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The infrared singularity structure of QCD amplitudes

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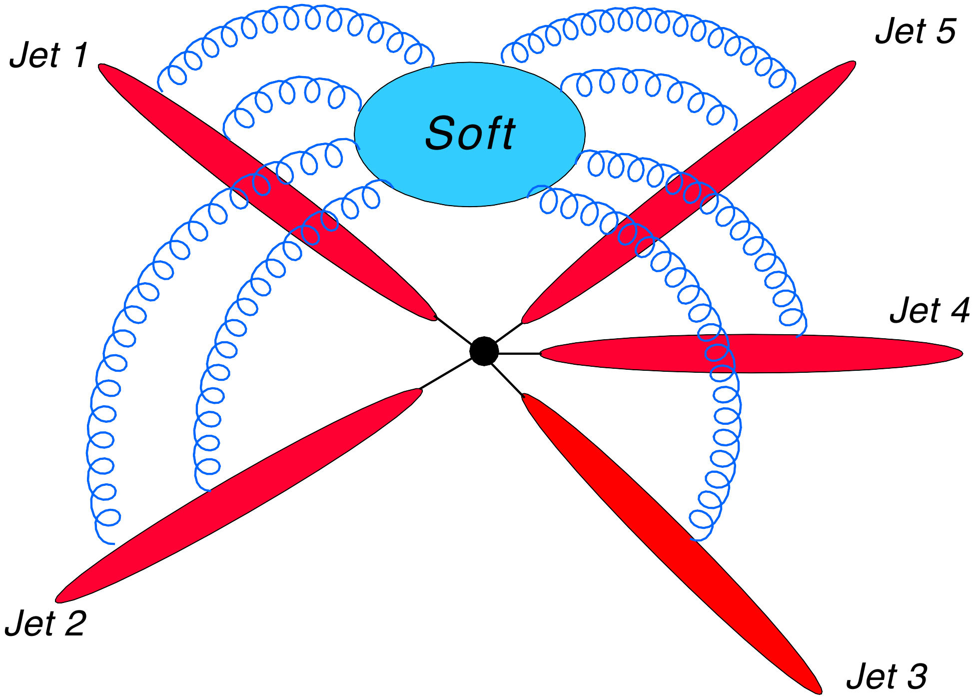


Prelude: the sum-over-dipoles formula

It is possible that all IR singularities in any massless gauge-theory amplitude are given by:

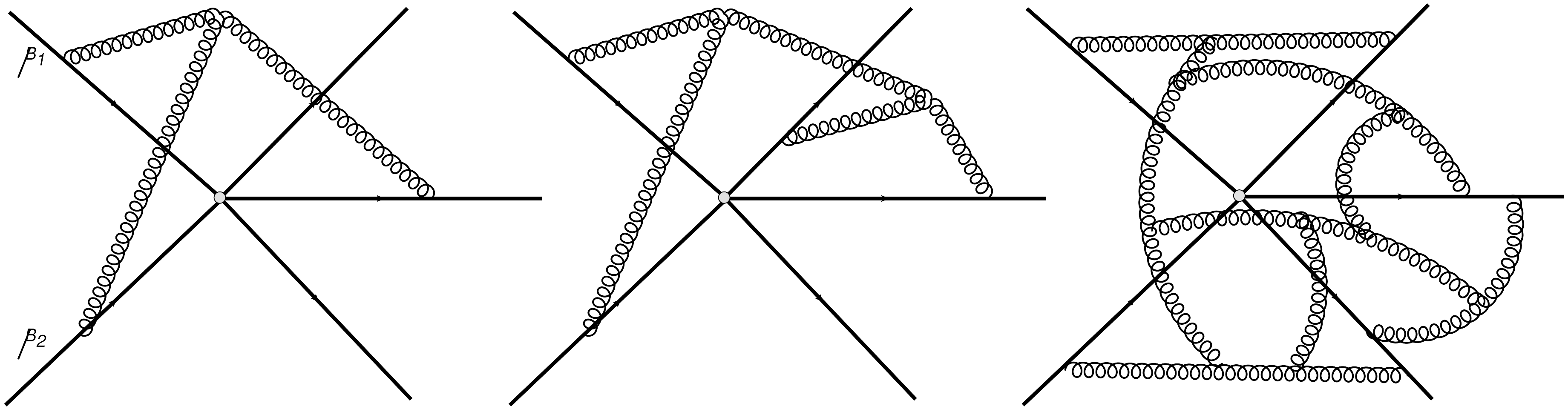
$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s, \epsilon\right) = \exp\left\{ -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma\left(\lambda, \alpha_s(\lambda^2, \epsilon)\right) \right\} \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s\right)$$

