Resummation of large-x and small-x double logarithms in deep-inelastic scattering & semi-inclusive annihilation

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partly with G. Soar, A. Almasy (UoL), S. Moch (DESY), J. Vermaseren (NIKHEF)

- Splitting and coefficient functions and their endpoint behaviour
- 4^{th}-order$/$all-order large-x logs from physical evolution kernels
- Large-x & small-x via unfactorized D-dim. structure functions

Galileo Galilei Institute, Florence, 08-09-11

Conventions and references

Double-log enhancement: **two additional logs L per additional order in α_s**

$$Q\big|_{\alpha_s^n} \sim L^{\ell_0}(\,\#L^{2n} + \#L^{2n-1} + \#L^{2n-2} + \ldots) + \ldots$$

LL **NLL** **NNLL**

LL, NLL, . . . : leading logarithms, next-to-leading logarithms, . . .

Counting of a resummation, cf. small-x, not of a (stronger) exponentiation, cf. soft gluons: **NNLL resummation $\Leftrightarrow$ (re-expanded) NLL exponentiation**

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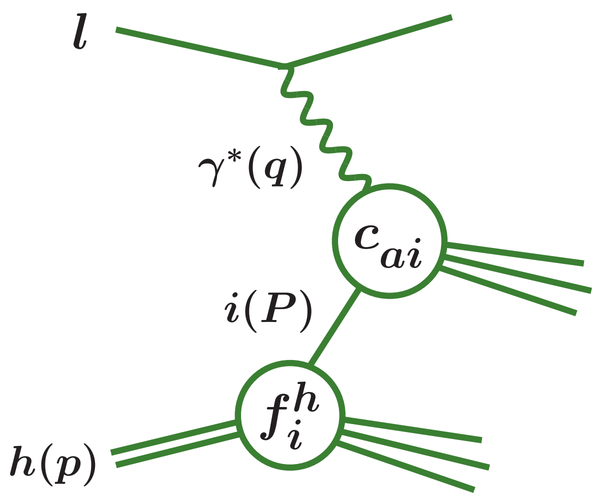
- Non-singlet NNLL (NLL for DY) resummation from physical kernels
  MV, arXiv:0902.2342 (JHEP), arXiv:0909.2124 (JHEP)
- Singlet NNLL for fourth-order splitting functions and F_L in DIS
  SMVV, arXiv:0912.0369 (NPB), arXiv:1008.0952 (proc. LL 2010)
- Large-x resummation of splitting & coefficient functions in DIS and SIA*
  A.V., arXiv:1005.1606 (PLB); ASV, arXiv:1012.3352 (JHEP); $*$ to appear
- Small-x resummation of splitting & coefficient functions in SIA and DIS*
  A.V., arXiv:1108.2993 (JHEP); $*$ to appear

Hard lepton-hadron processes in pQCD (I)

Inclusive deep-inelastic scattering (DIS), semi-incl. l^+l^- annihilation (SIA)



Left $\longrightarrow$ right: DIS, q spacelike, $Q^2 = -q^2$

$P = \xi p\,,\ f_i^h =$ parton distributions

Top $\longrightarrow$ bottom: l^+l^-, q timelike, $Q^2 = q^2$

$p = \xi P\,,$ fragmentation distributions

Drell-Yan (DY) l^+l^- production: bottom $\longrightarrow$ top, 2$^{\text{nd}}$ hadron from right ($\{\ldots\}$)

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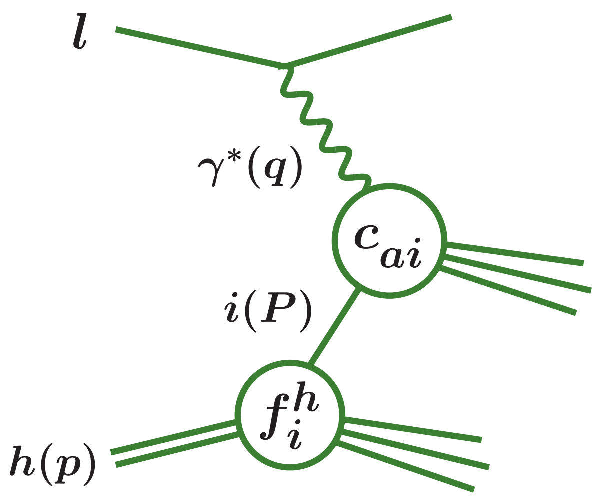

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DIS and SIA structure functions, DY cross section F_a: coefficient functions

$$F_a(x, Q^2) \; = \; \Big[C_{a,i\{j\}}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2)\{\otimes f_j^{h'}(\mu^2)\} \Big](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables: $x = Q^2/(2p\cdot q)$ in DIS etc. μ: renorm./mass-fact. scale

Hard lepton-hadron processes in pQCD (II)

Parton / fragmentation distributions f_i : (renorm. group) evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) \;=\; \left[P_{ik}^{(S,T)}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

$\otimes$ = Mellin convolution. Initial conditions incalculable in perturbative QCD.

$\Rightarrow$ predictions: fit-analyses of reference processes, universality of $f_i(\xi,\mu^2)$

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Expansion in α_s: splitting functions P, coefficient fct's c_a of observables

$$\begin{aligned} P &= \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \ldots \\ C_a &= \underbrace{\alpha_s^{n_a} \Big[c_a^{(0)} + \alpha_s c_a^{(1)}}_{} + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \ldots \Big] \end{aligned}$$

NLO: first real prediction of size of cross sections

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NLO: first real prediction of size of cross sections

NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate of pQCD predictions

New: $P_{ik}^{(2)T}$ now (almost) completely known AMV, arXiv:1107.2263 (NPB)

$\overline{\text{MS}}$ splitting functions at large x / large N

Mellin trf. $f(N) = \int_0^1 dx\, (x^{N-1}\{-1\})\, f(x)_{\{+\}}$: M-convolutions $\rightarrow$ products

$$\frac{\ln^n(1-x)}{(1-x)_+} \stackrel{\mathrm{M}}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \ \ldots\ , \quad \ln^n(1-x) \stackrel{\mathrm{M}}{=} \frac{(-1)^n}{N} \ln^n N + \ \ldots$$

Diagonal splitting functions: no higher-order enhancement at N^0, N^{-1}

$$P^{(\ell-1)}_{\mathrm{qq/gg}}(N) = A^{(\ell)}_{\mathrm{q/g}} \ln N + B^{(\ell)}_{\mathrm{q/g}} + C^{(\ell)}_{\mathrm{q/g}} \frac{1}{N} \ln N + \ \ldots\ , \quad A_{\mathrm{g}} = C_A/C_F\, A_{\mathrm{q}}$$

...; Korchemsky (89); Dokshitzer, Marchesini, Salam (05)

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Off-diagonal: double-log behaviour, colour structure with $C_{AF} = C_A - C_F$

$$C_F^{-1} P^{(\ell)}_{\mathrm{gq}} / n_f^{-1} P^{(\ell)}_{\mathrm{qg}} = \tfrac{1}{N} \ln^{2\ell} N \ \# C_{AF}^{l}$$
$$+ \tfrac{1}{N} \ln^{2\ell-1} N \,(\# C_{AF} + \# C_F + \# n_f)\, C_{AF}^{l-1} + \ldots$$

Double logs $\ln^n N$, $\ell+1 \le n \le 2\ell$ vanish for $C_F = C_A$ ($\rightarrow$ SUSY case)

$\overline{\text{MS}}$ coefficient functions at large x / large N

‘Diagonal’ [$\mathcal{O}(1)$] coeff. fct’s for $F_{2,3,\phi}$ in DIS, $F_{T,A,\phi}$ in SIA, $F_{\mathrm{DY}} = \frac{1}{\sigma_0}\frac{d\sigma_{q\bar{q}}}{dQ^2}$

$$C^{(\ell)}_{2,q/\phi,g/\ldots} = \# \ln^{2\ell} N + \ldots + N^{-1}(\# \ln^{2\ell-1} N + \ldots) + \ldots$$

N^0 parts: threshold exponentiation   Sterman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log (N^{3}LL) accuracy - mod. $A^{(4)}$

$\Rightarrow$ highest seven (DIS, SIA), six (DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05) (+ SCET papers, from 06), SIA: Blümlein, Ravindran (06); MV, arXiv:0908.2746 (PLB)

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'Off-diagonal' [$\mathcal{O}(\alpha_s)$] quantities: leading N^{-1} double logarithms

$$C^{(\ell)}_{\phi,q/2,g/\ldots} \;=\; N^{-1}(\#\;\ln^{2\ell-1} N + \ldots) + \ldots$$

Longitudinal DIS/SIA structure functions [convention: ℓ = order in α_s – 1]

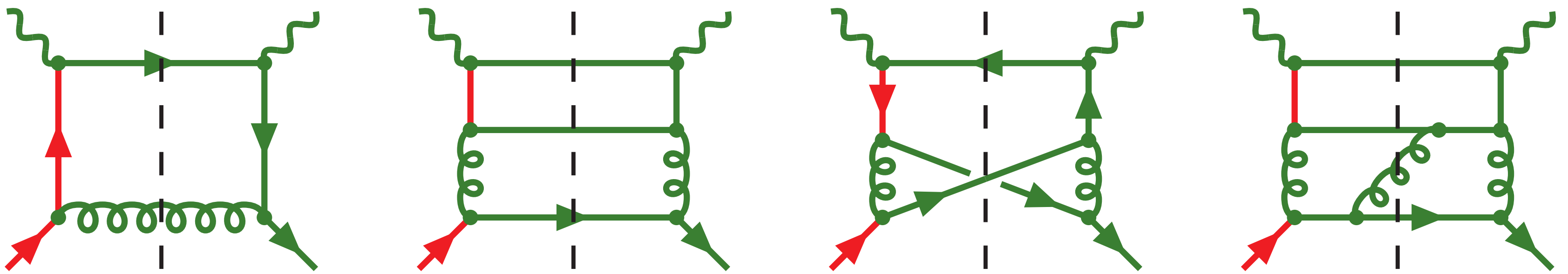
$$C^{(\ell)}_{L,q} \;=\; N^{-1}(\#\;\ln^{2\ell} N + \ldots) + \ldots\;,\quad C^{(\ell)}_{L,g} \;=\; N^{-2}(\#\;\ln^{2\ell} N + \ldots) + \ldots$$

Flavour singlet – non-singlet decomposition

Quark-quark splitting functions:

$$P_{\mathrm{q}_i\mathrm{q}_k} = P_{\bar{\mathrm{q}}_i\bar{\mathrm{q}}_k} = \delta_{ik} P^{\mathrm{v}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\mathrm{qq}}$$

$$P_{\mathrm{q}_i\bar{\mathrm{q}}_k} = P_{\bar{\mathrm{q}}_i\mathrm{q}_k} = \delta_{ik} P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} + P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}}$$



$P^{\mathrm{v}}_{\mathrm{qq}} = \mathcal{O}(\alpha_{\mathrm{s}})$ $\quad$ $P^{\mathrm{s}}_{\mathrm{qq}},\, P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \,:\, \alpha_{\mathrm{s}}^2$ $\quad$ $P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} \,:\, \alpha_{\mathrm{s}}^2$ $\quad$ $P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \neq P^{\mathrm{s}}_{\mathrm{qq}} \,:\, \alpha_{\mathrm{s}}^3$

Three types of difference (non-singlet) combinations: $P^{\pm}_{\mathrm{ns}} = P^{\mathrm{v}}_{\mathrm{qq}} \pm P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}}\,,\, P^{\mathrm{v}}_{\mathrm{ns}}$

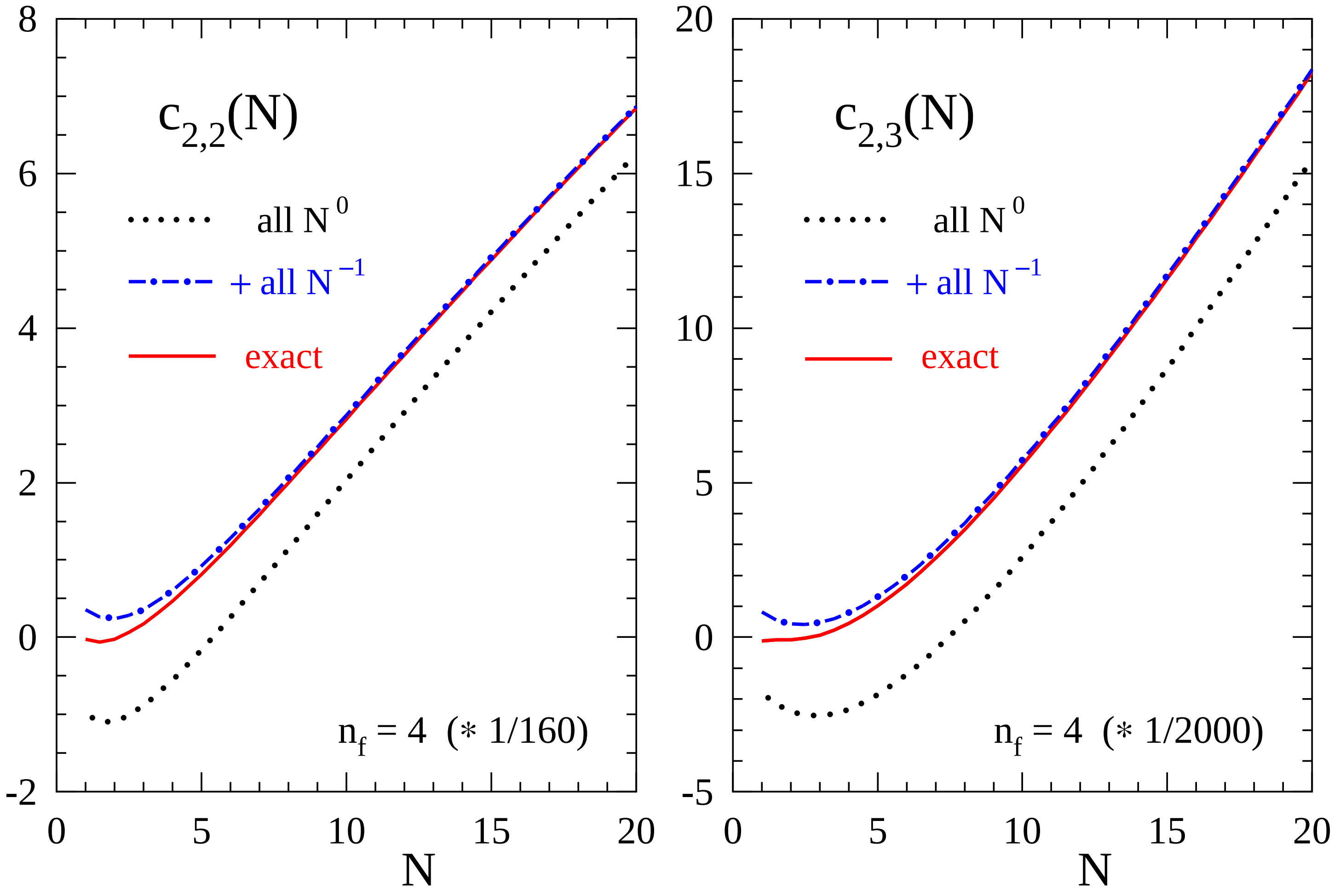
Evolution of gluon and flavour-singlet quark distributions g and q_{s}

$$q_{\mathrm{s}} = \sum_{r=1}^{n_f} (q_r + \bar{q}_r)\,, \quad \frac{d}{d\ln\mu^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

with (ps = 'pure singlet') $\quad P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + n_f (P^{\mathrm{s}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\bar{\mathrm{q}}\mathrm{q}}) \equiv P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$

Quark coefficient fct's: analogous decomposition $C_{a,\mathrm{q}\{\bar{\mathrm{q}}\}} = C_{a,\mathrm{ns}} + C_{a,\mathrm{ps}}$

