Möbius representation of BFKL equation

V.S. Fadin

Budker Institute of Nuclear Physics

Novosibirsk

Contents

Introduction

The BFKL (Balitsky-Fadin-Kuraev-Lipatov) equation is based on the remarkable property of QCD – gluon reggeization. The high-energy QCD can be reformulated in terms of the gauge-invariant effective field theory for reggeized gluon interactions,
L.N. Lipatov, 1995,
so that the primary reggeon in QCD is not the Pomeron, but the reggeized gluon.
But for phenomenological applications, the most interesting is the Pomeron (colour singlet) exchange.

For scattering of colourless particles in the leading order (LO) the BFKL equation can be written in the form which is invariant under the conformal (Möbius) transformations in impact-parameter space
L.N. Lipatov, 1986.
We call this form Möbius representation. When it is taken in the impact parameter space, We call it Möbius form.

Introduction

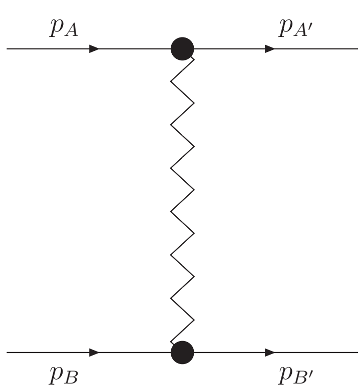
Conformal invariance of the BFKL equation is extremely important for its integrability. Therefore, it is very useful to know conformal properties of the BFKL kernel in the next-to-leading order.

Evidently, in QCD the conformal invariance is violated by running of α_s. But one could expect that the kernel is quasi-conformal, i.e. that the Möbius invariance is violated only by terms proportional to the β–function, and it is conserved in $N = 4$ SUSY Yang-Mills theories. One could expect also coincidence of the Möbius form of the BFKL kernel and the kernel of the colour dipole model.

However, the situation is not so simple, because the NLO kernels are not unambiguously defined. It turns out that this ambiguity makes possible to get agreement between the BFKL approach and the colour dipole model and to find the quasi-conformal representation of the BFKL kernel. It is done now in theories containing fermions and scalars in arbitrary representations of the colour group.

Basics of the BFKL approach

For elastic scattering processes $A+B \to A'+B'$ in the Regge kinematical region:
$s \simeq -u \to \infty$, t fixed (i.e. not growing with s) the Reggeization means that scattering amplitudes with the gluon quantum numbers in the t-channel has the form

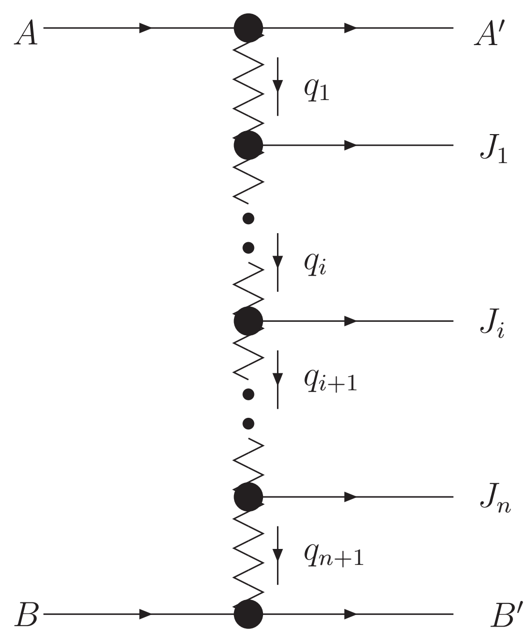


$$\mathcal{A}_{AB}^{A'B'} = \Gamma^c_{A'A}\left[\left(\frac{-s}{-t}\right)^{\omega(t)} - \left(\frac{s}{-t}\right)^{\omega(t)}\right]\Gamma^c_{B'B} \; ;$$

$\Gamma^c_{P'P}$–particle-particle-Reggeon (PPR) vertices or scattering vertices ("c" are color indices); $j(t)=1+\omega(t)$ – Reggeon trajectory.

Basics of the BFKL approach

The Reggeization means definite form not only of elastic amplitudes, but of inelastic amplitudes in the multi-Regge kinematics (MRK) as well. It can be presented by the picture



and written as

Basics of the BFKL approach

$$\Re\mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} = 2s\,\Gamma_{\tilde{A}A}^{c_1}\left(\prod_{i=1}^{n}\gamma_{c_i c_{i+1}}^{J_i}(q_i,q_{i+1})\left(\frac{s_i}{s_0}\right)^{\omega(t_i)}\frac{1}{t_i}\right)$$

$$\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_0}\right)^{\omega(t_{n+1})}\Gamma_{\tilde{B}B}^{c_{n+1}}$$

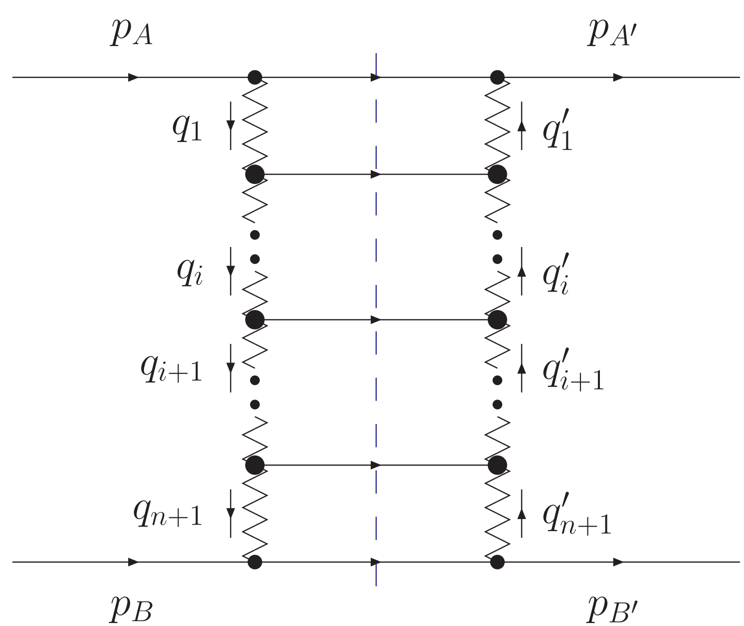
Here $\gamma_{c_i c_{i+1}}^{J_i}(q_i,q_{i+1})$ – the Reggeon-Reggeon-particle (RRP) or production vertices.

MRK is the kinematics where all particles have limited (not growing with s) transverse momenta and are combined into jets with limited invariant mass of each jet and large (growing with s) invariant masses of any pair of the jets.

The MRK gives dominant contributions to cross sections of QCD processes at high energy $\sqrt{s}$. In the LLA only a gluon can be produced. In the NLA one has to account production of $Q\bar{Q}$ and GG jets.