$$\Gamma(\lambda, \alpha_s) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right) \sum_{(i,j)} \ln\left(\frac{\lambda^2}{-s_{ij}}\right) \mathrm{T}_i \cdot \mathrm{T}_j + \sum_{i=1}^{n} \gamma_{J_i}\left(\alpha_s\right)$$

Becher & Neubert,
EG & Magnea

Soft anomalous dimension: multiparton correlations

At the multi-loop level the soft anomalous dimension can be very complicated:



Is it?

The IR singularity structure of QCD amplitudes

<u>plan of the talk</u>

- Non-Abelian exponentiation in multileg amplitudes:
  renormalization of Wilson-line operators

- The soft anomalous dimension.
  Factorization and rescaling symmetry constraints.
  The sum-over-dipoles formula and potential corrections.

- Webs: the diagrammatic approach to exponentiation

The IR singularity structure of QCD amplitudes

References

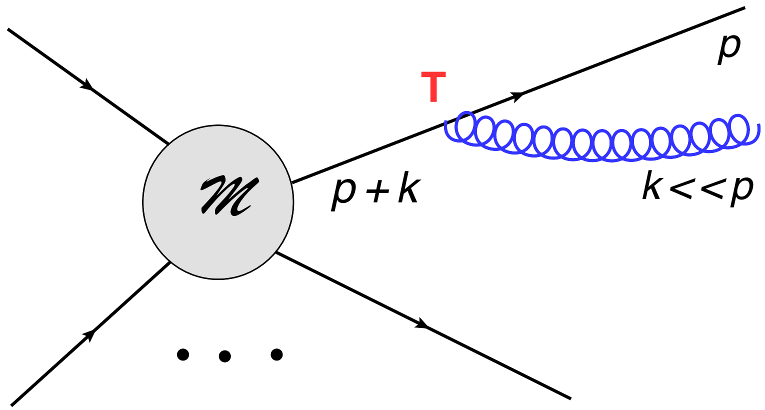
- *Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes,* EG and Magnea, **JHEP 0903 (2009) 079**
- *On soft singularities at 3 loops and beyond,* Dixon, EG and Magnea, **JHEP 1002 (2010) 081**

- *An infrared approach to Reggeization,* Del Duca, Duhr, EG, Magnea and White, **1108.5947**
- *The infrared structure of massless gauge theory amplitudes in the high-energy limit,* Del Duca, Duhr, EG, Magnea and White, **to appear.**

- *Webs in multiparton scattering using the replica trick,* EG, Laenen, Stavenga and White, **JHEP 1011 (2010) 155**
- *General properties of multiparton webs: Proofs from combinatorics,* EG and White, **JHEP 1103 (2011) 079**
- *On the renormalization of multiparton webs*, EG, Smillie and White, **1108.1357, to be published in JHEP.**

The soft approximation and rescaling invariance

Eikonal Feynman rules:



When all momentum components are small:

$$\bar{u}(p)\left(-\mathrm{i}g_s T^{(a)}\gamma^\mu\right)\frac{\mathrm{i}(\not{p}+\not{k}+m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon} \quad\longrightarrow\quad \bar{u}(p)g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}$$

Rescaling invariance: only the direction and the colour charge matter:

