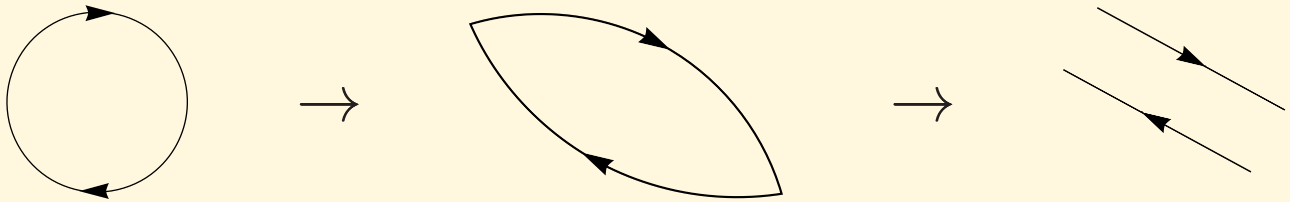

Generalized quark–antiquark potential at weak and strong coupling

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Based on `arXiv:1105.5144` - N.D. and Valentina Forini

Workshop on "Large-N Gauge Theories"

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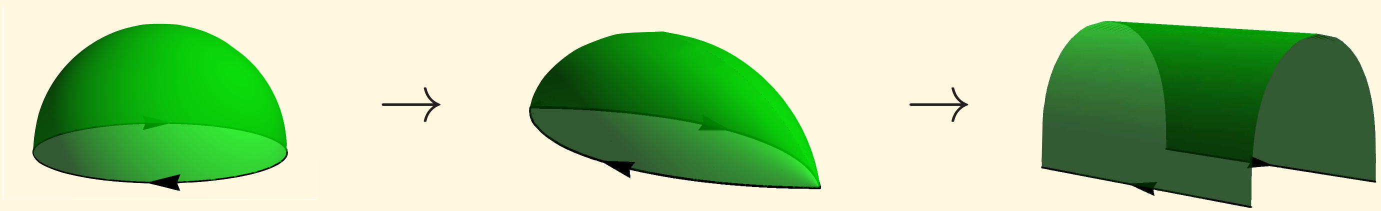

Introduction and motivation

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- Can we do any better?
- Shouldn't integrability allow us to calculate this for all values of the coupling (in the planar approximation)?

Wilson loops in $\mathcal{N} = 4$ super Yang-Mills

[Maldacena][Rey, Yee]

- The usual Wilson loop is

$$W = \operatorname{Tr} \mathcal{P} \exp \left[\oint i A_\mu \dot{x}^\mu \, ds \right]$$

- The most natural Wilson loops in $\mathcal{N} = 4$ SYM include a coupling to the scalar fields

$$W = \operatorname{Tr} \mathcal{P} \exp \left[\oint \left(i A_\mu \dot{x}^\mu + |\dot{x}| \theta^I \Phi_I \right) ds \right]$$

  θ^I do not have to be constant.

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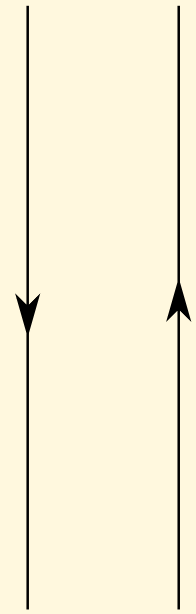
- The scalar coupling is natural for calculating the potential between W-bosons.
- For a pair of antiparallel lines

$$\langle W \rangle \approx \exp\left[-T\,V(L,\lambda)\right]$$

- In a conformal theory we expect

$$V(L,\lambda) = \frac{f(\lambda)}{L}$$



- Explicit calculations at weak and at strong coupling:

$$V(L,\lambda)=\begin{cases}-\dfrac{\lambda}{4\pi L}+\dfrac{\lambda^2}{8\pi^2 L}\ln\dfrac{T}{L}+\cdots & \lambda\ll 1\\[2ex] \dfrac{4\pi^2\sqrt{\lambda}}{\Gamma(\frac{1}{4})^4\,L}\left(1-\dfrac{1.3359\ldots}{\sqrt{\lambda}}+\cdots\right) & \lambda\gg 1\end{cases}$$

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- Hard to guess how to connect these two regimes.
- Could go to $O(\lambda^3)$ and $O(\lambda^4)$.
- We will add extra parameters and study a larger family of observables.
- Thus gather more information to help guess an exact interpolating function.

Outline

- Introduction and motivation
- Generalized quark-antiquark potential
- Perturbation theory calculation
- Classical string surfaces
- One loop string determinants
- Expansions in small angles
- Summary

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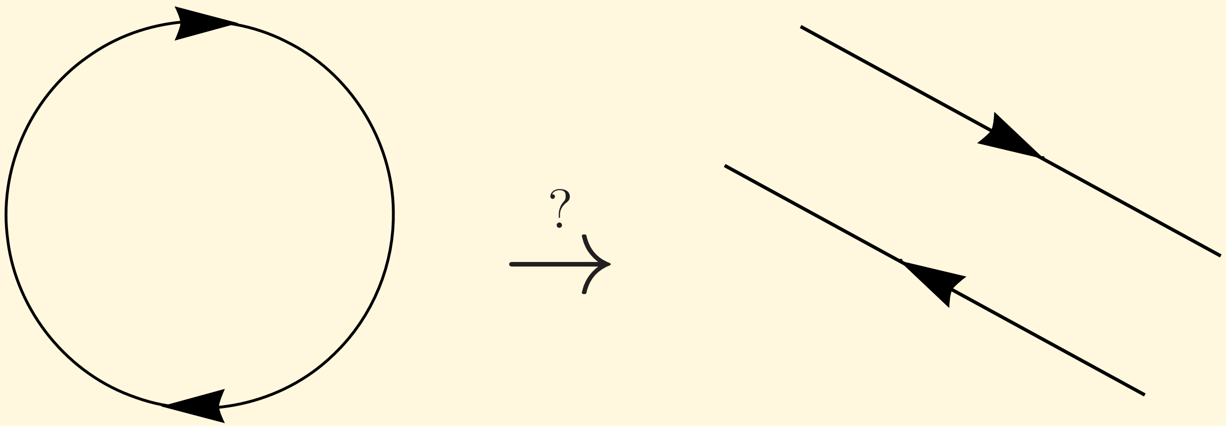
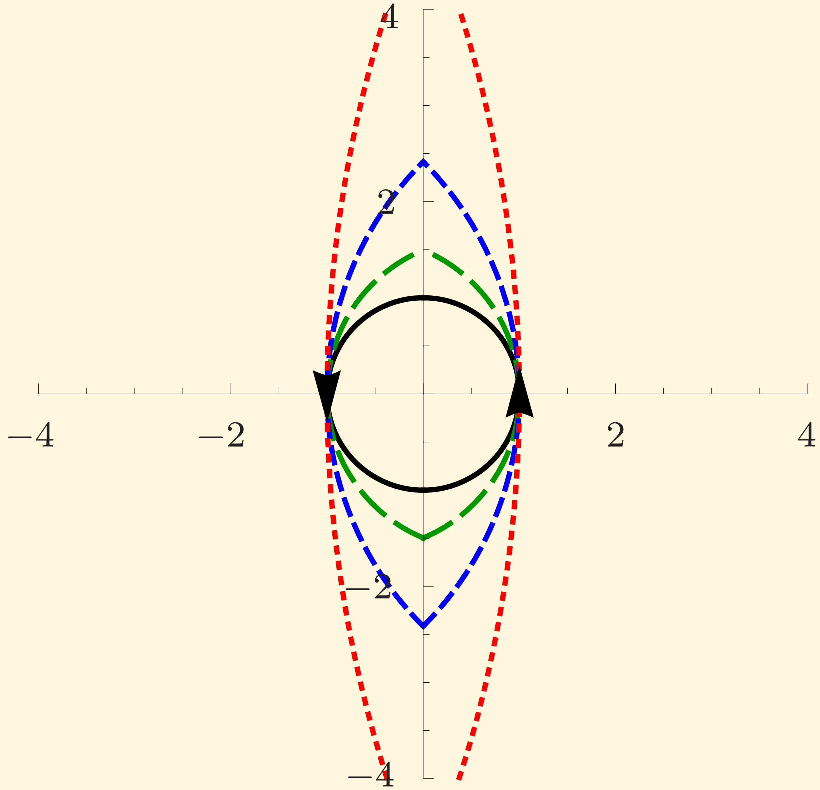
- Their expectation value is known exactly.
- Can we somehow view the antiparallel lines as a deformation of the circle/line?

