Threshold resummation far from threshold

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Plan of the talk:

1. **When is threshold resummation relevant?**
2. **Ambiguities in resummed results**

Results obtained in collaboration with Marco Bonvini Stefano Forte.

Generic observable in hadron collisions:

$$\sigma(\tau, Q^2) = \int_\tau^1 \frac{dz}{z}\, \mathcal{L}\left(\frac{\tau}{z}\right)\, C(z, \alpha_{\mathrm{S}}(Q^2)); \qquad \mathcal{L}(z) = \int_z^1 \frac{dx_1}{x_1} f_1(x_1) f_2\left(\frac{z}{x_1}\right)$$

(factorization of collinear singularities).

Example: Higgs production at the LHC. In this case

$$Q^2 = m_H^2, \qquad \tau = \frac{m_H^2}{s}, \qquad f_1(z) = f_2(z) = g(z)$$

QCD provides a perturbative expansion for $C(z, \alpha_{\mathrm{S}})$:

$$C(z, \alpha_{\mathrm{S}}) = \sum_{n=0}^{\infty} C_n(z) \alpha_{\mathrm{S}}^n$$

When s is close to Q^2 (threshold production), $\tau \to 1$ and therefore z is close to 1. Since

$$C_n(z) \sim \left[\frac{\log^{2n-1}(1-z)}{1-z}\right]_+$$

the perturbative expansion is unreliable in this region:

$$\alpha_s^n \int_\tau^1 \frac{dz}{z}\, \mathcal{L}\left(\frac{\tau}{z}\right)\, C_n(z) \sim \mathcal{L}(\tau)\alpha_s^n \log^{2n}(1-\tau)$$

All-order resummation techniques are available (more on this in the second part of the talk).

However, $\tau \ll 1$ in most cases of present interest. For example

$$\tau = \frac{m_H^2}{s} \simeq 8 \times 10^{-4}$$

for a 200 GeV Higgs boson at the LHC 7 TeV.

Is Sudakov resummation any useful in such cases?

No need of resummation in the usual sense: the expansion parameter

$$\alpha_{\mathrm{S}} \log^2(1-\tau)$$

is small as long as α_{S} is small.

Recall the general expression

$$\sigma(\tau, Q^2) = \int_\tau^1 \frac{dz}{z} \mathcal{L}\left(\frac{\tau}{z}\right) C(z, \alpha_{\rm S}(Q^2))$$

The partonic cross-section is computed as a function of the partonic center-of-mass energy

$$\hat{s} = \frac{Q^2}{z}; \qquad \tau \le z \le 1$$

Resummation relevant when $\hat{s}$ is not much larger than Q^2, or $z \sim 1$.

Whether or not resummation is relevant depends on which region gives the dominant contribution to the convolution integrals.

Go to Mellin moments:

$$\sigma(N, Q^2) = \int_0^1 d\tau\, \tau^{N-1}\, \sigma(\tau, Q^2)$$

with inverse

$$\sigma(\tau, Q^2) = \frac{1}{2\pi i} \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, \tau^{-N}\, \sigma(N, Q^2) = \frac{1}{2\pi i} \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, e^{E(\tau,N;Q^2)}$$

$$E(\tau, N; Q^2) \equiv N \log \frac{1}{\tau} + \log \sigma(N, Q^2).$$

Typically, $\sigma(N, Q^2)$ is a decreasing function of N on the real axis, with a singularity on the real positive axis because of the parton luminosity.

Hence $E(\tau, N; Q^2)$ always has a minimum on the real positive N axis at some $N = N_0(\tau)$, and the inversion integral is dominated by the region of N around $N_0(\tau)$ (saddle-point approximation).

Explicitly, N_0 **is defined by**

$$E'(\tau, N_0; Q^2) = \log\frac{1}{\tau} + \frac{\sigma'(N_0, Q^2)}{\sigma(N_0, Q^2)} = 0$$

and

$$\sigma(\tau, Q^2) \approx \frac{1}{\sqrt{2\pi}} \frac{e^{E(\tau, N_0; Q^2)}}{\sqrt{E''(\tau, N_0; Q^2)}}$$

after expanding

$$E(\tau, N; Q^2) = E(\tau, N_0; Q^2) + \frac{1}{2}E''(\tau, N_0; Q^2)(N - N^0)^2 + O((N - N_0)^3)$$

and a gaussian integration.

We expect $N_0(\tau)$ to be an increasing function of τ, because the slope of $N \log \frac{1}{\tau}$ decreases as $\tau \to 1$.

A simple example:

$$\sigma(N) = \frac{1}{N^k}$$

$$E(\tau, N) = N \log \frac{1}{\tau} - k \log N$$

$$\frac{dE(\tau, N)}{dN} = \log \frac{1}{\tau} - \frac{k}{N}$$

$$N_0(\tau) = \frac{k}{\log \frac{1}{\tau}}$$

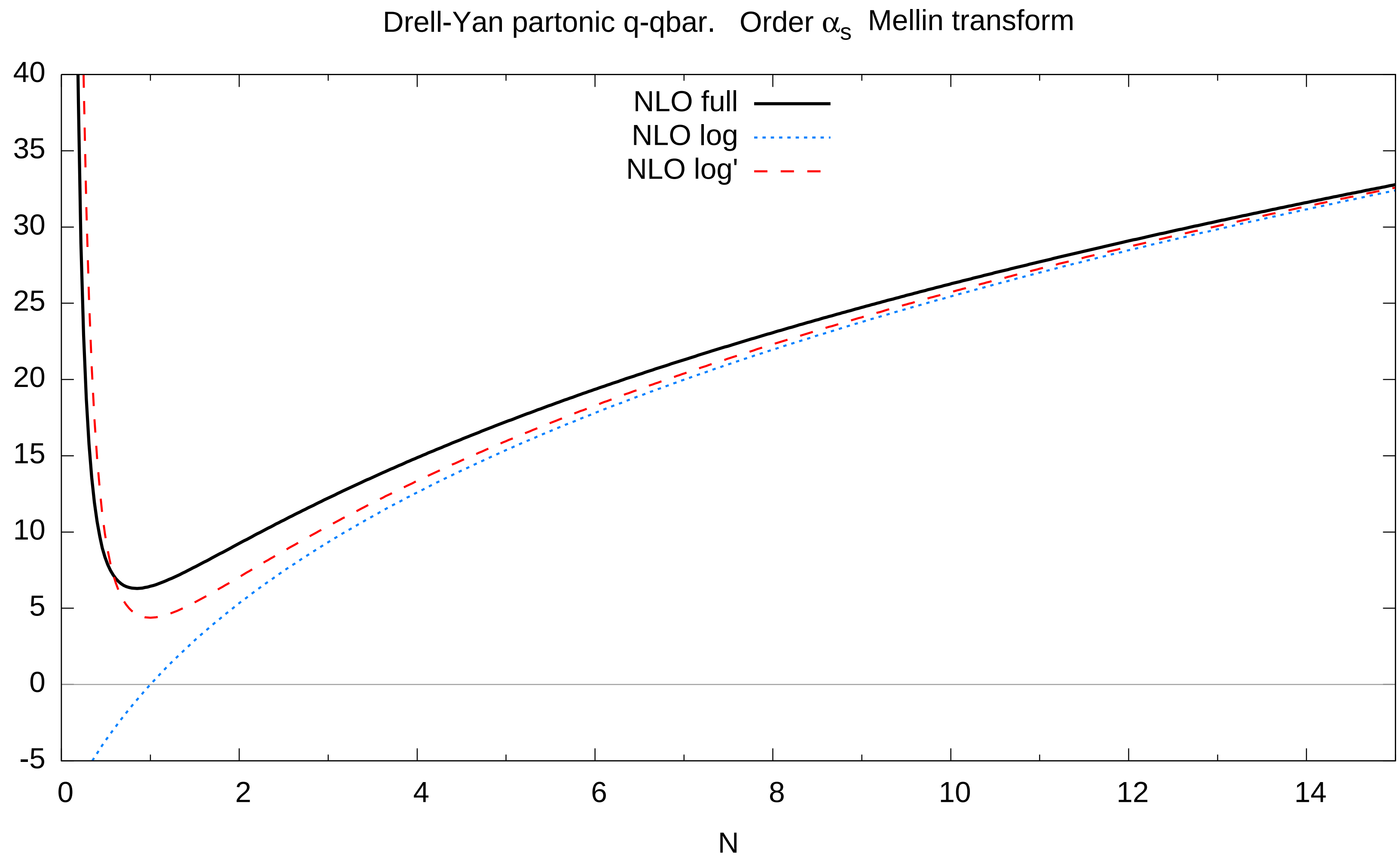
This shows that the Mellin transform maps the large-τ region onto the large-N region.

The value of N_0 depends strongly on the rate of decrease of

$$\sigma(N,Q^2) = \mathcal{L}(N,Q^2)\, C(N,\alpha_{\rm S}(Q^2))$$

with N, which in turn is only due to the parton luminosity $\mathcal{L}(N,Q^2)$: the partonic cross section is a distribution, its Mellin transform grows with N:

$$\int_0^1 dx\, x^{N-1} \left[\frac{\log^k(1-x)}{1-x}\right]_+ = \frac{1}{k+1}\,\log^{k+1}\frac{1}{N} + O(\log^k N)$$



[M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

An estimate of the position of the saddle point: to leading log

$$\mathcal{L}(N,Q^2) = \exp\left[\frac{\gamma(N)}{\beta_0}\log\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right]\mathcal{L}(N,Q_0^2)$$

Thus

$$\begin{aligned} E(\tau,N;Q^2) \quad = \quad & N\log\frac{1}{\tau} \\ & +\frac{\gamma(N)}{\beta_0}\log\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \\ & +\log\mathcal{L}(N,Q_0^2)+\log C(N,\alpha_s(Q^2)) \end{aligned}$$

The first term dominates at large N.

Second term: we have

$$\gamma(N) = \gamma_i(N) + \gamma_j(N)$$

for partons i, j in the initial state. Expanding the anomalous dimension about its rightmost singularity at leading order we have

$$\gamma_+(N) = \frac{N_c}{\pi}\frac{1}{N-1}\left[1 + O(N-1)\right]; \qquad \gamma_{\rm ns}(N) = \frac{C_F}{2\pi}\frac{1}{N}\left[1 + O(N)\right]$$

This pattern persists to all perturbative orders: singlet quark and gluon distributions have a steeper small–N and thus small–z behaviour.

We expect the small-N approximation to break down around $N \approx 2$ for γ_+, and $N = 1$ for $\gamma_{\rm ns}$, because $\gamma_+(2) = \gamma_{\rm ns}(1) = 0$.

Third line: assuming a power behaviour for the parton densities at Q_0^2,

$$f_i(z, Q_0^2) = z^{\alpha_i}(1 - z)^{\beta_i}$$

we find

$$\log \mathcal{L}(N, Q_0^2) \sim \log N$$

both at large and small N, and hence subdominant with respect to the anomalous dimension term and to the τ dependent term.

A similar argument holds for the partonic cross-section term $\log \hat{\sigma}(N)$.

These approximations are expected to be more accurate at moderate values of τ.