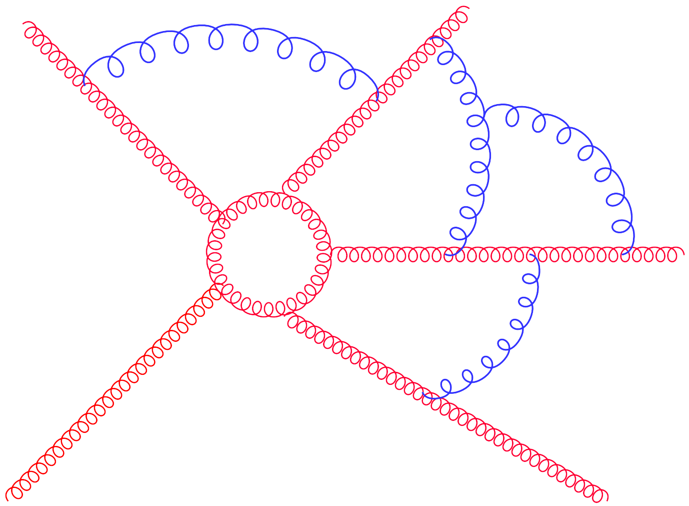
$$g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}=g_s T^{(a)}\frac{\beta^\mu}{\beta\cdot k+\mathrm{i}\varepsilon}$$

Equivalent to emission from a Wilson line:

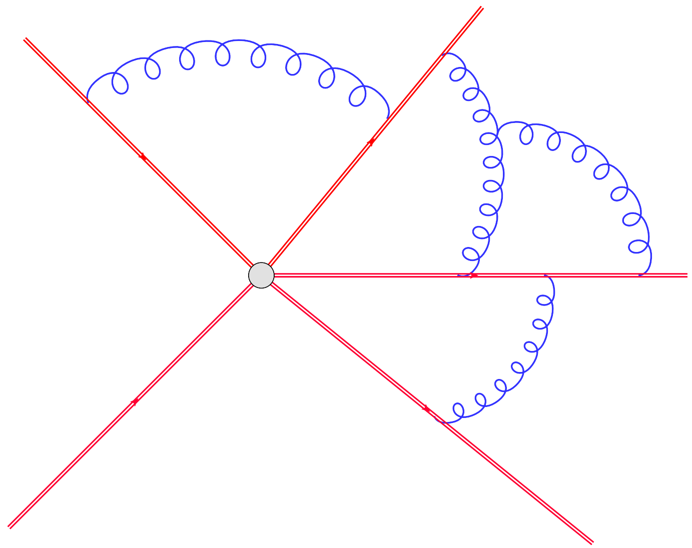
$$\Phi_{\beta_i}(\infty,0)\equiv P\exp\left\{\mathrm{i}g_s\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right\}$$

Factorization at fixed angles

5 hard gluon amplitude



5 Wilson line amplitude



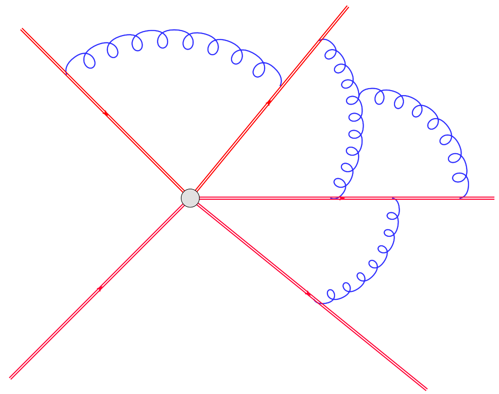
Factorization: soft gluons do not resolve the hard interaction

$$\mathcal{M}_J\left(p_i, \alpha_s, \epsilon\right) = \sum_K \mathcal{S}_{JK}\left(\frac{(\beta_i \cdot \beta_j)^2}{\beta_i^2\, \beta_j^2}, \alpha_s, \epsilon\right) H_K\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{m_i^2}{\mu^2}, \alpha_s, \epsilon\right)$$

All singularities sit here!

Wilson-line correlator: renormalization

$$\mathcal{S}\left(\frac{(\beta_i\cdot\beta_j)^2}{\beta_i^2\,\beta_j^2},\alpha_s(\mu^2),\epsilon_{\rm IR}\right)\equiv\left\langle 0\left|\prod_{i=1}^{L}\left[\Phi_{\beta_i}(\infty,0)_{a_k b_k}\right]\right|0\right\rangle_{\rm ren.}$$