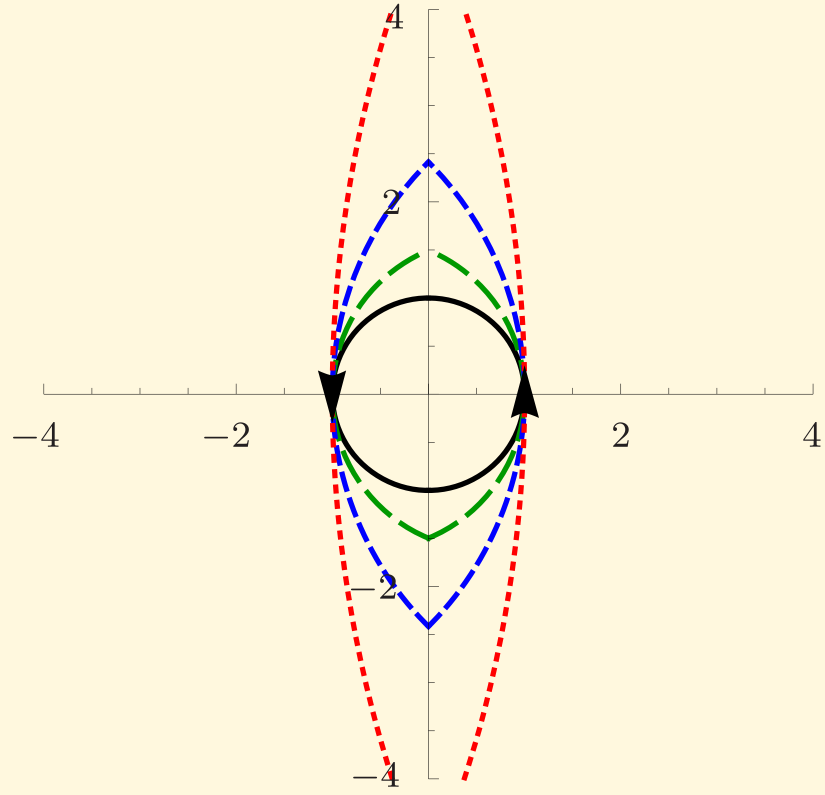


- We take the following family of curves:



- We take the following family of curves:

- These are pairs of arcs with opening angle $\pi - \phi$.
- $\phi = 0$ is the 1/2 BPS circle.
- $\phi \to \pi$ gives the antiparallel lines.



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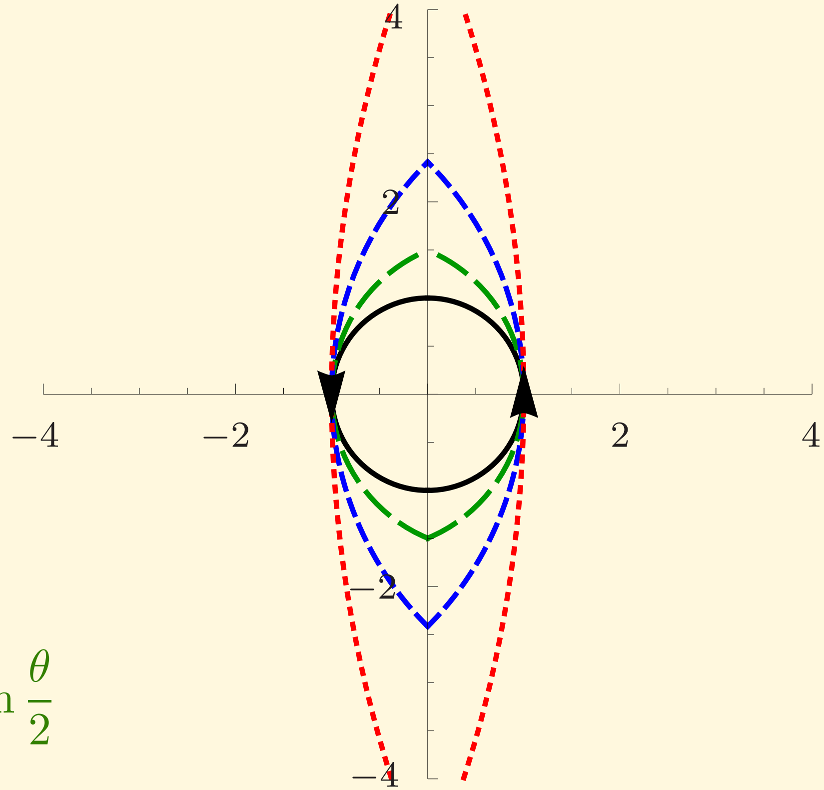

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- Can have each line couple to a different scalar field

$$\Phi_1 \cos\frac{\theta}{2} + \Phi_2 \sin\frac{\theta}{2} \qquad \text{and} \qquad \Phi_1 \cos\frac{\theta}{2} - \Phi_2 \sin\frac{\theta}{2}$$