Basics of the BFKL approach

In the BFKL approach imaginary part of the amplitude for the processes $A + B \longrightarrow A' + B'$ at large center of mass energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$ $s \gg |t|$ is calculated using the unitarity as the sum:

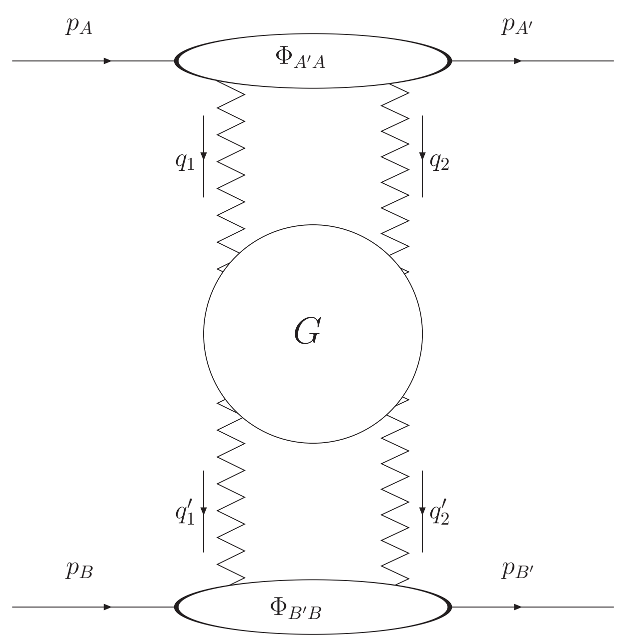
$\sum_n$ 

and the real part is restored using the analyticity.

Basics of the BFKL approach

The scattering amplitudes are represented by the convolution

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}$$



Basics of the BFKL approach

where the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe the transitions $A \to A'$ and $B \to B'$ due to scattering on the Reggeized gluons, while G is the Green's function for the two interacting Reggeized gluons. All the dependence of the amplitude on properties of particles $A,\ A'$ $\ (B,\ B')$ is contained in the impact factors $\Phi_{A'A}$ $\ (\Phi_{B'B})$, which are energy independent. The dependence on energy is determined by the universal (process independent) Greens's function G, which satisfies to the BFKL equation

$$\frac{\partial}{\partial Y}\hat{\mathcal{G}} = \hat{\mathcal{K}}\hat{\mathcal{G}}$$

and can be presented as

$$\hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}},$$

$\hat{\mathcal{K}}$ is the BFKL kernel, Y is the total rapidity $\ (Y = \ln(s/s_0))$.

Basics of the BFKL approach

Both the BFKL kernel and the impact factors are unambiguously defined in terms of the Reggeon vertices and the gluon Regge trajectory. For any colour representation $\mathcal{R}$ in the t-channel the BFKL kernel $\mathcal{K}^{(\mathcal{R})}\left(\vec{q}_1, \vec{q}_2; \vec{q}\right)$ is given as

$$\left[\omega\left(-\vec{q}_1^{\,2}\right) + \omega\left(-(\vec{q}_1 - \vec{q}\,)^{\,2}\right)\right] \delta^{(D-2)}\left(\vec{q}_1 - \vec{q}_2\right) + \mathcal{K}_r^{(\mathcal{R})}\left(\vec{q}_1, \vec{q}_2; \vec{q}\right)$$

The most interesting representations are the colour singlet (Pomeron channel, $\mathcal{R} = 1$) and the antisymmetric colour octet (gluon channel $\mathcal{R} = 8_a$).

For scattering of physical (colourless) particles only the channels exists. Nevertheless the gluon channel plays an role. It is caused by a possibility to use this channel for check of self-consistency, and, finally, for a proof of gluon Reggeization.

Basics of the BFKL approach

The LO BFKL kernel is known since 1975. The NLO kernel was calculated also long ago.

For the forward scattering (i.e. for $t = 0$ and color singlet in the t-channel) it is known more than 10 years

V.S. F., L.N. Lipatov, 1998,

M. Ciafaloni, G. Camicci, 1998.

More than five years ago the kernel was found also for any fixed (not growing with energy) momentum transfer t and any possible color state in the t-channel

V.S. F., R. Fiore, 2005.

All these results were obtained in the momentum space.

The direct way of finding of the Möbius representation of the BFKL kernel is to transform it from momentum to coordinate space.

Möbius representation of LO BFKL

In the LO such transformation makes evident conformal invariance of the Möbius representation of the BFKL kernel and coincidence of this representation with the kernel of the colour dipole approach

N.N. Nikolaev and B.G. Zakharov, 1994,

A.H. Mueller, 1994.

Starting from the papers

Yu.V. Kovchegov, H. Weigert, 2006

I. Balitsky, 2006

the comparison became possible also in the NLO for the quark contribution to the kernel.

In the NLO one could also expect coincidence of the Möbius representation of the BFKL kernel and the kernel of the colour dipole approach.

Möbius representation of LO BFKL

For colourless objects the impact factors in the representation

$$\delta(\vec{q}_A - \vec{q}_B) disc_s \mathcal{A}_{AB}^{A'B'} = \frac{i}{4(2\pi)^{D-2}} \langle A'\bar{A}| e^{Y\hat{\mathcal{K}}} \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle$$

are "gauge invariant":

$$\langle A'\bar{A}|\vec{q}, 0\rangle = \langle A'\bar{A}|0, \vec{q}\rangle = 0 \ .$$

Therefore $\langle A'\bar{A}|\Psi\rangle = 0$ if $\langle \vec{r}_1, \vec{r}_2|\Psi\rangle$ does not depend either on $\vec{r}_1$ or on $\vec{r}_2$. $\langle A'\bar{A}|\hat{\mathcal{K}}$ is "gauge invariant" as well, because $\langle \vec{q}_1, \vec{q}_2|\hat{\mathcal{K}}_r|\vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime}\rangle$ vanishes at $\vec{q}_1^{\,\prime} = 0$ or $\vec{q}_2^{\,\prime} = 0$.
It means that we can change $|In\rangle \equiv \left(\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}\right)^{-1} |\bar{B}'B\rangle$ for $|In_d\rangle$, where $|In_d\rangle$ has the "dipole " property $\langle \vec{r}, \vec{r}|In_d\rangle = 0$.
After this one can omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the "dipole " property.

Möbius representation of LO BFKL

The kernel obtained in this way is called Möbius form of the BFKL kernel. It can be written as

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_M | \vec{r}_1{}' \vec{r}_2{}' \rangle = \delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}) \int d\vec{r}_0 \, g_0(\vec{r}_1, \vec{r}_2; \vec{r}_0)$$

$$+ \delta(\vec{r}_{11'}) g_1(\vec{r}_1, \vec{r}_2; \vec{r}_2{}') + \delta(\vec{r}_{22'}) g_1(\vec{r}_2, \vec{r}_1; \vec{r}_1{}') + \frac{1}{\pi} g_2(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}')$$

with the functions $g_{1,2}$ turning into zero when their first two arguments coincide. The first three terms contain ultraviolet singularities which cancel in their sum, as well as in the LO, with account of the "dipole" property of the "target" impact factors. The coefficient of $\delta(\vec{r}_{11'}) \delta(\vec{r}_{22'})$ is written in the integral form in order to make the cancellation evident.

The term $g(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}')$ is absent in the LO because the LO kernel in the momentum space does not contain terms depending on all three independent momenta simultaneously.