Wilson-line correlators are multiplicatively renormalizable

[Dotsenko-Vergeles, Brandt et al. (1980)]

in pure dimensional regularization

$$\mathcal{S}\left(\frac{(\beta_i\cdot\beta_j)^2}{\beta_i^2\,\beta_j^2},\alpha_s(\mu_R^2),\epsilon_{\rm IR}\right)=\underbrace{\mathcal{S}_{\rm UV+IR}}_{=1}\;Z\left(\frac{(\beta_i\cdot\beta_j)^2}{\beta_i^2\,\beta_j^2},\alpha_s(\mu_R^2),\epsilon_{\rm UV}\right)$$

IR singularities of the amplitude $\Longleftrightarrow$ UV singularities of Wilson lines

[Korchemsky-Radyushkin (1985), Korchemsky-Korchemskaya (1995), …, Becher-Neubert (SCET)]

Wilson-line correlator: renormalization

DEFINITION: anomalous dimension matrix: $$\Gamma_{\mathcal{S}} \equiv -Z^{-1}\frac{dZ}{d\ln\mu}$$

To compute $Z(\epsilon_{\rm UV},\mu)$ consider the Wilson-line correlator with an IR cutoff m:

Multiplicative renormalizability $\mathcal{S}_{\rm ren}(\mu/m) = \mathcal{S}(\epsilon_{\rm UV}, m)\, Z(\epsilon_{\rm UV},\mu)$

implies that $\dfrac{\mathcal{S}_{\rm ren}}{d\ln\mu} = -\mathcal{S}_{\rm ren}\,\Gamma_{\mathcal{S}}$ so $\Gamma_{\mathcal{S}}$ itself must be finite!

Exponentiation:

$$Z = \exp\left\{\zeta(\epsilon)\right\} = {\rm P}\,\exp\left\{\frac{1}{2}\int_{\mu^2}^{\infty}\frac{d\lambda^2}{\lambda^2}\,\Gamma_{\mathcal{S}}(\alpha_s(\lambda^2))\right\}$$

where at each in the coupling, $\Gamma_{\mathcal{S}}$ gives rise to an $\mathcal{O}(1/\epsilon)$ contribution to $\zeta(\epsilon)$

Soft anomalous dimension in multi-parton scattering

Massless hard partons: all IR (soft/collinear) singularities are generated by a sum over colour dipoles (plus corrections??)

$$\Gamma(\{p_i\},\mu) = \sum_{(i,j)} \frac{T_i \cdot T_j}{4} \,\widehat{\gamma}_K(\alpha_s)\, \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s) + \mathcal{O}(\alpha_s^3)$$

- Dipole formalism (one-loop) [Catani-Seymour (1996)]
- "The singular behavior of QCD amplitudes at two loop order" [Catani (1998)]
- Colour exchange in hard scattering [Kidonkis-Oderda-Sterman (1998)]
- Two-loop calculation reveals: $\Gamma_{\mathcal{S}}^{(2)} = \frac{K}{2}\,\Gamma_{\mathcal{S}}^{(1)}$, there are no three-body correlations [Aybat-Dixon-Sterman (2006)]
- Factorization + rescaling symmetry constraints suggesting all-order generalization as a sum over colour dipoles

  [Gardi-Magnea, Becher-Neubert (2009)]

- Very specific corrections may appear at higher-loop orders:
  - ✓ Functions of conformal-invariant-cross-ratios starting at 3-loops(?)
  - ✓ Correlations associate with higher Casimirs starting at 4-loops(?)

  [Gardi-Magnea, Becher-Neubert (2009), Gardi-Magnea-Dixon (2010)]

Soft anomalous dimension – massive case

Massive partons: dependence on kinematics through $\xi_{IJ} = \cosh^{-1}\left(\frac{\beta_I \cdot \beta_J}{\sqrt{\beta_I^2\,\beta_J^2}}\right)$

multi-parton correlations starting at two-loops:

$$\Gamma(\{p_i\},\{m_i\},\mu) = \sum_{(i,j)} \frac{T_i \cdot T_j}{4}\,\widehat{\gamma}_K(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$
$$- \sum_{(I,J)} \frac{T_I \cdot T_J}{2}\,\widehat{\gamma}_{\rm cusp}(\xi_{IJ},\alpha_s) + \sum_I \gamma^I(\alpha_s) + \sum_{I,j} \frac{T_I \cdot T_j}{2}\,\widehat{\gamma}_K(\alpha_s)\,\ln\frac{m_I\mu}{-s_{Ij}}$$
$$+ \sum_{(I,J,K)} i f^{abc}\, T_I^a\, T_J^b\, T_K^c\, F_1(\xi_{IJ},\xi_{JK},\xi_{KI})$$
$$+ \sum_{(I,J)} \sum_k i f^{abc}\, T_I^a\, T_J^b\, T_k^c\, f_2\Big(\xi_{IJ}, \ln\frac{-\sigma_{Jk}\, v_J \cdot p_k}{-\sigma_{Ik}\, v_I \cdot p_k}\Big) + \mathcal{O}(\alpha_s^3)$$

three-parton correlations at $\mathcal{O}(\alpha_s^2)$

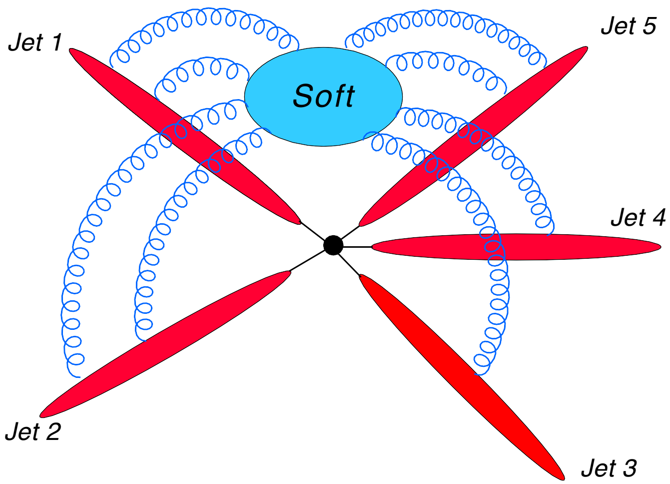
[Mitov-Sterman-Sung (2009),
Becher-Neubert (2009),
Kidonakis (2009),
Ferroglia-Neubert-Pecjak-Yan (2009),
...]

Factorization of a massless multi-leg amplitude

- All IR singularities of massless amplitudes, $p_i^2 = 0$, at fixed angles, $|p_i \cdot p_j| \gg \Lambda^2$, can be factorized:

$$\mathcal{M}_N\left(p_i/\mu, \epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \epsilon\right) H_L\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}\right)$$

$$\times \prod_{i=1}^{n} J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)$$

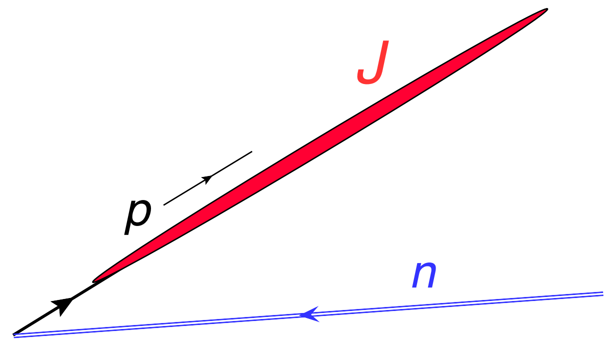


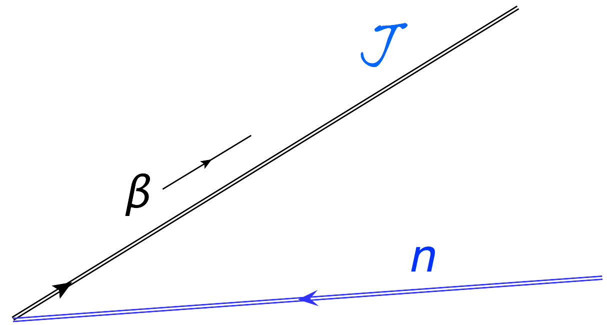
- Double counting of the soft-collinear region is avoided by dividing by the eikonal jets $\mathcal{J}_i$ removing from the jet J_i the eikonal part, which is already in $\mathcal{S}$

The jet function: definition

- Introduce an auxiliary vector n (with $n^2 \neq 0$) so that "collinear gluons" belonging to $\mathcal{J}$ have: $k \cdot n < p \cdot n$
- Define a gauge-invariant jet with a Wilson line ray along the direction n
- The factorization formula involves two "jet functions":