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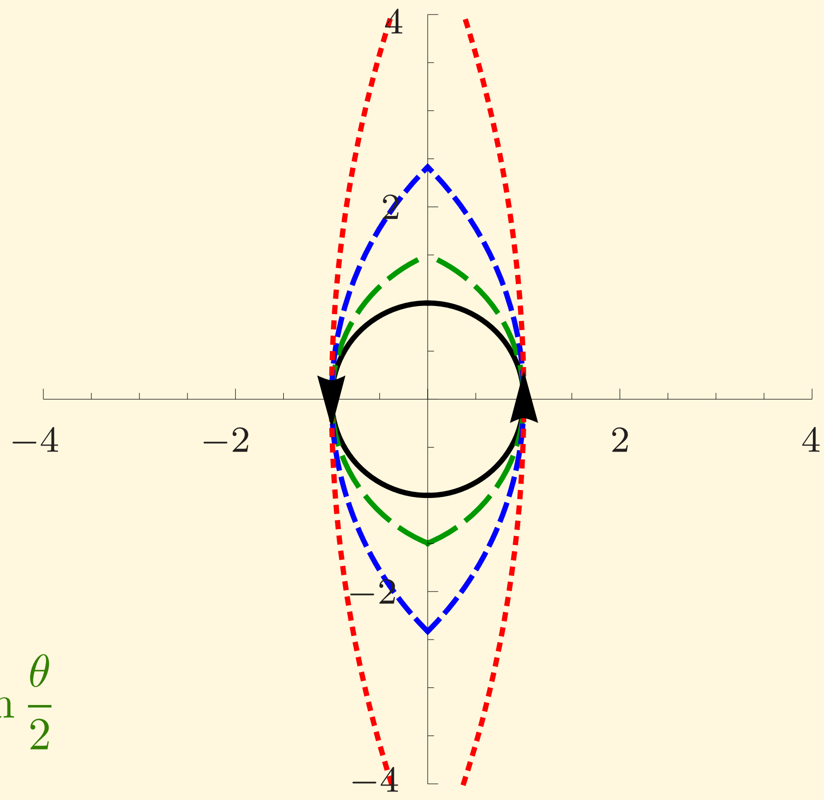
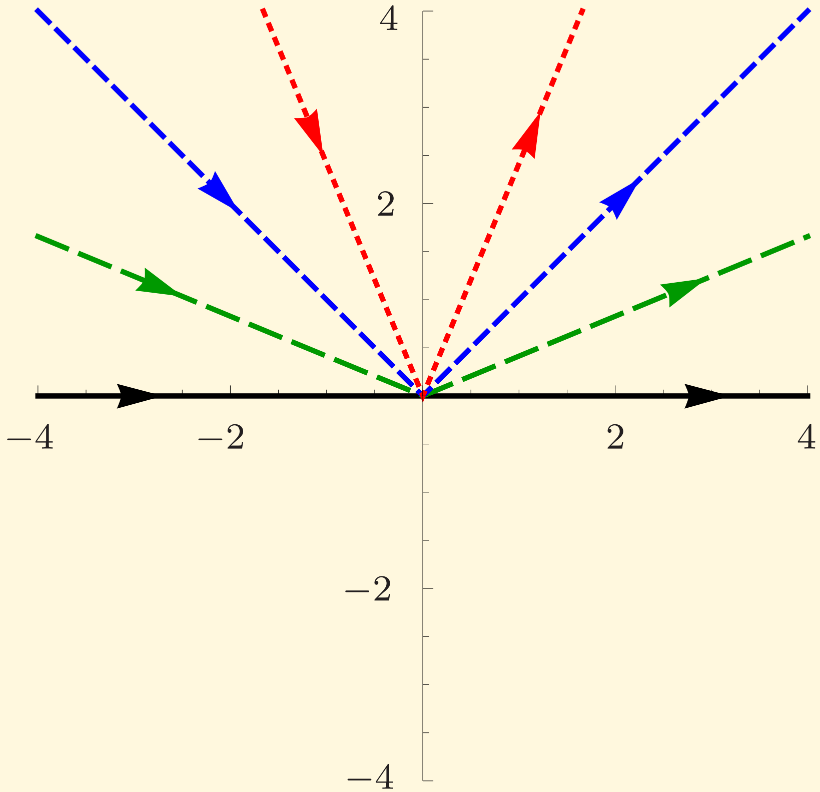
- Crucial point: Calculations are no harder than for the antiparallel case!

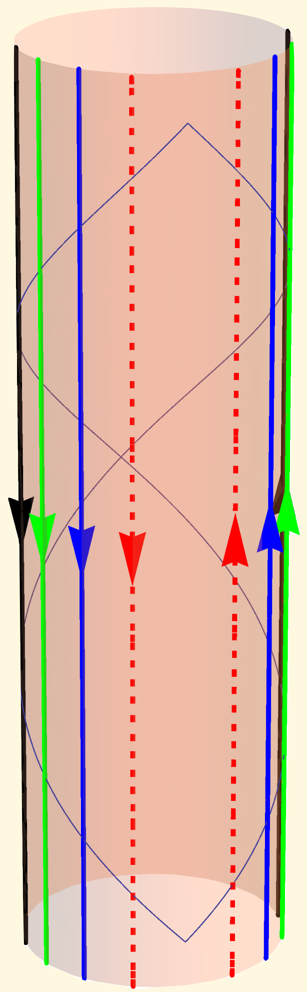


- By a conformal transformation which maps one cusp to infinity:



- This is a cusp in Euclidean space.
- Taking $\phi = iu$ and $u \to \infty$ gives the Lorenzian null cusp.

- By the inverse exponential map we get the gauge theory on $\mathbb{S}^3 \times \mathbb{R}$



- These are parallel lines on $\mathbb{S}^3 \times \mathbb{R}$.

- From this last picture we expect

$$\langle W\rangle \approx \exp\Big[-T\,V(\phi,\theta,\lambda)\Big]$$

- The same is true for the cusp in $\mathbb{R}^4$ with

$$T=\log\frac{R}{\epsilon}$$

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- Expanding at weak coupling

$$V(\phi,\theta,\lambda)=\sum_{n=1}^{\infty}\left(\frac{\lambda}{16\pi^2}\right)^n V^{(n)}(\phi,\theta)$$

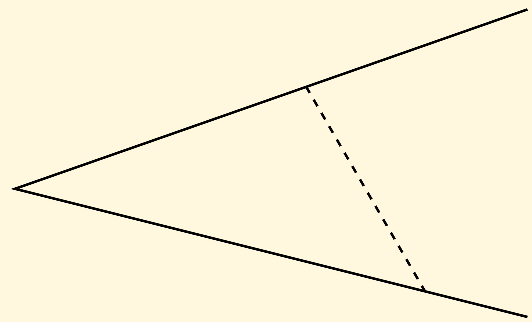
- And at strong coupling

$$V(\phi,\theta,\lambda)=\frac{\sqrt{\lambda}}{4\pi}\sum_{l=0}^{\infty}\left(\frac{4\pi}{\sqrt{\lambda}}\right)^l V^{(l)}_{AdS}(\phi,\theta)$$

Weak coupling

1–loop graphs

- Just the exchange of a gluon and scalar field



- This graph is given by the integral

$$
\begin{aligned}
\partial_\lambda \langle W\rangle\Big|_{\lambda=0} &= \int ds\, dt\, \langle -A(s)\cdot A(t) + \Phi(s)\cdot\Phi(t)\rangle \\
&= \frac{\lambda}{8\pi^2}\int ds\, dt\, \frac{-\dot x_\mu(s)\dot x^\mu(t)+\theta^I(s)\theta^I(t)}{|x(s)-x(t)|^2} \\
&= \frac{\lambda}{8\pi^2}\int ds\, dt\, \frac{\cos\theta-\cos\phi}{s^2+t^2+2st\cos\phi} = \frac{\lambda}{8\pi^2}\,\frac{\cos\theta-\cos\phi}{\sin\phi}\,\phi\log\frac{R}{\epsilon}
\end{aligned}
$$

- Therefore

$$
V^{(1)}(\phi,\theta) = -2\,\frac{\cos\theta-\cos\phi}{\sin\phi}\,\phi
$$

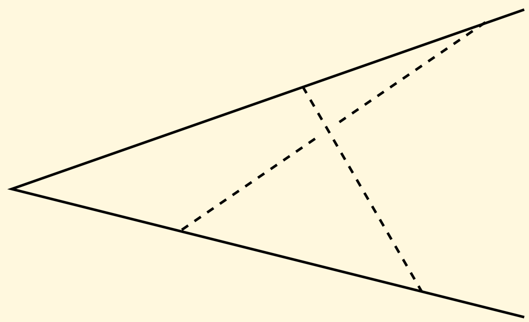
2–loop graphs

[Makeenko, Olesen, Semenoff]

- Ladder graphs are quite easy.

$$V^{(2)} = \frac{1}{2\log\frac{R}{\epsilon}}\partial_\lambda^2\Big[\log\langle W\rangle\Big]_{\lambda=0} = \frac{1}{2\log\frac{R}{\epsilon}}\Big[\partial_\lambda^2\langle W\rangle - (\partial_\lambda\langle W\rangle)^2\Big]_{\lambda=0}$$

- In $\langle W\rangle$ we include only planar graphs.
- $V^{(2)}_{\text{ladder}}$ is therefore minus the non-planar graphs.