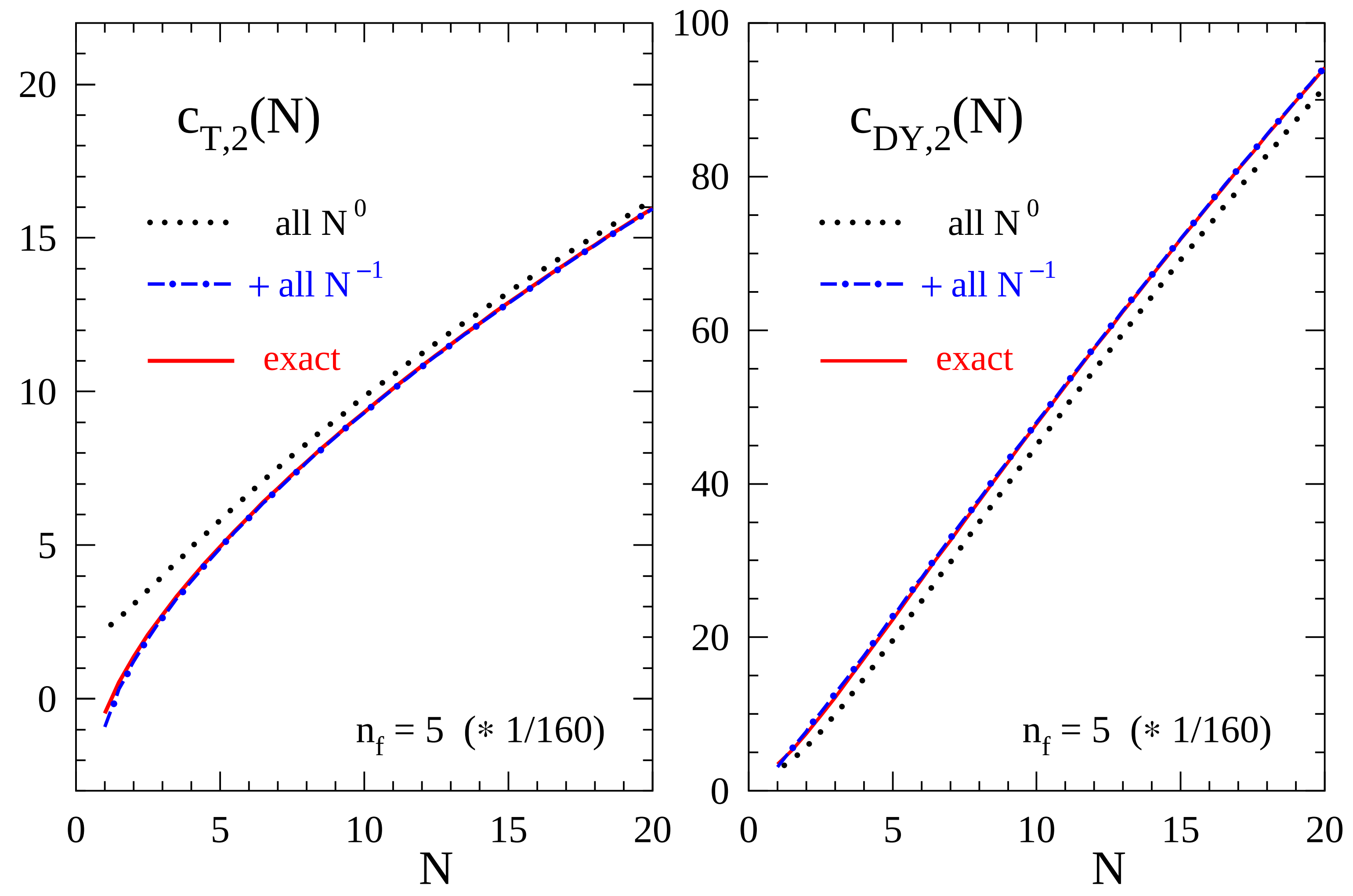
Second- and third-order N-space $C_{2,\mathrm{ns}}$ in DIS



N^{-1} terms relevant over full range shown, $\mathcal{O}(N^{-2})$ sizeable only at $N < 5$

Sum of $N^{-1} \ln^k N$ looks almost constant: half of maximum only at $N \simeq 150$

Second-order C_T in SIA and C_{DY} in N-space



DIS $\longrightarrow$ SIA $\longrightarrow$ DY : increase of the N^0 terms, N^{-1} corrections less important

$\overline{\text{MS}}$ splitting functions at small $x\,/\,N\!\to 1$ or 0

Logs in x-space $\Leftrightarrow$ poles in N-space, $\quad x^a \ln^n x \;\stackrel{\mathrm{M}}{=}\; \dfrac{(-1)^n n!}{(N+a)^{n+1}}$

Spacelike case: x^{-1} terms single-log enhanced

$$P_{\rm ij}^{(\ell)S} \;=\; x^{-1}(\,\#\ln^{\ell-\delta_{\rm iq}}x + \ldots\,) + (\,\#\ln^{2\ell}x + \ldots\,) \,+\, \ldots$$

x^{-1} **part:** BFKL (77/78); Jaroszewicz (82); Catani, Fiorani, Marchesini (89); Catani, Hautmann (94); ..., Fadin, Lipatov; Camici, Ciafaloni (98)

Non-singlet combinations: no x^{-1} terms, leading x^0 double logarithms :

Kirschner, Lipatov (83); Blümlein, A.V. (95)

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Timelike case: huge x^{-1} double logarithms

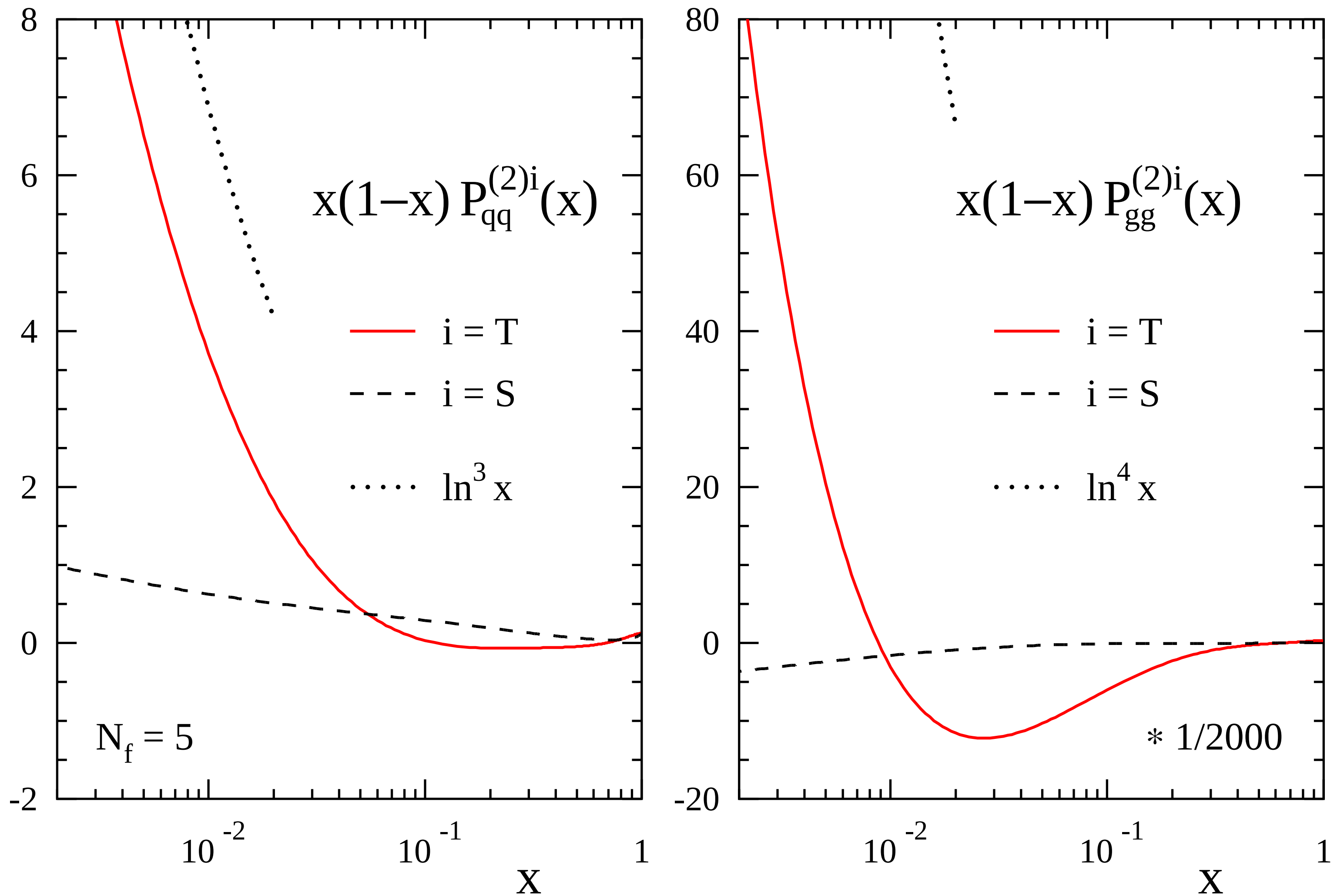
$$P_{\rm ij}^{(\ell)T} \;=\; x^{-1}(\#\,\ln^{2\ell-\delta_{\rm iq}}x + \ldots\,) + (\#\,\ln^{2\ell}x + \ldots\,) \,+\, \ldots$$

LL: Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82). NLL: Mueller (83)

– but latter not in $\overline{\text{MS}}$, see Albino, Bolzoni, Kniehl, Kotikov (2011)

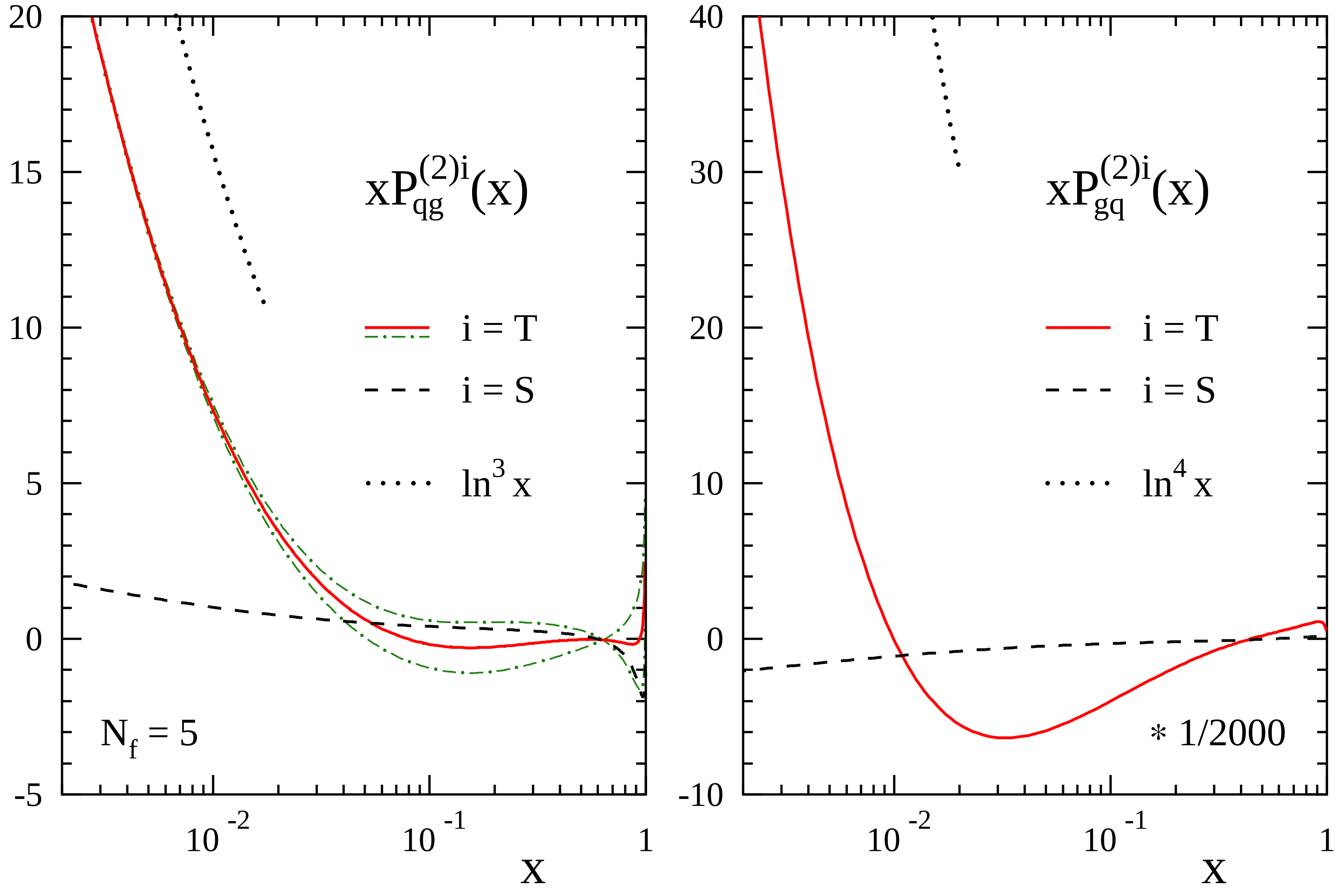
Behaviour of gauge-boson exchange coefficient functions analogous

Third-order diagonal splitting functions



T : small-x double logs, extreme rise from $x \approx 10^{-2}$ **Moch, A.V. (2007)**

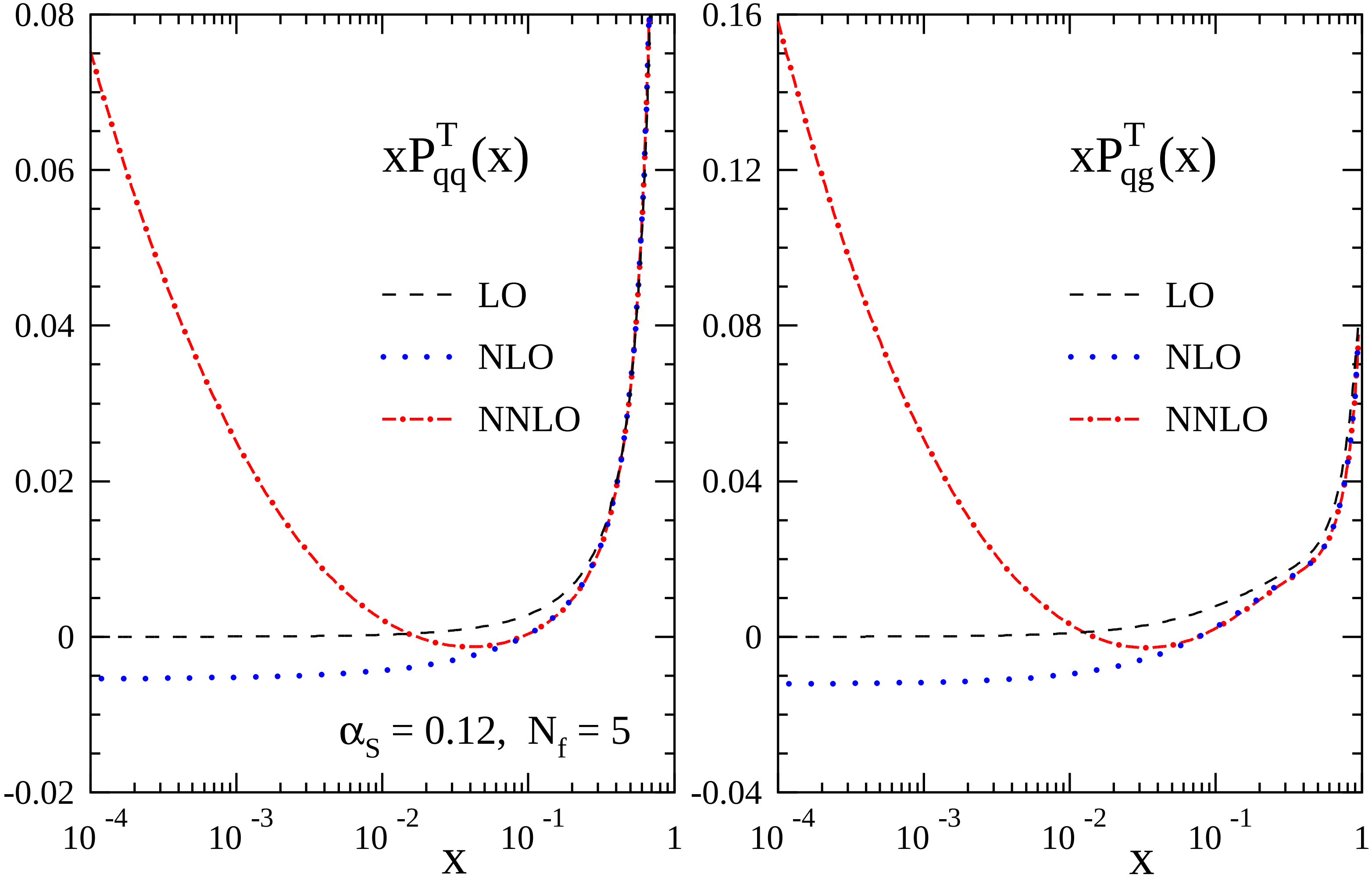
Third-order off-diagonal splitting functions