Partonic jet $\qquad p^2 = 0 \quad \Longrightarrow \quad \beta^2 = 0 \qquad$ Eikonal jet

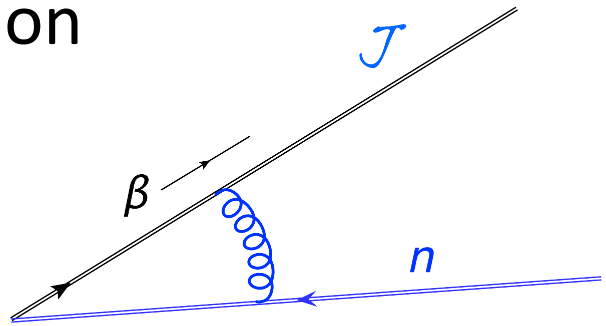




$$\overline{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \epsilon\right) = \langle p\,|\,\overline{\psi}(0)\,\Phi_n(0,-\infty)\,|0\rangle$$

$$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2}, \epsilon\right) = \langle 0|\,\Phi_\beta(\infty,0)\,\Phi_n(0,-\infty)\,|0\rangle$$

The eikonal jet function



- $\mathcal{J}$ is scaleless: all corrections vanish (before renormalization)

- If $\beta^2 \neq 0$ the renormalized $\mathcal{J}$ is rescaling invariant, depending on $\dfrac{(\beta\cdot n)^2}{\beta^2\, n^2}$

- If $\beta^2 = 0$ rescaling is broken by the lightlike cusp anomaly:

  - ✓ $\mathcal{J}$ features double poles
  - ✓ $\mathcal{J}$ depends on $(\beta\cdot n)^2/n^2$

$$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2},\epsilon\right) = \exp\left\{\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\frac{1}{4}\delta_{\mathcal{J}}\Big(\alpha_s(\lambda^2,\epsilon)\Big) - \frac{1}{8}\gamma_K\Big(\alpha_s(\lambda^2,\epsilon)\Big)\,\ln\left(\frac{2(\beta\cdot n)^2\mu^2}{n^2\lambda^2}\right)\right]\right\}$$

Factorization in terms of the reduced soft function

$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\epsilon\right) H_L \prod_{i=1}^{n} \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$$= \sum_L \overline{\mathcal{S}}_{NL}\left(\rho_{ij},\epsilon\right) H_L \prod_{i=1}^{n} J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)$$

- The lightlike cusp anomaly affecting the eikonal jet, has precisely the same effect of the soft function:
  double poles come together with violation of rescaling symmetry at the single-pole level, controlled by γ_K

- The dependence on the normalization of β_i must cancel in $\overline{\mathcal{S}}$

$$\Longrightarrow\ \overline{\mathcal{S}}\ \text{depends only on} \qquad \rho_{ij} \equiv \frac{(\beta_i\cdot\beta_j)^2}{\left[2(\beta_i\cdot n_i)^2/n_i^2\right]\left[2(\beta_j\cdot n_j)^2/n_j^2\right]}$$

The equations for $\mathbf{\Gamma}^{\overline{\mathcal{S}}}$

- Factorization + rescaling symmetry imply:

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \left[\mathbf{\Gamma}^{\overline{\mathcal{S}}}\right]_{NM} (\rho_{ij}, \alpha_s) = \frac{1}{4}\, \gamma_K^{(i)} (\alpha_s) \;\; \delta_{NM}, \qquad \forall i$$

[EG & Magnea (2009)]

- These equations hold for <u>any</u> massless amplitude, to <u>any</u> loop order!

- They relate the soft anomalous dimension matrix to the much simpler function, the cusp anomalous dimension $\gamma_K^{(i)}$

Solving for $\mathbf{\Gamma}^{\overline{\mathcal{S}}}$

- Do the factorization constraints have a unique solution?