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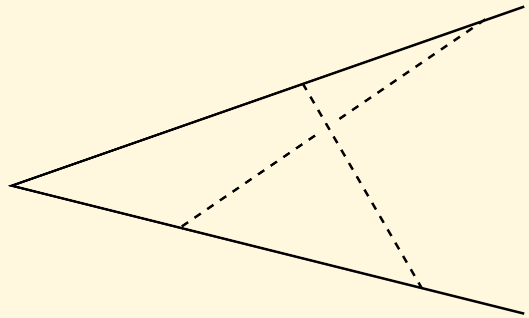
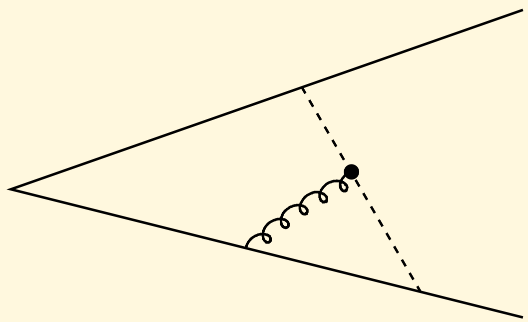


- This graph is given by the integral

$$\frac{\lambda^2}{(4\pi)^4}\int_{s_1<s_2}ds_1\,ds_2\int_{t_1<t_2}dt_1\,dt_2\,\frac{(\cos\phi-\cos\theta)^2}{(s_1^2+t_2^2+2s_1t_2\cos\phi)(s_2^2+t_1^2+2s_2t_1\cos\phi)}$$
$$=\frac{\lambda^2}{64\pi^4}\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\left[\mathrm{Li}_3\left(e^{2i\phi}\right)-\zeta(3)-i\phi\left(\mathrm{Li}_2\left(e^{2i\phi}\right)+\frac{\pi^2}{6}\right)+\frac{i}{3}\phi^3\right]\log\frac{R}{\epsilon}$$

- Dividing by $-\frac{\lambda^2}{(4\pi)^4}\log\frac{R}{\epsilon}$ we get $V^{(2)}_{\text{ladder}}$

- Interacting graphs are a bit more complicated.
- There are bubble graphs and the single cubic vertex graphs.



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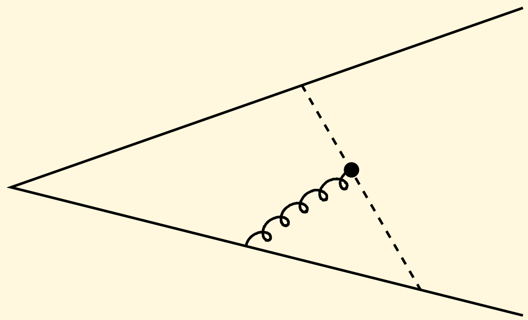
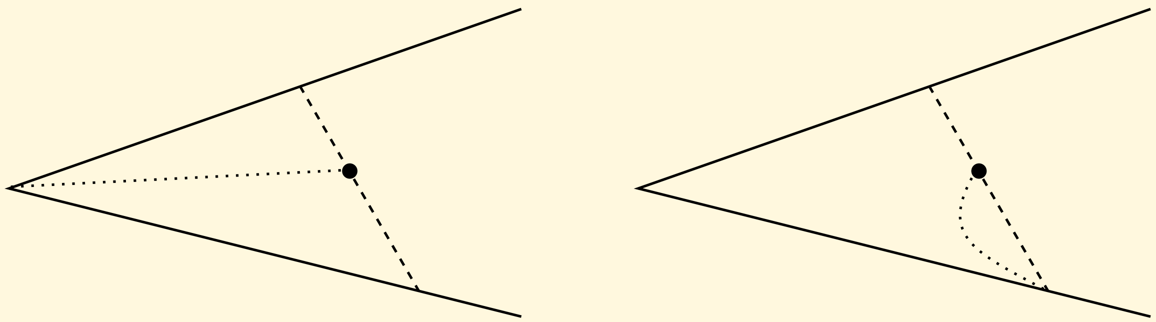


- One of the lines is always a gluon. It is a total derivative, giving two contributions



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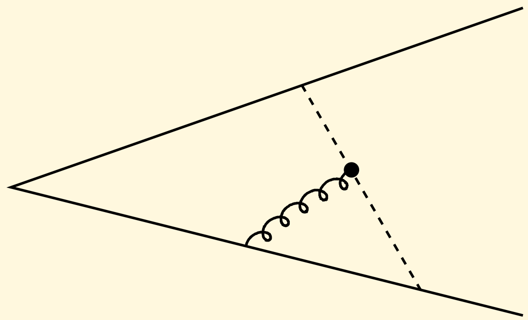

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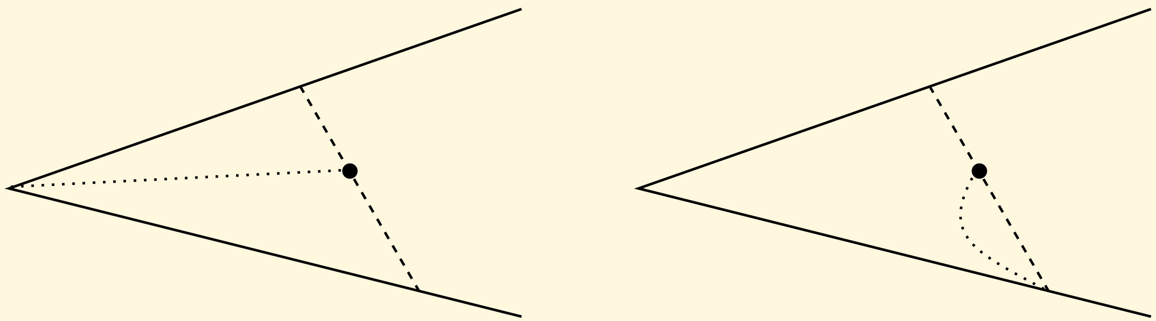
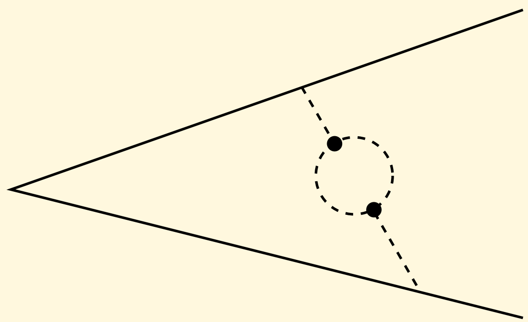
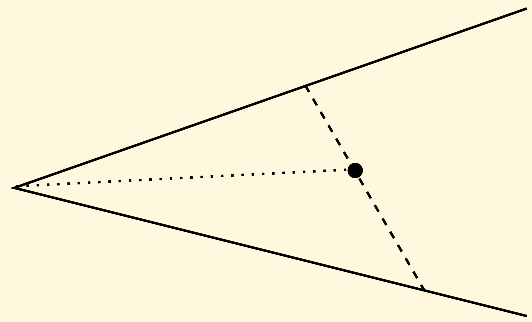