$q \rightarrow g$: not entirely fixed by Crewther-like ST-relation, $N = 2$, SUSY limit

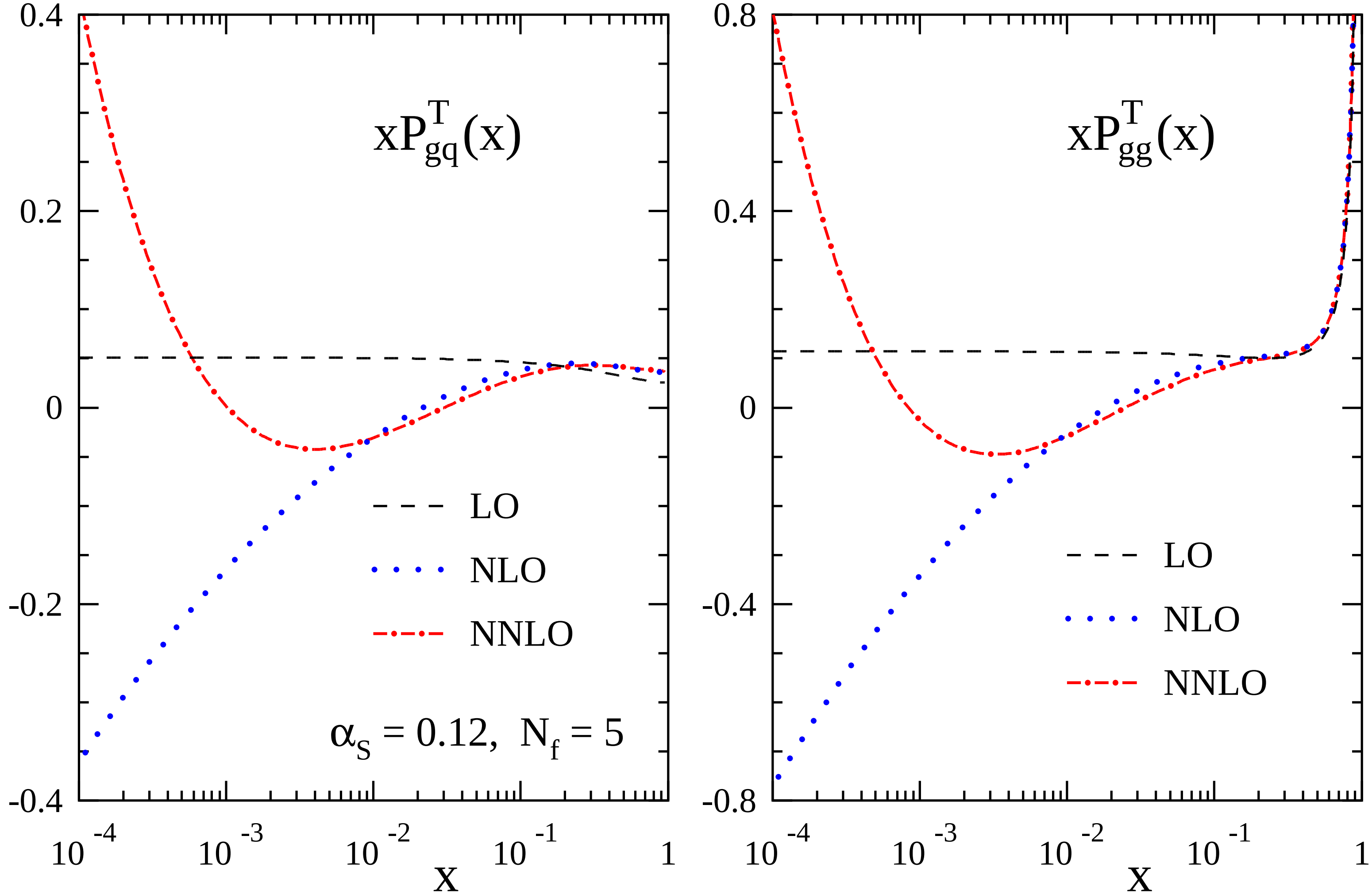
Dash-dotted: $\delta P_{\mathrm{qg}}^{(2)T}(x) = \pm\, 2\zeta_2 \beta_0\, (C_A - C_F)\, (11 + 24\, \ln x)\, P_{\mathrm{qg}}^{(0)T}(x)$

Fixed-order approximations for $P_{\mathrm{qi}}^{(2)T}(x, \alpha_s)$



NLO: no $x^{-1} \ln x$ terms. NNLO: up to $x^{-1} \ln^3 x$. Unstable at $x \lesssim 0.02$

Fixed-order approximations for $P^{(2)T}_{\mathrm{gi}}(x, \alpha_s)$



NLO/NNLO: terms up to $x^{-1}\ln^2 x$ / $x^{-1}\ln^4 x$. Unstable at $x \lesssim 0.005$

Non-singlet (NS) physical evolution kernels

Eliminate quark densities from scaling violations of observables ($\mu = Q$)

$$\frac{dF_a}{d\ln Q^2} \;=\; \frac{d\,C_a}{d\ln Q^2}\,q \,+\, C_a\,P\,q \;=\; \Big(\beta(a_s)\,\frac{d\,C_a}{da_s} \,+\, C_a\,P\Big)C_a^{-1}\,F_a$$

$$\;=\; \Big(P_a \,+\, \beta(a_s)\,\frac{d\ln C_a}{da_s}\Big)F_a \;=\; K_a\,F_a \;\equiv\; \sum_{\ell=0} a_s^{\ell+1} K_{a,\ell}\,F_a$$

K_a: physical kernel for the NS observable F_a in N-space.

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$$=\; \Big(P_a \,+\, \beta(a_s)\,\frac{d\ln C_a}{da_s}\Big) F_a \;=\; K_a F_a \;\equiv\; \sum_{\ell=0} a_s^{\,\ell+1} K_{a,\ell}\, F_a$$

K_a: physical kernel for the NS observable F_a in N-space. For $c_{a,0} = 1$:

$$K_a \;=\; a_s P_{a,0} \,+\, \sum_{l=1} a_s^{\,\ell+1} \Big(P_{a,\ell} - \sum_{k=0}^{\ell-1} \beta_k\, \tilde{c}_{a,\ell-k}\Big), \qquad a_s \equiv \alpha_s/(4\pi)$$

with

$$\tilde{c}_{a,1} \;=\; c_{a,1} \;\;,\quad \tilde{c}_{a,2} = 2\,c_{a,2} - c_{a,1}^{\,2}$$
$$\tilde{c}_{a,3} \;=\; 3\,c_{a,3} - 3\,c_{a,2}\,c_{a,1} + c_{a,1}^{\,3}$$
$$\tilde{c}_{a,4} \;=\; 4\,c_{a,4} - 4\,c_{a,3}\,c_{a,1} - 2\,c_{a,2}^{\,2} + 4\,c_{a,2}\,c_{a,1}^{\,2} - c_{a,1}^{\,4} \;\;,\quad \ldots$$

Manipulations of harmonic sums/polylogarithms, (inverse) Mellin transform

FORM3 + packages: Vermaseren (00); TFORM: Tentyukov, Vermaseren (07)

Large-x logarithms in the physical kernels

Soft limit $1-x \ll 1 \;\Leftrightarrow\;$ **large** $L \equiv \ln N$**: threshold exponentiation**

$$C_a(N) \;=\; g_0 \exp\{Lg_1(a_{\rm s}L) + g_2(a_{\rm s}L) + \ldots\} \;+\; \mathcal{O}(1/N)$$

$\Rightarrow$ **single-logarithmic (SL) enhancement of physical evolution kernels** K_a

$$K_a(N) \;=\; -\sum_{\ell=1} A_\ell a_{\rm s}^\ell\, L \;+\; \beta(a_{\rm s})\,\frac{d}{da_{\rm s}}\,\{Lg_1(a_{\rm s}L) + g_2(a_{\rm s}L) + \ldots\} \;+\; \ldots$$

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Crucial observation: all K_a singly enhanced to all orders in N^{-1} or $(1-x)$

DIS/SIA $a \neq L$ leading-logarithmic kernels, with $p_{\rm qq}(x) = 2/(1-x)_+ - 1 - x$

$$\begin{aligned}
K_{a,0}(x) \;&=\; 2\, C_F \, p_{\rm qq}(x) \\
K_{a,1}(x) \;&=\; \ln(1-x)\, p_{\rm qq}(x) \;\left[-2\, C_F \beta_0 \;\mp\; 8\, C_F^2 \;\ln x\right] \\
K_{a,2}(x) \;&=\; \ln^2(1-x)\, p_{\rm qq}(x) \left[\;\; 2\, C_F \beta_0^2 \;\pm\; 12\, C_F^2\, \beta_0 \;\ln x + \mathcal{O}(\ln^2 x)\right] \\
K_{a,3}(x) \;&=\; \ln^3(1-x)\, p_{\rm qq}(x) \left[-2\, C_F \beta_0^3 \mp 44/3\; C_F^2\, \beta_0^2 \;\ln x + \mathcal{O}(\ln^2 x)\right] \\
K_{a,4}(x) \;&=\; \ln^4(1-x)\, p_{\rm qq}(x) \left[\;\; 2\, C_F \beta_0^4 \;\pm\; \xi_{K_4} C_F^2\, \beta_0^3 \;\ln x + \mathcal{O}(\ln^2 x)\right]
\end{aligned}$$

First term: leading large n_f, all orders via C_2 of Mankiewicz, Maul, Stein (97)

Higher-order non-singlet predictions

Conjecture: Single-log behaviour of K_a persists to (all) higher orders in α_s

$\Leftrightarrow$ **resummation of the coefficient functions beyond soft $(1-x)^{-1}$ terms**

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$\Leftrightarrow$ resummation of the coefficient functions beyond soft $(1-x)^{-1}$ terms

Recall
$$\underbrace{\tilde{c}_{a,4}}_{\text{SL}} = \underbrace{4\, c_{a,4}}_{\text{DL, new}} \underbrace{-\, 4\, c_{a,3}\, c_{a,1} - 2\, c_{a,2}^{2} + 4\, c_{a,2}\, c_{a,1}^{2} - c_{a,1}^{4}}_{\text{DL, known for DIS/SIA}}$$

$\Rightarrow$ coefficients of highest three powers of $\ln(1-x)$ from fourth order in α_s,

i.e., $\ln^{7,6,5}(1-x)$ at order α_s^4 for $F_{1,2,3}$ in DIS and $F_{\mathrm{T,I,A}}$ in SIA

Higher-order non-singlet predictions

Conjecture: Single-log behaviour of K_a persists to (all) higher orders in α_s

$\Leftrightarrow$ resummation of the coefficient functions beyond soft $(1-x)^{-1}$ terms

Recall $$\underbrace{\tilde{c}_{a,4}}_{\text{SL}} \;=\; \underbrace{4\, c_{a,4}}_{\text{DL, new}} \;\underbrace{-\, 4\, c_{a,3}\, c_{a,1} - 2\, c_{a,2}^{\,2} + 4\, c_{a,2}\, c_{a,1}^{\,2} - c_{a,1}^{\,4}}_{\text{DL, known for DIS/SIA}}$$

$\Rightarrow$ coefficients of highest three powers of $\ln(1-x)$ from fourth order in α_s,

i.e., $\ln^{7,6,5}(1-x)$ at order α_s^4 for $F_{1,2,3}$ in DIS and $F_{\mathrm{T,I,A}}$ in SIA

Leading terms: $K_1 = K_2$, $K_{\mathrm{T}} = K_{\mathrm{I}}$ [total ('integrated') fragmentation fct.]

$\Rightarrow$ also three logarithms for space- and timelike F_L: $\ln^{6,5,4}(1-x)$ at α_s^4 etc

Alternative derivation: physical kernels for F_L, agreement non-trivial check

Drell-Yan: only NNLO known $\Rightarrow$ only two logarithms fully predicted from α_s^3

Example: α_s^4 coefficient function for F_1 in DIS

$$
\begin{aligned}
c^{(4)}_{1,\mathrm{ns}}(x) \;=\; & \big(\ln^7(1-x)\, 8/3\, C_F^4 - \ln^6(1-x)\, 14/3\, C_F^3\, \beta_0 + \ln^5(1-x)\, 8/3\, C_F^2\, \beta_0^2\big)\, p_{\mathrm{qq}}(x) \\
& + \ln^6(1-x)\;\big[C_F^4\,\{\,p_{\mathrm{qq}}(x)\,(-14 - 68/3\,\mathrm{H}_0) + 4 + 8\,\mathrm{H}_0 - (1-x)(6 + 4\,\mathrm{H}_0)\}\big] \\
& + \ln^5(1-x)\;\Big[C_F^4\,\Big\{p_{\mathrm{qq}}(x)\,(-9 - 8\,\widetilde{\mathrm{H}}_{1,0} + 448/3\,\mathrm{H}_{0,0} + 84\,\mathrm{H}_0 - 64\,\zeta_2) + 48\,\widetilde{\mathrm{H}}_{1,0} \\
& \qquad -22 - 96\,\mathrm{H}_{0,0} - 104\,\mathrm{H}_0 - (1-x)(13 + 24\,\widetilde{\mathrm{H}}_{1,0} - 48\,\mathrm{H}_{0,0} - 84\,\mathrm{H}_0 - 16\,\zeta_2)\Big\} \\
& \quad + C_F^3\,\beta_0\,\{\,p_{\mathrm{qq}}(x)\,(41 + 316/9\,\mathrm{H}_0) - 10 - 32/3\,\mathrm{H}_0 + (1-x)(41/3 + 16/3\,\mathrm{H}_0)\} \\
& \quad + C_F^3\,C_A\,\Big\{p_{\mathrm{qq}}(x)\,(16 + 8\,\widetilde{\mathrm{H}}_{1,0} + 8\,\mathrm{H}_{0,0} - 24\,\zeta_2) + 4 + (1-x)(28 - 8\,\zeta_2)\Big\} \\
& \quad + C_F^3\,(C_A - 2\,C_F)\,p_{\mathrm{qq}}(-x)\,(16\,\widetilde{\mathrm{H}}_{-1,0} - 8\,\mathrm{H}_{0,0})\Big] \;+\; \mathcal{O}\big(\ln^4(1-x)\big)
\end{aligned}
$$

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& + \ln^6(1-x)\; \Big[C_F^4 \,\{ p_{\mathrm{qq}}(x)\, (-14 - 68/3\, \mathrm{H}_0) + 4 + 8\, \mathrm{H}_0 - (1-x)(6 + 4\, \mathrm{H}_0) \} \Big] \\
& + \ln^5(1-x)\; \bigg[C_F^4 \Big\{ p_{\mathrm{qq}}(x)\, (-9 - 8\, \widetilde{\mathrm{H}}_{1,0} + 448/3\, \mathrm{H}_{0,0} + 84\, \mathrm{H}_0 - 64\, \zeta_2) + 48\, \widetilde{\mathrm{H}}_{1,0} \\
& \qquad -22 - 96\, \mathrm{H}_{0,0} - 104\, \mathrm{H}_0 - (1-x)(13 + 24\, \widetilde{\mathrm{H}}_{1,0} - 48\, \mathrm{H}_{0,0} - 84\, \mathrm{H}_0 - 16\, \zeta_2) \Big\} \\
& \quad + C_F^3\, \beta_0 \,\{ p_{\mathrm{qq}}(x)\, (41 + 316/9\, \mathrm{H}_0) - 10 - 32/3\, \mathrm{H}_0 + (1-x)(41/3 + 16/3\, \mathrm{H}_0) \} \\
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& \quad + C_F^3\, (C_A - 2\, C_F)\, p_{\mathrm{qq}}(-x)\, (16\, \widetilde{\mathrm{H}}_{-1,0} - 8\, \mathrm{H}_{0,0}) \bigg] \; + \; \mathcal{O}\big(\ln^4(1-x)\big)
\end{aligned}
$$

