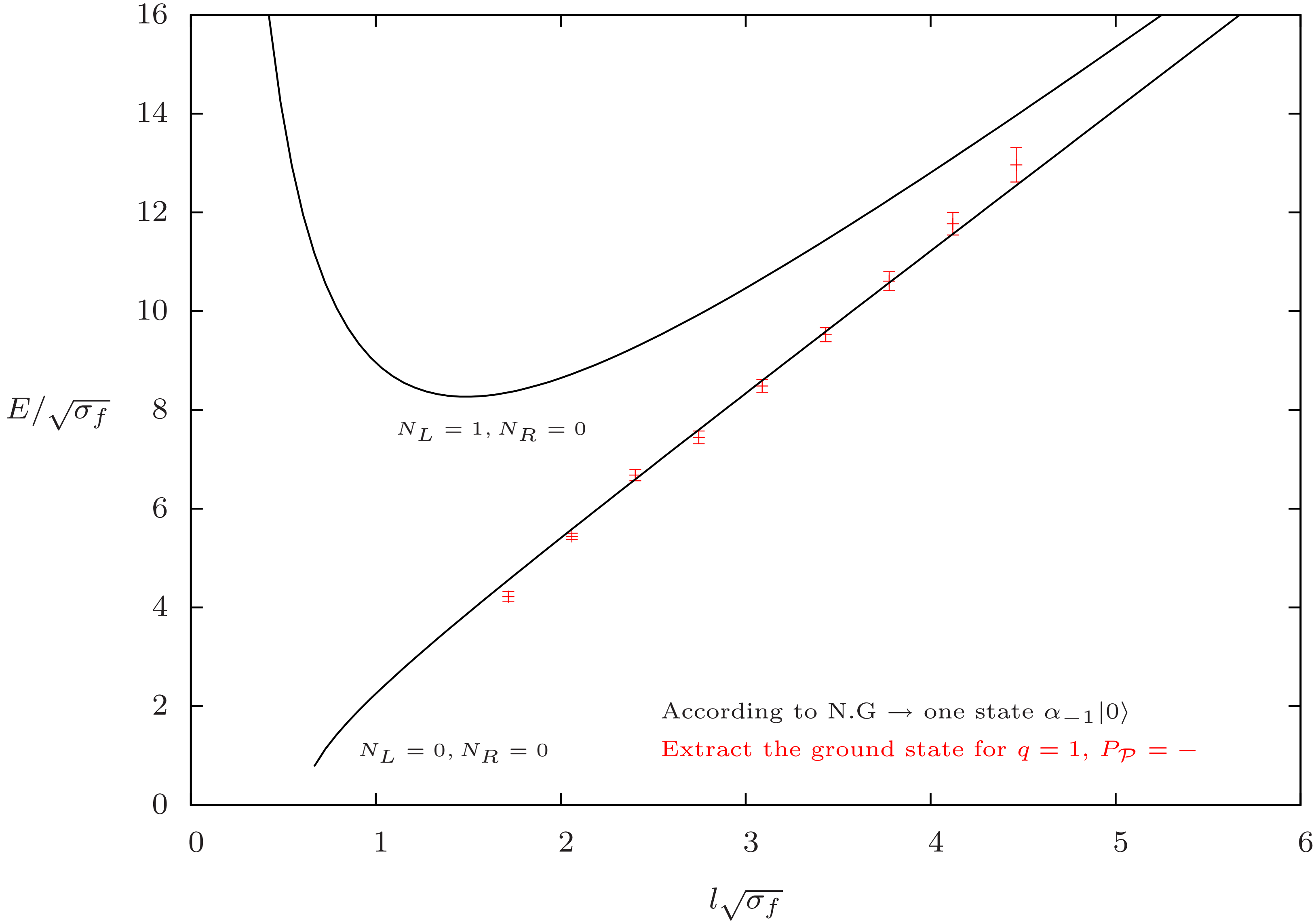
7. Results for $k = 3$ Mixed Representation