- The second graph cancels exactly against the bubble graphs



- Remaining graph involves the triangle graph



- It is given by the integral

$$\frac{\lambda^2}{64\pi^6}\int dt\,ds\int d^4w\,\frac{\cos\theta-\cos\phi}{|x(s)-w|^2\,|x(t)-w|^2\,|w|^2}$$

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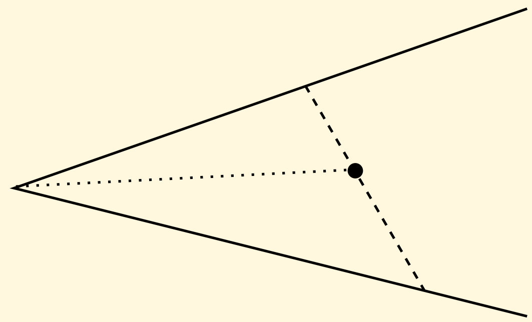

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- The integration over w can be done exactly and gives a function (with dilogarithms) of s/t and ϕ.
- Doing the integral over s and t and dividing by $-\log\frac{R}{\epsilon}$ gives

$$V^{(2)}_{\text{int}}(\phi,\theta)=\frac{4}{3}\,\frac{\cos\theta-\cos\phi}{\sin\phi}(\pi^2-\phi^2)\phi$$

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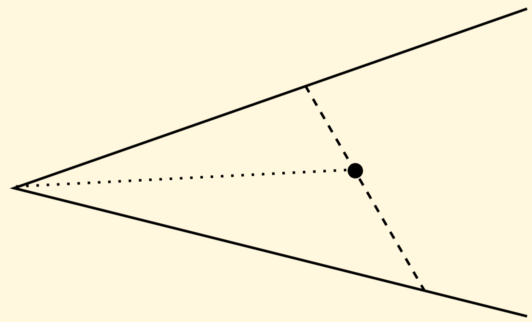

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$$V^{(2)}_{\rm int}(\phi,\theta)=\frac{4}{3}\,\frac{\cos\theta-\cos\phi}{\sin\phi}(\pi^2-\phi^2)\phi$$

- The result is simpler than the ladder graphs and closely related to 1–loop:

$$V^{(2)}_{\rm int}(\phi,\theta)=-\frac{2}{3}(\pi^2-\phi^2)V^{(1)}(\phi,\theta)$$

  First sign of simplification for this set of observables...

String theory calculation

Classical string in $AdS_3 \times \mathbb{S}^1$

- The boundary conditions are lines separated by $\pi - \phi$ on the boundary of AdS and θ on $\mathbb{S}^5$.
- All the string solutions fit inside $AdS_3 \times \mathbb{S}^1$

$$ds^2 = \sqrt{\lambda}\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2 + d\vartheta^2\right)$$

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- As world–sheet coordinates we can take t and ϑ rescaled

$$\sigma = \frac{\sqrt{b^4+p^2}}{b\,q}\,\vartheta \qquad \tau = \frac{\sqrt{b^4+p^2}}{b\,p}\,t$$

and then

$$\rho = \rho(\sigma)\,, \qquad \vartheta = \vartheta(\sigma)$$

- The Nambu-Goto action is

$$\mathcal{S}_{NG} = \frac{\sqrt{\lambda}}{2\pi}\int dt\,d\varphi\,\cosh\rho\sqrt{\sinh^2\rho\,\varphi'^2 + \rho'^2 + 1}$$

- Two conserved quantities are

$$E = \frac{\varphi'\sinh^2\rho\cosh\rho}{\sqrt{\sinh^2\rho\,\varphi'^2 + \rho'^2 + 1}} \qquad J = -\frac{\cosh\rho}{\sqrt{\sinh^2\rho\,\varphi'^2 + \rho'^2 + 1}}$$

- The resulting equations are elliptic.

$$\varphi'^2 = \frac{b^2}{(b^4+p^2)\sinh^4\rho}\,, \qquad \rho'^2 = \frac{(b^2\sinh^2\rho - 1)(b^2+p^2\sinh^2\rho)}{(b^4+p^2)\sinh^2\rho}$$

With

$$p = -\frac{1}{E} \qquad q = \frac{J}{E} \qquad b^2 = \frac{1}{2}\left(p^2 - q^2 + \sqrt{(p^2-q^2)^2+4p^2}\right) \qquad k^2 = \frac{b^2(b^2-p^2)}{b^4+p^2}$$

- The solution is

$$\cosh^2\rho = \frac{1+b^2}{b^2\,\mathrm{cn}^2(\sigma)}$$

$$\varphi = \frac{\pi}{2} + \frac{p^2}{b\sqrt{b^4+p^2}}\left(\sigma - \Pi\big(\tfrac{b^4}{b^4+p^2}, \mathrm{am}(\sigma+\mathbb{K})|k^2\big) + \Pi\big(\tfrac{b^4}{b^4+p^2}|k^2\big)\right),$$

where $\mathrm{am}(x)$ is the Jacobi amplitude and $\mathbb{K}$ the complete elliptic integral.

- The initial value is then

$$\frac{\phi}{2} = \frac{\pi}{2} - \frac{p^2}{b\sqrt{b^4+p^2}}\left(\mathbb{K} - \Pi\big(\tfrac{b^4}{b^4+p^2}|k^2\big)\right) \qquad \text{and} \qquad -\mathbb{K} < \sigma < \mathbb{K}$$

- These are transcendental equations for p, q in terms of θ, ϕ

- The induced metric is

$$ds^2_{\rm ind} = \sqrt{\lambda}\,\frac{1-k^2}{{\rm cn}^2(\sigma)}\left[-d\tau^2 + d\sigma^2\right].$$

- The classical action can also be calculated

$$\mathcal{S}_{\rm cl} = \frac{\sqrt{\lambda}}{2\pi}\int dt\,d\varphi\,p\cosh^2\rho\sinh^2\rho = \frac{T\sqrt{\lambda}}{\pi}\,\frac{\sqrt{b^4+p^2}}{b\,p}\left[\frac{(b^2+1)p^2}{b^4+p^2}\mathbb{K} - \mathbb{E}\right]$$

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- This determines $V^{(0)}_{AdS}$ as a function of p, q and implicitly in term of ϕ, θ.
- We can also expand around $\phi = \theta = 0$

$$\begin{aligned} V^{(0)}_{AdS}(\phi,\theta) = {}& \frac{1}{\pi}(\theta^2-\phi^2) - \frac{1}{8\pi^3}(\theta^2-\phi^2)\left(\theta^2-5\phi^2\right) \\ &+ \frac{1}{64\pi^5}(\theta^2-\phi^2)\left(\theta^4-14\theta^2\phi^2+37\phi^4\right) \\ &- \frac{1}{2048\pi^7}(\theta^2-\phi^2)\left(\theta^6-27\theta^4\phi^2+291\theta^2\phi^4-585\phi^6\right) + O((\phi,\theta)^{10}) \end{aligned}$$

1–loop determinant

- At one–loop we should consider the 8 transverse bosonic and 8 fermionic fluctuation modes.
- Such a calculation was done long ago for a confining string by Lüscher.
- The "Lüscher term" is proportional to the number of transverse dimensions and always has a Coulomb behavior.
- We have to repeat the calculation in the $AdS_5 \times \mathbb{S}^5$ sigma model.