Möbius representation of LO BFKL

The kernel $\hat{\mathcal{K}}$ is presented as

$$\hat{\mathcal{K}} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\mathcal{K}}_r \ , \quad \langle \vec{q}_i | \hat{\omega}_i | \vec{q}_i^{\,\prime} \rangle = \delta(\vec{q}_i - \vec{q}_i^{\,\prime}) \omega(-\vec{q}_i^{\,2}) \ ,$$

$\omega(t)$– the gluon Regge trajectory, $\hat{\mathcal{K}}_r$–real part.
In the leading order $\langle \vec{q}_1, \vec{q}_2 | \hat{\mathcal{K}} | \vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime} \rangle = \delta(\vec{q} - \vec{q}^{\,\prime}) \mathcal{K}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q})$, $\vec{q}_1 - \vec{q}_1^{\,\prime} = \vec{k}$

$$\mathcal{K}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = \frac{\alpha_s N_c}{2\pi^2} \left[2 \left(\frac{1}{\vec{k}^{\,2}} - \frac{\vec{q}_1 \vec{k}}{\vec{q}_1^{\,2} \vec{k}^{\,2}} + \frac{\vec{q}_2 \vec{k}}{\vec{q}_2^{\,2} \vec{k}^{\,2}} - \frac{\vec{q}_1 \vec{q}_2}{\vec{q}_1^{\,2} \vec{q}_2^{\,2}} \right) \right.$$

$$\left. - \delta(\vec{k}) \int d^2 l \left(\frac{2}{\vec{l}^{\,2}} - \frac{\vec{l}(\vec{l} - \vec{q}_1)}{\vec{l}^{\,2} (\vec{l} - \vec{q}_1)^2} - \frac{\vec{l}(\vec{l} - \vec{q}_2)}{\vec{l}^{\,2} (\vec{l} - \vec{q}_2)^2} \right) \right] .$$

Möbius representation of LO BFKL

The kernel in the coordinate representation is given by the integral

$$\langle \vec{r}_1, \vec{r}_2 | \hat{\mathcal{K}} | \vec{r}_1{}', \vec{r}_2{}' \rangle = \int \frac{d\vec{q}_1 \; d\vec{q}_2 \; d\vec{k}}{(2\pi)^4} \mathcal{K}(\vec{q}_1, \vec{q}_2; \vec{k}) \; e^{i\,(\vec{q}_1 \vec{r}_{11'} + \vec{q}_2 \vec{r}_{22'} + \vec{k} \vec{r}_{1'2'})} \, .$$

The last term in the real part gives $\delta(\vec{r}_{1'2'})$ and is omitted in the Möbius form. The first terms in the real and virtual parts give $2\alpha_s N_c \delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}) v(\vec{r}_{12})/(2\pi)^2$,

$$v(\vec{r}_{12}) = 2 \int d\vec{k} \, \frac{1}{\vec{k}^{\,2}} \left(e^{i\,\vec{k}\,\vec{r}_{12}} - 1 \right)$$

and contains ultraviolet divergence. This divergence is artificial. To make evident its cancelation $v(\vec{r}_{12})$ is represented in the integral form:

$$v(\vec{r}_{12}) = - \int \frac{d\vec{k}_1 \, d\vec{k}_2 \, d\vec{r}_0}{(2\pi)^2} \frac{(\vec{k}_1 \, \vec{k}_2)}{\vec{k}_1^{\,2} \, \vec{k}_2^{\,2}} \left(2 e^{i(\vec{k}_1 \, \vec{r}_{10} + \vec{k}_2 \, \vec{r}_{20})} - e^{i(\vec{k}_1 + \vec{k}_2) \vec{r}_{10}} - e^{(\vec{k}_1 + \vec{k}_2) \vec{r}_{20}} \right)$$

Möbius representation of LO BFKL

$$v(\vec{r}_{12}) = -\int d\vec{r}_0 \, \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} \; .$$

The second terms in the real and virtual parts give

$$-\frac{2\alpha_s N_c}{(2\pi)^2}\delta(\vec{r}_{22'})\int \frac{d\vec{q}_1\, d\vec{k}}{(2\pi)^2}\frac{(\vec{k}\,\vec{q}_1)}{\vec{k}^{\,2}\,\vec{q}_1^{\,2}} e^{i\,\vec{q}_1\,\vec{r}_{11'}}\left(2e^{-i\vec{k}\,\vec{r}_{21'}} + e^{i\vec{k}\,\vec{r}_{11'}}\right)$$

$$= \frac{2\alpha_s N_c}{(2\pi)^2}\,\delta(\vec{r}_{22'})\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{21'}^{\,2}} - \frac{1}{\vec{r}_{21'}^{\,2}}\right) \; .$$

In the Möbius form the last term must be omitted. The contribution of the third terms is obtained by the substitution $1 \to 2$. Finally,

$$\langle \vec{r}_1, \vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1^{\,\prime}, \vec{r}_2^{\,\prime}\rangle = \frac{\alpha_s N_c}{2\pi^2}\int d\vec{r}_0\,\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}}\,(\delta(\vec{r}_{11'})\delta(\vec{r}_{02'}) + \delta(\vec{r}_{22'})\delta(\vec{r}_{01'}) - \delta(\vec{r}_{11'})\delta(\vec{r}_{22'})$$

where the subscript M denotes the Möbius form. Remarkably, that this form exactly coincides with the kernel of the dipole approach.

Möbius representation of LO BFKL

The transformations of the Möbius group in the two-dimensional space $\vec{r} = (x, y)$ can be written as

$$z \to \frac{az + b}{cz + d} \ , \tag{1}$$

where $z = x + iy, \;\; a, b, c, d$ are complex numbers, with $ad - bc \neq 0$. Under these transformations, one has

$$z_1 - z_2 \to \frac{z_1 - z_2}{(cz_1 + d)(cz_2 + d)}(ad - bc) \ ,$$

$$dzdz^* \to dzdz^* \frac{|ad - bc|^2}{\left|(cz + d)^2\right|^2} \ , \tag{2}$$

so that the conformal invariance of $\langle \vec{r}_1, \vec{r}_2 | \hat{\mathcal{K}} | \vec{r}_1{}', \vec{r}_2{}' \rangle d\vec{r}_1{}' d\vec{r}_2{}'$ is evident.

Möbius representation of LO BFKL

For the forward scattering, there is a remarkable functional identity between the Möbius form and the BFKL kernel in the momentum representation.

$$\langle \vec{q}_1 | \hat{\mathcal{K}} | \vec{q}_1^{\,\prime} \rangle = \mathcal{K}(\vec{q}_1, \vec{q}_1^{\,\prime}; 0)$$

$$\langle \vec{q} | \hat{\mathcal{K}} | \vec{q}^{\,\prime} \rangle = \frac{\alpha_s N_c}{2\pi^2} \left[\frac{2\vec{q}^{\,\prime 2}}{(\vec{q}-\vec{q}^{\,\prime})^2\, \vec{q}^{\,2}} - \delta(\vec{q}-\vec{q}^{\,\prime}) \int \frac{d\vec{l}\; \vec{q}^{\,2}}{(\vec{q}-\vec{l})^2 \vec{l}^{\,2}} \right].$$