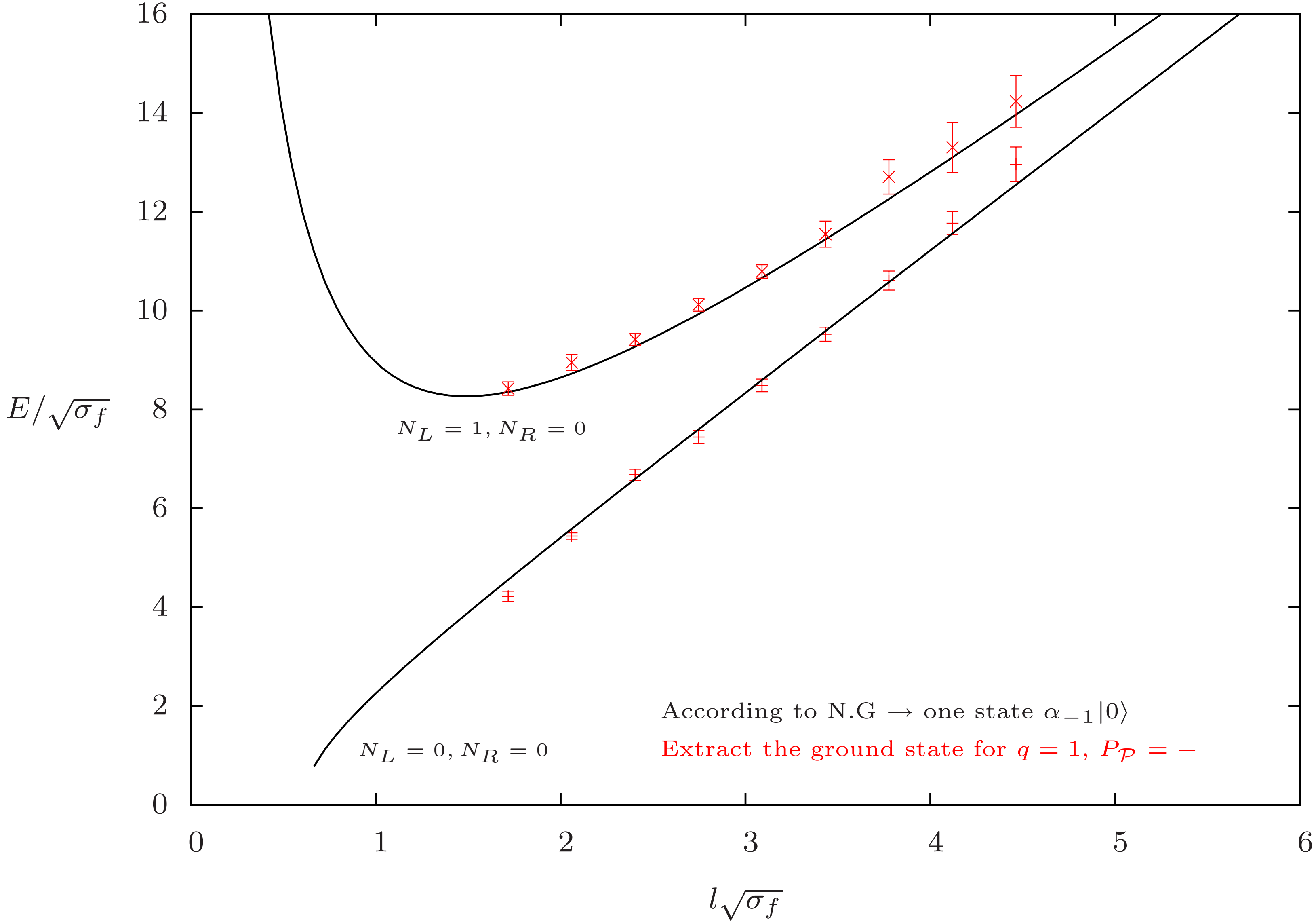
7. Results for $k = 3$ Mixed Representation