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- We have to repeat the calculation in the $AdS_5 \times \mathbb{S}^5$ sigma model.
- We need the full metric

$$ds^2 = \Big(\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \left(dx_1^2 + \cos^2 x_1 (dx_2^2 + \cos^2 x_2 \, d\varphi^2) \right)$$

$$+ dx_3^2 + \cos^2 x_3 \Big(dx_4^2 + \cos^2 x_4 \Big(dx_5^2 + \cos^2 x_5 (dx_6^2 + \cos^2 x_6 \, d\vartheta^2) \Big) \Big) \Big).$$

- We define the fluctuation modes

$$\rho = \rho(\sigma) + \delta\rho \,, \qquad \varphi = \varphi(\sigma) + \delta\varphi \,, \qquad \vartheta = \vartheta(\sigma) + \delta\vartheta \,, \qquad x_i \,, \quad i = 1, \cdots, 6$$

- After fixing the static gauge it results in the bosonic Lagrangean

$$\mathcal{L}_B = \frac{1}{2}\sqrt{g}\Big[g^{ab} \, \partial_a \zeta_P \, \partial_b \zeta_P + M_{PQ} \zeta_P \zeta_Q \Big], \qquad P, Q = 1, \cdots, 8$$

with a complicated mass–matrix M_{PQ}.

- Generically the mass matrix is nondiagonal.
- If we set either $\theta = 0$ or $\phi = 0$, it is diagonal.
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The case of $\theta = 0$

- The resulting determinant is

$$Z = \frac{\det^4(i\gamma^i\,\hat\nabla_i - \gamma_3)}{\det(-\nabla^2 + 2)\,\det^{1/2}(-\nabla^2 + R^{(2)} + 4)\,\det^{5/2}(-\nabla^2)}$$

- All derivatives are with the world–sheet metric.
- This is formally the same for all ϕ, except for the different world–sheet metrics.
- The bosonic fluctuation operators are (after Fourier transform $\partial_\tau \to i\omega$)

$$\mathcal{O}_0 \equiv \sqrt{g}\left(-\nabla^2\right) = -\partial_\sigma^2 + \omega^2$$

$$\mathcal{O}_1 \equiv \sqrt{g}\left(-\nabla^2 + 2\right) = -\partial_\sigma^2 + \omega^2 + \frac{2(1-k^2)}{\mathrm{cn}^2(\sigma)}$$

$$\mathcal{O}_2 \equiv \sqrt{g}\left(-\nabla^2 + R^{(2)} + 4\right) = -\partial_\sigma^2 + \omega^2 + \frac{2(1-k^2)}{\mathrm{cn}^2(\sigma)} - 2k^2\,\mathrm{cn}^2(\sigma)$$

- All the differential operators can be written as Lamé operators

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- All the differential operators can be written as Lamé operators

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- Explicitly

$$\mathcal{O}_1 = (1-k^2)\Big[-\partial_{\sigma_1}^2 + \omega_1^2 + 2k_1^2\,\mathrm{sn}^2(\sigma_1 + i\mathbb{K}_1'|k_1^2)\Big]$$
$$\mathcal{O}_2 = (1-k^2)(1+k_1)^2\Big[-\partial_{\sigma_2}^2 + \omega_2^2 + 2k_2^2\,\mathrm{sn}^2(\sigma_2 + i\mathbb{K}_2'|k_2^2)\Big]$$

where

$$k_1^2 = \frac{k^2}{k^2-1} \qquad \sigma_1 = \sqrt{1-k^2}\,\sigma + \mathbb{K}_1 \qquad \omega_1^2 = \frac{\omega^2}{1-k^2}$$
$$k_2^2 = \frac{4k_1}{(1+k_1)^2} \qquad \sigma_2 = (1+k_1)(\sqrt{1-k^2}\,\sigma + \mathbb{K}_1) \qquad \omega_2^2 = \frac{\omega^2}{(1-k^2)(1+k_1)^2} - k_2^2$$

- A similar expression exists for the fermions.

1d determinants through the Gelfand-Yaglom method

- The general solution to the Lamé eigenvalue problem

$$\left[-\partial_x^2 + 2k^2\,\mathrm{sn}^2(x|k^2)\right] f(x) = \Lambda\, f(x)$$

is explicitly known

$$y_\pm(x) = \frac{H(x \pm \alpha)}{\Theta(x)}\, e^{\mp\, x\, Z(\alpha)} \qquad \mathrm{sn}(\alpha|k^2) = \frac{1}{k}\sqrt{1 + k^2 - \Lambda}$$

- We can write down the solution satisfying

$$u(-\mathbb{K}) = 0\,, \qquad u'(-\mathbb{K}) = 1$$

- Then

$$\det \mathcal{O} = u(\mathbb{K})$$

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- Actually need to worry about divergences from the boundary, so introduce a cutoff at $\sigma = \pm(\mathbb{K}-\epsilon)$
- The regularized u is

$$u(\sigma) = \frac{y_+(-\mathbb{K}+\epsilon)\,y_-(\sigma) - y_-(-\mathbb{K}+\epsilon)\,y_+(\sigma)}{y_+(-\mathbb{K}+\epsilon)\,y_-'(-\mathbb{K}+\epsilon) - y_+'(-\mathbb{K}+\epsilon)\,y_-(-\mathbb{K}+\epsilon)}$$

- This gives the explicit answers like

$$\det \mathcal{O}_1 = \frac{(k^2-1)\,\mathrm{ns}^2(\epsilon_1, k_1^2) - 2k^2 + \omega^2 + 1}{\sqrt{k^2-\omega^2}\sqrt{3k^2(\omega^2+1) - 2k^4 - (\omega^2+1)^2}}\,\sinh\big(2Z(\alpha_1)(\mathbb{K}_1 - \epsilon_1) + \Sigma_1\big)$$

with

$$\Sigma_1 = \ln \frac{\vartheta_4\big(\frac{\pi(\alpha_1+\epsilon)}{2\mathbb{K}_1}, q_1\big)}{\vartheta_4\big(\frac{\pi(\alpha_1-\epsilon)}{2\mathbb{K}_1}, q_1\big)} \qquad \epsilon_1 = \sqrt{1-k^2}\,\epsilon$$

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- The determinant depends only on the leading term of the expansion in ϵ

$$\det \mathcal{O}_0^\epsilon \cong \frac{\sinh(2\mathbb{K}\,\omega)}{\omega}$$

$$\det \mathcal{O}_1^\epsilon \cong -\frac{\sinh(2\mathbb{K}_1\,Z(\alpha_1))}{\epsilon^2\sqrt{(\omega^2-k^2)(\omega^2-k^2+1)(\omega-2k^2+1)}}$$