Three cases:

1. $\gamma_i = \gamma_j = \gamma_+$ **(e.g. Higgs production in gluon fusion)**

2. $\gamma_i = \gamma_+, \gamma_j = \gamma_{\rm ns}$ **(e.g. Drell-Yan production at the LHC)**

3. $\gamma_i = \gamma_j = \gamma_{\rm ns}$ **(e.g. Drell-Yan production at the Tevatron)**

We find

$$N_{ij}^0 = 1 - k_i k_j + \sqrt{\frac{\gamma_{ij}^{(0)}}{\beta_0 \log\frac{1}{\tau}} \log \frac{\alpha_{\rm s}(Q_0^2)}{\alpha_{\rm s}(Q^2)}}$$

where

$$k_+ = 0; \quad k_{\rm ns} = 1.$$

and

$$\gamma_{\rm ns\,ns}^{(0)} = \frac{C_F^2}{4\pi^2}; \qquad \gamma_{++}^{(0)} = \frac{N_c^2}{\pi^2}; \qquad \gamma_{+\,\rm ns}^{(0)} = \frac{N_c}{\pi}\frac{C_F}{2\pi}$$

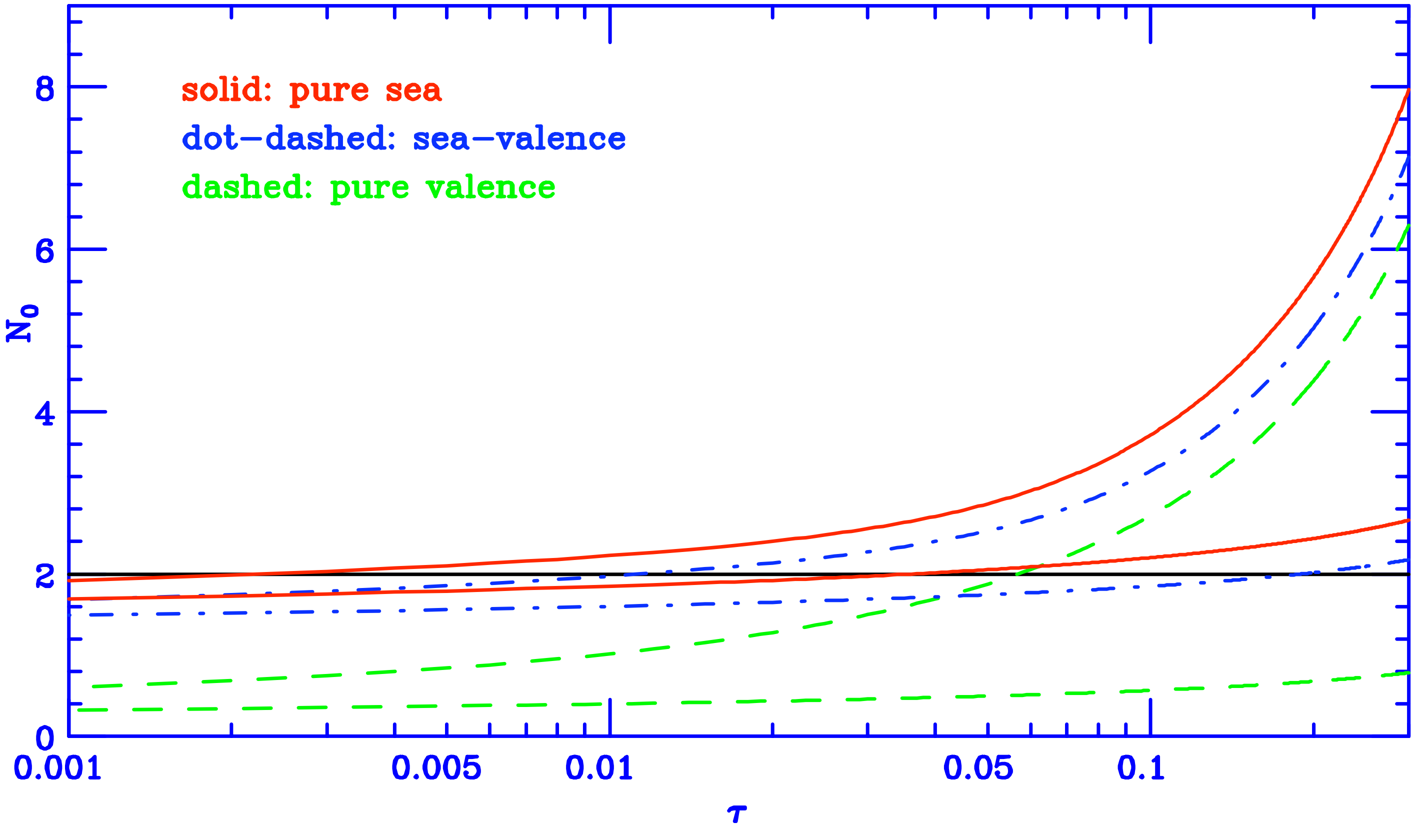


Figure 1: Position of N_0 as a function of τ ($\hat{\sigma}$ neglected, LO anomalous dimensions, $\alpha_{\rm ns} = 1/2, \beta_{\rm ns} = 3; \alpha_+ = 0, \beta_+ = 4$, $Q_0 = 1$ GeV, $Q = 100$ GeV.) Upper curves: exact LO an. dim.; lower curves: approximated LO an. dim.

[M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

Comments:

- **In cases 1. and 2., $N_0 \gtrsim 2$ down to fairly low values of $\tau \sim 0.01$, due to the rise of the anomalous dimension related to the pole at $N = 1$ in the singlet sector.**

- **At larger τ, say above 0.1, the rapid drop of PDFs raises the position of the saddle.**

A realistic calculation: Drell-Yan production at NLO

Consider the $q\bar{q}$ channel for Drell-Yan production. The coefficient function admits the perturbative expansion

$$C(z,\alpha_{\rm S}) = \left[\delta(1-z) + \frac{\alpha_{\rm S}}{\pi} C_1(z) + \left(\frac{\alpha_{\rm S}}{\pi}\right)^2 C_2(z) + \dots\right];$$

with

$$\begin{aligned}
C_1(z) &= C_F \left\{ 4\left[\frac{\log(1-z)}{1-z}\right]_+ - \frac{4}{1-z}\log\sqrt{z} \right. \\
&\left. -2(1+z)\log\frac{1-z}{\sqrt{z}} + \left(\frac{\pi^2}{3} - 4\right)\delta(1-z) \right\} \\
C_1(N) &= C_F \left\{ \frac{2\pi^2}{3} - 4 + 2\gamma_E^2 + 2\psi_0^2(N) - \psi_1(N) + \psi_1(N+2) + 4\gamma_E\psi_0(N) \right. \\
&\left. + \frac{2}{N}\left[\gamma_E + \psi_0(N+1)\right] + \frac{2}{N+1}\left[\gamma_E + \psi_0(N+2)\right] \right\}
\end{aligned}$$

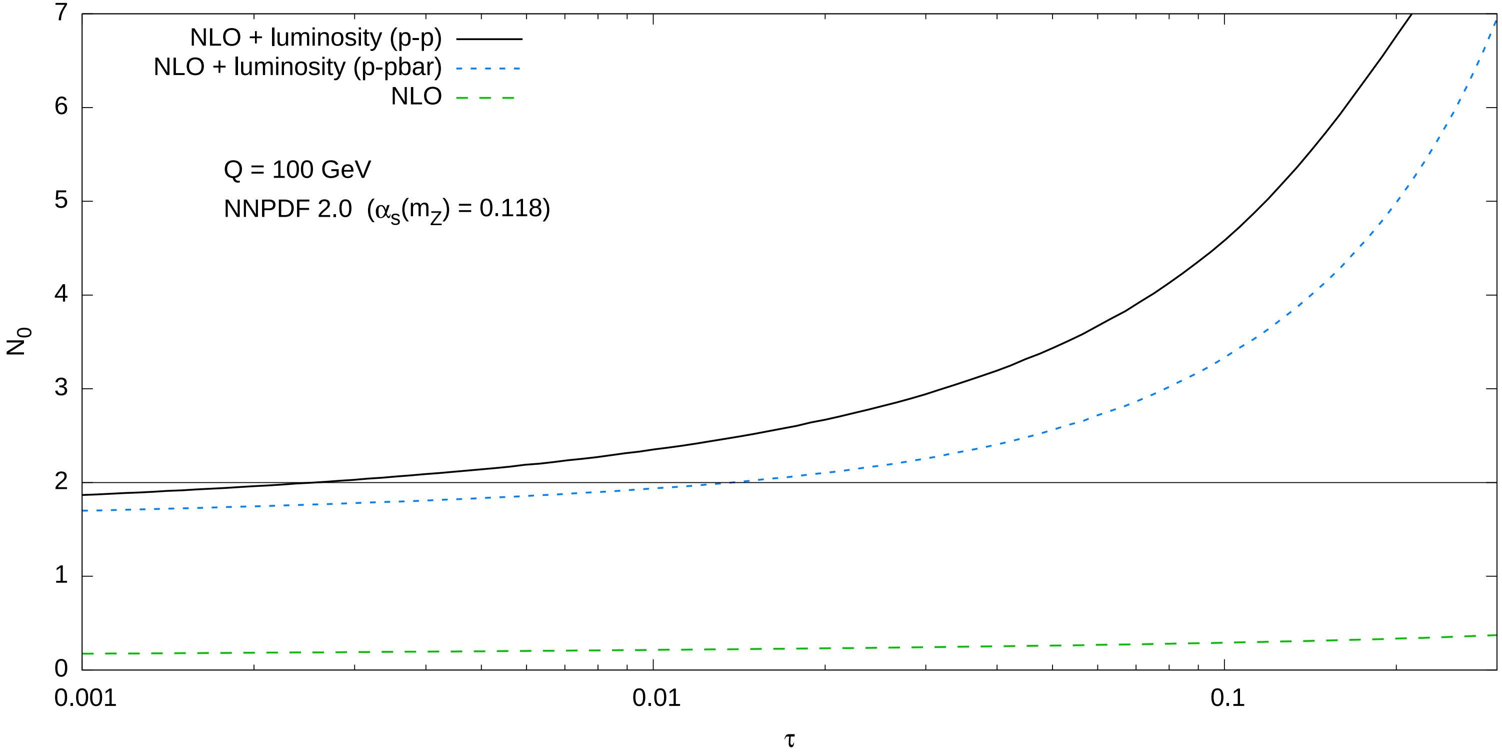


Figure 2: N_0 as a function of τ for NLO neutral Drell-Yan pairs.