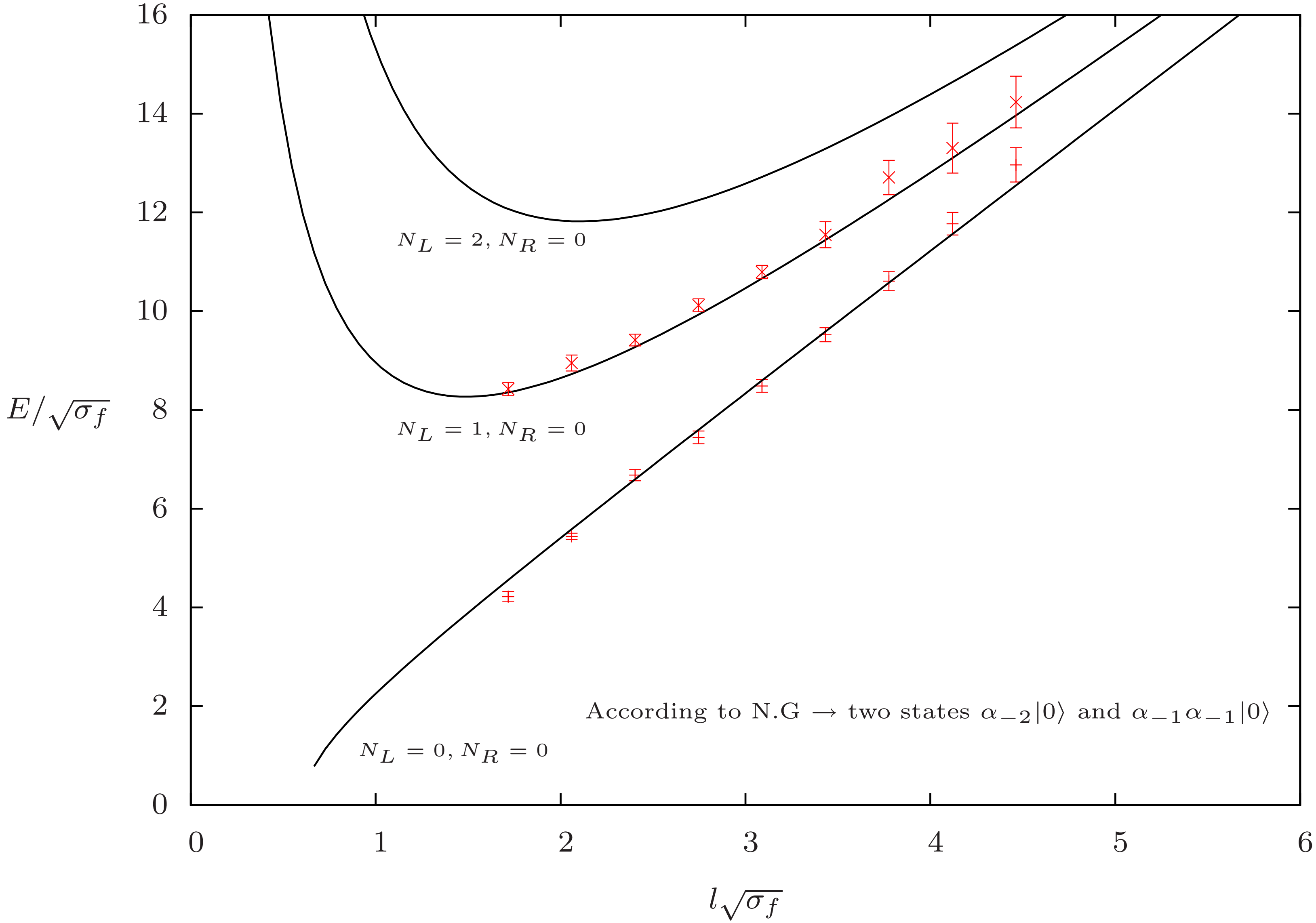
7. Results for $k = 3$ Mixed Representation