In the coordinate space

$$\langle \vec{r} | \hat{\mathcal{K}} | \vec{r}^{\,\prime} \rangle = \int \langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}} | \vec{r}_1^{\,\prime} \vec{r}_2^{\,\prime} \rangle \delta(\vec{r}_{12} - \vec{r}) d\vec{r}_1 d\vec{r}_2 \; ; \quad \vec{r}^{\,\prime} = \vec{r}_1^{\,\prime} - \vec{r}_2^{\,\prime};$$

Möbius representation of LO BFKL

$$\langle \vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle = \int \frac{d\vec{r}}{2\pi}\frac{d\vec{r}\,'}{2\pi} e^{-i\vec{q}\vec{r}+i\vec{q}\,'\vec{r}\,'} \langle \vec{r}|\hat{\mathcal{K}}|\vec{r}\,'\rangle \ ,$$

$$\langle \vec{r}|\hat{\mathcal{K}}_d|\vec{r}\,'\rangle = \frac{\alpha_s N_c}{2\pi^2}\left[\frac{2\vec{r}^{\,2}}{(\vec{r}-\vec{r}\,')^2\,\vec{r}\,'^2} - \delta(\vec{r}-\vec{r}\,')\int \frac{d\vec{\rho}\,\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^2\vec{\rho}^{\,2}}\right] \ .$$

$$\frac{\vec{r}\,'^{\,2}}{\vec{r}^{\,2}}\langle \vec{r}|\hat{\mathcal{K}}_M|\vec{r}\,'\rangle = \frac{\vec{q}^{\,2}}{\vec{q}\,'^2}\langle \vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle\Big|_{\vec{q}\to\vec{r},\ \vec{q}\,'\to\vec{r}\,'} \ ,$$

In the NLO this relation is violated only by the charge renormalization

Möbius representation of NLO BFKL

The infrared behaviour of the virtual and real parts of the kernel is far worse in the NLO. However, it turns out possible to perform explicit cancelation of infrared singularities and to write the kernel in the physical space.

$$\mathcal{K}(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \mathcal{K}_p(\vec{q}_1, \vec{q}_1{}'; \vec{q}) + \mathcal{K}_f(\vec{q}_1, \vec{q}_1{}'; \vec{q}),$$

where

$$\mathcal{K}_p\left(\vec{q}_1, \vec{q}_1{}'; \vec{q}\,\right)$$

$$= \left[\omega\left(-\vec{q}_1{}^2\right) + \omega\left(-\left(\vec{q}_1 - \vec{q}\,\right)^2\right)\right] \vec{q}_1{}^2 \left(\vec{q}_1 - \vec{q}\,\right)^2 \delta^{(D-2)}\left(\vec{q}_1 - \vec{q}_1{}'\right) + 2\mathcal{K}_r^{(8_a)}\left(\vec{q}_1, \vec{q}_1{}'; \vec{q}\right),$$

and the finite part $\mathcal{K}_f(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ contains neither ultraviolet nor infrared singularities and therefore can be written from the beginning in the physical space with $D = 4$.

Möbius representation of NLO BFKL

$$\mathcal{K}_{p,f}(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \mathcal{K}^G_{p,f}(\vec{q}_1, \vec{q}_1{}'; \vec{q}) + \mathcal{K}^Q_{p,f}(\vec{q}_1, \vec{q}_1{}'; \vec{q}),$$

$$\mathcal{K}^Q_p(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \frac{\alpha_s^2(\mu)}{16\pi^3}\frac{2N_c n_f}{3}\left\{2\left(\ln\left(\frac{\vec{k}^{\,2}}{\mu^2}\right) - \frac{5}{3}\right)\left(\frac{\vec{q}_2{}'^{\,2}\vec{q}_1^{\,2} + \vec{q}_1{}'^{\,2}\vec{q}_2^{\,2}}{\vec{k}^2} - \vec{q}^{\,2}\right)\right.$$

$$+\vec{q}^{\,2}\ln\left(\frac{\vec{q}^{\,4}\vec{k}^4}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{q}_1{}'^{\,2}\vec{q}_2{}'^{\,2}}\right) - \frac{\left(\vec{q}_2{}'^{\,2}\vec{q}_1^{\,2} - \vec{q}_1{}'^{\,2}\vec{q}_2^{\,2}\right)}{\vec{k}^2}\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2}}{\vec{q}_2^{\,2}\vec{q}_1{}'^{\,2}}\right)$$

$$-\vec{q}_1^{\,2}\vec{q}_2^{\,2}\delta(\vec{k})\int d^2l\left[\left(\frac{2}{\vec{l}^{\,2}} + \frac{2\vec{l}(\vec{q}_1 - \vec{l})}{\vec{l}^{\,2}(\vec{q}_1 - \vec{l})^2}\right)\left(\ln\left(\frac{\vec{l}^2(\vec{q}_1 - \vec{l})^2}{\mu^2\vec{q}_1^{\,2}}\right) - \frac{5}{3}\right)\right.$$

$$\left.\left.+\left(\frac{2}{\vec{l}^{\,2}} + \frac{2\vec{l}(\vec{q}_2 - \vec{l})}{\vec{l}^{\,2}(\vec{q}_2 - \vec{l})^2}\right)\left(\ln\left(\frac{\vec{l}^2(\vec{q}_2 - \vec{l})^2}{\mu^2\vec{q}_2^{\,2}}\right) - \frac{5}{3}\right)\right]\right\}\ .$$

Here and below $\vec{q}_2 = \vec{q} - \vec{q}_1,\ \ \vec{q}_2{}' = \vec{q} - \vec{q}_1{}',\ \ \vec{k} = \vec{q}_1 - \vec{q}_1{}' = \vec{q}_2{}' - \vec{q}_2$.

Möbius representation of NLO BFKL

$$\mathcal{K}_p^G(\vec{q}_1, \vec{q}_1{}'; \vec{q}) =$$

$$\frac{\alpha_s(\mu)N_c}{4\pi^2}\left\{\left(\frac{\vec{q}_1{}^2\vec{q}_2{}'^{\,2}+\vec{q}_1{}'^{\,2}\vec{q}_2{}^2}{\vec{k}^{\,2}}-\vec{q}^{\,2}\right)\left(1+\frac{\alpha_s(\mu)N_c}{4\pi}\left[-\frac{11}{3}\ln\left(\frac{\vec{k}^{\,2}}{\mu^2}\right)+\frac{67}{9}-2\zeta(2)\right]\right)\right.$$

$$+\frac{\alpha_s(\mu)N_c}{4\pi}\left[\vec{q}^{\,2}\left(\frac{11}{3}\ln\left(\frac{\vec{q}_1{}^2\vec{q}_1{}'^{\,2}}{\vec{q}^{\,2}\vec{k}^{\,2}}\right)+\frac{1}{2}\ln\left(\frac{\vec{q}_1{}^2}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2{}^2}{\vec{q}^{\,2}}\right)+\frac{1}{2}\ln\left(\frac{\vec{q}_1{}'^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2{}'^2}{\vec{q}^{\,2}}\right)\right.\right.$$