[M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

Comments:

- **Our simple model works well in the case of** pp **collisions: always at least one sea (antiquark) PDF.**
- $p\bar{p}$**: OK for** $\tau \gtrsim 0.1$ **For smaller** τ**, the actual value of** N_0 **decreases much more slowly: when** $N \lesssim 2$ **the contribution** γ_+ **rapidly grows due to the pole so that even the valence distribution is dominated by it. Also in this case, the relevance of log terms extends to lower** τ **values.**
- **If the parton luminosity is omitted,** N_0 **is much smaller. Saddle determined by PDFs, which tend to extend the importance of resummation to a wider kinematic region.**

In summary:

- $N_0 \gtrsim 2$ **for** $\tau \gtrsim 0.003$ **in** pp **collisions, and** $\tau \gtrsim 0.02$ **in** $p\bar{p}$ **collisions.**
- **For** $\tau \lesssim 0.1$ **the position of the saddle is determined by the pole in the anomalous dimension**
- **For larger values of** τ **the large** x **drop of PDFs, due both to their initial shape and to perturbative evolution, very substantially enhances the impact of resummation.**

Very weak dependence on Q^2**.**

The resummation region for the Drell-Yan process

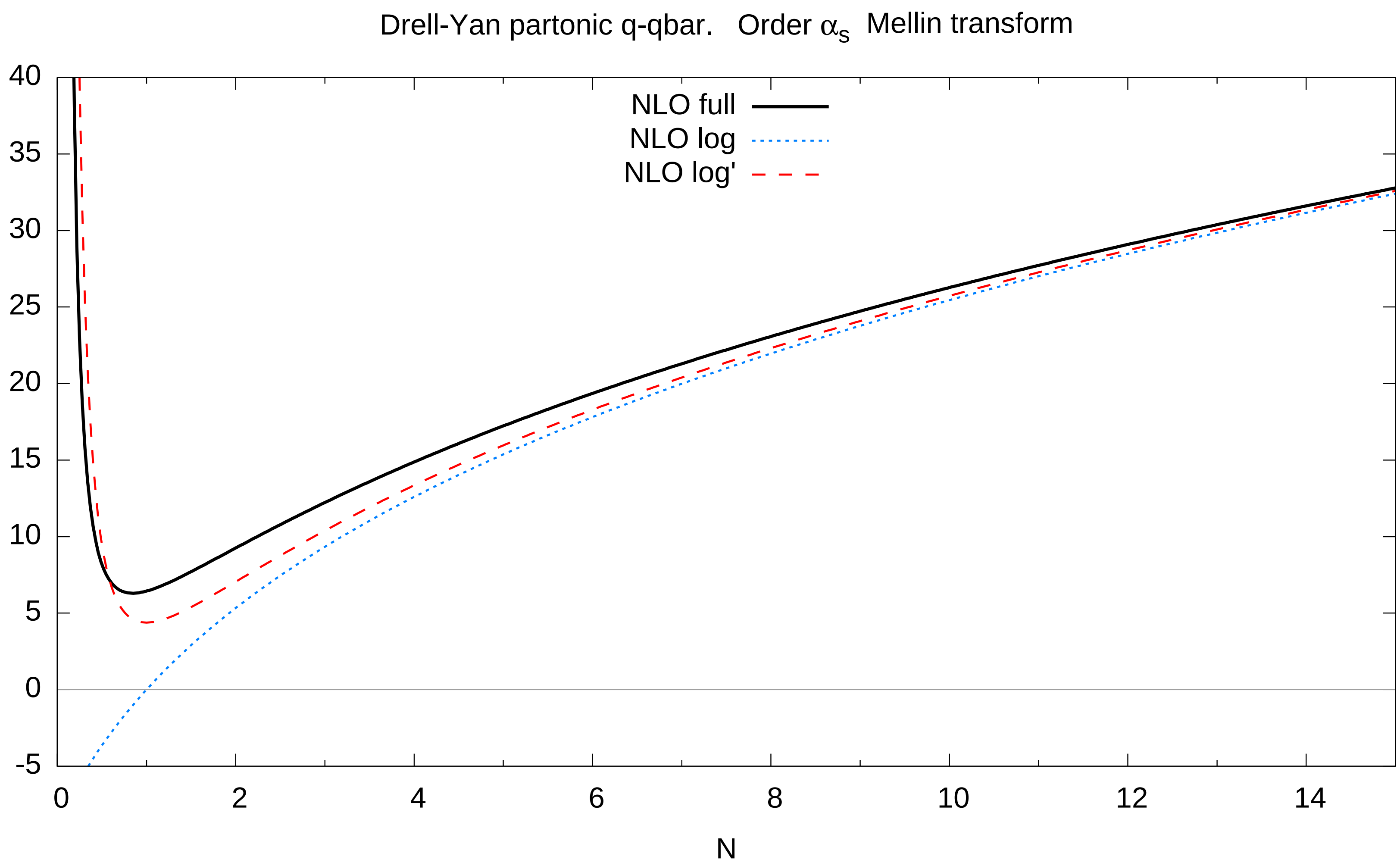
We now want to establish quantitatively the value of N at which logarithmically enhanced contributions give a sizable contribution to the cross-section.

Compare $C_1(N)$ to its logarithmic approximation

$$C_1^{\log}(z) = 4C_F \left[\frac{\log(1-z)}{1-z}\right]_+$$

whose Mellin transform is

$$C_1^{\log}(N) = C_F \left[2\psi_0^2(N) - 2\psi_1(N) + 4\gamma_E\psi_0(N) + \frac{\pi^2}{3} + 2\gamma_E^2\right]$$


Drell-Yan partonic q-qbar. Order α_s Mellin transform
NLO full
NLO log
NLO log'
40
35
30
25
20
15
10
5
0
-5
0
2
4
6
8
10
12
14
N

Good agreement at large N, up to a small constant shift:

$$\lim_{N\to\infty}\left[C_1(N) - C_1^{\log}(N)\right] = C_F\left(\frac{\pi^2}{3} - 4\right).$$

For $N \gtrsim 2$ the logarithmic contribution is already about 50% of the full result. This suggests that indeed the logarithmic contribution is sizable for $N \gtrsim 2$.

The definition of the log contrbution is quite arbitrary. For example, should we include constant terms?

In general, logarithmically enhanced contributions in N–space also contain subleading terms when transformed to z–space, and conversely.

Since $\psi_1 \underset{N\to\infty}{\sim} \frac{1}{N}$, an equally good choice would be

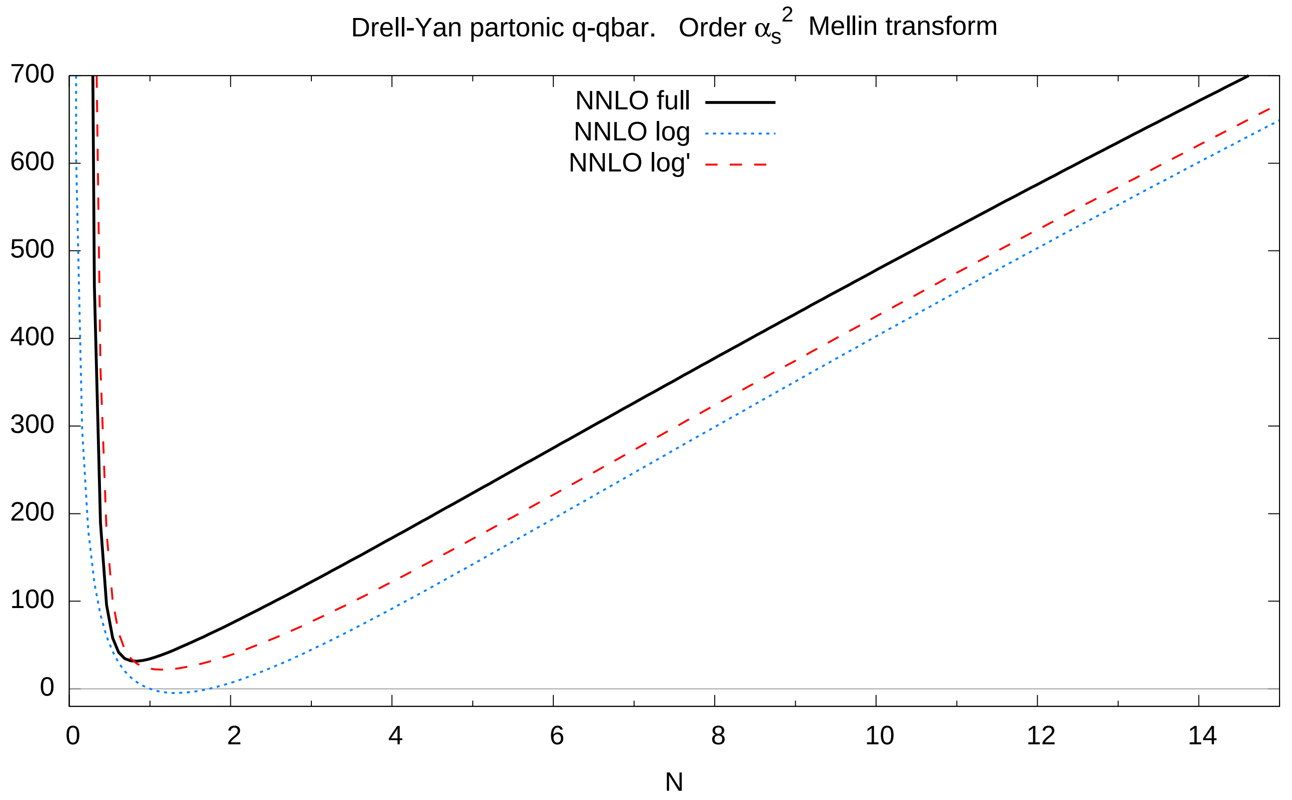
$$C_1^{\log'}(N) = C_F\left[2\psi_0^2(N) + 4\gamma_E\psi_0(N) + \frac{\pi^2}{3} + 2\gamma_E^2\right],$$

which is the Mellin transform of

$$C_1^{\log'}(z) = 4C_F\left\{\left[\frac{\log(1-z)}{1-z}\right]_+ - \frac{\log\sqrt{z}}{1-z}\right\}.$$

Essentially the same for $N \gtrsim 2$, closer to the full result at small N (more on this later).

Does this pattern persist at higher orders? At NNLO the situation is similar, but not quite the same:



Log terms are sizable for $N \gtrsim 2-3$, depending on the choice of subleading terms. [M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

Final comment: **logarithmic effects turn out to be important in a region where $\alpha_{\rm S} \log^2 N \ll 1$ as long as $\alpha_{\rm S} \ll 1$. Resummation has therefore a perturbative character.**

Ambiguities in resummed results

Resummation usually performed in the space of Mellin transformed quantities:

$$f(N) = \int_0^1 dx\, x^{N-1}\, f(x); \qquad f(x) = \frac{1}{2\pi i} \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, x^{-N}\, f(N)$$

- **well defined and analytic in the half-plane** $\mathrm{Re}\, N > A$ **if** $f(x)$ **is at most as singular as** x^{-A}
- **Convolution products are turned into ordinary products.**
- **The region** $x \to 1$ **is mapped in the region** $N \to \infty$**:**

$$\int_0^1 dx\, x^{N-1} \left[\frac{\log^k(1-x)}{1-x} \right]_+ = \frac{1}{k+1} \log^{k+1} \frac{1}{N} + O(\log^k N)$$

Why Mellin moments?

$$C(z, \alpha_S) = \delta(1 - z) + \sum_{n=1}^{\infty} \int_0^1 dz_1 \ldots dz_n \, \frac{dw_n(z_1, \ldots, z_n)}{dz_1 \ldots dz_n} \, \Theta_{PS}(z; z_1, \ldots, z_n)$$

The multi-gluon emission probability factorizes in the soft limit,

$$\frac{dw_n(z_1, \ldots, z_n)}{dz_1 \ldots dz_n} \simeq \frac{1}{n!} \prod_{i=1}^{n} \frac{dw(z_i)}{dz_i}$$

(easily seen in QED in the eikonal approximation) but the phase space factor

$$\Theta_{PS}(z; z_1, \ldots, z_n) = \delta(z - z_1 z_2 \cdots z_n)$$

does not ...