First line includes identity of coefficients of leading $\ln^k(1-x)$ and $\dfrac{\ln^k(1-x)}{x-1}$ terms

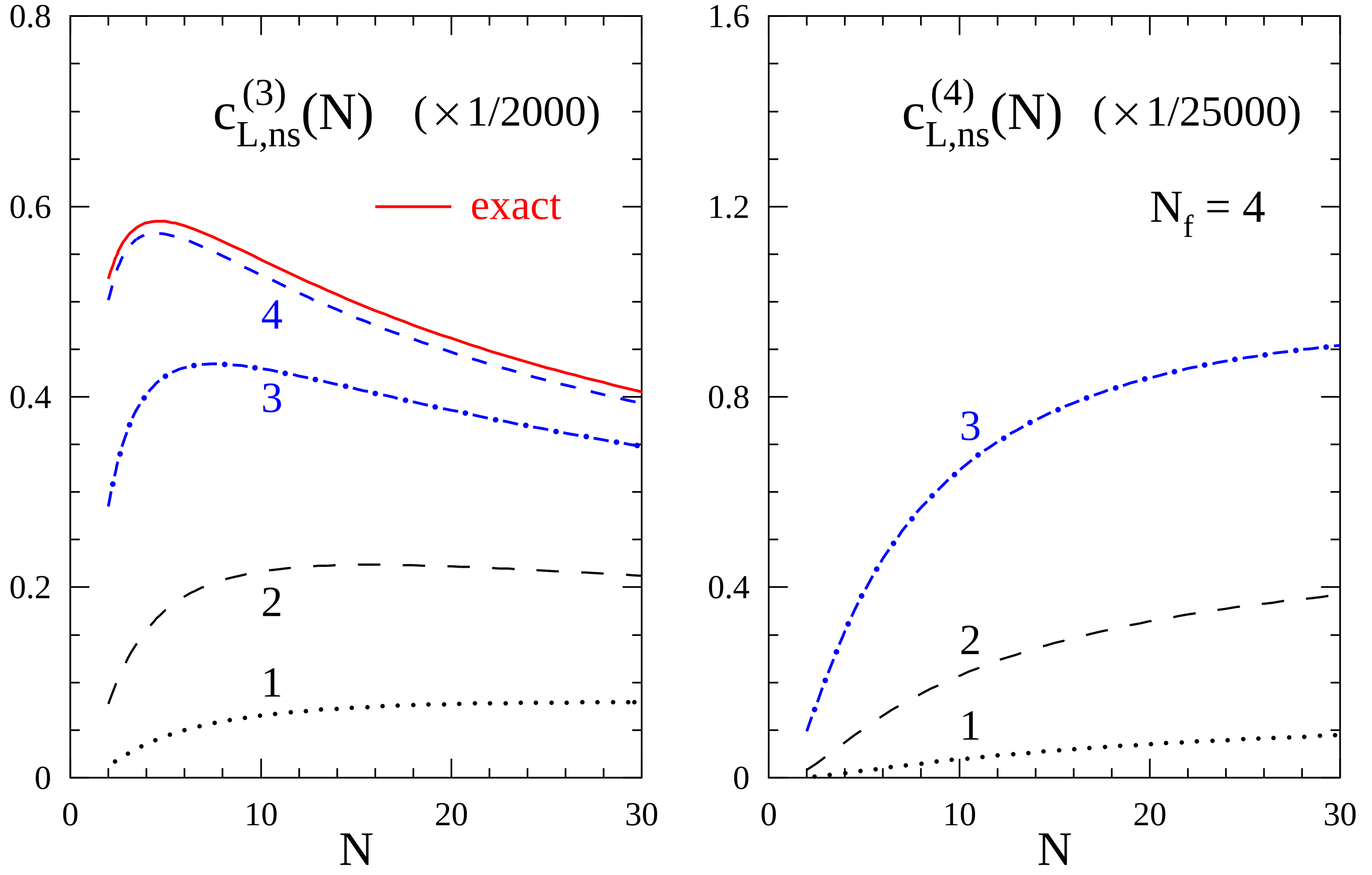
Conjectured by Krämer, Laenen, Spira (97)

Modified basis $\widetilde{\mathrm{H}}_{m_1,m_2,\ldots} \equiv \widetilde{\mathrm{H}}_{m_1,m_2,\ldots}(x)$ of harmonic polylogarithms, e.g.,

$$\widetilde{\mathbf{H}}_{1,0} \;=\; \mathbf{H}_{1,0} + \zeta_2 \;\;,\;\; \widetilde{\mathbf{H}}_{1,1,0} \;=\; \mathbf{H}_{1,1,0} - \zeta_2\, \ln(1-x) - \zeta_3$$

All $\ln(1-x)$ terms and ζ-functions taken out of expansions to all orders in $1-x$

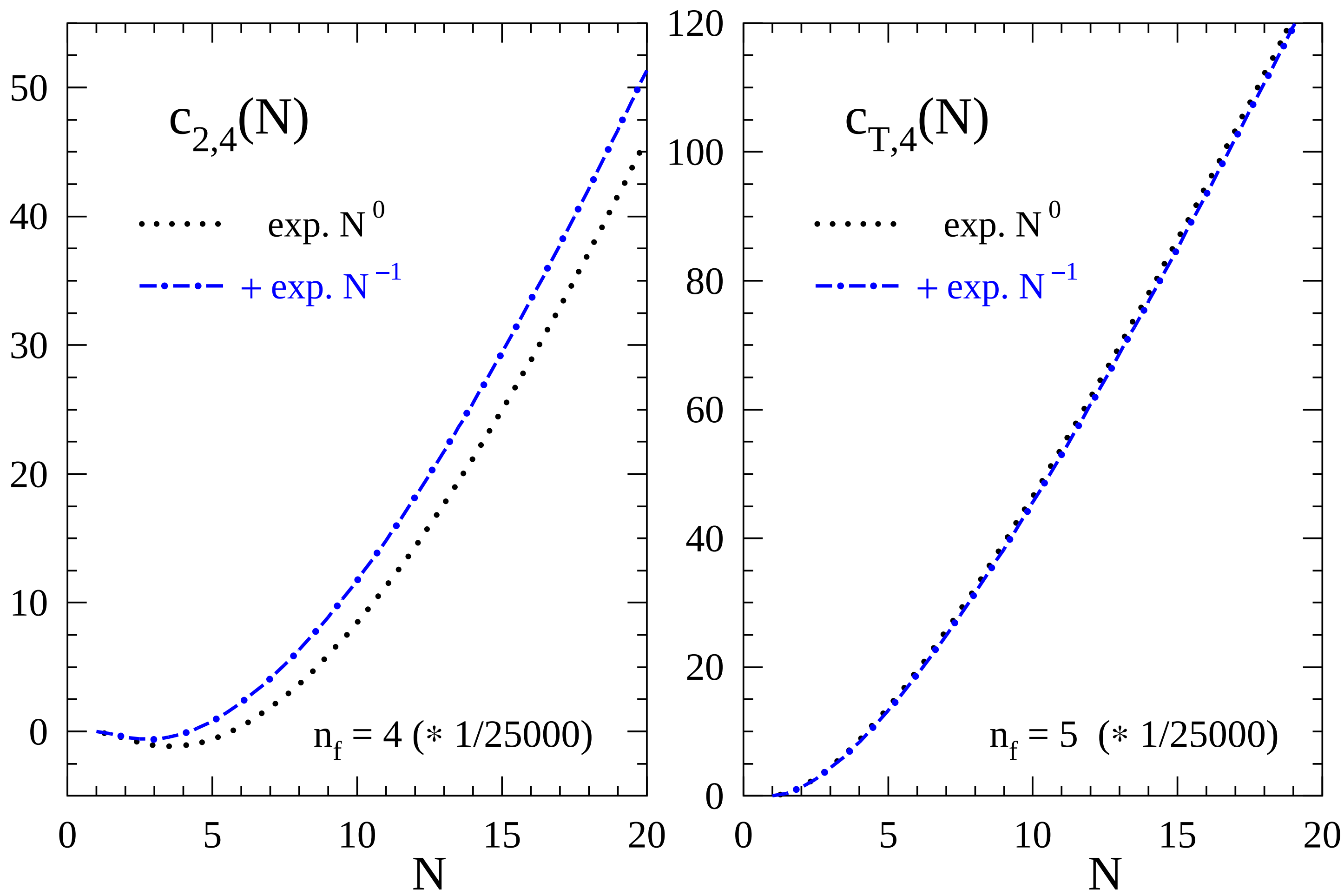
Third- and fourth-order C_L in DIS in N-space



1 = leading log etc. Good α_s^3 approximation by all four N^{-1} logarithms

As usual, cf. small-x: leading logs do not lead. Padé: ≈ 2.0 at $N = 20$

Fourth-order C_2 (DIS) and C_T (SIA) at large N



Exp. N^0: 7 of 8 logs, exp. N^{-1}: 4 of 7 logs – ξ_{K_4} numerically suppressed

N^{-1} contributions again relevant for F_2, but small for F_T at least at $N > 5$

All-order resummation of the $1/N$ terms (I)

For $F_{1,2,3}$, $F_{\mathrm{T,I,A}}$ and F_{DY}, up to terms of order $1/N^2$, with $L \equiv \ln N$

$$C_a(N) - C_a\Big|_{N^0 L^k} \;=\; \frac{1}{N}\left(\left[d_{a,1}^{(1)} L + d_{a,0}^{(1)}\right] a_{\mathsf{s}} + \left[\widetilde{d}_{a,1}^{(2)} L + d_{a,0}^{(2)}\right] a_{\mathsf{s}}^{\,2} + \ldots\right)$$

$$\exp\left\{L h_1(a_{\mathsf{s}} L) + h_2(a_{\mathsf{s}} L) + a_{\mathsf{s}} h_3(a_{\mathsf{s}} L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_{\mathsf{s}} L) \;\equiv\; \sum_{n=1} h_{kn}\,(a_{\mathsf{s}} L)^n$

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$$\exp\left\{L h_1(a_{\mathrm{s}}L) + h_2(a_{\mathrm{s}}L) + a_{\mathrm{s}} h_3(a_{\mathrm{s}}L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_{\mathrm{s}}L) \;\equiv\; \sum_{n=1} h_{kn}(a_{\mathrm{s}}L)^n$

Coefficients for DIS/SIA (upper/lower sign) relative to $N^0 L^k$ resummation

$g_{lk} =$ **coefficients in soft-gluon exponentiation**

$$h_{1k} \;=\; g_{1k}$$
$$h_{21} \;=\; g_{21} + \frac{1}{2}\beta_0 \pm 6\,C_F$$
$$h_{22} \;=\; g_{22} + \frac{5}{24}\beta_0^2 \pm \frac{17}{9}\beta_0\,C_F - 18\,C_F^2$$
$$h_{23} \;=\; g_{23} + \frac{1}{8}\beta_0^3 \pm \left(\frac{\xi_{\mathrm{K}_4}}{8} - \frac{53}{18}\right)\beta_0^2 C_F - \frac{34}{3}\beta_0\,C_F^2 \pm 72\,C_F^3$$

First term of h_3 also known, but non-universal within DIS and SIA ($\Leftrightarrow F_L$)

All-order resummation of the $1/N$ terms (II)

For space-like (-) and time-like (+) structure/fragmentation functions F_L

$$C_L^{(\pm)}(N) \;=\; N^{-1}(d_1^{(\pm)} a_s + d_2^{(\pm)} a_s^2 + \ldots)\;\exp\{L h_1(a_s L) + h_2(a_s L) + \ldots\}$$

with

$$\begin{aligned}
h_{11} &= 2\,C_F \;, \quad h_{12} \;=\; \frac{2}{3}\,\beta_0\,C_F \;, \quad h_{13} \;=\; \frac{1}{3}\,\beta_0^2\,C_F \\
h_{21} &= \beta_0 \;+\; 4\,\gamma_e C_F \;-\; C_F \;+\; (4 - 4\,\zeta_2)(C_A - 2C_F) \\
h_{22} &= \underbrace{\frac{1}{2}\,(\,\beta_0\,h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} \;-\; \underbrace{8\,(C_A - 2C_F)^2(1 - 3\,\zeta_2 + \zeta_3 + \zeta_2^{\,2}\,)}_{\text{Who ordered THIS?}}
\end{aligned}$$

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For space-like (-) and time-like (+) structure/fragmentation functions F_L

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with

$$h_{11} \;=\; 2\,C_F \;, \quad h_{12} \;=\; \frac{2}{3}\,\beta_0\,C_F \;, \quad h_{13} \;=\; \frac{1}{3}\,\beta_0^2\,C_F$$

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$$h_{22} \;=\; \underbrace{\frac{1}{2}\,(\,\beta_0\,h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} \;-\; \underbrace{8\,(C_A - 2C_F)^2(1 - 3\,\zeta_2 + \zeta_3 + \zeta_2^{\,2}\,)}_{\text{Who ordered THIS?}}$$

Remarks/questions

- Less predictive than $N^0 L^k$ exponentiation: nothing new, but A_2, in g_{22}
- NLL exponentiation – complete $h_2(a_s L)$ – could be feasible for $F_{a\neq L}$
- NNLL exponentiation for $F_{1,2,3}$ etc, NLL for F_L: possible at all?

Singlet physical evolution kernel for (F_2, F_ϕ)

F_ϕ: Higgs-exchange DIS in heavy-top limit, to order α_s^2 also by

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (09)

As in the non-singlet case above, but with 2-vectors$/2\times 2$ matrices P_{ij} and

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,\mathrm{q}} & C_{2,\mathrm{g}} \\ C_{\phi,\mathrm{q}} & C_{\phi,\mathrm{g}} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}$$

Furmanski, Petronzio (81); ...

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Furmanski, Petronzio (81); ...

$$\frac{dF}{d\ln Q^2} = \frac{dC}{d\ln Q^2}\, q + CP\, q = \Big(\beta(a_s)\,\frac{dC}{da_s} + CP\Big) C^{-1} F$$

$$= \Big(\underbrace{\beta(a_s)\,\frac{d\ln C}{da_s}} + \underbrace{[C,P]\,C^{-1} + P}\Big) F = KF$$

DL (ns + ps) DL (singlet only)

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$$= \Big(\underbrace{\beta(a_\mathrm{s})\,\frac{d\ln C}{da_\mathrm{s}}}_{\text{DL (ns + ps)}} + \underbrace{[C,P]\,C^{-1} + P}_{\text{DL (singlet only)}}\Big)F = KF$$

Observation at NLO, NNLO: single-log enhancement to all powers of $1-x$

$$K_{ab}^{(n)} \sim \ln^n(1-x) + \ \ldots\ , \quad \text{leading } K_{22/\phi\phi}^{(n)} \text{ same as NS}/C_F = 0$$

Conjecture: this behaviour persists to N^{3}LO

$\Rightarrow$ prediction of $\ln^{6,5,4}(1-x)$ of $P^{(3)}_{\mathrm{qg,gq}}$ [and $\ln^{5,4,3}(1-x)$ of $P^{(3)}_{\mathrm{ps,gg}|_{C_F}}$]

Example: α_s^4 splitting function $P_{\rm qg}^{(3)}(x)$

For brevity: only $(1-x)^0$ part shown – known to all powers, $C_{AF} \equiv C_A - C_F$

$$
\begin{aligned}
P_{\rm qg}^{(3)}(x) \quad = \quad & \ln^6(1-x) \;\cdot 0 \\
& + \ln^5(1-x)\left[\,\frac{22}{27}\,C_{AF}^3 n_f - \frac{14}{27}\,C_{AF}^2 C_F n_f + \frac{4}{27}\,C_{AF}^2 n_f^2\right] \\
& + \ln^4(1-x)\left[\,\left(\frac{293}{27} - \frac{80}{9}\,\zeta_2\right) C_{AF}^3 n_f + \left(\frac{4477}{16} - 8\zeta_2\right) C_{AF}^2 C_F n_f \right. \\
& \qquad \left. - \frac{13}{81}\,C_{AF} C_F^2 n_f - \frac{116}{81}\,C_{AF}^2 n_f^2 + \frac{17}{81}\,C_{AF} C_F n_f^2 - \frac{4}{81}\,C_{AF} n_f^3\right] \\
& + \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}
$$