$$\left.+\frac{1}{2}\ln^2\left(\frac{\vec{q}_1{}^2}{\vec{q}_1{}'^{\,2}}\right)\right)-\frac{\vec{q}_1{}^2\vec{q}_2{}'^{\,2}+\vec{q}_2{}^2\vec{q}_1{}'^{\,2}}{\vec{k}^{\,2}}\ln^2\left(\frac{\vec{q}_1{}^2}{\vec{q}_1{}'^{\,2}}\right)+\frac{\vec{q}_1{}^2\vec{q}_2{}'^{\,2}-\vec{q}_2{}^2\vec{q}_1{}'^{\,2}}{\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1{}^2}{\vec{q}_1{}'^{\,2}}\right)$$

$$\times\left(\frac{11}{3}-\frac{1}{2}\ln\left(\frac{\vec{q}_1{}^2\vec{q}_1{}'^{\,2}}{\vec{k}^{\,4}}\right)\right)+\left[\vec{q}^{\,2}(\vec{k}^{\,2}-\vec{q}_1{}^2-\vec{q}_1{}'^{\,2})+2\vec{q}_1{}^2\vec{q}_1{}'^{\,2}-\vec{q}_1{}^2\vec{q}_2{}'^{\,2}-\vec{q}_2{}^2\vec{q}_1{}'^{\,2}\right.$$

$$\left.\left.\left.+\frac{\vec{q}_1{}^2\vec{q}_2{}'^{\,2}-\vec{q}_2{}^2\vec{q}_1{}'^{\,2}}{\vec{k}^{\,2}}(\vec{q}_1{}^2-\vec{q}_1{}'^{\,2})\right]I(\vec{q}_1{}^2,\vec{q}_1{}'^{\,2},\vec{k}^{\,2})\right]\right\}-\delta(\vec{k})\vec{q}_1{}^2\vec{q}_2{}^2\left(\int d^2l\,\left(\frac{2}{\vec{l}^{\,2}}+2\frac{\vec{l}(\vec{q}_1-\vec{l})}{\vec{l}^{\,2}(\vec{q}_1-\vec{l})^2}\right.\right.$$

$$\left.\left..+\frac{\alpha_s(\mu)N_c}{\pi}\left(V(\vec{l})+V(\vec{l},\vec{l}-\vec{q}_1)\right)\right)-3\alpha_s(\mu)N_c\zeta(3)\right)+\left(\vec{q}_1\leftrightarrow\vec{q}_2,\ \ \vec{q}_1{}'\leftrightarrow\vec{q}_2{}'\right),$$

Möbius representation of NLO BFKL

$$V(\vec{k}) = \frac{1}{2\vec{k}^{\,2}} \left(\frac{67}{9} - 2\zeta(2) - \frac{11}{3} \ln\left(\frac{\vec{k}^{\,2}}{\mu^2} \right) \right) \ ,$$

$$V(\vec{k},\vec{q}) = \frac{\vec{k}\vec{q}}{2\vec{k}^{\,2}\vec{q}^{\,2}} \left(\frac{11}{3} \ln\left(\frac{\vec{k}^{\,2}\vec{q}^{\,2}}{\mu^2(\vec{k}-\vec{q})^2} \right) - \frac{67}{9} + 2\zeta(2) \right)$$

$$- \frac{11}{12\vec{k}^{\,2}} \ln\left(\frac{\vec{q}^{\,2}}{(\vec{k}-\vec{q})^2} \right) - \frac{11}{12\vec{q}^{\,2}} \ln\left(\frac{\vec{k}^{\,2}}{(\vec{k}-\vec{q})^2} \right) \ ,$$

$$I(a,b,c) = \int_0^1 \int_0^1 \int_0^1 \frac{dx_1 dx_2 dx_3 \delta(1-x_1-x_2-x_3)}{(ax_1+bx_2+cx_3)(x_1x_2+x_1x_3+x_2x_3)} \ .$$

Möbius representation of NLO BFKL

Although the piece $\mathcal{K}_f(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ contains neither ultraviolet nor infrared singularities, it is the most complicated part of the kernel in the momentum representation. The quark part of this piece, which is called abelian part of the quark contribution, differs from the QED Pomeron kernel

V. N. Gribov, L. N. Lipatov, and G. V. Frolov, 1970

H. Cheng and T. T. Wu, 1970

only by the coefficient. For finding the Möbius form of the kernel it is convenient to use the expression for this part before integration over the momenta of produced $q\bar{q}$ pair.

$$\mathcal{K}_f^Q(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = -2\frac{\alpha_s^2(\mu) n_f}{(2\pi)^2 N_c}\int_0^1 dx \int \frac{d^2 k_1}{(2\pi)^2} F(\vec{q}_1, \vec{q}_2; \vec{k}_1, \vec{k}_2) \ ,$$

Möbius representation of NLO BFKL

$$F(\vec{q}_1,\vec{q}_2;\vec{k}_1,\vec{k}_2) = x(1-x)\left(\frac{2(\vec{q}_1\vec{k}_1)-\vec{q}_1^{\,2}}{\sigma_{11}}+\frac{2(\vec{q}_1\vec{k}_2)-\vec{q}_1^{\,2}}{\sigma_{21}}\right)$$

$$\times\left(\frac{2(\vec{q}_2\vec{k}_1)+\vec{q}_2^{\,2}}{\sigma_{12}}+\frac{2(\vec{q}_2\vec{k}_2)+\vec{q}_2^{\,2}}{\sigma_{22}}\right)+\frac{x\vec{q}^{\,2}(2(\vec{q}_1\vec{k}_1)-\vec{q}_1^{\,2})}{2\sigma_{11}}\left(\frac{1}{\sigma_{22}}-\frac{1}{\sigma_{12}}\right)$$

$$+\frac{x\vec{q}^{\,2}(2(\vec{q}_2\vec{k}_1)+\vec{q}_2^{\,2})}{2\sigma_{12}}\left(\frac{1}{\sigma_{11}}-\frac{1}{\sigma_{21}}\right)+\frac{1}{\sigma_{11}\sigma_{22}}\Bigg(-2(\vec{q}_1\vec{k}_1)(\vec{q}_2\vec{q}_2^{\,\prime}\,)$$

$$-2(\vec{q}_2\vec{k}_1)(\vec{q}_1\vec{q}_1^{\,\prime})+(\vec{q}_2^{\,2}-\vec{q}_1^{\,2})(\vec{k}_1\vec{k})+\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}-\frac{\vec{k}^2\vec{q}^{\,2}}{2}\Bigg)\,,$$

where $\vec{k}_1+\vec{k}_2=\vec{k}=\vec{q}_1-\vec{q}_1^{\,\prime}=\vec{q}_2^{\,\prime}-\vec{q}_2$,

$$\sigma_{11}=(\vec{k}_1-x\vec{q}_1)^2+x(1-x)\vec{q}_1^{\,2},\;\;\sigma_{21}=(\vec{k}_2-(1-x)\vec{q}_1)^2+x(1-x)\vec{q}_1^{\,2}\;,$$

$$\sigma_{12}=(\vec{k}_1+x\vec{q}_2)^2+x(1-x)\vec{q}_2^{\,2},\;\;\sigma_{22}=(\vec{k}_2+(1-x)\vec{q}_2)^2+x(1-x)\vec{q}_2^{\,2}\;.$$