... unless one goes to Mellin moments:

$$C(N,\alpha_S) = \int_0^1 dz\, z^{N-1}\, \hat\sigma(z,\alpha_S)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 dz_1 \cdots dz_n \prod_{i=1}^{n} \frac{dw(z_i)}{dz_i} \int_0^1 dz\, z^{N-1} \delta(z - z_1 \cdots z_n)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_0^1 dz_1\, z_1^{N-1} \frac{dw(z_1)}{dz_1} \right] \ldots \left[\int_0^1 dz_n\, z_n^{N-1} \frac{dw(z_n)}{dz_n} \right]$$

Hence

$$C(N,\alpha_S) = \exp \int_0^1 dz\, z^{N-1} \frac{dw}{dz}$$

Multigluon emission exponentiates in the soft limit.

One can prove the generalized formula

$$\begin{aligned} C^{\rm res}(N,\alpha_{\rm S}(Q^2)) &= g_0(\alpha_{\rm S})\,\exp\mathcal{S}\,(L,\bar\alpha) \\ \mathcal{S}(L,\bar\alpha) &= \frac{1}{\bar\alpha}\,g_1(L)+g_2(L)+\bar\alpha\,g_3(L)+\bar\alpha^2\,g_4(L)+\ldots \\ \bar\alpha &= a\,\alpha_{\rm S}(Q^2)\,\beta_0; \qquad\qquad L=\bar\alpha\log\frac{1}{N} \end{aligned}$$

which defines an improved expansion (in powers of $\alpha_{\rm S}$ with $\alpha_{\rm S}\log N$ fixed) for $C^{\rm res}(N,\alpha_{\rm S})$: g_1 gives the leading-log (LL) approximation, g_1 and g_2 give the next-to-leading-log approximation (NLL), and so on.

A difficulty immediately arises. Define $\tilde{\Sigma}(L, \alpha_S)$ by

$$C^{\rm res}(N, \alpha_S(Q^2)) = 1 + \tilde{\Sigma}(L, \alpha_S(Q^2)) = 1 + \sum_{k=1}^{\infty} h_k(\alpha_S(Q^2)) L^k$$

$\tilde{\Sigma}$ arises as an expansion in powers of $\alpha_S(Q^2)$ of a function of $\alpha_S(Q^2/N^a)$. To NLL we have

$$\alpha_S\left(\frac{Q^2}{N^a}\right) = \frac{\alpha_S(Q^2)}{1+L}\left[1 - \alpha_S(Q^2)\frac{\beta_1}{\beta_0}\frac{\log(1+L)}{1+L}\right]; \qquad L = a\alpha_S(Q^2)\beta_0 \log\frac{1}{N}$$

which has a branch cut on the real positive N axis for $L \leq -1$, or

$$N \geq N_L \equiv e^{\frac{1}{a\beta_0\alpha_S(Q^2)}}.$$

because of the Landau singularity.

The inverse Mellin transform of $C^{\rm res}(N, \alpha_S(Q^2))$ does not exist.

One possible way out: take the term-by-term inverse Mellin transform of $\tilde{\Sigma}(L, \alpha_{\rm S})$:

$$\Sigma(z, \alpha_{\rm S}(Q^2)) = \sum_{k=1}^{\infty} h_k \bar{\alpha}^k \frac{1}{2\pi i} \oint_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N} \log^k \frac{1}{N}$$

but the series is divergent! Proof:

$$\frac{1}{2\pi i} \oint_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N} \log^k \frac{1}{N} = \frac{k!}{2\pi i} \left[\oint \frac{d\xi}{\xi^{k+1}} \frac{\log^{\xi-1} \frac{1}{z}}{\Gamma(\xi)} \right]_+$$

$$\Sigma(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \left[\frac{1}{\log \frac{1}{z}} \oint \frac{d\xi}{\xi} \frac{\log^{\xi} \frac{1}{z}}{\Gamma(\xi)} \sum_{k=1}^{\infty} k! h_k \left(\frac{\bar{\alpha}}{\xi} \right)^k \right]_+$$

A second possible way out: taking the inverse Mellin transform of each $\log^k N$ term at the relevant (leading, next-to-leading...) logarithmic level, the perturbative series converges. For example, to leading log accuracy one has

$$\frac{1}{2\pi i}\int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N}\, \log^k \frac{1}{N} = k\left[\frac{\log^{k-1}(1-z)}{1-z}\right]_+ + \mathrm{NLL}$$

The series now converges to

$$\Sigma_{\mathrm{LLx}}(z,\alpha_s(Q^2)) = \bar{\alpha}\left[\frac{1}{1-z}\tilde{\Sigma}'(\bar{\alpha}\log(1-z),\alpha_s(Q^2))\right]_+$$

but only for $z < z_L = 1 - e^{-\frac{1}{\bar{\alpha}}}$ again because of the Landau pole at $z = z_L$.

The minimal prescription

An idea of S. Catani, M. Mangano, P. Nason and L. Trentadue*: the minimal prescription. A very simple recipe: just take

$$\sigma(\tau, Q^2) = \frac{1}{2\pi i} \int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN\, \tau^{-N}\, \mathcal{L}(N, Q^2)\, C^{\rm res}(N, \alpha_{\rm S}(Q^2))$$

with $0 < N_{\rm MP} < N_L$**.**

This is not a true inverse Mellin: the integrand is not analytical in any right half-plane, because of the branch cut due to the Landau pole.

*[**NPB 478(1996)273, hep-ph/9604351**]

Nonetheless, the MP has a number of good properties:

- **it is well defined for all values of τ**
- **it is an asymptotic sum of the original, divergent perturbative expansion**
- **the difference between the original series, truncated at the best-approximation term, and the minimal prescription, is suppressed more strongly than any power of Λ^2/Q^2.**

A closer look at the minimal prescription:

$$\sigma(\tau, Q^2) = \frac{1}{2\pi i} \int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN \, \tau^{-N} \, C^{\rm res}(N, \alpha_{\rm S}(Q^2)) \int_0^1 dy \, y^{N-1} \, \mathcal{L}(y, Q^2)$$

$$= \int_0^1 \frac{dy}{y} \, \mathcal{L}(y, Q^2) \, C^{\rm res}\left(\frac{\tau}{y}, \alpha_{\rm S}(Q^2)\right)$$

Looks like a convolution, but the integration region $0 \leq y \leq \tau$ **cannot be excluded: indeed**

$$C^{\rm res}(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN \, z^{-N} \, C^{\rm res}(N, \alpha_{\rm S}(Q^2))$$

does not vanish for $z > 1$ **because of the Landau cut.**

This is reflected in a difficulty in the numerical implementation of the minimal prescription formula: $\hat{\sigma}(\tau/y, \alpha_S)$ oscillates in the region $y \sim \tau$, where the luminosity is smooth, and large cancellations take place.

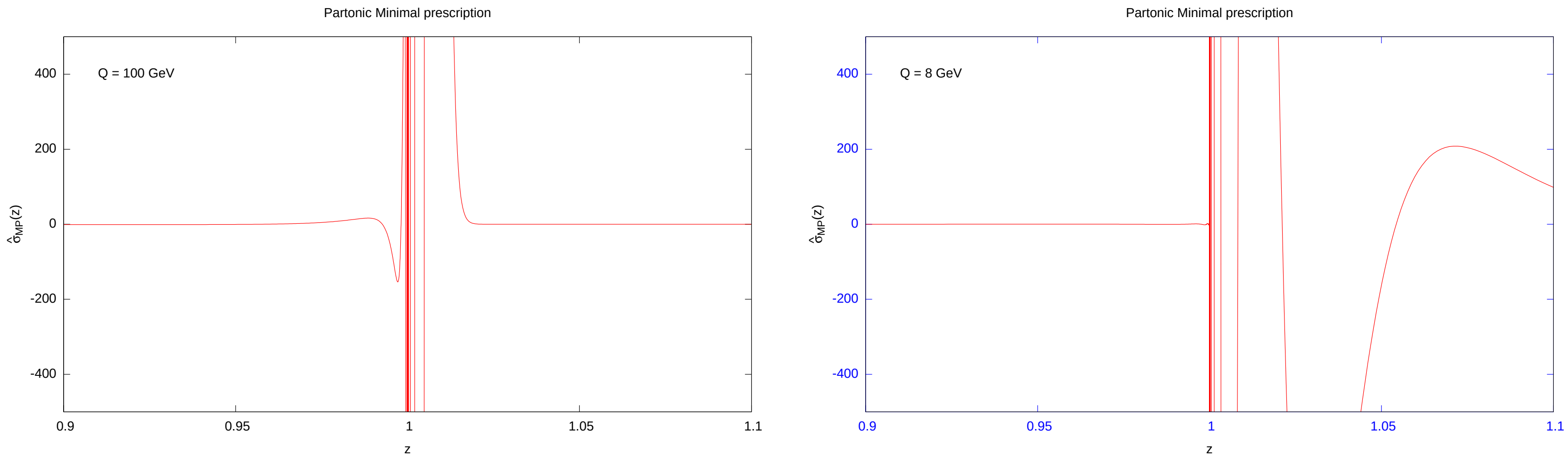


Figure 3: The partonic cross-section $C^{\rm res}(z, \alpha_S(Q^2))$ computed using the minimal prescription at $\sqrt{Q^2} = 8$ GeV and $\sqrt{Q^2} = 100$ GeV (Drell-Yan NLL).

[M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

One might avoid the problem by simply going back to the original formulation of the MP,

$$\sigma(x, Q^2) = \frac{1}{2\pi i} \int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN\, x^{-N}\, \mathcal{L}(N, Q^2)\, \hat{\sigma}(N, \alpha_{\rm S}(Q^2))$$

but $\mathcal{L}(N, Q^2)$ is typically not available.

Different techniques have been developed to overcome this problem.

[CMNT, NPB 478(1996)273, hep-ph/9604351]

[M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

The Borel prescription

Is it possible to sum the divergent series

$$\Sigma(z,\alpha_s(Q^2)) = \frac{1}{2\pi i}\left[\frac{1}{\log\frac{1}{z}}\oint\frac{d\xi}{\xi}\,\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\sum_{k=1}^{\infty}k!\,h_k\left(\frac{\bar{\alpha}}{\xi}\right)^k\right]_+$$

using the Borel technique?

S. Forte, J. Rojo, M. Ubiali, GR, PLB635(2006)313, hep-ph/0601048

R. Abbate, S. Forte, GR, PLB657(2007)55, arXiv:0707.2452

Consider a generic power series, not necessarily convergent in the Cauchy sense:

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} f_k \, \bar{\alpha}^k$$

and define its Borel transform

$$\hat{f}(w) = \sum_{k=1}^{\infty} f_k \, \frac{w^{k-1}}{(k-1)!}$$

Because of the factor $(k-1)!$, the Borel transformed series $\hat{f}(w)$ has much better convergence properties. An inverse transformation exists, since

$$\int_0^{+\infty} dw \, e^{-\frac{w}{\bar{\alpha}}} \, w^{k-1} = (k-1)! \, \bar{\alpha}^k \to f_B(\bar{\alpha}) = \int_0^{+\infty} dw \, e^{-\frac{w}{\bar{\alpha}}} \, \hat{f}(w)$$

Various cases:

1. **The original series is convergent in the usual sense. Then $f_B(\bar{\alpha}) = f(\bar{\alpha})$, but the Borel sum may enlarge the convergence region. Example:**

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} \bar{\alpha}^k = \frac{\bar{\alpha}}{1-\bar{\alpha}}, \qquad |\bar{\alpha}| < 1$$

$$f_B(\bar{\alpha}) = \int_0^{+\infty} dw\, e^{-\frac{w}{\bar{\alpha}}}\, e^w = \frac{\bar{\alpha}}{1-\bar{\alpha}} \qquad \mathrm{Re}\,\bar{\alpha} < 1$$

2. **The original series is divergent, but the Borel sum exists:**

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} (-1)^{k-1}\,(k-1)!\,\bar{\alpha}^k \qquad \hat{f}(w) = \frac{1}{1+w}$$

$$f_B(\bar{\alpha}) = \int_0^{+\infty} dw\, e^{-\frac{w}{\bar{\alpha}}}\, \frac{1}{1+w} < \infty \qquad \mathrm{Re}\,\bar{\alpha} > 0$$

3. The original series is divergent, its Borel transform exists, but it has a singularity in the range of the inversion integral:

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} (k-1)!\, \bar{\alpha}^k \qquad \hat{f}(w) = \frac{1}{1-w}$$

$$f_B(\bar{\alpha}) = \int_0^{+\infty} dw\, e^{-\frac{w}{\bar{\alpha}}}\, \frac{1}{1-w}$$

This is the case e.g. of renormalons.