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$$
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&+ \ln^5(1-x)\left[\frac{22}{27}\,C_{AF}^3 n_f - \frac{14}{27}\,C_{AF}^2 C_F n_f + \frac{4}{27}\,C_{AF}^2 n_f^2\right] \\
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&+ \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}
$$

- Vanishing of the coefficient of the leading term at order α_s^4 :
  accidental (??) cancellation of contributions, for all four splitting fct's
- Remaining terms vanish in the supersymmetric case $C_A = C_F (= n_f)$
  Nontrivial check: same as for $P_{qg}^{(2)}$, not obvious from above construction

Singlet physical evolution kernel for (F_2, F_L)

As above, but with $F_\phi \to \widehat{F}_L = F_L / a_s c_{L,q}^{(0)}$, hence $\widehat{c}_{L,\mathrm{q/g}}^{(n)} \sim \{1/\frac{1}{N}\} \ln^{2n} N$

$$F = \begin{pmatrix} F_2 \\ \widehat{F}_L \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1 & \widehat{c}_{L,\mathrm{g}}^{(0)} \end{pmatrix} + \sum_{n=1} a_{\mathrm{s}}^n \begin{pmatrix} c_{2,\mathrm{q}}^{(n)} & c_{2,\mathrm{g}}^{(n)} \\ \widehat{c}_{L,\mathrm{q}}^{(n)} & \widehat{c}_{L,\mathrm{g}}^{(n)} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\mathrm{L}} \\ K_{\mathrm{L}2} & K_{\mathrm{LL}} \end{pmatrix}$$

Catani (96), Blümlein, Ravindran, van Neerven (00) [different normalization]

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Catani (96), Blümlein, Ravindran, van Neerven (00) [different normalization]

Observation: single-log enhancement of N^0 part of K at NLO and NNLO

N^{3}LO conjecture + above $P^{(3)}_{qg}$: prediction of three double logs in $c^{(3)}_{L,q/g}$, e.g.

$$\begin{aligned} N^2 c^{(3)}_{L,g}(N) \quad = \quad & \ln^6 N \, \frac{32}{3} C_A^3 n_f \\ & + \ln^5 N \left[\frac{1504}{9} C_A^3 n_f - \frac{64}{9} C_A^2 n_f^2 - \frac{104}{3} C_A^2 n_f C_F - \frac{40}{3} n_f C_F^2 \right] \\ & + \ln^4 N \left[\text{known coefficients} \right] + \mathcal{O}\left(\ln^3 N\right) \end{aligned}$$

Agrees with/extends results [NS-like $C_F = 0$ part of $C_{L,g}$ only] of MV (02/09)

Threshold logarithms before factorization (I)

Unfactorized partonic structure functions in $D = 4-2\varepsilon$ dimensions

$$T_{a,j} = \widetilde{C}_{a,i}\, Z_{ij}\;, \quad -\gamma \equiv P = \frac{dZ}{d\ln Q^2}\, Z^{-1}\;, \quad \frac{da_s}{d\ln Q^2} = -\varepsilon a_s + \beta_{D=4}$$

a_s^n**:** $\varepsilon^{-n} \ldots \varepsilon^{-2}$**: lower-order terms,** ε^{-1}**:** n**-loop splitting functions + . . . ,**
ε^0**:** n**-loop coefficient fct's + . . . ,** ε^k**,** $0 < k < l$**: required for order** $n+l$

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N^0 and N^{-1} transition functions Z to next-to-leading log (NLL) accuracy

$$Z\Big|_{a_{\rm s}^n} = \frac{1}{\varepsilon^n}\frac{\gamma_0^{n-1}}{n!}\left[\gamma_0 - \frac{\beta_0}{2} n(n-1)\right] + \sum_{\ell=1}^{n-1} \frac{1}{\varepsilon^{n-\ell}} \sum_{k=1}^{n-\ell-1} \gamma_0^{n-\ell-k-1} \gamma_\ell \gamma_0^k \frac{(\ell+k)!}{n!\,\ell!}$$
$$- \frac{\beta_0}{2} \sum_{\ell=1}^{n-2} \frac{1}{\varepsilon^{n-\ell}} \sum_{k=1}^{n-\ell-2} \gamma_0^{n-\ell-k-2} \gamma_\ell \gamma_0^k \frac{(\ell+k)!}{n!\,\ell!} (n(n-1) - \ell(\ell+k+1))$$
$$+\ \textbf{NNLL contributions (explicit expressions)}\ +\ \ldots$$

$\varepsilon^{-n+\ell}$ **off-diagonal entries: contributions up to** $N^{-1} \ln^{n+\ell-1} N$
Diagonal cases: γ_0 **only for** N^0 **part, second term with** $\ell = 1$ **for** N^{-1} **NLL**

Threshold logarithms before factorization (II)

D-dimensional coefficient functions $\widetilde{C}_a$: finite for $\varepsilon \to 0$

$$\widetilde{C}_{a,i} = 1_{\text{(diagonal cases)}} + \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} a_{\mathrm{s}}^{n} \, \varepsilon^{\ell} \, c_{a,i}^{(n,\ell)}$$

$c_{a,i}^{(n,\ell)}$: ℓ additional factors $\ln N$ relative to $c_{a,i}^{(n,0)} \equiv c_{a,i}^{(n)}$ discussed above

Full N^{m}LO calc. of $T_{a,j}$: highest $m+1$ powers of ε^{-1} to all orders in α_{s}

Extension to all powers of ε: all-order resummation of highest $m+1$ logs

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Full N^{m}LO calc. of $T_{a,j}$: highest $m+1$ powers of ε^{-1} to all orders in $\alpha_{\rm s}$

Extension to all powers of ε: all-order resummation of highest $m+1$ logs

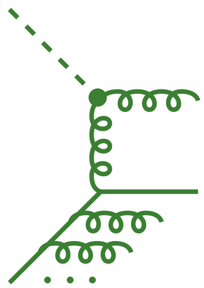
Example: Leading-log (LL) $1/N$ terms of $T_{\phi,\rm q}^{(n)}$ and $T_{2,\rm g}^{(n)}$, with $L \equiv \ln N$

$$\frac{1}{C_F} T_{\phi,\rm q}^{(n)} = \frac{1}{n_f} T_{2,\rm g}^{(n)} = \frac{L^{n-1}}{N \varepsilon^{n}} \sum_{k=0}^{\infty} (\varepsilon L)^{k} \mathcal{L}_{n,k} \left(C_F^{n-1} + C_F^{n-2} C_A + \ldots + C_A^{n-1} \right)$$

to all orders in ε (calc. + D-dim. structure), with same coefficients $\mathcal{L}_{n,k}$

$\Rightarrow$ all-order relation for one colour structure of either amplitude sufficient

All-order off-diagonal leading-log amplitudes



$$T_{\phi,\mathrm{q}}^{(n)}\Big|_{C_F\ \mathrm{only}} \stackrel{\mathrm{LL}}{=} \frac{1}{n}\, T_{\phi,\mathrm{q}}^{(1)}\, \underbrace{T_{2,\mathrm{q}}^{(n-1)}}_{\frac{1}{(n-1)!}\,(T_{2,\mathrm{q}}^{(1)})^{n-1}} \stackrel{\mathrm{LL}}{=} \frac{1}{n!}\, T_{\phi,\mathrm{q}}^{(1)}\, (T_{2,\mathrm{q}}^{(1)})^{n-1}$$

Three-loop diagram calculation + $P_{\mathrm{gq}}^{(3)} \stackrel{\mathrm{LL}}{=} 0$ + general mass factorization:

first four powers in ε known at any order. Rest $\longrightarrow$ higher-order predictions

$$T_{\phi,\mathrm{q}}\Big|_{C_F\ \mathrm{only}} \stackrel{\mathrm{LL}}{=} T_{\phi,\mathrm{q}}^{(1)}\, \frac{\exp(a_{\mathrm{s}} T_{2,\mathrm{q}}^{(1)}) - 1}{T_{2,\mathrm{q}}^{(1)}}$$

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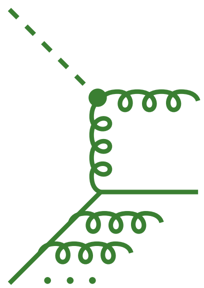

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Exact D-dimensional leading-log expressions for the one-loop amplitudes

$$T_{\phi,\mathrm{q}}^{(1)} \stackrel{\mathrm{LL}}{=} -2C_F\, \frac{1}{\varepsilon}(1-x)^{-\varepsilon} \stackrel{\mathrm{M}}{=} -\frac{2C_F}{N}\, \frac{1}{\varepsilon}\exp(\varepsilon \ln N)$$

$$T_{2,\mathrm{q}}^{(1)} \stackrel{\mathrm{LL}}{=} -4C_F\, \frac{1}{\varepsilon}(1-x)^{-1-\varepsilon} + \text{virtual} \stackrel{\mathrm{M}}{=} 4C_F\, \frac{1}{\varepsilon^2}(\exp(\varepsilon \ln N) - 1)$$

$\Rightarrow$ **leading-log expression for $T_{\phi,\mathrm{q}}$ and $T_{2,\mathrm{g}}$ completely determined**

Leading-log splitting and coefficient functions

Expansions and iterative mass factorization to 'any' order [done in FORM]

$\Rightarrow$ **All-order expressions for LL off-diagonal splitting and coefficient fct's**

$$\boxed{P_{\mathrm{qg}}^{\mathrm{LL}}(N, \alpha_{\mathrm{s}}) \;=\; \frac{n_f}{N}\,\frac{\alpha_{\mathrm{s}}}{2\pi}\sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,\tilde{a}_{\mathrm{s}}^{\;n}\,, \qquad \tilde{a}_{\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{\pi}\,(C_A - C_F)\,\ln^2 N}$$

Bernoulli numbers B_n: zero for odd $n \geq 3$ $\;\Rightarrow\;$ $P_{\mathrm{gq}}^{(3)}(N) \stackrel{\mathrm{LL}}{=} 0$ not accidental

$$B_0 = 1\,,\; B_1 = -\frac{1}{2}\,,\; B_2 = \frac{1}{6}\,,\; B_4 = -\frac{1}{30}\,,\; B_6 = \frac{1}{42}\,,\; \ldots\,,\; B_{12} = -\frac{691}{2730}\,,\; \ldots$$

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$$\boxed{C_{2,\rm g}^{\rm LL} \;=\; \frac{1}{2N\ln N}\,\frac{n_f}{C_A - C_F}\left\{\exp(2C_F\, a_{\rm s}\ln^2 N)\,\mathcal{B}_0(\tilde{a}_{\rm s}) - \exp(2C_A\, a_{\rm s}\ln^2 N)\right\}}$$

$\exp(\ldots)$: LL soft-gluon exponentials   Parisi; Curci, Greco; Amati et al. (80)

$$\mathcal{B}_0(x) \;=\; \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,x^n$$

$P_{\rm gq}^{\rm LL}$, $C_{\phi,\rm q}^{\rm LL}$: same functions but with $C_F \leftrightarrow C_A$ (also in $\tilde{a}_{\rm s}$), then $n_f \to C_F$

First properties of the new $\mathcal{B}$-functions

Relation between even-n Bernoulli numbers and the Riemann ζ-function

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\,\zeta_{2n}\left(\frac{x}{2\pi}\right)^{2n}$$

$\mathcal{B}_0(2\pi i)$ **numerically known (Wolfram MathWorld, Sloane's A093721), no closed form**

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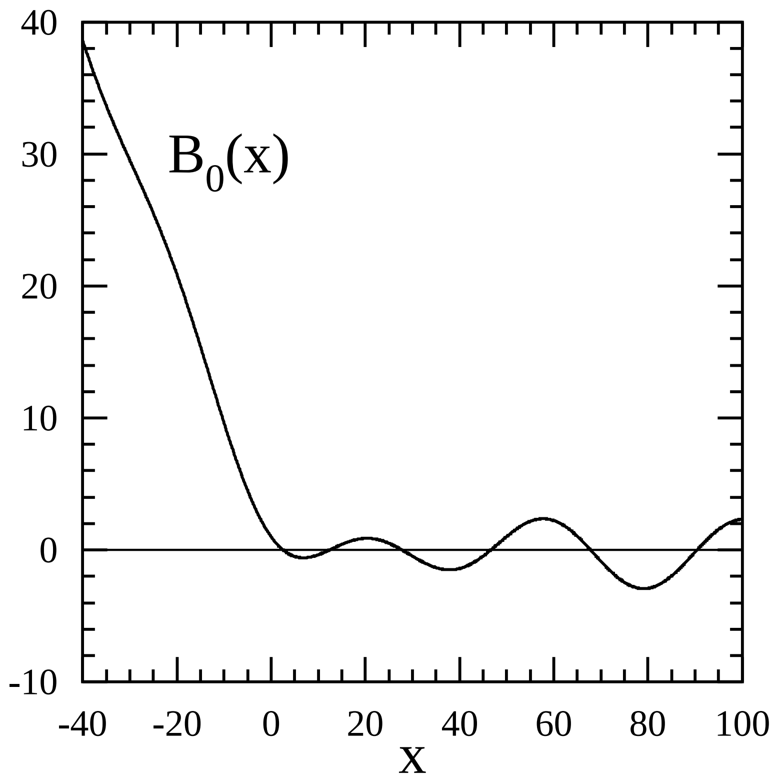


Further $\mathcal{B}$-functions for later use

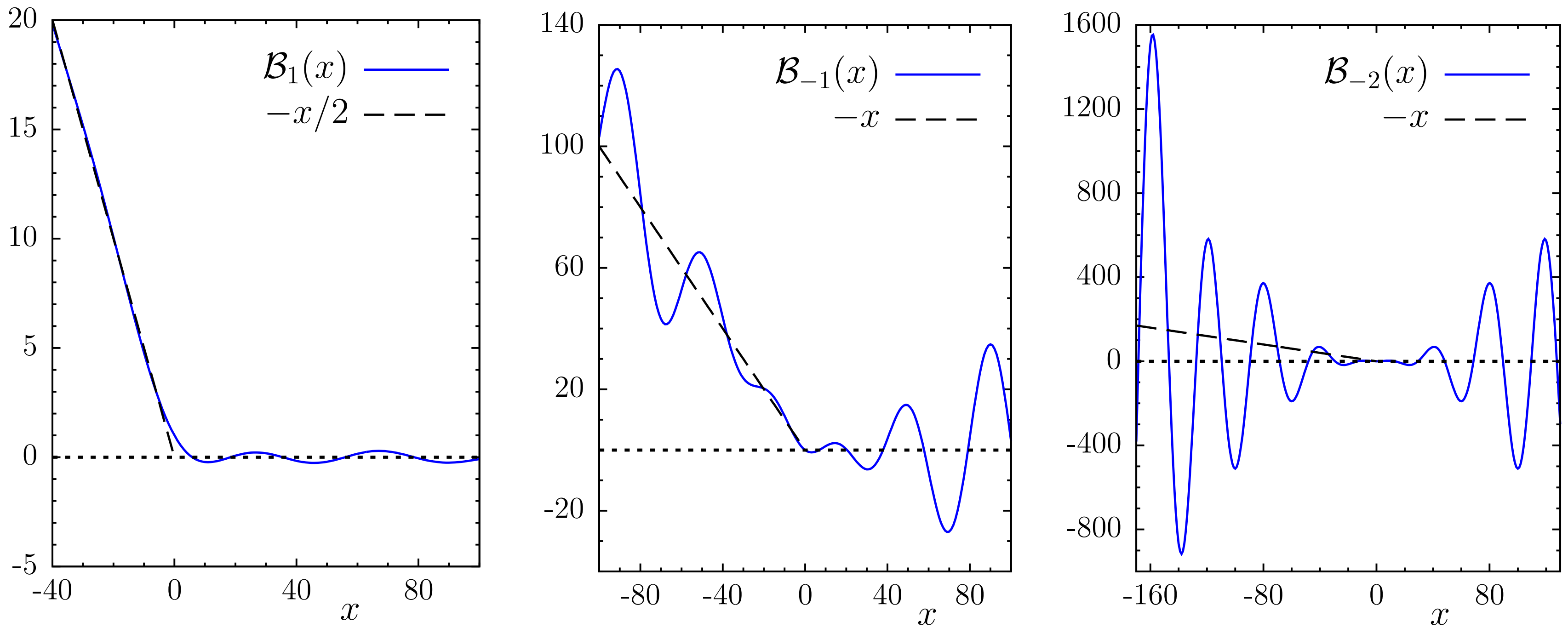
$$\mathcal{B}_k(x) \;=\; \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!}\,x^n$$

$$\mathcal{B}_{-k}(x) \;=\; \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!}\,x^n$$