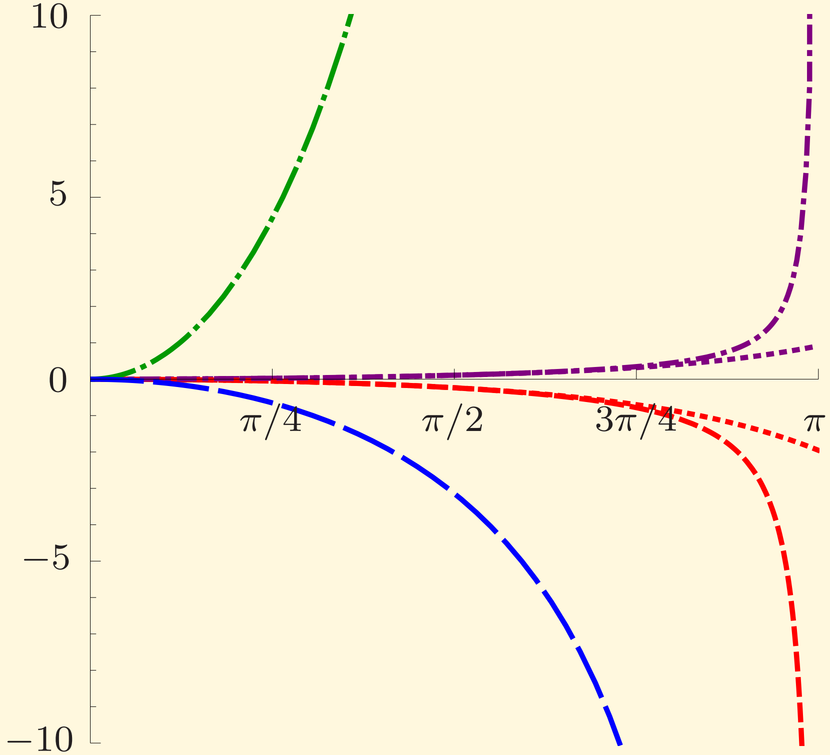
$$\det \mathcal{O}_2^\epsilon \cong \frac{\sinh(2\mathbb{K}_2\,Z(\alpha_2))}{\epsilon^2(1-k^2)^{3/2}(k_1+1)^3\sqrt{(\omega_2^2+k_2^2)(\omega_2^2+1)(\omega_2^2+k_2^2+1)}}$$

$$\det \mathcal{O}_F^\epsilon \cong \frac{8\mathbb{K}_2\sqrt{\omega_3^2+k_2^2}\,\sinh(\mathbb{K}_2\,Z(\alpha_F))}{\epsilon\pi(1-k^2)(k_1+1)^2\sqrt{(\omega_3^2+1)(\omega_3^2+k_2^2+1)}}\,\frac{\vartheta_2(0,q_2)\,\vartheta_4\big(\frac{\pi\alpha_F}{2\mathbb{K}_2},q_2\big)}{\vartheta_1'(0,q_2)\,\vartheta_3\big(\frac{\pi\alpha_F}{2\mathbb{K}_2},q_2\big)}$$

- After removing a divergence we find ($\mathcal{T}$ is a cutoff on τ)

$$\Gamma_{\mathrm{reg}} = -\frac{\mathcal{T}}{2}\lim_{\epsilon\to0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln\frac{\epsilon^2\omega^2\det^8\mathcal{O}_F^\epsilon}{\det^5\mathcal{O}_0^\epsilon\,\det^2\mathcal{O}_1^\epsilon\,\det\mathcal{O}_2^\epsilon}$$

- This can be integrated numerically to high precision



Blue: $V^{(1)}(\phi, 0)$

Green: $V^{(2)}(\phi, 0)$

Red: $V^{(0)}_{AdS}(\phi, 0)$

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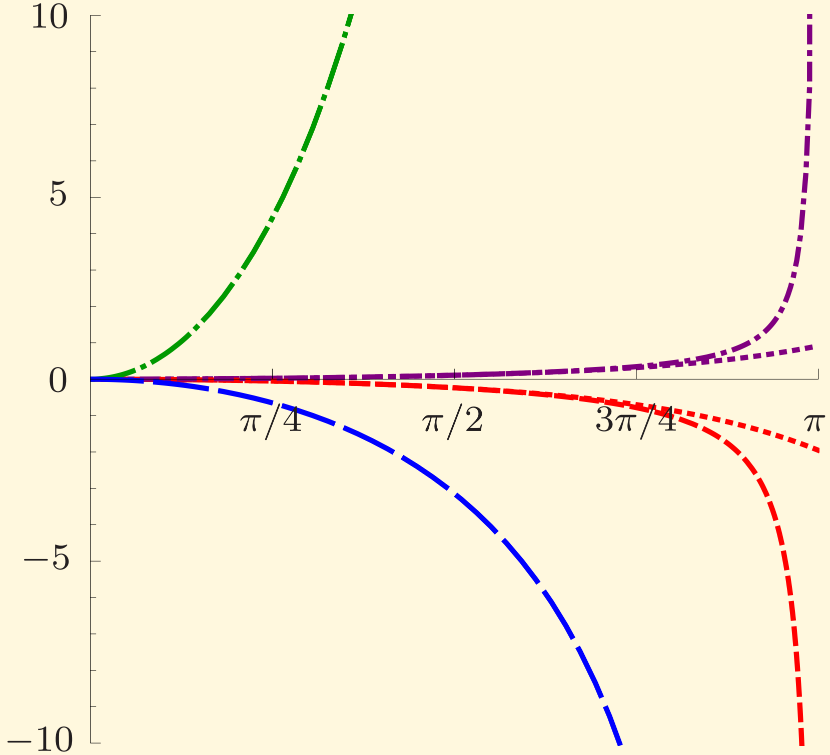

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- The 1d determinants can also be expanded about $\phi=0$ and evaluated analytically

$$V^{(1)}_{AdS}(\phi,0)=\frac{3}{2}\frac{\phi^2}{4\pi^2}+\left(\frac{53}{8}-3\,\zeta(3)\right)\frac{\phi^4}{16\pi^4}+\left(\frac{223}{8}-\frac{15}{2}\zeta(3)-\frac{15}{2}\zeta(5)\right)\frac{\phi^6}{64\pi^6}$$
$$+\left(\frac{14645}{128}-\frac{229}{8}\zeta(3)-\frac{55}{4}\zeta(5)-\frac{315}{16}\zeta(7)\right)\frac{\phi^8}{256\pi^8}+O(\phi^{10})$$

The case of $\phi = 0$

- Everything can be done in that case too.

The case of $\phi = 0$

- Everything can be done in that case too.
- At the end the small θ expansion gives

$$V^{(1)}_{AdS}(0,\theta) = -\frac{3}{2}\frac{\theta^2}{4\pi^2} + \left(\frac{5}{8} - 3\,\zeta(3)\right)\frac{\theta^4}{16\pi^4} + \left(\frac{1}{8} + \frac{3}{2}\zeta(3) - \frac{15}{2}\zeta(5)\right)\frac{\theta^6}{64\pi^6}$$
$$+ \left(-\frac{11}{128} - \frac{5}{8}\zeta(3) + \frac{25}{4}\zeta(5) - \frac{315}{16}\zeta(7)\right)\frac{\theta^8}{256\pi^8} + O(\theta^{10})$$

Our main result:

Explicit expressions for these families of Wilson loops at weak and strong coupling.

$\phi \to \pi$ limit

- $V^{(1)}$, $V^{(2)}$, $V_{AdS}^{(0)}$ and $V_{AdS}^{(1)}$ all have poles at $\phi = \pi$
- In perturbation theory

$$V(\phi,\theta) \to -\frac{\lambda}{8\pi}\frac{1+\cos\theta}{\pi-\phi} + \frac{\lambda^2}{32\pi^3}\frac{(1+\cos\theta)^2}{\pi-\phi}\log\frac{e}{2(\pi-\phi)} + O(\lambda^3)$$

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- In the case of $\theta = 0$ we get essentially the same as the antiparallel lines with $L \to \pi - \phi$

$$V(L,\lambda) = \begin{cases} -\dfrac{\lambda}{4\pi L} + \dfrac{\lambda^2}{8\pi^2 L}\ln\dfrac{T}{L} + \cdots & \lambda \ll 1 \\[2ex] \dfrac{4\pi^2\sqrt{\lambda}}{\Gamma(\frac{1}{4})^4\, L}\left(1 - \dfrac{1.3359\ldots}{\sqrt{\lambda}} + \cdots\right) & \lambda \gg 1 \end{cases}$$

- The strong coupling calculations also agree in the limit.