Back to

$$C^{\text{res}}(N, \alpha_{\text{S}}(Q^2)) = 1 + \tilde{\Sigma}(L, \alpha_{\text{S}}(Q^2))$$

We have

$$\Sigma(z, \alpha_{\text{S}}(Q^2)) \equiv \frac{1}{2\pi i}\int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N}\tilde{\Sigma}(L, \alpha_{\text{S}}(Q^2)) = \left[\frac{R(z)}{\log\frac{1}{z}}\right]_+$$

$$R(z) = \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\sum_{k=1}^{\infty} k!\, h_k \left(\frac{\bar{\alpha}}{\xi}\right)^k$$

The Borel transform of $R(z)$ with respect to $\bar{\alpha}$ is found replacing

$$\bar{\alpha}^k \to \frac{w^{k-1}}{(k-1)!}$$

and it is convergent:

$$\hat{R}(w, z) = \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\sum_{k=1}^{\infty} k\, h_k \frac{w^{k-1}}{\xi^k} = \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\frac{d}{dw}\tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\text{S}}(Q^2)\right)$$

The branch cut of $\tilde{\Sigma}(L, \alpha_{\rm S}(Q^2))$, $-\infty < L \leq -1$, is mapped onto the range $-w \leq \xi \leq 0$ on the real axis of the complex ξ plane. Hence, the ξ integration path is any closed curve which encircles the cut.

As a consequence, the inverse Borel transform of $\hat{R}$ does not exist, because w integration is divergent at $+\infty$.

We introduce a cutoff:

$$
\begin{aligned}
R_C(z) &= \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}} \hat{R}(w,z) \\
&= \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{\log^{\xi} \frac{1}{z}}{\Gamma(\xi)} \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}} \frac{d}{dw} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \\
\Sigma_I(z, \alpha_{\rm S}(Q^2)) &= \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}} \frac{d}{dw} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \left[\log^{\xi-1} \frac{1}{z}\right]_+
\end{aligned}
$$

Remarks:

- **The original divergent series for Σ is asymptotic to the function $\Sigma_I(z, \alpha_{\rm S}(Q^2))$**
- **For any finite-order truncation of the divergent series, the full and cutoff results differ by a twist–$\left(2 + \frac{2C}{a}\right)$.**
- **C can be chosen freely in the range $C \geq a$; different choices differ by power suppressed terms. The minimal choice is $C = a$.**

A somewhat simpler result is obtained if the Borel transform is performed through the replacement

$$\bar{\alpha}^k \to \frac{1}{\bar{\alpha}} \frac{w^k}{k!}$$

In this case one gets

$$R_C(z) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{\log^{\xi} \frac{1}{z}}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right)$$

$$\Sigma_I(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \left[\log^{\xi-1} \frac{1}{z}\right]_+$$

which provides an equally good resummation prescription; the difference is in practice very small.

If we only wish to retain terms which do not vanish as $z \to 1$ we may expand

$$\log \frac{1}{z} = 1 - z + O((1-z)^2)$$

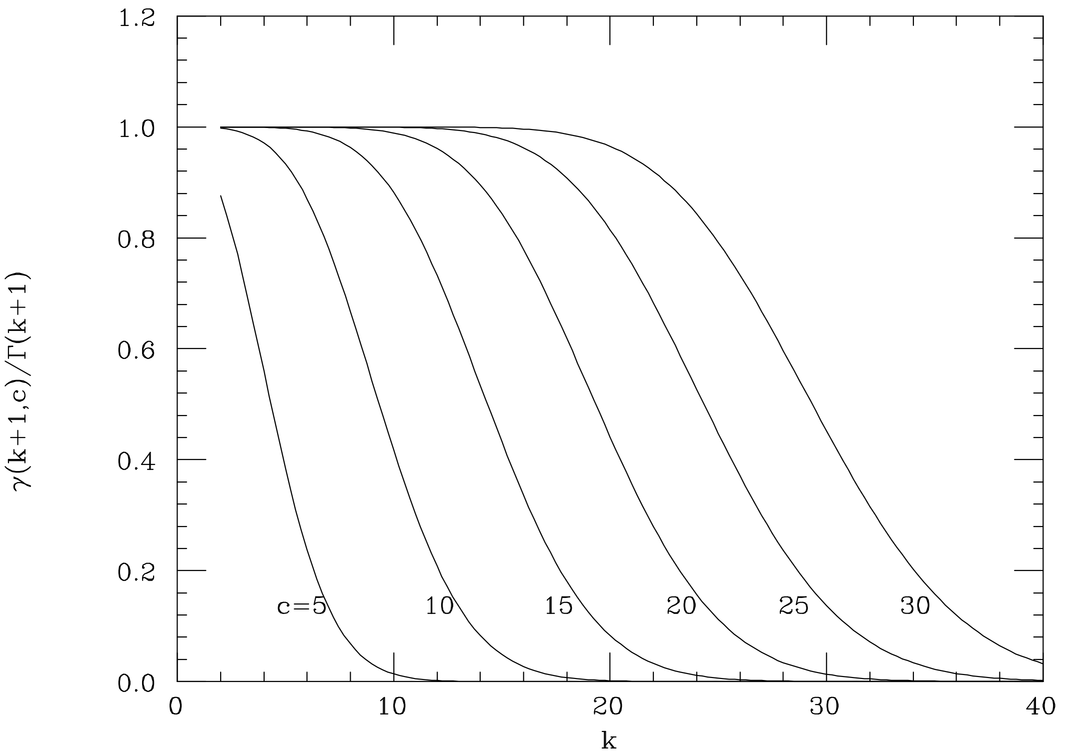
with the result

$$\Sigma_{II}(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma} \left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2) \right) \left[(1-z)^{\xi - 1} \right]_+$$

The Borel prescription effectively replaces

$$h_k \to h_k \frac{\gamma(k+1, C/\bar{\alpha})}{\Gamma(k+1)}; \quad \gamma(k+1,c) = \int_0^c dt\, e^{-t} t^k$$

thereby damping high orders:



Subleading terms

Different prescriptions give different results for two reasons:

- **The different way they handle the high-order behaviour of the divergent series. This makes in practice a small difference unless τ is close to the Landau pole (very rare).**

  Example: Borel prescription with $C = 2$, $\alpha_S = 0.11$, then $c \approx 15$, and the perturbative expansion is truncated around $k \sim 15$.

- **Prescriptions also differ in the subleading terms which are introduced when performing the resummation.**

  Example: the minimal prescription just gives the exact Mellin inverse of any truncation of the series. Because this result depends on z through $\log \frac{1}{z}$, in z space it generates a series of power suppressed terms.

We have now two versions of the Borel prescription:

$$\Sigma_I(z, \alpha_S(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_S(Q^2)\right) \left[\log^{\xi-1}\frac{1}{z}\right]_+$$

$$\Sigma_{II}(z, \alpha_S(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_S(Q^2)\right) \left[(1-z)^{\xi-1}\right]_+$$

They differ by non-logarithmically-enhanced terms (their Mellin transforms differ by terms suppressed by powers of $\frac{1}{N}$).

Version I is closer to the minimal prescription: when applied to individual $\log^k \frac{1}{N}$ terms, it gives back the exact Mellin transform (apart from the suppression factor).

Two opposite extreme choices in the treatment of subleading terms:

Version I: no $1/N$ power-suppressed terms, hence $1 - z$ power suppressed terms in z space appear;

Version II: the opposite.

However, with the Borel prescription the z dependence is under analytic control: it is entirely contained in the factor $\log^{\xi-1} \frac{1}{z}$ and can be modified at will.

We may therefore consider intermediate options. We already observed that at NLO and NNLO the inclusion of some subleading terms by replacing

$$\log(1-z) \to \log \frac{1-z}{\sqrt{z}}$$

provides better agreement with the full result

This can be understood: soft resummation arises from the kinematic fact that as $z \to 1$ the dependence of partonic cross-sections on z is always in the combination $Q^2(1-z)^2$, which is the upper limit of the integral over the energy of radiated gluons for DY. However, the actual value is

$$k^0_{\max} = \sqrt{\frac{Q^2(1-z)^2}{4z}}$$

It is easy to do so by the Borel prescription: we define

$$\Sigma_{III}(z, \alpha_s(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_s(Q^2)\right) \left[(1-z)^{\xi-1}\right]_+ z^{-\frac{\xi}{2}}.$$

With this choice, the kinematic correction is automatically included to all orders.

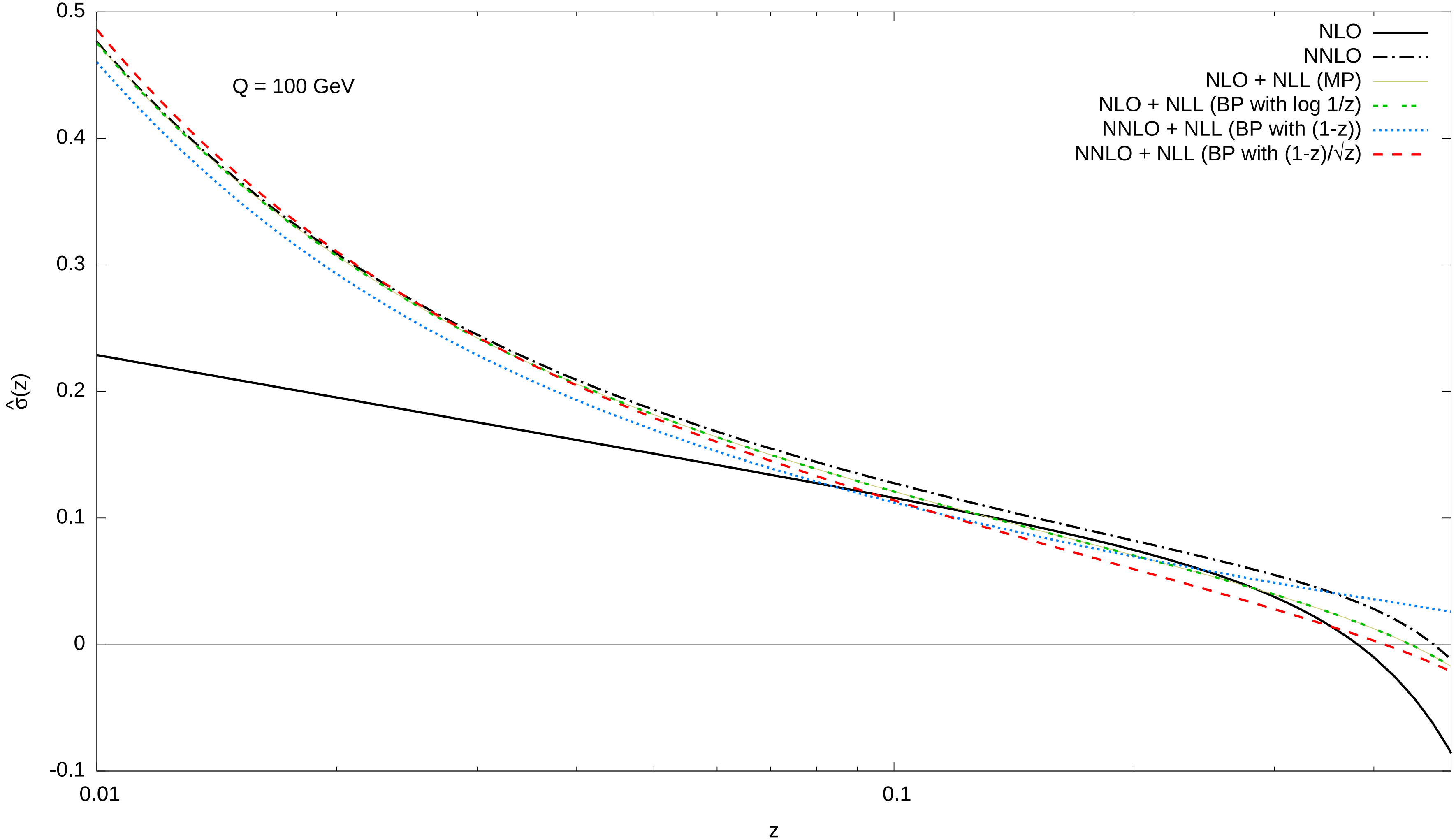
In fact, III closer than I to the MP because