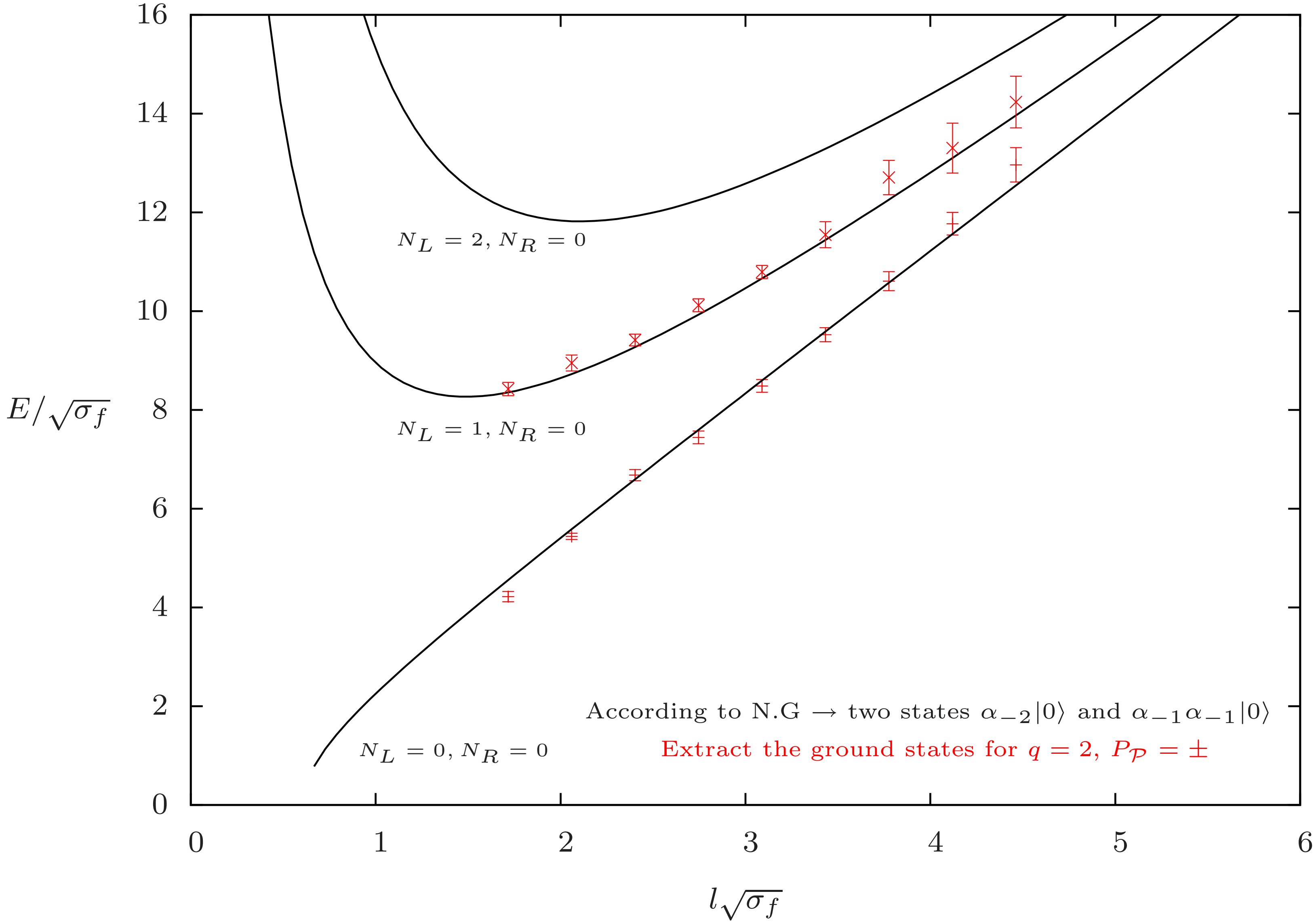
7. Results for $k = 3$ Mixed Representation