Möbius representation of NLO BFKL

The gluon part $\mathcal{K}_f^G$ of the finite piece $\mathcal{K}_f(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ is called symmetric part of the gluon contribution. As well as in the case of $\mathcal{K}_f^Q$, for finding the Möbius form of $\mathcal{K}_f^G$, it is better to start from its expression in the momentum space before integration over the momenta of the produced particles. Contrary to the abelian part of the quark contribution, the symmetric part of the gluon contribution contains the subtraction term. Therefore it is decomposed into two pieces:

$$\mathcal{K}_f^G(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \mathcal{K}_{s1}(\vec{q}_1, \vec{q}_1{}'; \vec{q}) + \mathcal{K}_{s2}(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \ ,$$

where

$$\mathcal{K}_{s1}(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \frac{\alpha_s^2(\mu) N_c^2}{2\pi^3} \int_0^1 dx \int \frac{d^2 k_1}{2\pi} \left(\frac{F_s(k_1, k_2)}{x(1-x)} \right)_+$$

Möbius representation of NLO BFKL

$$\mathcal{K}_{s2}(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = -\int \frac{d^2k_1}{4} \frac{\mathcal{K}_r^B(\vec{q}_1, \vec{q}_1 - \vec{k}_1; \vec{q}) \mathcal{K}_r^B(\vec{q}_1 - \vec{k}_1, \vec{q}_1{}'; \vec{q})}{(\vec{q}_1 - \vec{k}_1)^2 (\vec{q}_2 + \vec{k}_1)^2} \ln\left(\frac{\vec{k}_2{}^2}{\vec{k}_1{}^2}\right) .$$

Here $\vec{k}_1 + \vec{k}_2 = \vec{k}$, the subscript $_+$ means

$$\left(\frac{f(x)}{x(1-x)}\right)_+ \equiv \frac{1}{x}[f(x) - f(0)] + \frac{1}{(1-x)}[f(x) - f(1)] ,$$

$\mathcal{K}_r^B$ is the real part of the leading order kernel,

$$\mathcal{K}_r^B(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \frac{\alpha_s(\mu) N_c}{2\pi^2} \left(\frac{\vec{q}_1^{\,2} \vec{q}_2{}'^{\,2} + \vec{q}_1{}'^{\,2} \vec{q}_2^{\,2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right) ,$$

Möbius representation of NLO BFKL

$$F_s(k_1,k_2) = c^{\alpha\beta}(q_1;k_1,k_2)c^{\alpha\beta}(q_1;k_2,k_1),$$

where

$$c^{\alpha\beta}(q_1;k_1,k_2) = \frac{q_{1\perp}^2 k_{1\perp}^\alpha k_{2\perp}^\beta}{k_{1\perp}^2 k_{2\perp}^2}$$

$$+\frac{1}{2}g_\perp^{\alpha\beta}x_2\frac{q_{1\perp}^2-2q_{1\perp}k_{1\perp}}{\sigma_{11}}+\frac{x_2k_{1\perp}^\alpha q_{1\perp}^\beta - x_1 q_{1\perp}^\alpha (q_1-k_1)_\perp^\beta}{x_1\sigma_{11}}+\frac{q_{1\perp}^2 k_{1\perp}^\alpha (q_1-k_1)_\perp^\beta}{k_{1\perp}^2\sigma_{11}}$$

$$+\frac{1}{2}g_\perp^{\alpha\beta}x_1\frac{q_{1\perp}^2-2q_{1\perp}k_{2\perp}}{\sigma_{21}}+\frac{x_1 q_{1\perp}^\alpha k_{2\perp}^\beta - x_2 (q_1-k_2)_\perp^\alpha q_{1\perp}^\beta}{x_2\sigma_{21}}+\frac{q_{1\perp}^2 (q_1-k_2)_\perp^\alpha k_{2\perp}^\beta}{k_{2\perp}^2\sigma_{21}}\ .$$

Möbius representation of NLO BFKL

In QCD the NLO kernel contains quark and gluon contributions. In ones turn, the quark contribution is divided into two pieces: non-Abelian" (leading in N_c) and Abelian" (suppressed by N_c^{-2}). Their Möbius forms

V.S. F., R. Fiore, A. Papa, 2006, 2007

agrees, with account of the ambiguity of the kernel, with the results

V.V. Kovchegov, H. Weigert, 2006,

I. Balitsky, 2006,

obtained by direct calculation in the dipole picture. The Abelian part is greatly simplified in comparison with the momentum representation. Moreover, this part is conformal invariant. It could be important for the QED Pomeron.

Möbius representation of NLO BFKL

The most important contribution to the BFKL kernel is the gluon one. In the momentum representation in the NLO for arbitrary momentum transfer it is very complicated
V.S. F, R. Fiore 2005.
The Möbius form of this contribution
V.S. F, R. Fiore, A.V. Grabovsky, A. Papa, 2007
turned out strikingly simple.

However, the conformal invariance is broken not only by the terms related to the renormalization.

Moreover, it occurred afterwards that the NLO gluon contribution to the kernel of the colour dipole approach
I. Balitsky, G.A. Chirilli, 2008
does not agree with the Möbius form of the same contribution to the BFKL kernel.

Möbius representation of NLO BFKL

Supersymmetric Yang-Mills theories contain gluons and Maiorana fermions in the adjoint representation of the colour group. The gluon contribution is the same as in QCD. The fermion one can be obtained by change of the group coefficients: $n_f \to n_M N_c$ for the "non-Abelian" part, and $n_f \to -n_M N_c^3$ for the "Abelian" part; n_M is the number of flavours of Maiorana quarks. For N–extended SUSY $n_M = N$.

At $N > 1$ besides quarks there are n_S scalar particles; $n_S = 2$ at $N = 2$ and $n_S = 6$ at $N = 4$. At $N = 4$ $\beta_0 = \frac{11}{3} - \frac{2}{3}n_M - \frac{1}{6}n_S = 0$ and α_s is not running. Nevertheless, the Möbius form of the NLO kernel

V.S. F, R. Fiore, 2007

is not conformal invariant.

Möbius representation of NLO BFKL

In the theory with n_M Maiorana fermions and n_S scalars in the adjoint representation we have

$$g_1(\vec{r}_1,\vec{r}_2;\vec{r}_2{}') = \frac{\alpha_s(\frac{4e^{-2C}}{\vec{r}^{\,2}})N_c}{2\pi^2}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\left[1+\frac{\alpha_s N_c}{2\pi}\left(\frac{67}{18}-\zeta(2)-\frac{5n_M}{9}-\frac{2n_S}{9}\right.\right.$$

$$\left.\left.+\frac{\beta_0}{2N_c}\frac{\vec{r}_{12'}^{\,2}-\vec{r}_{22'}^{\,2}}{\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)-\frac{1}{2}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)+\frac{\vec{r}_{12'}^{\,2}}{2\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)\right)\right].$$

Since only the integral of g^0 is fixed, it can be written in different forms. One of them is

$$g_0(\vec{r}_1,\vec{r}_2;\vec{r}_0) = -g(\vec{r}_1,\vec{r}_2;\rho)+\frac{\alpha_s^2N_c^2}{4\pi^3}\delta(\vec{r}_0)2\pi\zeta(3)\,.$$

The function $g_1(\vec{r}_1,\vec{r}_2;\vec{\rho})$ vanish at $\vec{r}_1=\vec{r}_2$. Then, these functions turn into zero for $\vec{\rho}^{\,2}\to\infty$ faster than $(\vec{\rho}^{\,2})^{-1}$ to provide the infrared safety. The ultraviolet singularities of this function at $\vec{\rho}=\vec{r}_2$ and $\vec{\rho}=\vec{r}_1$ cancel with the singularities of $g^0(\vec{r}_1,\vec{r}_2;\vec{\rho})$ on account of the “dipole” property of the “target” impact factors.