$$\log\frac{1}{z} = \frac{1-z}{\sqrt{z}}\left(1 + O((1-z)^2)\right),$$

so that

$$\frac{\log^k \log\frac{1}{z}}{\log\frac{1}{z}} = \frac{\sqrt{z}}{1-z} \log^k \frac{1-z}{\sqrt{z}}\left(1 + O((1-z)^2)\right).$$

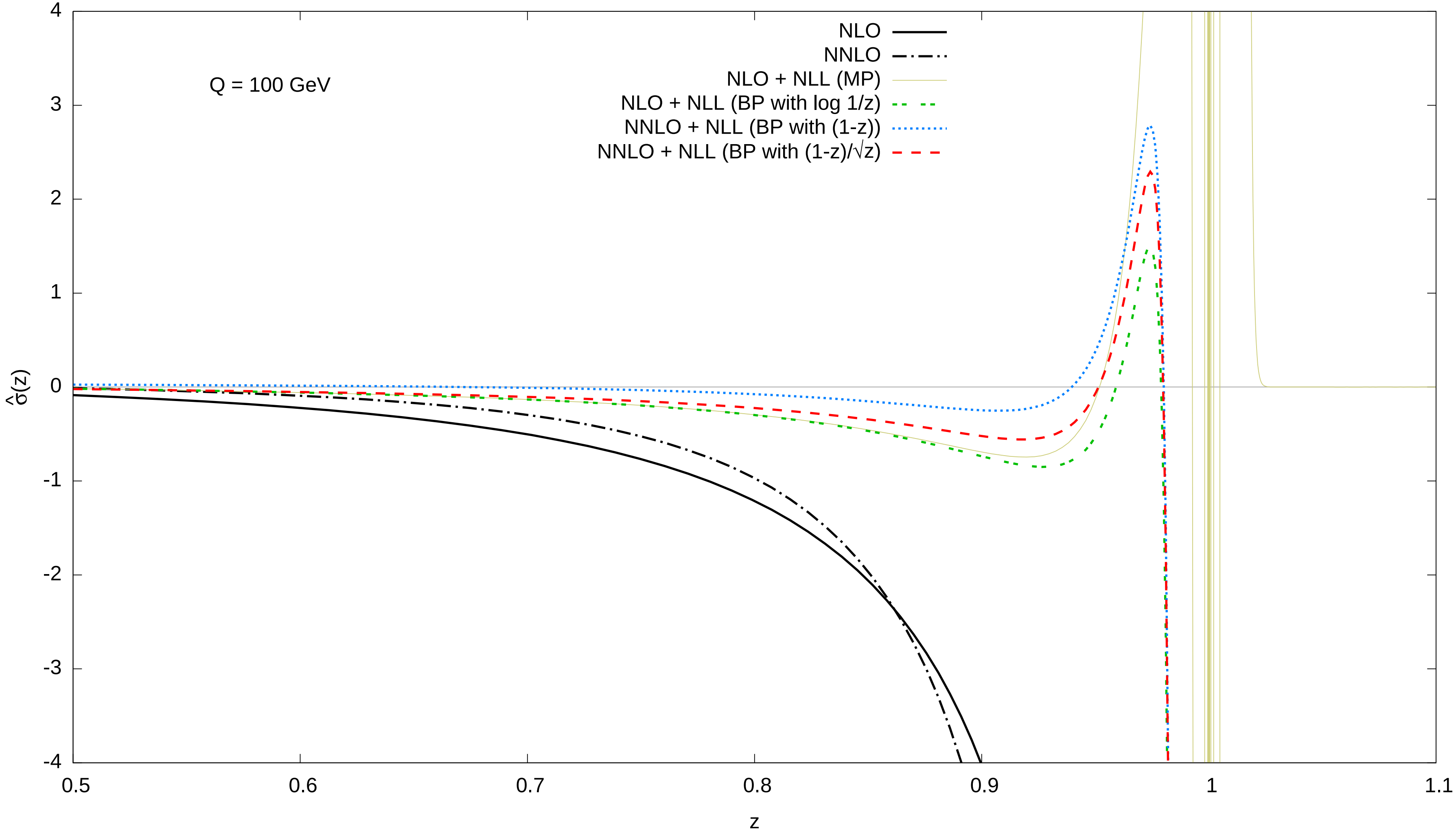
This shows that, up to terms suppressed by two powers of $1-z$, the MP effectively performs the kinematic subleading replacement (but at the same time introduces an overall factor $\sqrt{z}$ which is absent in the true coefficients).



0.5
0.4
0.3
0.2
0.1
0
-0.1
0.01
0.1
z
σ̂(z)
Q = 100 GeV
NLO
NNLO
NLO + NLL (MP)
NLO + NLL (BP with log 1/z)
NNLO + NLL (BP with (1-z))
NNLO + NLL (BP with (1-z)/√z)

Comments:

- **MP and BP-I (BP with $\log 1/z$) essentially indistinguishable (for values of $z \lesssim 0.9$, where the MP starts oscillating)**
- **BP-II (BP with $1 - z$) rather different from them (and unconfortably large in the intermediate region).**
- **BP-III prescription (BP with $(1 - z)/\sqrt{z}$) as expected differs less from the MP in the wholw range. The difference between MP and BP-III sizable, but smaller than the size of resummation where resummation is relevant, and induces small corrections at small z. A reliable estimate of the ambiguity in the resummation.**
- **Interesting interplay between large-z and small-z behaviour.**

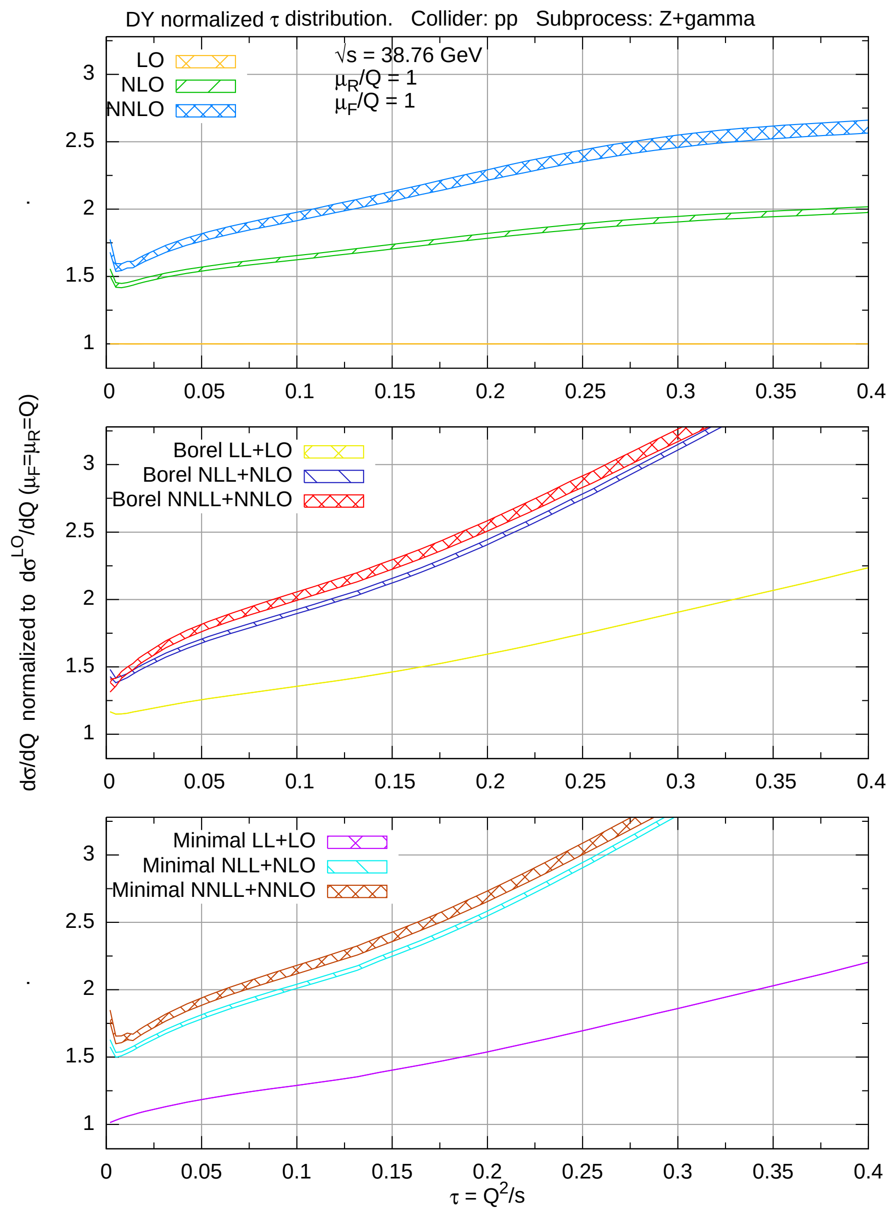


NLO
NNLO
NLO + NLL (MP)
NLO + NLL (BP with log 1/z)
NNLO + NLL (BP with (1-z))
NNLO + NLL (BP with (1-z)/√z)
Q = 100 GeV
σ̂(z)
z
4
3
2
1
0
-1
-2
-3
-4
0.5
0.6
0.7
0.8
0.9
1
1.1