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} [\mathbf{\Gamma}^{\overline{\mathcal{S}}}]_{NM} \left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \gamma_K^{(i)} \left(\alpha_s\right) \; \delta_{NM} \, , \qquad \forall i$$

- For amplitudes with 2 or 3 coloured partons – yes!
  For general multi-leg amplitudes – no

- For an L-leg amplitude there are L equations and L(L-1)/2 variables.
  Indeed, for 4-legs or more:
  particular solution induced by $\gamma_K^{(i)}$
  + homogeneous solution, a function of conformal-invariant cross ratios:

$$\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left(\frac{\rho_{ij}\,\rho_{kl}}{\rho_{ik}\,\rho_{jl}}\right)^{1/2}$$

The sum-over-dipoles solution $\Gamma^{\overline{\mathcal{S}}}$

- If $\gamma_K^{(i)}$ admits Casimir scaling (known to hold at least to 3-loops), $C_i \equiv \mathrm{T}_i \cdot \mathrm{T}_i$,

$$\gamma_K{}^{(i)} = 2C_i\frac{\alpha_s}{\pi} + KC_i\left(\frac{\alpha_s}{\pi}\right)^2 + K^{(2)}C_i\left(\frac{\alpha_s}{\pi}\right)^3 + \cdots = C_i\,\widehat{\gamma}_K\left(\alpha_s\right) + \underbrace{\widetilde{\gamma}_K^{(i)}\left(\alpha_s\right)}_{\text{Higher Casimirs}}$$

there is a simple solution to the factorization equations – sum over dipoles:

$$\Gamma^{\overline{\mathcal{S}}}_{\text{dip.}}\left(\rho_{ij},\alpha_s\right) = -\frac{1}{8}\,\widehat{\gamma}_K\left(\alpha_s\right)\sum_{i\neq j}\ln(\rho_{ij})\,\mathrm{T}_i\cdot\mathrm{T}_j \;+\; \frac{1}{2}\,\widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s)\sum_{i=1}^{n}\mathrm{T}_i\cdot\mathrm{T}_i$$

- Corrections going beyond this solution fall into two categories:

  1) those induced by higher Casimirs $\widetilde{\gamma}_K^{(i)}\left(\alpha_s\right)$ – may first appear at 4 loops

  2) homogeneous solutions - functions of conformal-invariant cross ratios – may first appear at 3 loops

Additional constraints on soft singularities

Dedicated effort to exclude corrections beyond the dipole formula – or determine their possible form.

[Gardi-Magnea, Becher-Neubert (2009), Gardi-Magnea-Dixon (2010)]

Several powerful constraints were found (beyond those discussed above):

A. Non-Abelian exponentiation: "maximally non-Abelian" colour structure

B. Bose symmetry – interplay between colour and kinematic dependence

C. Transcendentality – upper bound always applies.
It is saturated when there is a direct relation to $\mathcal{N} = 4$

D. Collinear limits:
relating L parton amplitude with a collinear pair to (L-1) parton amplitude

E. Regge limit. [Del Duca, Duhr, Gardi, Magnea and White, **1108.5947]**

The soft anomalous dimension in fixed angle scattering amplitudes of massless partons – conclusions

- IR singularities are known to 2 loops for any number of legs and general Nc.

- At higher orders their structure is highly constrained by factorization and rescaling symmetry, as well as by non-abelian exponentiation, Bose symmetry, transcendentality, collinear limits and the Regge limit.

- The sum-over-dipoles formula – an all-order ansatz for the singularities of amplitudes with massless partons – is consistent with all present constraints and calculations.

- Sources of potential corrections beyond the sum-over-dipoles formula have been analysed.
  Presently we have no example of a function that is consistent with all constraints are three loops, yet no proof that this is an empty set.

- A 3-loop calculation is needed.

Application: An infrared approach to reggeization

- Knowledge of IR singularities in fixed-angle scattering can be applied to study the Regge limit.

  [Sterman-Sotiropoulos, Korchemsky-Korchemskaya, Dokshitzer-Marchesini]

- Using the "dipole formula" we can derive Reggeization (controlling only the IR-singular contributions to the Regge trajectory).

- We find that Reggeization holds at LL for general colour representations provided that leading order is dominated by the cross channel exchange.

- Reggeization breaks at NLL for the imaginary part of the amplitude and at NNLL for the real part.

- The method easily generalises to multi-Regge kinematics with any number of partons.

[Del Duca, Duhr, Gardi, Magnea and White, **1108.5947**]