Small θ and ϕ expansions

- Consider the expansion of $V(\phi,\theta,\lambda)$ at small ϕ or θ

$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0}=-\frac{1}{2}\frac{\partial^2}{\partial\phi^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0}=\begin{cases}\dfrac{\lambda}{16\pi^2}-\dfrac{\lambda^2}{384\pi^2}+\cdots & \lambda\ll 1\\[2ex] \dfrac{\sqrt{\lambda}}{4\pi^2}-\dfrac{3}{8\pi^2}+\cdots & \lambda\gg 1\end{cases}$$

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  - What does this calculate?
  - How do we calculate this?
  - Can we find an exact interpolating function?

- In terms of the Wilson loop

$$\frac{\partial^2}{\partial\theta^2}V(0,0) = -\frac{1}{\ln\frac{R}{\epsilon}}\frac{\partial^2}{\partial\theta^2}\log\langle W\rangle \approx -\frac{1}{\ln\frac{R}{\epsilon}}\frac{\partial^2}{\partial\theta^2}\langle W\rangle.$$

- Write the Wilson loop as

$$W = \mathrm{Tr}\,\mathcal{P}\left[\exp\left(\int_{-\infty}^{0}(iA_1+\Phi_1)ds\right)\exp\left(\int_{0}^{\infty}(iA_1+\Phi_1\cos\theta+\Phi_2\sin\theta)ds\right)\right]$$

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$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V = &-\frac{1}{\ln(L/\epsilon)}\frac{1}{2N}\int_0^\infty ds_1\int_0^\infty ds_2\left\langle\operatorname{Tr}\mathcal{P}\left[\Phi_2(s_1)\Phi_2(s_2)\,e^{\int_{-\infty}^{\infty}(iA_1+\Phi_1)ds}\right]\right\rangle\\ &+\frac{1}{\ln(L/\epsilon)}\frac{1}{2N}\int_0^\infty ds_1\left\langle\operatorname{Tr}\mathcal{P}\left[\Phi_1(s_1)\,e^{\int_{-\infty}^{\infty}(iA_1+\Phi_1)ds}\right]\right\rangle.\end{aligned}$$

- These are insertions of adjoint valued local operators into the loop.

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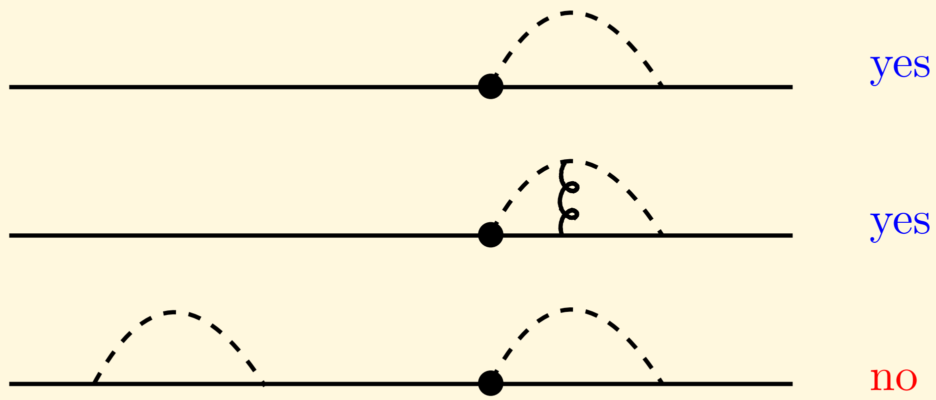
- These are insertions of adjoint valued local operators into the loop.
- The double insertion is related to a BPS quantity. It gives no log divergence and is not renormalized.

- It is easy to see that some graphs will contribute and some not to this correlator



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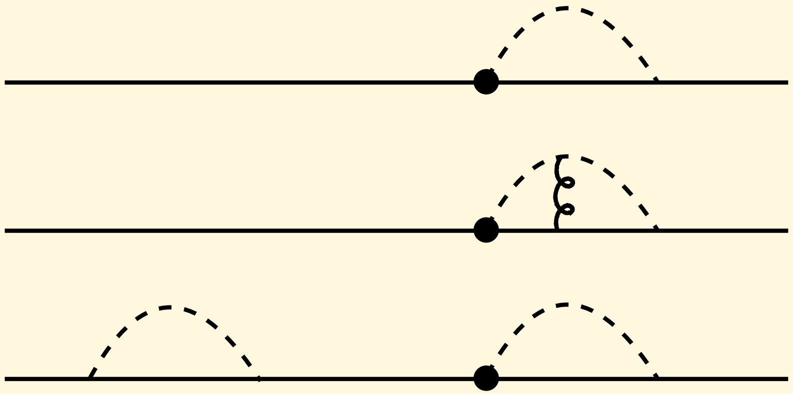

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- Indeed the 2–loop ladder graphs

$$V^{(2)}_{\text{ladder}} = -\frac{1}{64\pi^4}\frac{(\cos\theta - \cos\phi)^2}{\sin^2\phi}\left[\mathrm{Li}_3\left(e^{2i\phi}\right) - \zeta(3) - i\phi\left(\mathrm{Li}_2\left(e^{2i\phi}\right) + \frac{\pi^2}{6}\right) + \frac{i}{3}\phi^3\right]$$

  contributes only from $O((\theta,\phi)^4)$.

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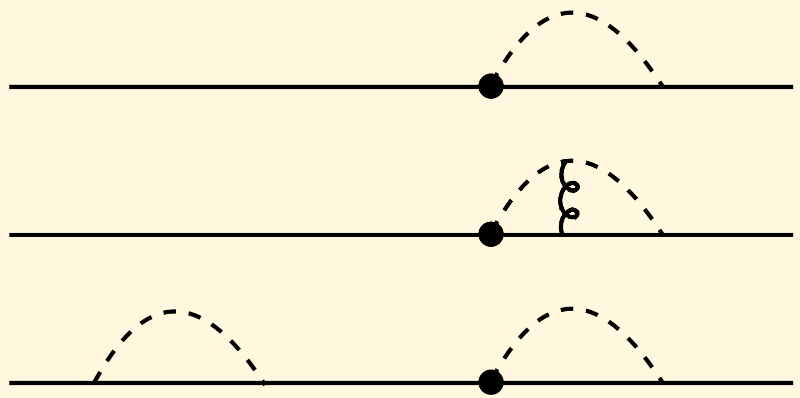

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- What is the sum of all these graphs?

Summary

- A two–parameter family of Wilson loop going between the circle and the antiparallel lines.
- The antiparallel lines is the residue at $\phi \to \pi$.
- They are no more complicated than the antiparallel lines.
  - Explicit expression to order λ^2.
  - Classical sting solution given by elliptic integrals.
  - Differential operators for two one–parameter families, are of Lamé type.
  - One loop determinant known in these examples.
- New expansion parameters: ϕ and θ.
- Natural separation of perturbative calculation into graphs with more and less connected components.
- The two–loop connected graphs give a simple result.

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- New expansion parameters: ϕ and θ.
- Natural separation of perturbative calculation into graphs with more and less connected components.
- The two–loop connected graphs give a simple result.
- Would be good to get the result at $O(\lambda^3)$.
- Can we guess an interpolating function for $\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0}$

Will there be a gauge theory derivation of the strong coupling potential:

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