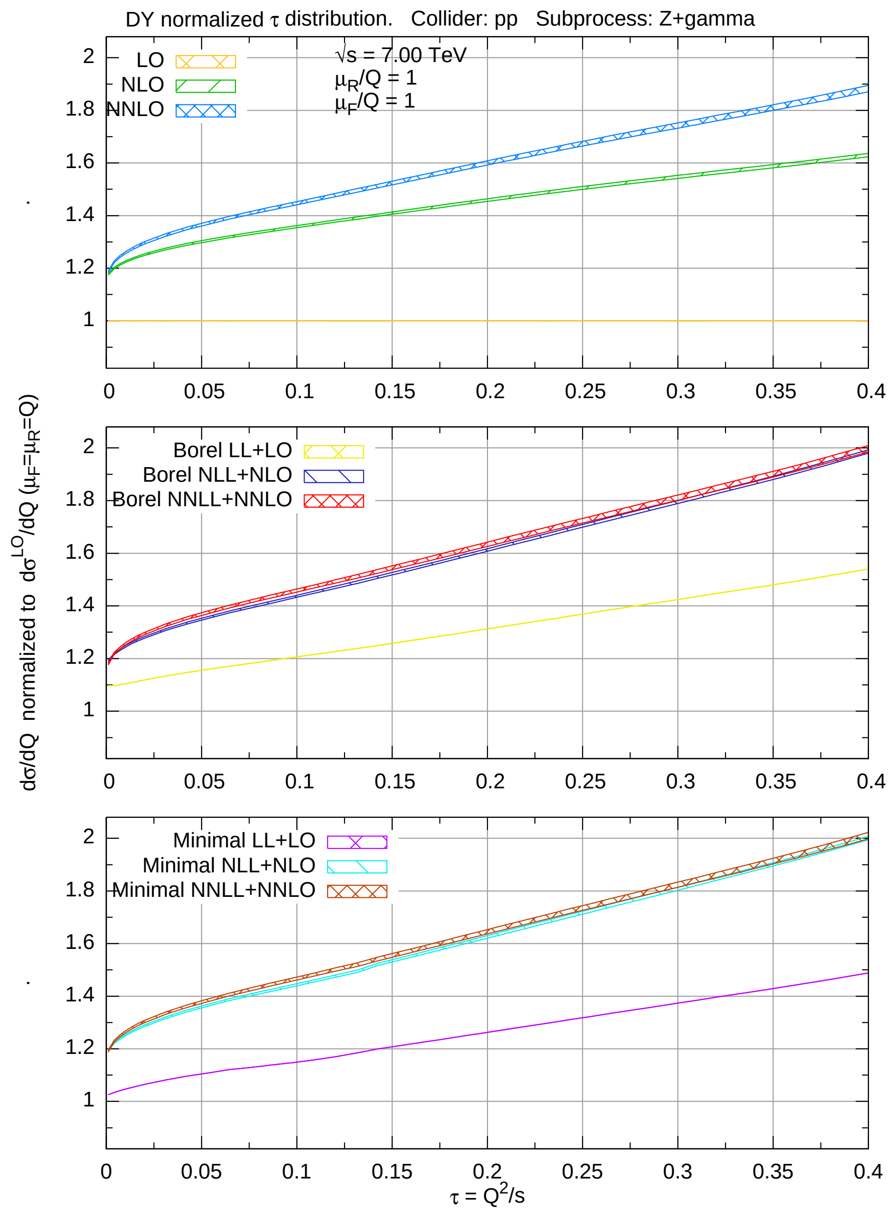
Summary and outlook

- Resummation of threshold logarithms provides an improvement in the theoretical predictions even at relatively small values of τ. This statement can be made quantitative.
- There are ambiguities in the computation of observables from resummed quantities in QCD, to be ascribed to the presence of a Landau singularity in the running coupling.
- A prescription based on Borel sum and twist expansion can be given; it gives better control over subleading terms with respect to the minimal prescription.
- Work in progress: combined small-x and large-x resummation. Comparison with other resummation approaches.

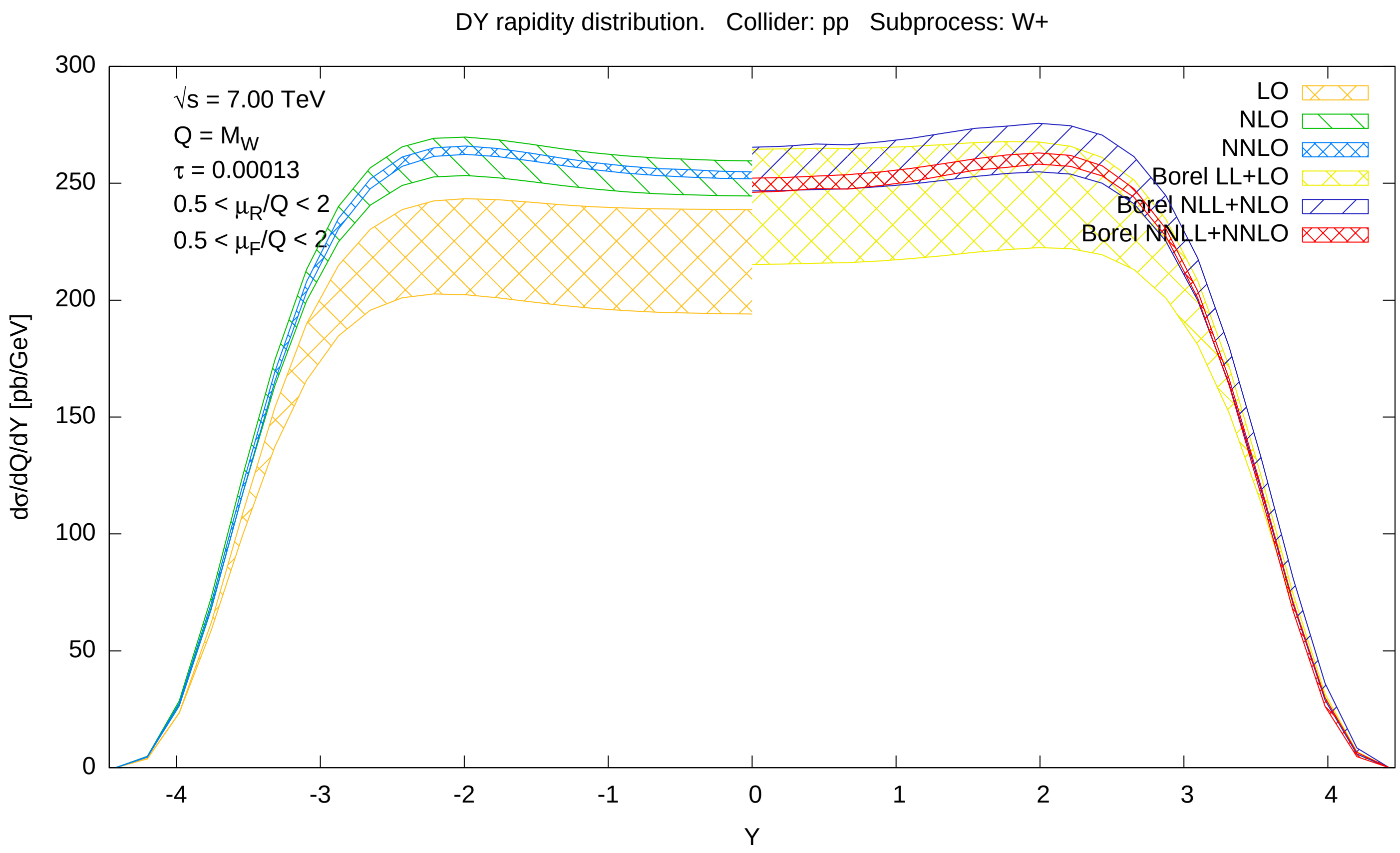
† M. Bonvini, S. Forte, T. Peraro, GR, in preparation



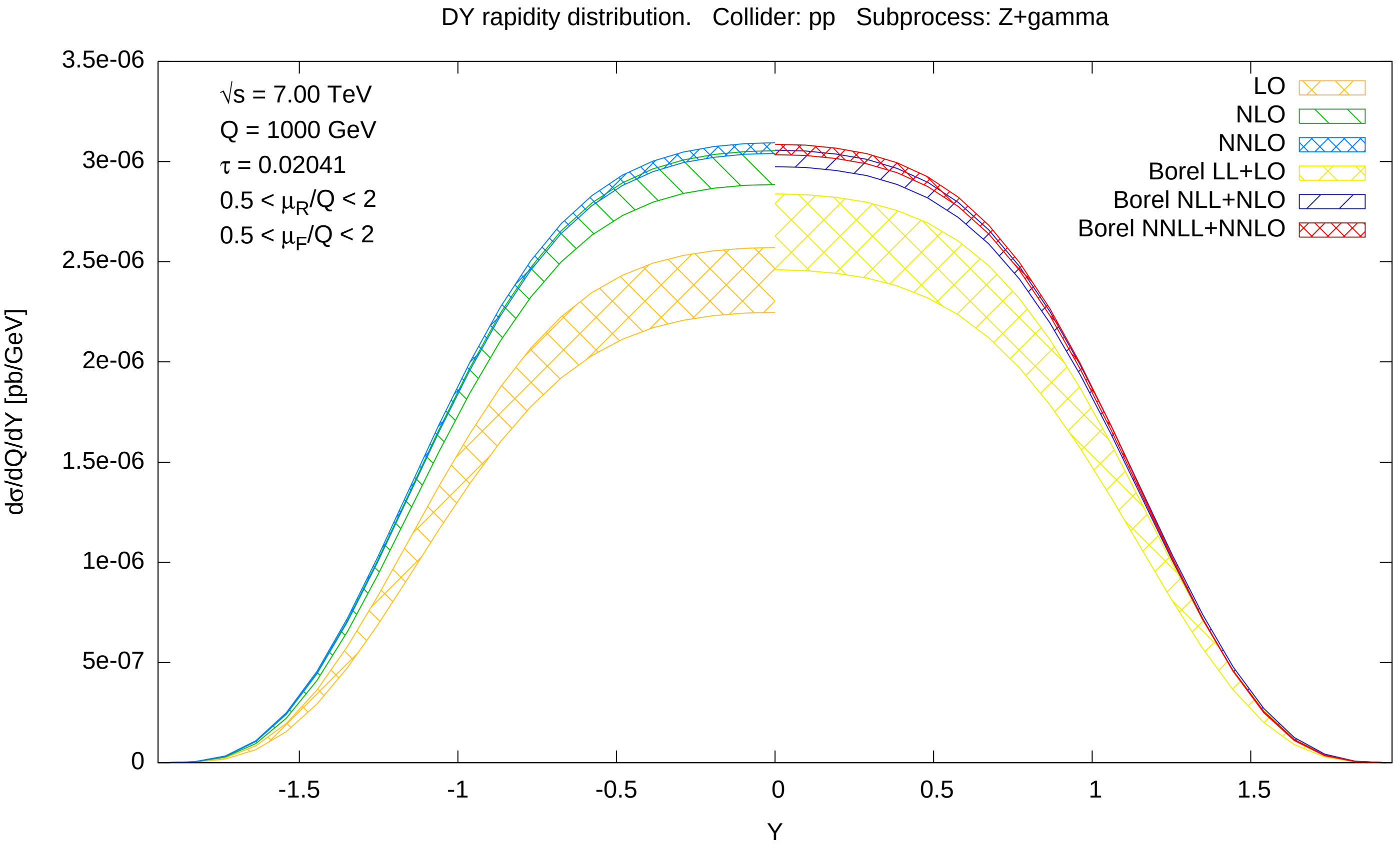
DY normalized τ distribution. Collider: pp Subprocess: Z+gamma
LO
NLO
NNLO
√s = 38.76 GeV
μR/Q = 1
μF/Q = 1
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
Minimal LL+LO
Minimal NLL+NLO
Minimal NNLL+NNLO
dσ/dQ normalized to dσLO/dQ (μF=μR=Q)
τ = Q2/s