Relations to $\mathcal{B}_0(x)$

$$\frac{d^k}{dx^k}\left(x^k\mathcal{B}_k\right) = \mathcal{B}_0\,, \quad x^k\frac{d^k}{dx^k}\,\mathcal{B}_0 = \mathcal{B}_{-k}$$

NLL: $\mathcal{B}$-functions with index unequal zero



$x > 0$: all functions $\mathcal{B}_k(x)$ oscillate about $y = 0$

$x < 0$: oscillations about $y = -\frac{x}{(k+1)!}$ for $k \geq 0$ and $y = -x$ for $k < 0$

Amplitudes increase very rapidly with decreasing k

Oscillation of $\mathcal{B}_0$ continuous (much more irregularly) to very large x

D. Broadhurst, private communication

Next-to-leading logarithmic iteration for $T_{\phi,\mathrm{q}}^{(n)}$

Ansatz for $T_{\phi,\mathrm{q}}^{(n)}$ in terms of first-order quantity and diagonal amplitudes

$$T_{\phi,\mathrm{q}}^{(n)} \stackrel{\mathrm{NL}}{=} \frac{1}{n}\, T_{\phi,\mathrm{q}}^{(1)} \left\{ \sum_{i=0}^{n-1} T_{\phi,\mathrm{q}}^{(i)}\, T_{2,\mathrm{q}}^{(n-i-1)} f(n,i) - \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} T_{\phi,\mathrm{q}}^{(i)}\, T_{2,\mathrm{q}}^{(n-i-2)} g(n,i) \right\}$$

All-order agreement with known highest four powers of ε^{-1} for

$$\begin{aligned} f(n,i) &= \binom{n-1}{i}^{-1} \left[1 + \varepsilon \left(\frac{\beta_0}{8C_A}\, (i+1)(n-i)\, \theta_{i1} - \frac{3}{2}\, (1 - n\, \delta_{i0}) \right) \right] \\ g(n,i) &= \binom{n}{i+1}^{-1} \end{aligned}$$

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$$g(n,i) = \binom{n}{i+1}^{-1}$$

Soft-gluon exponentiation: also $T_{\phi,\mathrm{g}}^{(n)}$ and $T_{2,\mathrm{q}}^{(n)}$ known at all powers of ε

$\Rightarrow$ next-to-leading logarithmic expression for $T_{\phi,\mathrm{q}}$ completely predicted

Mass factorization $\Rightarrow$ $P_{\mathrm{gq}}^{\mathrm{NLL}}$, $c_{\phi,\mathrm{q}}^{\mathrm{NLL}}$ to all orders. $P_{\mathrm{qg}}^{\mathrm{NLL}}$, $c_{2,\mathrm{g}}^{\mathrm{NLL}}$ analogous

Extension of this approach to higher-log accuracy (at least) cumbersome

D-dim. structure of unfactorized observables

Maximal phase space for deep-inelastic scattering/semi-incl. annihilation

$\mathbf{NLO:}\ \ \mathbf{2 \rightarrow 2\ /\ 1 \rightarrow 1 + 2}$ $\quad \mathbf{(1-x)^{-1-\varepsilon}\ x^{\cdots} \int_0^1}$ **one other variable**

$\mathbf{N^2LO:}\ \ \mathbf{2 \rightarrow 3\ /\ 1 \rightarrow 1 + 3}$ $\quad \mathbf{(1-x)^{-1-2\varepsilon}\ x^{\cdots} \int_0^1}$ **four other variables**

$\mathbf{N^3LO:}\ \ \mathbf{2 \rightarrow 4\ /\ 1 \rightarrow 1 + 4}$ $\quad \mathbf{(1-x)^{-1-3\varepsilon}\ x^{\cdots} \int_0^1}$ **seven other variables**

$\ldots$

N^{2}LO: Matsuura, van Neerven (88), Rijken, vN (95), N$^{n\geq 3}$LO, indirectly: MV[V] (05)

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Purely real contributions to unfactorized structure functions

$$T_{a,j}^{(n)\mathrm{R}} = (1-x)^{-1-n\varepsilon} \sum_{\xi=0} (1-x)^{\xi}\, \frac{1}{\varepsilon^{2n-1}} \left\{ R_{a,j,\xi}^{(n)\mathrm{LL}} + \varepsilon R_{a,j,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Mixed contributions ($2 \to r+1$ with $n-r$ loops in DIS)

$$T_{a,j}^{(n)\mathrm{M}} = \sum_{l=r}^{n} (1-x)^{-1-l\varepsilon} \sum_{\xi=0} (1-x)^{\xi}\, \frac{1}{\varepsilon^{2n-1}} \left\{ M_{a,j,l,\xi}^{(n)\mathrm{LL}} + \varepsilon M_{a,j,l,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Purely virtual part (diagonal cases, $\eta = 0$ present): $\gamma^* qq$, Hgg form factors

$$T_{a,j}^{(n)\mathrm{V}} = \delta(1-x)\, \frac{1}{\varepsilon^{2n}} \left\{ V_{a,j}^{(n)\mathrm{LL}} + \varepsilon V_{a,j}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Resulting resummation of large-x double logs

KLN cancellation between purely real, mixed and purely virtual contributions

$$\boldsymbol{T_{a,j}^{(n)} = T_{a,j}^{(n)\mathrm{R}} + T_{a,j}^{(n)\mathrm{M}} \left(+ T_{a,j}^{(n)\mathrm{V}} \right) = \frac{1}{\varepsilon^{n}} \left\{ T_{a,j}^{(n)0} + \varepsilon T_{a,j}^{(n)1} + \ldots \right\}}$$

$\Rightarrow$ **Up to $n-1$ relations between the coeff's of $(1-x)^{-1-l\varepsilon}$, $l = 1, \ldots, n$**

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Log expansion: N^{k}LL higher-order coefficients completely fixed, if first $k+1$ powers of ε known to all orders – provided by N^{k}LO calculation, see above

Present situation: (a) N^{3}LO for non-singlet $F_{a \neq L}$ in DIS – recall DMS (05)
(b) N^{2}LO for SIA, non-singlet F_L in DIS, and singlet DIS

$\Rightarrow$ **resummation of the (a) four and (b) three highest $N^{-1} \ln^{k} N$ terms to all orders in α_{s}: consistent with, and extending, our previous results**

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Soft-gluon exponentiation of the $(1-x)^{-1}/N^0$ diagonal coefficient functions:

$(1-x)^{-1-\varepsilon}, \ldots, (1-x)^{-1-(n-1)\varepsilon}$ at order n: products of lower-order quantities

$\Rightarrow$ **N^{n}LO [+$A^{(n+1)}$] $\rightarrow$ N^{n}LL exponentiation; $2n[+1]$ highest logs predicted**

Reminder: soft limits of $q\bar{q} \to \gamma^*$, $gg \to H$

a_s^n expansion coefficients of bare partonic cross sections to $n = 3$

cf. Matsuura, van Neerven (88)

$$
\begin{aligned}
W_0^{\mathrm{b}} &= \delta(1-x) \\
W_1^{\mathrm{b}} &= 2\,\mathrm{Re}\,\mathcal{F}_1\,\delta(1-x) + \mathcal{S}_1 \\
W_2^{\mathrm{b}} &= (2\,\mathrm{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\delta(1-x) + 2\,\mathrm{Re}\,\mathcal{F}_1\mathcal{S}_1 + \mathcal{S}_2 \\
W_3^{\mathrm{b}} &= (2\,\mathrm{Re}\,\mathcal{F}_3 + 2\,|\mathcal{F}_1\mathcal{F}_2|)\delta(1-x) + (2\,\mathrm{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\mathcal{S}_1 + 2\,\mathrm{Re}\,\mathcal{F}_1\mathcal{S}_2 + \mathcal{S}_3
\end{aligned}
$$

$\mathcal{F}_\ell$: bare ℓ-loop time-like q or g form factor, $\mathcal{S}_\ell$ includes soft real emissions

$$
\mathcal{S}_k = \mathsf{S}_k(\varepsilon)\cdot\varepsilon[\,(1-x)^{-1-2k\varepsilon}\,]_+ = \mathsf{S}_k(\varepsilon)\Big[-\frac{1}{2k}\,\delta(1-x) + \sum_{i=0}\frac{(-2k\varepsilon)^i}{i\,!}\,\varepsilon\,\mathcal{D}_i\Big]
$$

Poles in $\varepsilon = 2 - D/2$: KLN, renormalization, mass factorization

$1/\varepsilon$ pieces of $\mathcal{F}_n$ + n-loop splitting functions $\to$ $1/\varepsilon$ coefficients of S_n

$\to$ $\mathcal{D}_{2n,\ldots,0}$ terms of coefficient fct's c_n $\to$ N^{n}LL resummation coeff's D_n

$n = 3$: Moch, A.V. (2005)

NS results, off-diagonal splitting fct's and $C_{L,\mathrm{g}}$

NS cases: $K_{a,4}(x)$ **of p. 15 confirmed with** $\xi_{K_4} = \frac{100}{3}$**: fourth log for** $c_{a,\mathrm{ns}}^{(n \geq 4)}$

also: Grunberg (2010)

Off-diagonal splitting functions

$$\tilde{a}_{\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{\pi}\,(C_A - C_F)\,\ln^2 \widetilde{N}$$

$$N P_{\mathrm{qg}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;=\; 2 a_{\mathrm{s}}\, n_f\, \mathcal{B}_0(\tilde{a}_{\mathrm{s}})$$
$$+\; a_{\mathrm{s}}^2 \ln \widetilde{N}\, n_f \left\{ (6C_F - \beta_0)\left(\frac{2}{\tilde{a}_{\mathrm{s}}} \mathcal{B}_{-1}(\tilde{a}_{\mathrm{s}}) + \mathcal{B}_1(\tilde{a}_{\mathrm{s}}) \right) + \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\, \mathcal{B}_{-2}(\tilde{a}_{\mathrm{s}}) \right\}$$

$$N P_{\mathrm{gq}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;=\; 2 a_{\mathrm{s}}\, C_F\, \mathcal{B}_0(-\tilde{a}_{\mathrm{s}}) \;+\; a_{\mathrm{s}}^2 \ln \widetilde{N}\, C_F \left\{ (12 C_F - 6\beta_0)\, \frac{1}{\tilde{a}_{\mathrm{s}}}\, \mathcal{B}_{-1}(-\tilde{a}_{\mathrm{s}}) \right.$$
$$\left. -\frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\, \mathcal{B}_{-2}(-\tilde{a}_{\mathrm{s}}) + (14 C_F - 8 C_A - \beta_0)\, \mathcal{B}_1(-\tilde{a}_{\mathrm{s}}) \right\}$$

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Off-diagonal splitting functions

$$\tilde{a}_\mathrm{s} = \frac{\alpha_\mathrm{s}}{\pi}\,(C_A - C_F)\,\ln^2\widetilde{N}$$

$$NP_{\mathrm{qg}}^{\mathrm{NL}}(N,\alpha_\mathrm{s}) \;=\; 2a_\mathrm{s}\,n_f\,\mathcal{B}_0(\tilde{a}_\mathrm{s}) + a_\mathrm{s}^2\ln\widetilde{N}\,n_f\left\{(6C_F-\beta_0)\left(\frac{2}{\tilde{a}_\mathrm{s}}\mathcal{B}_{-1}(\tilde{a}_\mathrm{s}) + \mathcal{B}_1(\tilde{a}_\mathrm{s})\right) + \frac{\beta_0}{\tilde{a}_\mathrm{s}}\,\mathcal{B}_{-2}(\tilde{a}_\mathrm{s})\right\}$$

$$NP_{\mathrm{gq}}^{\mathrm{NL}}(N,\alpha_\mathrm{s}) \;=\; 2a_\mathrm{s}\,C_F\,\mathcal{B}_0(-\tilde{a}_\mathrm{s}) \;+\; a_\mathrm{s}^2\ln\widetilde{N}\,C_F\left\{(12C_F - 6\beta_0)\,\frac{1}{\tilde{a}_\mathrm{s}}\,\mathcal{B}_{-1}(-\tilde{a}_\mathrm{s}) - \frac{\beta_0}{\tilde{a}_\mathrm{s}}\,\mathcal{B}_{-2}(-\tilde{a}_\mathrm{s}) + (14C_F - 8C_A - \beta_0)\,\mathcal{B}_1(-\tilde{a}_\mathrm{s})\right\}$$

Gluon contribution to F_L – 'non-singlet' $C_F = 0$ part done before

MV (09)

$$N^2 C_{L,\mathrm{g}}^{\mathrm{NL}}(N,\alpha_\mathrm{s}) \;=\; 8a_\mathrm{s}\,n_f\,\exp(2C_A\,a_\mathrm{s}\ln^2\widetilde{N}) \;+\; 4a_\mathrm{s}C_F\,NC_{2,\mathrm{g}}^{\mathrm{LL}}(N,\alpha_\mathrm{s}) + 16a_\mathrm{s}^2\ln\widetilde{N}\,n_f\left\{4C_A - C_F + \frac{1}{3}a_\mathrm{s}\ln^2\widetilde{N}\,C_A\beta_0\right\}\exp(2C_A\,a_\mathrm{s}\ln^2\widetilde{N})$$

NNLL terms known to 'any' order, but no closed expressions (except $C_{L,\mathrm{g}}$)

Resummed gluon coefficient function for F_2

$$N C_{2,\mathrm{g}}(N, \alpha_{\mathrm{s}}) \; = $$

$$\frac{1}{2 \ln \widetilde{N}} \; \frac{n_f}{C_A - C_F} \Big[\exp(2 a_{\mathrm{s}} C_F \ln^2 \widetilde{N}) \mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - \exp(2 a_{\mathrm{s}} C_A \ln^2 \widetilde{N}) \Big]$$

$$- \frac{1}{8 \ln^2 \widetilde{N}} \; \frac{n_f (3 C_F - \beta_0)}{(C_A - C_F)^2} \Big[\exp(2 a_{\mathrm{s}} C_F \ln^2 \widetilde{N}) \mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - \exp(2 a_{\mathrm{s}} C_A \ln^2 \widetilde{N}) \Big]$$

$$- \frac{a_{\mathrm{s}}}{4} \; \frac{n_f}{C_A - C_F} \; \exp\left(2 a_{\mathrm{s}} C_A \ln^2 \widetilde{N}\right) \left(8 C_A + 4 C_F - \beta_0\right)$$

$$- \frac{a_{\mathrm{s}}}{4} \; \frac{n_f}{C_A - C_F} \; \exp\left(2 a_{\mathrm{s}} C_F \ln^2 \widetilde{N}\right) \Big[- 6 C_F \mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - (6 C_F - \beta_0) \mathcal{B}_1(\tilde{a}_{\mathrm{s}})$$