Möbius representation of NLO BFKL

$$g_2(\vec r_1,\vec r_2;\vec r_1{}',\vec r_2{}') = \frac{\alpha_s^2 N_c^2}{4\pi^3}\left[\frac{1}{2\vec r^{\,4}_{1'2'}}\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{d}\ln\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}\right)-1\right)\left(1-n_M+\frac{n_S}{2}\right)\right.$$

$$-\left(\frac{(4-n_M)}{4\vec r^{\,4}_{1'2'}}\frac{\vec r^{\,2}_{12}\,\vec r^{\,2}_{1'2'}}{d}-\frac{1}{4\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}}\left(\frac{\vec r^{\,4}_{12}}{d}-\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{1'2'}}\right)\right)\ln\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}\right)$$

$$+\frac{\ln\left(\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{1'2'}}\right)}{4\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\,\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}\right)}{2\vec r^{\,2}_{12'}\vec r^{\,2}_{21'}}\left(\frac{\vec r^{\,2}_{12}}{2\vec r^{\,2}_{1'2'}}+\frac{1}{2}-\frac{\vec r^{\,2}_{22'}}{\vec r^{\,2}_{1'2'}}\right)+\frac{\vec r^{\,2}_{12}\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}\right)}{4\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}}$$

$$+\frac{\ln\left(\frac{\vec r^{\,2}_{22'}}{\vec r^{\,2}_{12}}\right)}{2\vec r^{\,2}_{11'}\,\vec r^{\,2}_{12'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12'}\vec r^{\,2}_{22'}}\right)}{2\vec r^{\,2}_{11'}\,\vec r^{\,2}_{1'2'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{11'}}{\vec r^{\,2}_{22'}\,\vec r^{\,2}_{1'2'}}\right)}{2\vec r^{\,2}_{12'}\vec r^{\,2}_{1'2'}}$$

$$\left.+\frac{\vec r^{\,2}_{12}\ln\left(\frac{\vec r^{\,2}_{11'}}{\vec r^{\,2}_{1'2'}}\right)}{2\vec r^{\,2}_{11'}\vec r^{\,2}_{12'}\,\vec r^{\,2}_{22'}}+\frac{\vec r^{\,2}_{21'}\ln\left(\frac{\vec r^{\,2}_{21'}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12}\,\vec r^{\,2}_{11'}}\right)}{2\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}}+(1\leftrightarrow 2)\right],\quad d=\vec r^{\,2}_{12'}\vec r^{\,2}_{21'}-\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}.$$

Ambiguities of NLO kernels

However, the hope for conformal invariance still remained.

The reason is the ambiguity of the NLO kernel.

The BFKL kernel has an evident ambiguity connected with impact factors.

The discontinuity

$$\langle A'\bar{A}|e^{Y\widehat{\mathcal{K}}}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle$$

remains intact under the transformation

$$\hat{\mathcal{K}} \to \hat{\mathcal{O}}^{-1}\hat{\mathcal{K}}\hat{\mathcal{O}}\ , \quad \langle A'\bar{A}| \to \langle A'\bar{A}|\hat{\mathcal{O}}\ , \quad \frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle \to \hat{\mathcal{O}}^{-1}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle .$$

If the LO kernel is fixed, one can take $\hat{\mathcal{O}} = 1 - \alpha_s\hat{U}$, and get

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - \alpha_s[\hat{\mathcal{K}}^{(B)}, \hat{U}].$$

Secondly, there is a freedom in the energy scale s_0. At first sight, it can lead to an additional ambiguity of the NLO kernel. However, it is not so.

Ambiguities of NLO kernels

It was shown

V.F., 1986

that any change of the energy scale can be compensated by the corresponding redefinition of the impact factors. Alternatively, we can leave one of the impact factored unchanged, changing the kernel. In this case the change will have the form $\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} - \alpha_s[\widehat{\mathcal{K}}^B \hat{U}]$ with a specific form of the operator $\hat{U}$

V. S. F., R. Fiore, A. V. Grabovsky, 2009.

Therefore, the freedom in a choice of the energy scale does not give anything new. In the NLO dependence on s_0 of the energy factor is cancelled by the dependence of the impact factors, so that s_0 can be taken as a free parameter. This freedom can be used for optimization of perturbative results

D.Yu. Ivanov, A. Papa, 2006.

Reduction of the NLO kernel to quasi-conformal shape

To get a hint on possible form of the transformation which can eliminate the discrepancy with BC-2008 the forward scattering was considered V. S. F., R. Fiore, A. V. Grabovsky, 2009.

It was shown that in this case the discrepancy can be removed by the transformation

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} + \frac{1}{2}[\widehat{\mathcal{K}}^B, \ln \hat{\vec{q}}^{\,2} \widehat{\mathcal{K}}^B] \; ,$$

up to the term with $\zeta(3)$ and to the difference in the renormalization scales. In the BFKL approach the term with $\zeta(3)$ passed through a great number of verifications. In particular, it is necessary for fulfillment of the bootstrap conditions for the gluon reggeization. We have no doubt that this term is correct.

Fortunately, it was recognized

I. Balitsky, G.A. Chirilli, 2009

that there was an error in their calculation of this term.

Reduction of the NLO kernel to quasi-conformal shape

In principle, one can easily write a formal expression for the operator $\hat{U}$ eliminating the discrepancy. Indeed, let us denote $\hat{\mathcal{K}}_M - \hat{\mathcal{K}}_{BC} = \hat{\Delta}$, the Born kernel $\hat{\mathcal{K}}^B$ eigenstates $|\mu\rangle$, and corresponding eigenvalues ω_μ^B. Then, if $\hat{\Delta} = \alpha_s \left[\hat{\mathcal{K}}^B, \hat{U}\right]$, one has

$$\left(\omega_{\mu'}^B - \omega_\mu^B\right) \langle \mu' | \alpha_s \hat{U} | \mu \rangle = \langle \mu' | \hat{\Delta} | \mu \rangle.$$

It is seen from here that the operator $\hat{U}$ exists only if the operator $\hat{\Delta}$ has zero matrix elements between states of equal energies. If so, supposing that the states $|\mu\rangle$ form a complete set, one has

$$\langle \mu' | \alpha_s \hat{U} | \mu \rangle = \sum_{\mu, \mu'} \frac{|\mu'\rangle \langle \mu' | \hat{\Delta} | \mu \rangle \langle \mu |}{\omega_{\mu'}^B - \omega_{\mu'}^B}.$$

Reduction of the NLO kernel to quasi-conformal shape

Finally, it was found

V.S. F., R.Fiore, A.V. Grabovsky, 2009

that with account of the error in the $\zeta(3)$ term and the difference of the renormalization scheme used by Balitsky and Chirilli from the $\overline{MS}$ one, their result agree with the Möbius form of the BFKL kernel. The agreement is reached by the transformation