DY normalized τ distribution. Collider: pp Subprocess: Z+gamma
LO
NLO
NNLO
√s = 7.00 TeV
μR/Q = 1
μF/Q = 1
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
Minimal LL+LO
Minimal NLL+NLO
Minimal NNLL+NNLO
dσ/dQ normalized to dσLO/dQ (μF=μR=Q)
τ = Q2/s



DY rapidity distribution. Collider: pp Subprocess: W+
√s = 7.00 TeV
Q = M_W
τ = 0.00013
0.5 < μ_R/Q < 2
0.5 < μ_F/Q < 2
LO
NLO
NNLO
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
dσ/dQ/dY [pb/GeV]
Y
300
250
200
150
100
50
0
-4
-3
-2
-1
0
1
2
3
4



DY rapidity distribution. Collider: pp Subprocess: Z+gamma
√s = 7.00 TeV
Q = 1000 GeV
τ = 0.02041
0.5 < μR/Q < 2
0.5 < μF/Q < 2
LO
NLO
NNLO
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
dσ/dQ/dY [pb/GeV]
Y
3.5e-06
3e-06
2.5e-06
2e-06
1.5e-06
1e-06
5e-07
0
-1.5
-1
-0.5
0
0.5
1
1.5

Divergence of $R(z)$ and convergence of $\hat{R}(w, z)$

Back to the generic resummed quantity:

$$\tilde{\Sigma}(\alpha_{\rm S}, L) = \sum_{k=1}^{\infty} h_k \, L^k; \qquad L = \bar{\alpha} \, \log \frac{1}{N}$$

To log accuracy,

$$\frac{1}{2\pi i} \int_{\overline{N}-i\infty}^{\overline{N}+i\infty} dN \, z^{-N} \log^k \frac{1}{N} = \left[\frac{P_{k-1}(\ell)}{1-z} \right]_+$$

where $P_{k-1}(\ell)$ is a polynomial of degree $k-1$ in $\ell \equiv \log(1-z)$. Thus

$$\Sigma(\alpha_{\rm S}(Q^2), z) = \left[\frac{R(z)}{1-z} \right]_+ ; \qquad R(z) = \sum_{k=1}^{\infty} h_k \, \bar{\alpha}^k \, P_{k-1}(\ell)$$

The explicit form of $P_{k-1}(\ell)$ is

$$P_{k-1}(\ell) = \sum_{j=1}^{k} \binom{k}{j} \Delta^{(j)}(0)\, \ell^{k-j}; \qquad \Delta(\eta) = \frac{1}{\Gamma(\eta)}$$

Hence,

$$R_K(z) = \sum_{k=1}^{K} h_k\, \bar{\alpha}^k \sum_{j=1}^{k} \frac{\Delta^{(j)}(0)}{j!} \frac{k!}{(k-j)!}\, \ell^{k-j}$$

If the sum over j is truncated at $j = J$ (corresponding to $\mathbf{N}^{J-1}\mathbf{L}$ $\log(1-x)$ accuracy) we get a convergent result. This is because

$$\frac{k!}{(k-j)!}\, \ell^{k-j} = \frac{d^j\, \ell^k}{d\ell^j}$$

and therefore

$$R_K(z) = \sum_{j=1}^{J} \frac{\Delta^{(j)}(0)}{j!} \frac{d^j}{d\ell^j} \sum_{k=1}^{K} h_k\, \bar{\alpha}^k\, \ell^k \to \sum_{j=1}^{J} \frac{\Delta^{(j)}(0)}{j!} \frac{d^j}{d\ell^j} \tilde{\Sigma}(\bar{\alpha}\ell)$$

which is convergent for $|\bar{\alpha}\ell| < 1$, because of the Landau pole at $\bar{\alpha}\ell = 1$.

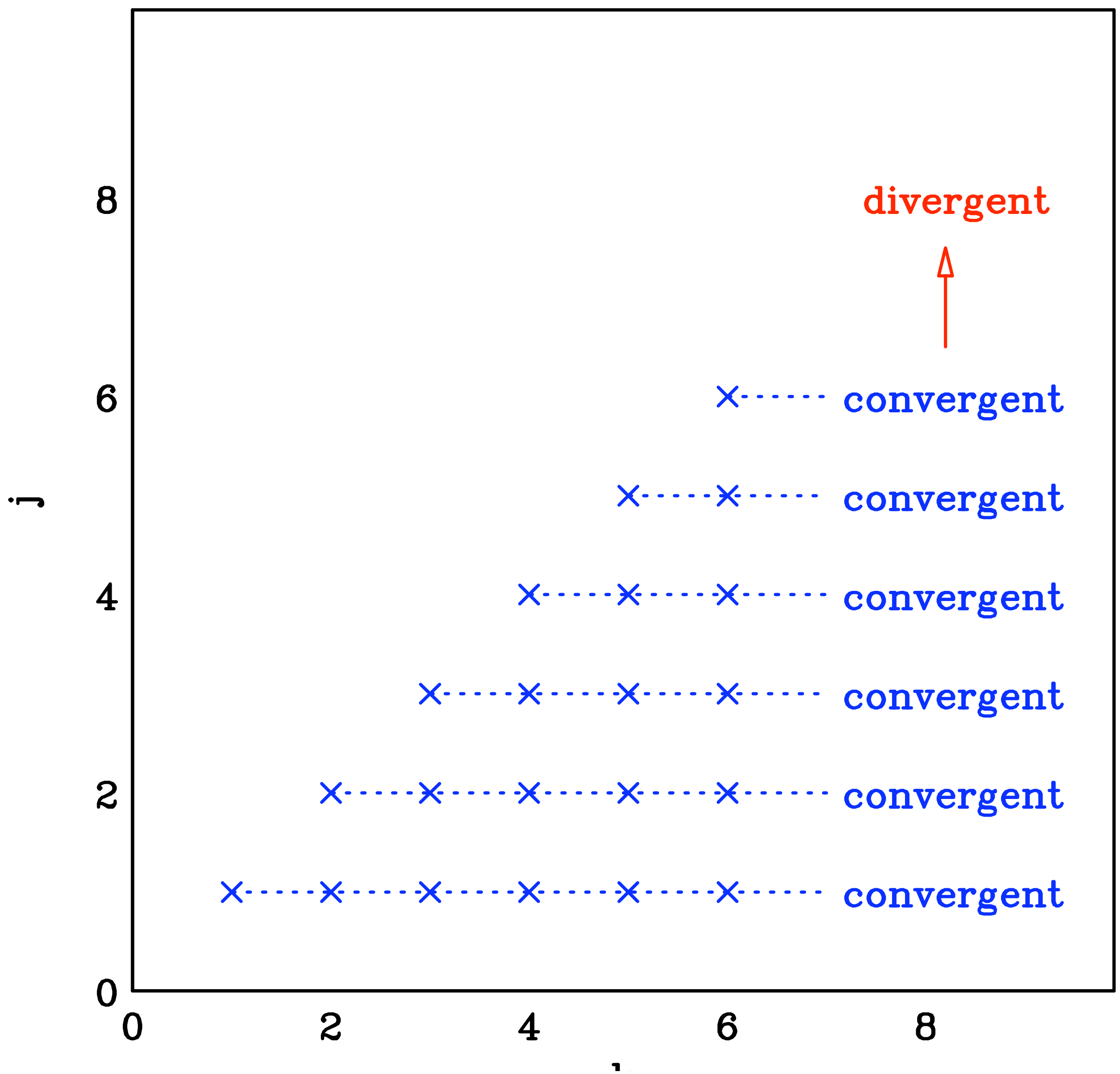
The full sum is however divergent. To see this, use the identity

$$\frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\, e^{\xi}\, \xi^{-(k-j)} = \begin{cases} \frac{1}{(k-j)!} & k-j \geq 0 \\ 0 & k-j < 0 \end{cases}$$

to get

$$\begin{aligned} R_K(z) &= \sum_{k=1}^{K} h_k\, \bar{\alpha}^k \sum_{j=1}^{k} \frac{\Delta^{(j)}(0)}{j!} \frac{k!}{(k-j)!}\, \ell^{k-j} \\ &= \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\, e^{\xi} \sum_{k=1}^{K} k!\, h_k\, \bar{\alpha}^k \sum_{j=1}^{\infty} \frac{\Delta^{(j)}(0)}{j!}\, \ell^{k-j}\, \xi^{-(k-j)} \\ &= \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\, e^{\ell\xi}\, \Delta(\xi) \sum_{k=1}^{K} k!\, h_k \left(\frac{\bar{\alpha}}{\xi}\right)^k \end{aligned}$$

Since $\sum_k h_k\, L^k$ has convergence radius 1, $\sum_k k!\, h_k\, L^k$ has convergence radius 0.



Terms in the expansion of $R(z)$ in powers $\bar{\alpha}^k \log^{k-j}(1-z)$.

By a similar manipulation one can show that the Borel transform of R wrt to $\bar{\alpha}$ is convergent. Indeed, replacing $\bar{\alpha}^k \to w^{k-1}/(k-1)!$ we get

$$\hat{R}_K(w,z) = \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\, e^{\ell\xi}\,\Delta(\xi)\sum_{k=1}^{K} k\, h_k\, \frac{w^{k-1}}{\xi^k}$$

which is convergent as $K \to \infty$:

$$\hat{R}(w,z) = \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\, e^{\ell\xi}\,\Delta(\xi)\frac{d}{dw}\tilde{\Sigma}\left(\alpha_{\rm S}, \frac{w}{\xi}\right)$$

So finally

$$
\begin{aligned}
\hat{R}(z,w) &= \frac{1}{2\pi i}\oint_H \frac{d\xi}{\xi}\, e^{\ell\xi}\,\Delta(\xi)\,\frac{d}{dw}\tilde{\Sigma}\left(\frac{w}{\xi}\right); \qquad |\xi| > w \text{ on } H \\
R(z) &= \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}}\,\hat{R}(z,w) \\
&= \frac{1}{2\pi i}\oint_H \frac{d\xi}{\xi}\, e^{\ell\xi}\,\Delta(\xi)\left[e^{-\frac{C}{\bar{\alpha}}}\,\tilde{\Sigma}\left(\frac{C}{\xi}\right) + \frac{1}{\bar{\alpha}}\int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}}\,\tilde{\Sigma}\left(\alpha_{\mathrm{S}}, \frac{w}{\xi}\right)\right]
\end{aligned}
$$

which is explicitly written in terms of the function $\tilde{\Sigma}$.

The resummed (Mellin-transformed) cross section $\tilde{\Sigma}(\alpha_{\rm S}, L)$ has a branch cut in the complex plane L in

$$-\infty < \mathrm{Re\,L} \leq -1; \qquad \mathrm{Im}\, L = 0$$

which is mapped into

$$-w \leq \mathrm{Re}\,\xi \leq 0; \qquad \mathrm{Im}\,\xi = 0$$

for $\tilde{\Sigma}(\alpha_{\rm S}, w/\xi)$. The contour H must be chosen so that it encloses the cut, and therefore is pushed to large negative values of $\mathrm{Re}\,\xi$ as $w \to +\infty$. In that region, $\Delta(\xi)$ oscillates with factorially growing amplitude, and the w integral does not converge.