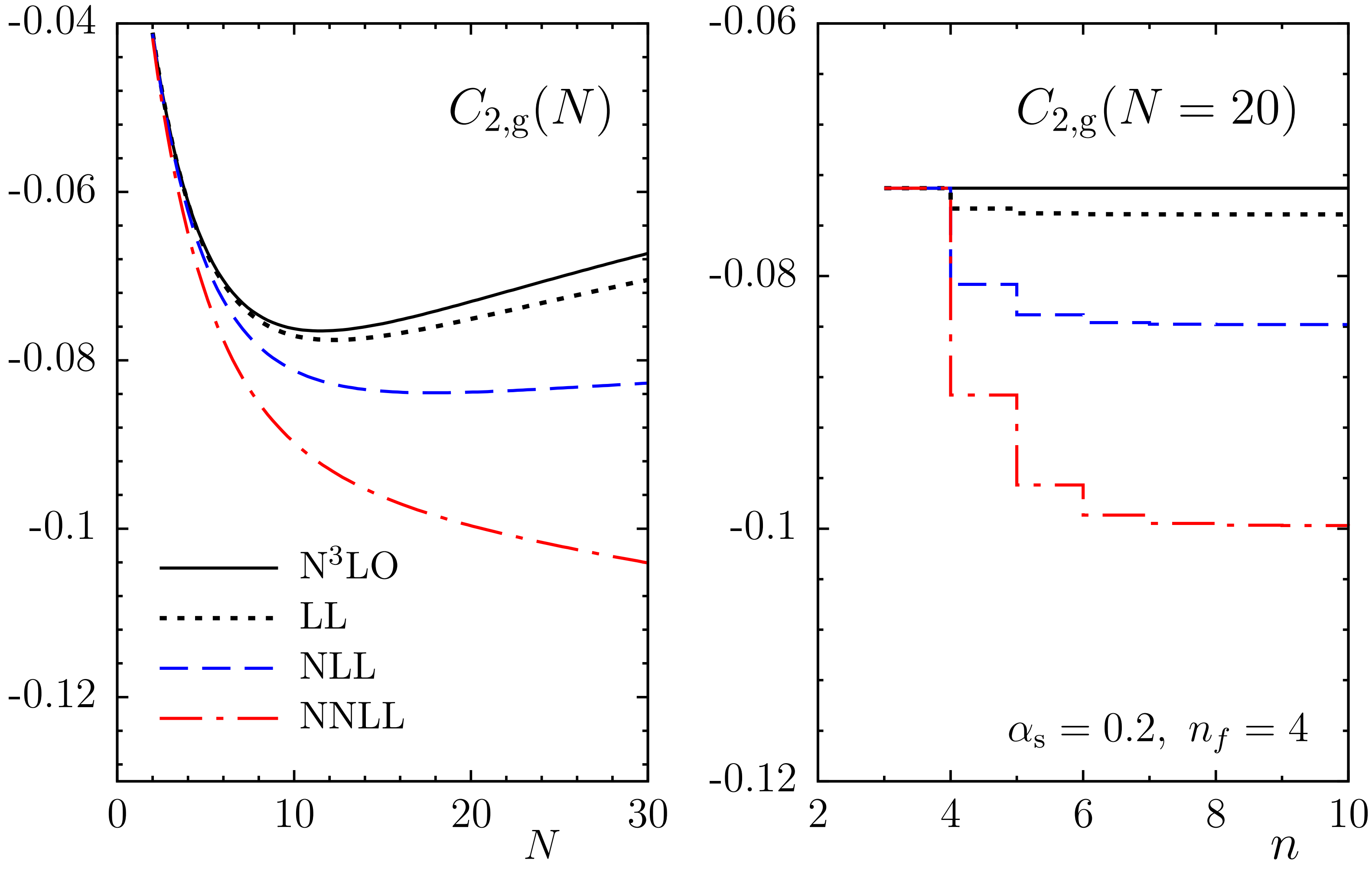
$$- (12 C_F - 4 \beta_0) \, \frac{1}{\tilde{a}_{\mathrm{s}}} \, \mathcal{B}_{-1}(\tilde{a}_{\mathrm{s}}) - \frac{\beta_0}{\tilde{a}_{\mathrm{s}}} \, \mathcal{B}_{-2}(\tilde{a}_{\mathrm{s}}) \Big]$$

$$- \frac{a_{\mathrm{s}}^2}{3} \, \beta_0 \ln^2 \widetilde{N} \, \frac{n_f}{C_A - C_F} \Big[C_A \exp(2 a_{\mathrm{s}} C_A \ln^2 \widetilde{N}) - C_F \exp(2 a_{\mathrm{s}} C_F \ln^2 \widetilde{N}) \mathcal{B}_0(\tilde{a}_{\mathrm{s}}) \Big]$$

$+$ **known NNLL contributions (tables)** $+ \; \ldots$

$C_{\phi,\mathrm{q}}$ **analogous. Analytic forms identified via the physical kernel for** (F_2, F_ϕ)

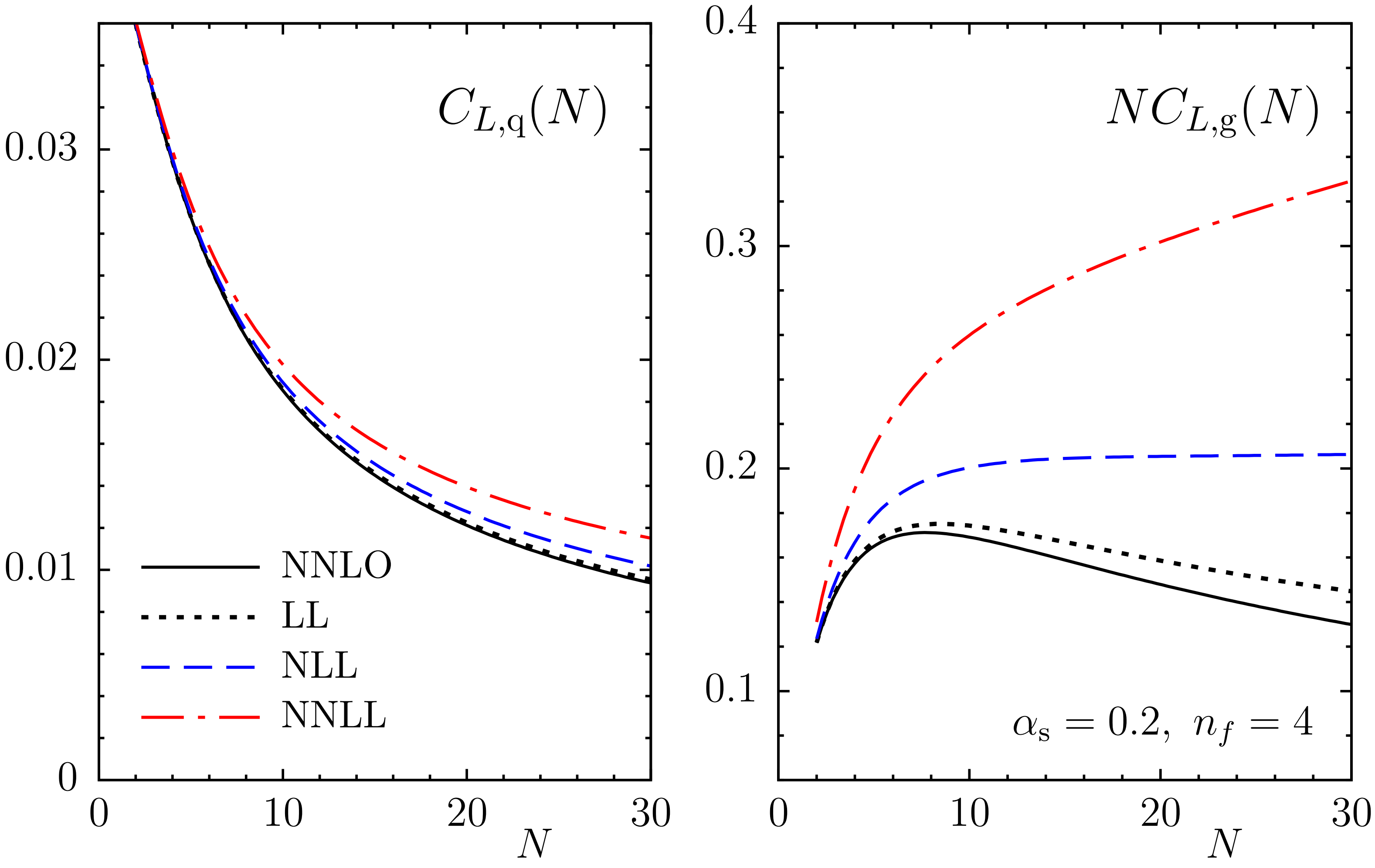
Numerical illustration of $C_{2,\mathrm{g}}$



NNLL terms dominate $\Rightarrow$ impact of high orders presumably underestimated

About 35% correction at $N = 20$, 4$^{\mathrm{th}}$-order coefficient $\approx$ Padé estimate

Numerical illustration of $C_{L,\mathrm{q}}$ and $C_{L,\mathrm{g}}$



Corrections smaller and convergence with order n faster in quark case(s)

$\simeq$ 15% NNLL correction at $N = 20$ for $C_{L,\mathrm{q}}$ vs. 100% for $C_{L,\mathrm{g}}$ ($\approx$ Padé)

Small-x resummation via unfactorized SIA (I)

Phase-space integrations: $x^{a\varepsilon}$ terms analogous to $(1-x)^{b\varepsilon}$ structures above

2nd order: Matsuura, van Neerven (88), Rijken, vN (95)

$\Rightarrow$ look for decomposition similar to that in the large-x case

Formalism for fragmentation functions (timelike structure functions) $F_{T,L,\phi}$: analogous to DIS cases (with singlet splitting-function matrix transposed)

$\Rightarrow$ NNLO results fix the highest three powers of $1/\varepsilon$ to all orders in α_s

2nd order $C_{\phi,\mathrm{g}}$ and $C_{\phi,\mathrm{q}}$: AMV (2010)

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$\Rightarrow$ look for decomposition similar to that in the large-x case

Formalism for fragmentation functions (timelike structure functions) $F_{T,L,\phi}$: analogous to DIS cases (with singlet splitting-function matrix transposed)

$\Rightarrow$ NNLO results fix the highest three powers of $1/\varepsilon$ to all orders in α_s

2nd order $C_{\phi,\mathrm{g}}$ and $C_{\phi,\mathrm{q}}$: AMV (2010)

Decomposition of the D-dim. partonic fragmentation functions for $a = T, \phi$

$$\widehat{F}^{(n)}_{a,\mathrm{g}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\varepsilon} \Big\{ A^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon B^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon^2 C^{(\ell,n)}_{a,\mathrm{g}} + \ldots \Big\}$$

$$\stackrel{\mathrm{M}}{=} \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} \frac{1}{N-1-2(n-\ell)\varepsilon} \Big\{ A^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon B^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon^2 C^{(\ell,n)}_{a,\mathrm{g}} + \ldots \Big\}$$

Small-x resummation via unfactorized SIA (II)

LL: $\widehat{F}_{a,\mathrm{g}}^{(n)}$ includes terms of the form $x^{-1}\ln^{n+\delta-1}x$ at all orders $\varepsilon^{-n+\delta}$ with $\delta = 0,\,1,\,2,\,\ldots,$ and is decomposed into n contributions of the form

$$\varepsilon^{-2n+1}\,x^{-1-k\,\varepsilon} \;=\; \varepsilon^{-2n+1}\,x^{-1}\Big[1 \,-\, k\,\varepsilon\ln x \,+\, \frac{1}{2}(k\,\varepsilon)^2\ln^2 x \,+\, \ldots\Big]\,, \qquad k = 2,\,4,\,\ldots,\,2n$$

$n-1$ KLN-type cancellations – $\widehat{F}_{a,\mathrm{g}}^{(n)}$ starts at order $1/\varepsilon^n$ – plus 3 constraints from the NNLO results $\Rightarrow$ $n+2$ linear equations for n coefficients $A_{a,\mathrm{g}}^{(\ell,n)}$

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Thus, again: N^{n}LO known $\Rightarrow$ **highest** $n+1$ **double logs fixed at all orders**

Quark cases: analogous with prefactor ε^{-n+2} **but one term missing in sums**

$$\widehat{F}^{(n)}_{a,\mathrm{g}} = \frac{1}{\varepsilon^{2n-2}} \sum_{\ell=0}^{n-2} \frac{1}{N-1-2(n-\ell)\,\varepsilon} \left\{ A^{(\ell,n)}_{a,\mathrm{q}} + \varepsilon B^{(\ell,n)}_{a,\mathrm{q}} + \varepsilon^2 C^{(\ell,n)}_{a,\mathrm{q}} + \ldots \right\}$$

$\Rightarrow$ **also here highest three logarithms at all orders fixed by NNLO results**

‘All-order’ mass factorization: NNLL timelike splitting & coefficient functions

Resummed splitting and coefficient functions

$$\frac{C_A}{C_F}\, P^T_{\mathrm{gq}}(N,\alpha_s) \stackrel{\mathrm{LL}}{=} P^T_{\mathrm{gg}}(N,\alpha_s) \stackrel{\mathrm{LL}}{=} \frac{1}{4}(N-1)\left\{(1-4\xi)^{1/2}-1\right\}, \quad \xi = -\frac{8C_A a_s}{(N-1)^2}$$

Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82)

NNL contributions to the splitting functions: only partially in closed form

$$\left[P^T_{\mathrm{gg}}\right]^{\mathrm{NLL}}_{C_F=0} = \left\{(1-4\xi)^{-1/2}+1\right\} a_s \left(\frac{11}{6}C_A + \frac{1}{3}n_f\right)$$

$$\left[\frac{C_A}{C_F}\, P^T_{\mathrm{gq}}\right]^{\mathrm{NLL}}_{C_F=0} = \left[P^T_{\mathrm{gg}}\right]^{\mathrm{NLL}}_{C_F=0} + \left\{(1-4\xi)^{1/2}-1\right\} \frac{1}{24}(N-1)^2(1+n_f/C_A)$$

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Leading logarithmic $\overline{\rm MS}$ coefficient functions for F_T and F_ϕ

$$C^{\rm LL}_{T,\,{\rm g}} \;=\; \frac{C_F}{C_A}\left(C^{T,\,{\rm LL}}_{\phi,{\rm g}}-1\right) \;=\; \frac{C_F}{C_A}\left\{(1-4\xi)^{-1/4}-1\right\}$$

also: Albino, Bolzoni Kniehl, Kotikov (2011)

'Everything else', including all of $P^T_{\rm qq}$, $P^T_{\rm qg}$, the quark coefficient fct's, $C_{L,{\rm i}}$:

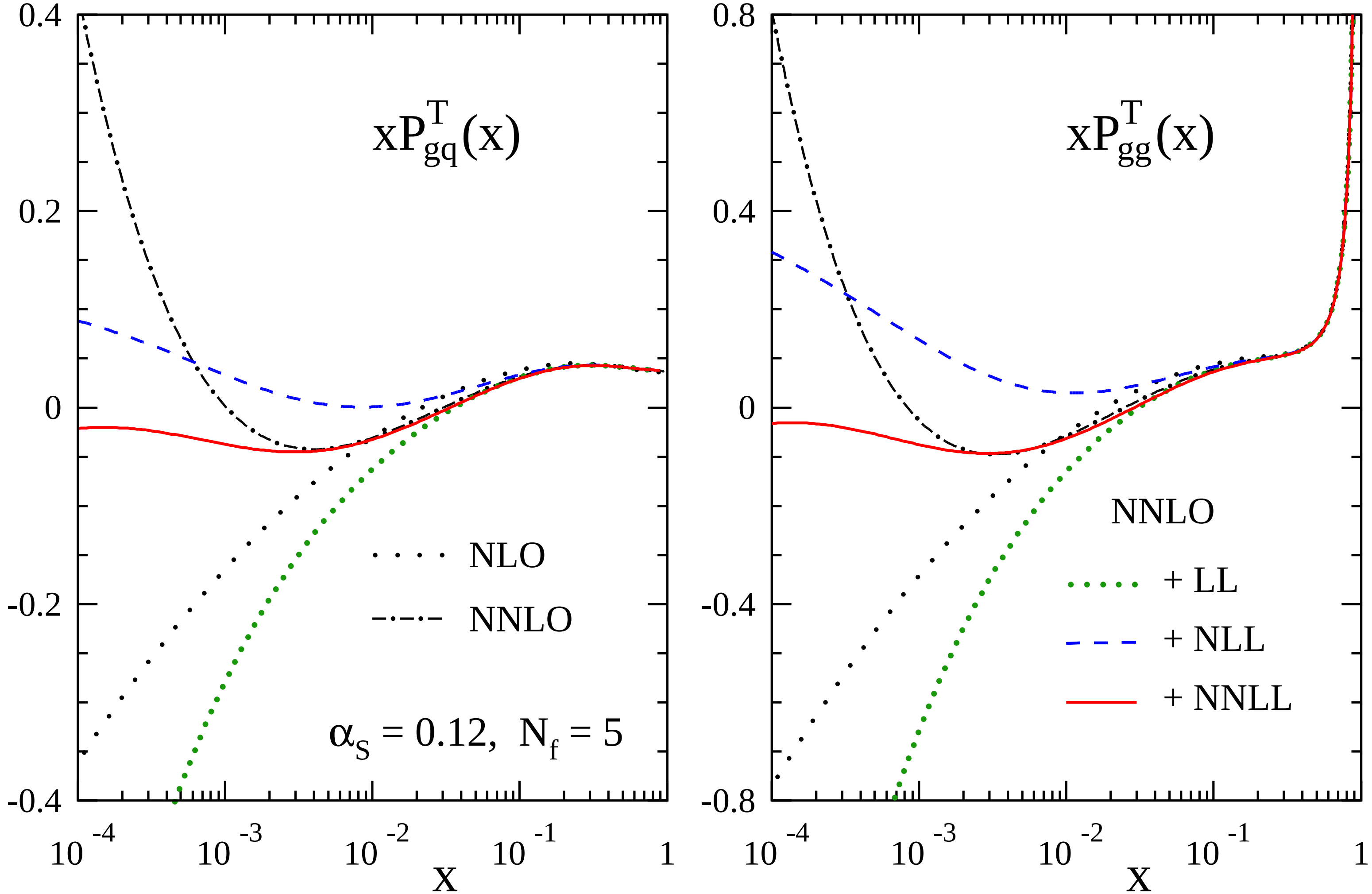
Tables of coefficients to order $\alpha_{\rm s}^{16}$ – numerically sufficient for $x \gtrsim 10^{-4}$