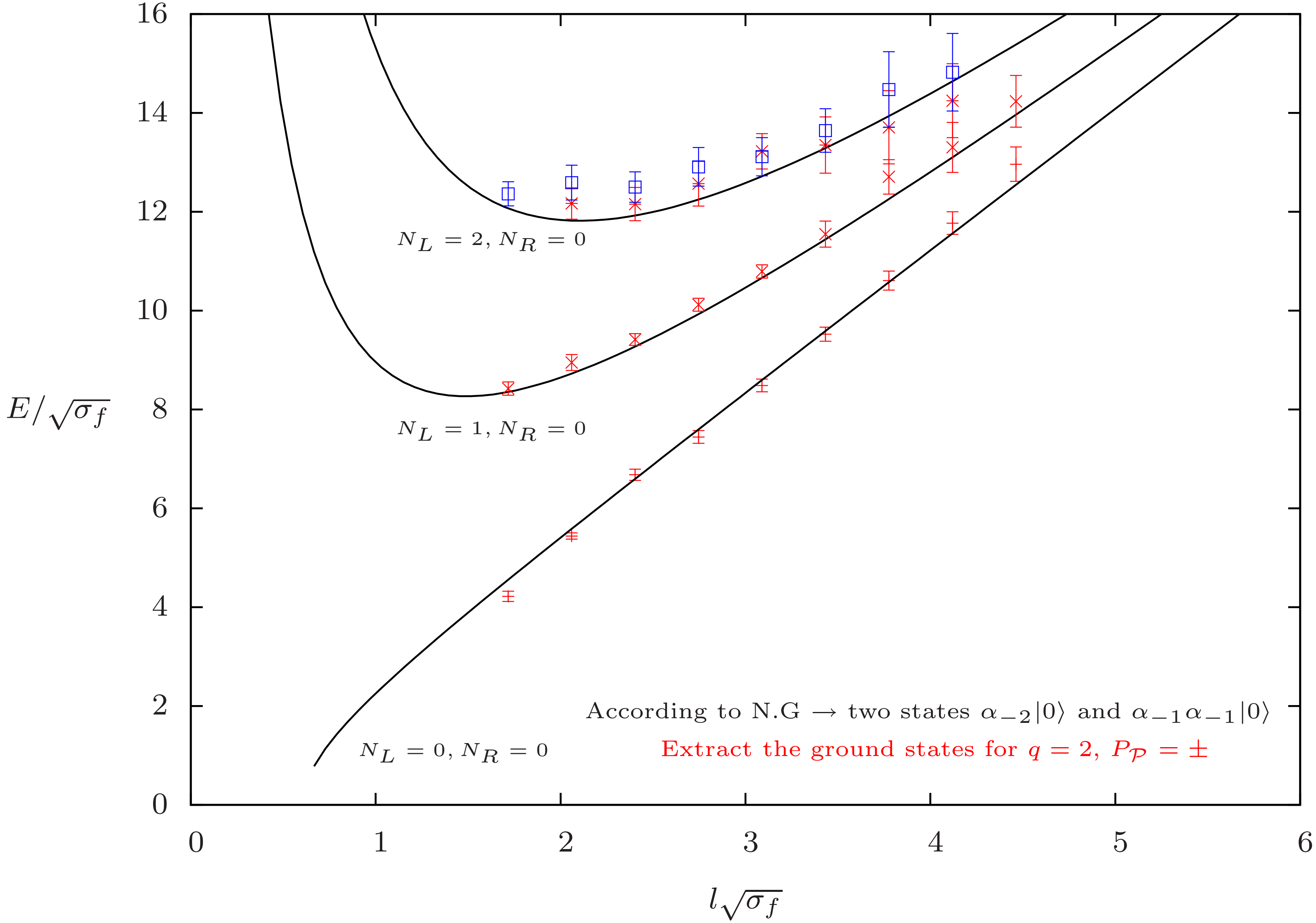
7. Results for $k = 3$ Mixed Representation