Normalized LL, NLL splitting-fct. coefficients

n	$A^{(n)}_{\rm gi}$	$B^{(n)}_{\rm gg,1}$	$B^{(n)}_{\rm gg,2}$	$B^{(n)}_{\rm gq,1}$	$B^{(n)}_{\rm gq,2}$	$B^{(n)}_{\rm gq,3}$	$A^{(n)}_{\rm qi}$
0	1	1	–	9	–	–	–
1	1	1	2	9	–	–	–
2	2	3	5	29	1	1	1
3	5	10	$\frac{49}{3}$	100	5	$\frac{19}{3}$	$\frac{11}{3}$
4	14	35	$\frac{347}{6}$	357	21	$\frac{179}{6}$	$\frac{73}{6}$
5	42	126	$\frac{6353}{30}$	1302	84	$\frac{3833}{30}$	$\frac{1207}{30}$
6	132	462	$\frac{11839}{15}$	4818	330	$\frac{7879}{15}$	$\frac{2021}{15}$
7	429	1716	$\frac{624557}{210}$	18018	1287	$\frac{444377}{210}$	$\frac{96163}{210}$
8	1430	6435	$\frac{316175}{28}$	67925	5005	$\frac{236095}{28}$	$\frac{44185}{28}$
9	4862	24310	$\frac{54324719}{1260}$	257686	19448	$\frac{42072479}{1260}$	$\frac{6936481}{1260}$

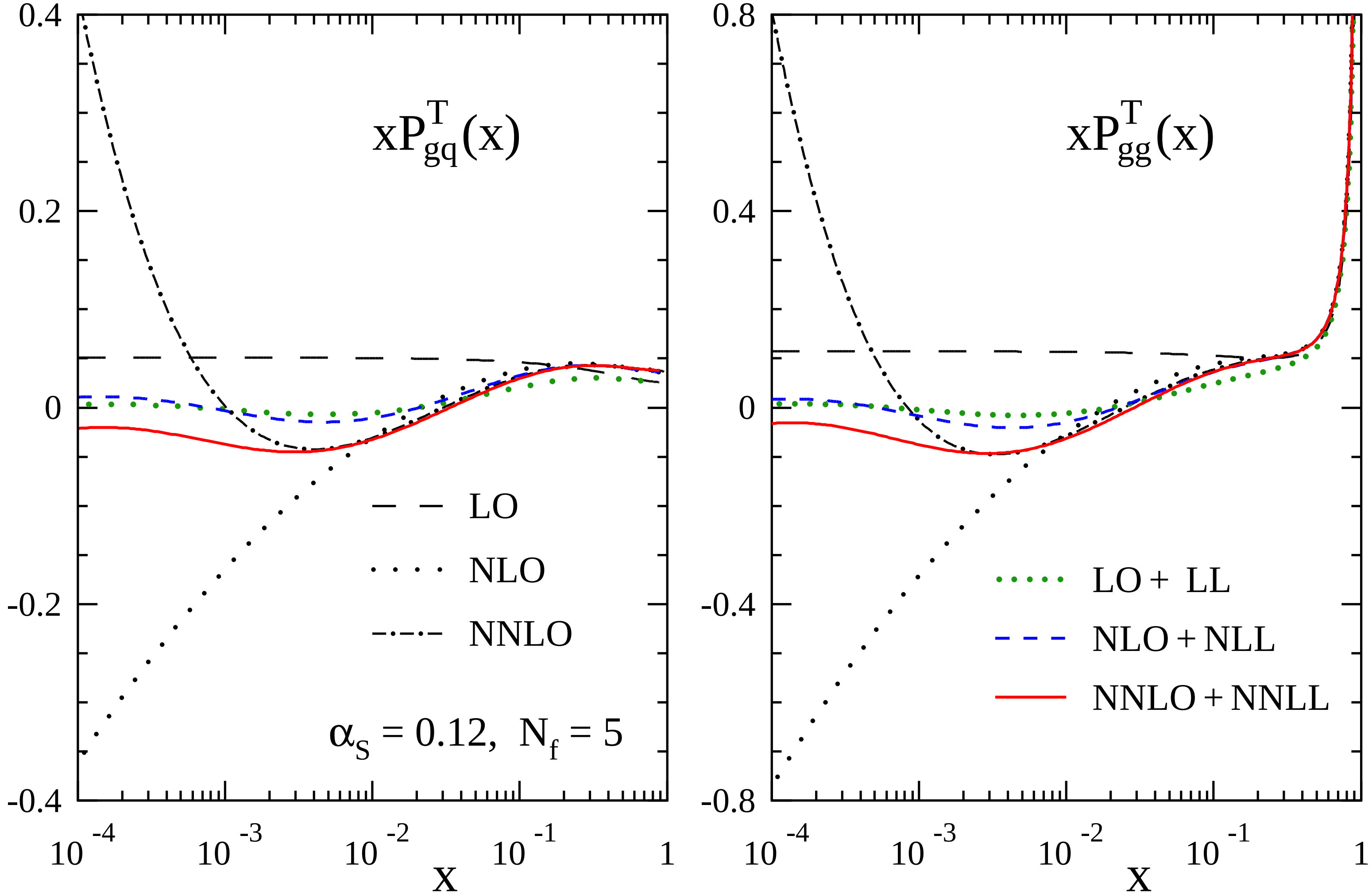
Solution of one non-integer series: analytic structure of all NLL contributions

Small-x gluon-parton splitting functions (I)



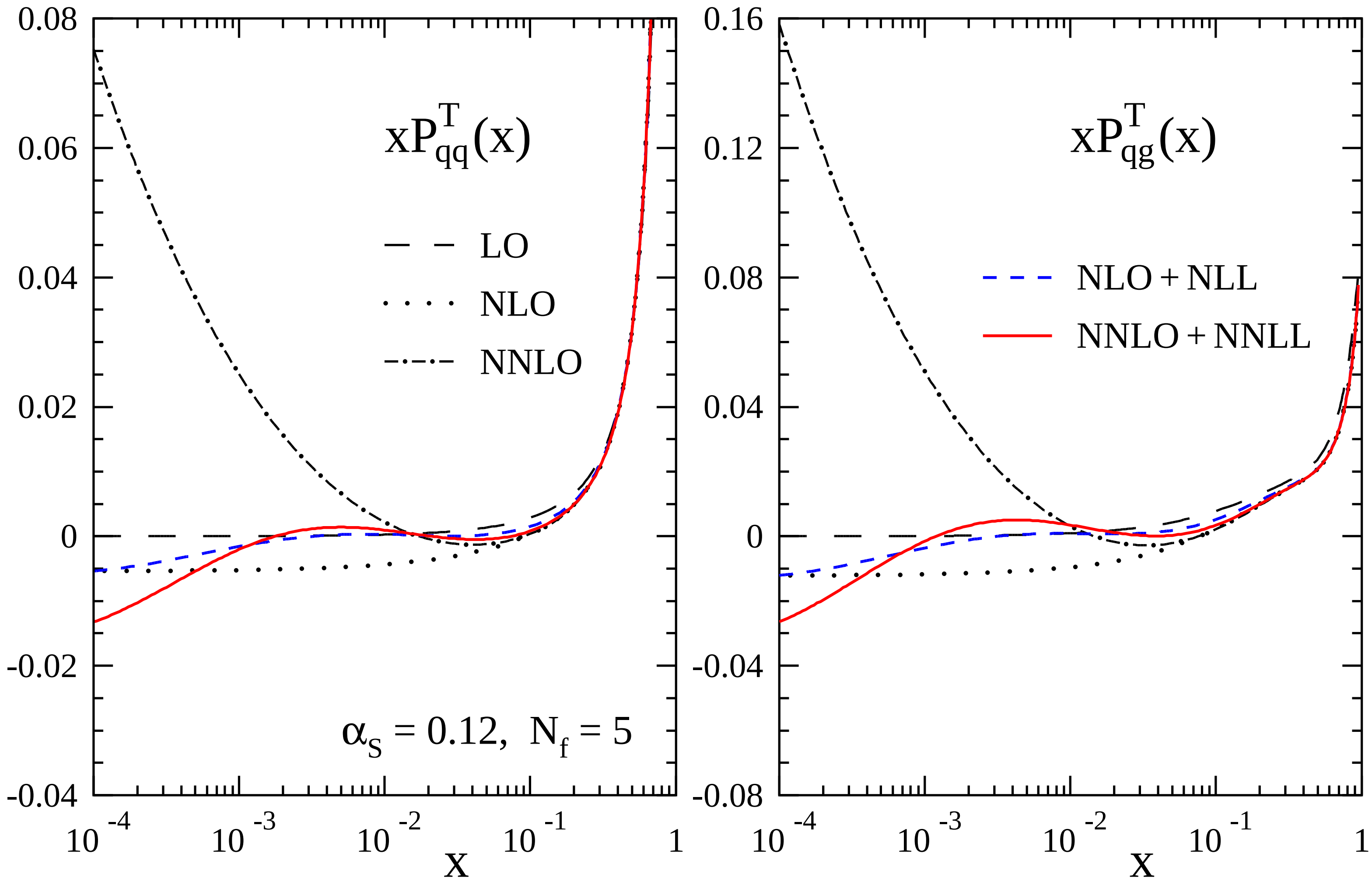
LL insufficient, near-perfect cancellation of NNLO rise by NNLL resummation

Small-x gluon-parton splitting functions (II)



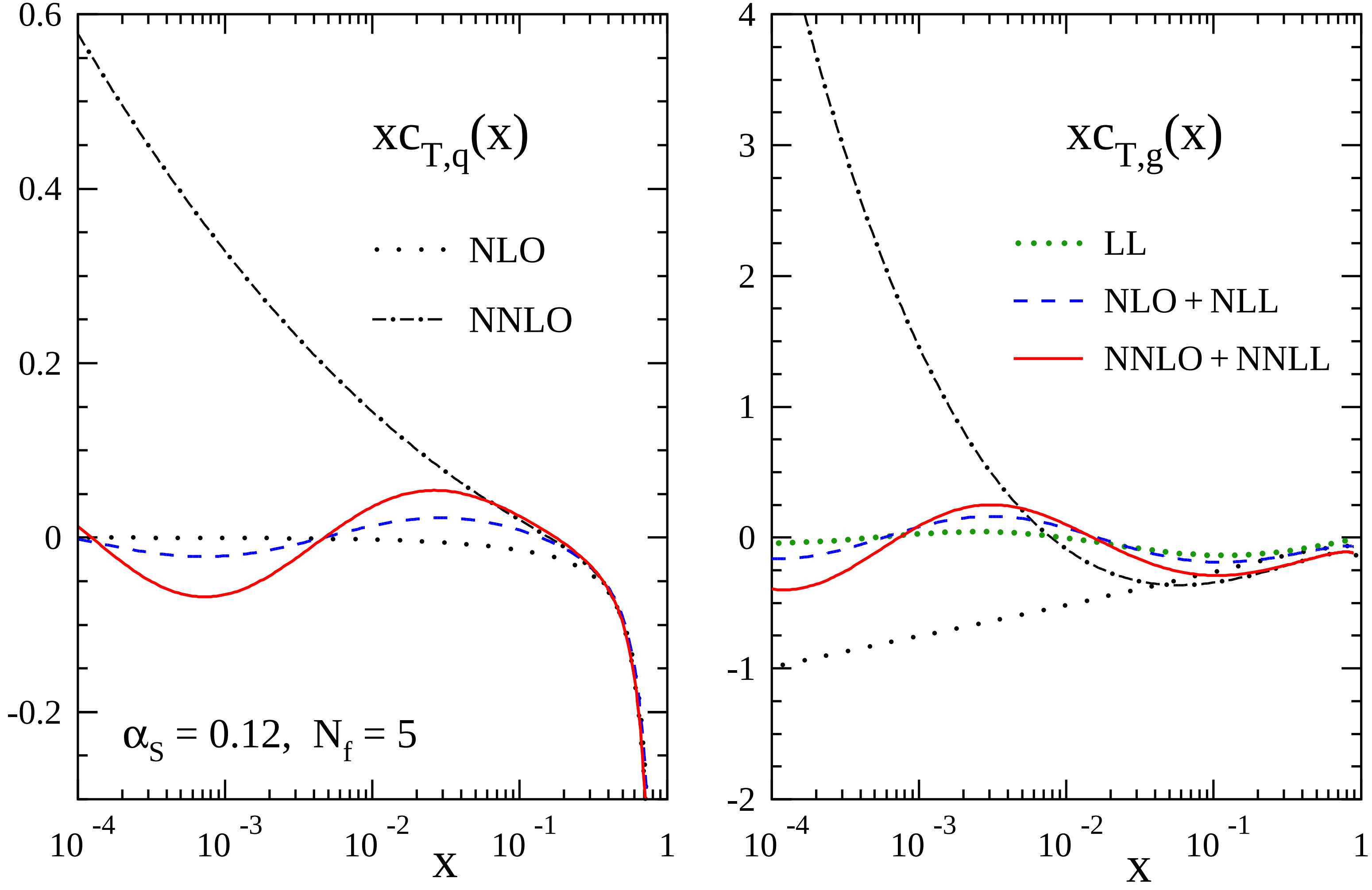
Approximation sequence LO+LL, NLO+NLL, NNLO+NNLL rather stable to very small x

Small-x quark-parton splitting functions



Also consistent with $xP^T_{ji} \approx 0$ at $x < 10^{-2}$ (N^{3}LL corr's known and positive)

Small-x coefficient functions for F_T



A bit worrying? But no errors found – and already LL oscillates down to extreme low x

Large-x summary and outlook

- **Non-singlet physical kernels for nine observables in DIS, SIA and DY:**
  single-log large-x enhancement at NNLO$/$N^{3}LO to all orders in $1-x$
  All-order conjecture $\Rightarrow$ leading three (DY: two) logs of higher-order C_a
- **Singlet kernels for (F_2, F_ϕ) and (F_2, F_L) in DIS also single-logarithmic**
  $\Rightarrow$ Prediction of three logs in N^{3}LO α_s^4 splitting and F_L coefficient fct's

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- Complementary: Grunberg; Laenen, Gardi, Magnea, Stavenga, White
- Limited phenomenol. relevance now: assess relevance of NS $1/N$ terms
- Near$/$mid future: combine with other results, esp. fixed-N calculations
  (close to) feasible now: 4-loop sum rules Baikov, Chetyrkin, Kühn (10)

Small-x summary and outlook

- D-dimensional structure of unfactorized SIA$/$DIS structure functions
  $\Rightarrow$ NNLL small-x resummation of timelike splitting & coefficient fct's
  Required for using NNLO results in SIA below $x \approx 10^{-2} \ldots 10^{-3}$
- Analogous results for (singlet case: subdominant) $x^0 \ln^\ell x$ terms in DIS
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- Unlike large-x case: no direct generalization to all (higher) a in $x^a \ln^\ell x$
  But works for higher even a in SIA – DIS case not checked yet

- Does not work for the odd-N quantities F_3 and g_1 in DIS, F_A in SIA
  E.g., leading logs with group factor $d_{abc}d^{abc}$ at third order in F_3 and F_A
  cf. Dokshitzer, Marchesini (2007)

All large-x and many, but not all, small-x double logarithms in SIA and DIS appear to be 'inherited' from lower-order results.