7. Results for $k = 3$ Mixed Representation

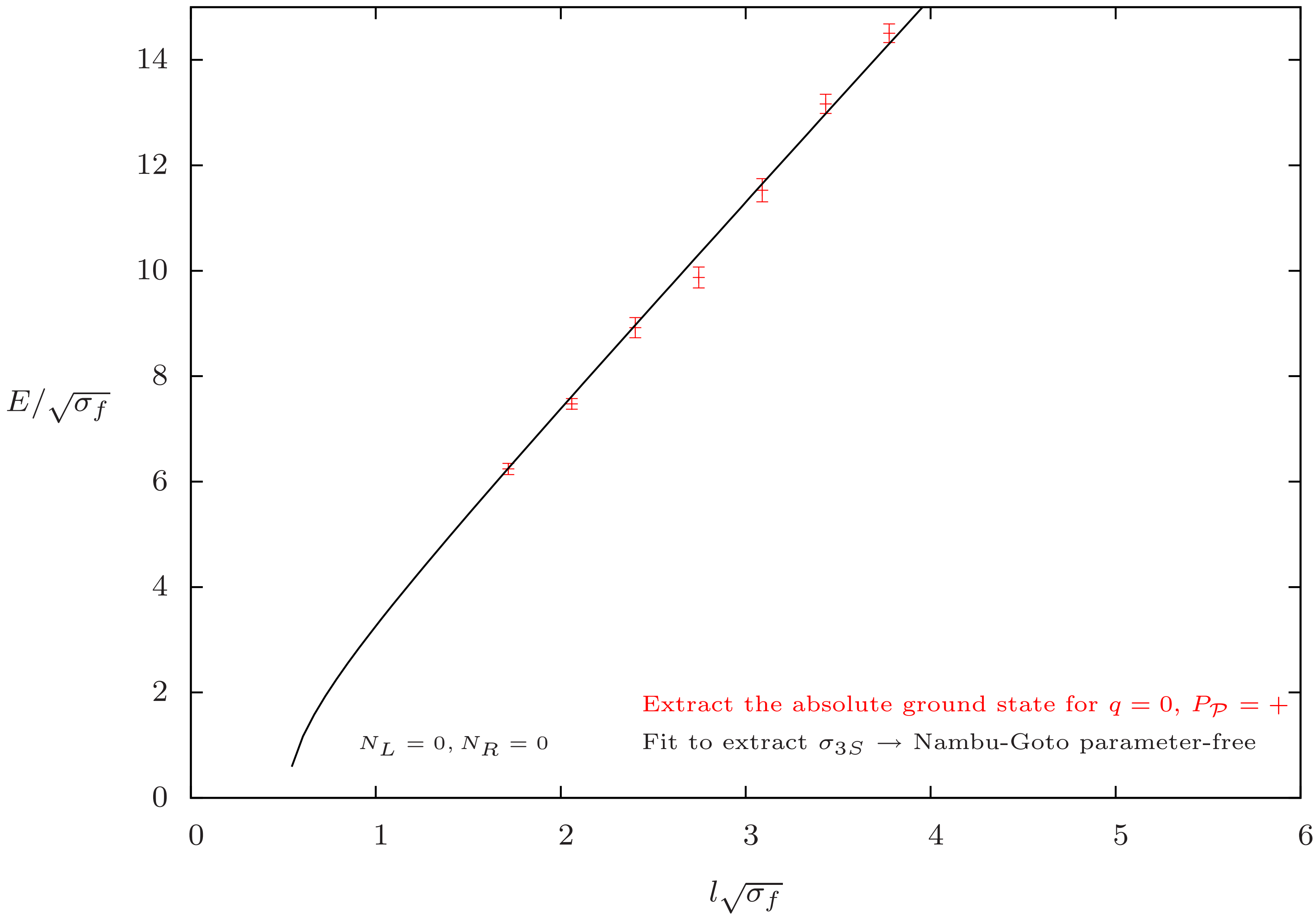


7. Results for $k = 3$ Mixed Representation



7. RESULTS FOR $k = 3$ SYMMETRIC REPRESENTATION

7. Results for $k = 3$ Symmetric Representation



7. STRING TENSION AND CASIMIR SCALING

7. String Tensions and Casimir Scaling.

Gauge Group	Representation $\mathcal{R}$	$a\sqrt{\sigma_{\mathcal{R}}}$	$\sigma_{\mathcal{R}}/\sigma_f$	$C_{\mathcal{R}}/C_f$
$SU(4)$	fundamental	0.13084(21)	1	1
$SU(4)$	$k = 2$ Antisymmetric	0.15242(17)	1.3570(53)	1.333 ...
$SU(4)$	$k = 2$ Symmetric	0.1989(11)	2.311(26)	2.4
$SU(5)$	fundamental	0.12976(11)	1	1
$SU(5)$	$k = 1$ **45**	0.20901(69)	2.595(18)	2.6833...
$SU(5)$	$k = 1$ **70**	0.2351(18)	3.282(50)	3.5
$SU(5)$	$k = 2$ Antisymmetric	0.16070(15)	1.5337(39)	1.5
$SU(5)$	$k = 2$ Symmetric	0.19476(85)	2.253(20)	2.333 ...
$SU(6)$	fundamental	0.08582(4)	1	1
$SU(6)$	$k = 1$ **84**	0.14244(29)	2.755(11)	2.7428
$SU(6)$	$k = 1$ **120**	0.15705(70)	3.349(30)	3.4
$SU(6)$	$k = 2$ Antisymmetric	0.10938(16)	1.6244(50)	1.6
$SU(6)$	$k = 2$ Symmetric	0.12895(30)	2.258(11)	2.2857
$SU(6)$	$k = 3$ Antisymmetric	0.11627(26)	1.8355(84)	1.8
$SU(6)$	$k = 3$ Mixed	0.14456(49)	2.837(19)	2.8286
$SU(6)$	$k = 3$ Symmetric	0.16785(55)	**3.825(25)**	**3.8571**

8. Conclusions

- Low-lying spectrum can be very well approximated by Nambu-Goto.
- Nambu-Goto much better than any other prediction.
- $D = 2 + 1$ spectrum only string like.