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} - \alpha_s[\widehat{\mathcal{K}}^B \hat{U}]$$

with

$$\langle \vec{q}_1, \vec{q}_2|\alpha_s\hat{U}|\vec{q}_1{}', \vec{q}_2{}'\rangle = -\delta(\vec{q}_{11'} + \vec{q}_{22'})\frac{\mathcal{K}_r^B(\vec{q}_1, \vec{q}_1{}'; \vec{q})}{2\vec{q}_1{}^2\vec{q}_2{}^2}\ln \vec{q}_{11'}^{\,2}$$

$$+\frac{\alpha_s N_c}{4\pi^2}\,\delta(\vec{q}_{22'})\delta\left(\vec{q}_{11'}\right)\int d^{2+2\epsilon}k \ln \vec{k}^{\ 2}\left(\frac{2}{\vec{k}^{\ 2}} - \frac{\vec{k}(\vec{k}-\vec{q}_1)}{\vec{k}^{\ 2}(\vec{k}-\vec{q}_1)^2} - \frac{\vec{k}(\vec{k}-\vec{q}_2)}{\vec{k}^{\ 2}(\vec{k}-\vec{q}_2)^2}\right).$$

Reduction of the NLO kernel to quasi-conformal shape

Moreover, an additional transformation

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} - [\widehat{\mathcal{K}}^B \hat{U}_1]$$

with

$$\langle \vec{r}_1 \vec{r}_2 | \hat{U}_1 | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{\alpha_s N_c}{4\pi^2} \int d\vec{r}_0 \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}^{\,2} \vec{r}_{20}^{\,2}} \ln \left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{10}^{\,2} \vec{r}_{20}^{\,2}} \right)$$

$$\times \left[\delta(\vec{r}_{11'}) \delta(\vec{r}_{2'0}) + \delta(\vec{r}_{1'0}) \delta(\vec{r}_{22'}) - \delta(\vec{r}_{11'}) \delta(r_{22'}) \right]$$

removes the pieces on the Möbius form of the BFKL kernel, which are not related to the renormalization, but nevertheless are non-conformal. Therefore, in N=4 SUSY it makes the kernel conformal invariant.

Reduction of the NLO kernel to quasi-conformal shape

Thus we find

$$g^0_{SUSY}(\vec r_1, \vec r_2; \vec\rho) = 6\pi\zeta(3)\,\delta(\vec\rho) - g_{SUSY}(\vec r_1, \vec r_2; \vec\rho)\ ,$$

$$g_{SUSY}(\vec r_1, \vec r_2; \vec r_2^{\,\prime}) = \frac{\vec r_{12}^{\,2}}{\vec r_{22'}^{\,2}\vec r_{12'}^{\,2}}\left[\frac{32}{9} - \zeta(2) - \frac{5n_M + 2n_S}{9} + \frac{\beta_0}{2N_c}\ln\left(\frac{\vec r_{12}^{\,2}\mu^2}{4e^{2\psi(1)}}\right)\right.$$
$$\left. + \frac{\beta_0}{2N_c}\frac{\vec r_{12'}^{\,2} - \vec r_{22'}^{\,2}}{\vec r_{12}^{\,2}}\ln\left(\frac{\vec r_{22'}^{\,2}}{\vec r_{12'}^{\,2}}\right)\right],$$

$$g_{SUSY}(\vec r_1, \vec r_2; \vec r_1^{\,\prime}, \vec r_2^{\,\prime}) = \frac{1}{\vec r_{1'2'}^{\,4}}\left(\frac{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2} - 2\vec r_{12}^{\,2}\,\vec r_{1'2'}^{\,2}}{d}\ln\left(\frac{\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right) - 1\right)\left(1 - n_M + \frac{n_S}{2}\right)$$
$$+ \left(\frac{(2n_S - 3n_M)}{2\vec r_{1'2'}^{\,2}}\frac{\vec r_{12}^{\,2}}{d} + \frac{1}{2\vec r_{11'}^{\,2}\,\vec r_{22'}^{\,2}}\left(\frac{\vec r_{12}^{\,4}}{d} - \frac{\vec r_{12}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)\right)\ln\left(\frac{\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right)$$
$$+ \frac{\vec r_{12}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}\vec r_{1'2'}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}\vec r_{1'2'}^{\,2}}{\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}\right),$$

$$d = \vec r_{12'}^{\,2}\vec r_{21'}^{\,2} - \vec r_{11'}^{\,2}\vec r_{22'}^{\,2}, \quad \beta_0 = \left(\frac{11}{3} - \frac{2n_M}{3} - \frac{n_S}{6}\right)N_c\ .$$

Reduction of the NLO kernel to quasi-conformal shape

And finally for $N=4$ SUSY theory, we put $n_S=6,\ n_M=4,\ \beta_0=0$ and write

$$\langle \vec r_1 \vec r_2 | \hat{\mathcal{K}}_{N=4} | \vec r_1{}' \vec r_2{}' \rangle = \frac{\alpha_s N_c}{2\pi^2}\left(1-\frac{\alpha_s N_c \zeta(2)}{2\pi}\right)\int d\vec\rho \frac{\vec r_{12}{}^2}{\vec r_{1\rho}^{\,2}\vec r_{2\rho}^{\,2}}$$

$$\times\left[\delta(\vec r_{11'})\delta(\vec r_{2'\rho})+\delta(\vec r_{1'\rho})\delta(\vec r_{22'})-\delta(\vec r_{11'})\delta(r_{22'})\right]$$

$$+\frac{\alpha_s^2 N_c^2}{4\pi^4}\left[\frac{\ln\left(\frac{\vec r_{12'}^{\,2}\,\vec r_{21'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right)}{2\vec r_{\ 11'}^{\ 2}\ \vec r_{\ 22'}^{\ 2}}\left(\frac{\vec r_{\ 12}^{\ 4}}{\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}-\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}-\frac{\vec r_{\ 12}^{\ 2}}{\vec r_{\ 1'2'}^{\ 2}}\right)\right.$$

$$\left.+\frac{\vec r_{12}^{\,2}\ln\left(\frac{\vec r_{12}^{\,2}\vec r_{1'2'}^{\,2}}{\vec r_{12'}^{\,2}\ \vec r_{21'}^{\,2}}\right)}{\vec r_{\ 11'}^{\ 2}\vec r_{\ 22'}^{\ 2}\vec r_{1'2'}^{\,2}}+6\pi^2\zeta\left(3\right)\delta(\vec r_{11'})\delta(r_{22'})\right].$$

Summary

- Originally the BFKL approach was developed in the momentum space.
- Transfer to impact parameter space reveals important properties of the approach.
- In the case of scattering of colourless objects the BFKL kernel can be written in the Möbius form.
- The Möbius form is greatly simplified in comparison with the BFKL kernel in the momentum space.

Summary

- There was an evident discrepancy between the Möbius form of the BFKL kernel and the BC kernel.
- It was recognized that the discrepancy can be removed using the ambiguities of the NLO kernels.
- The ambiguity is caused by the possibility to redistribute radiative corrections between the kernels and the impact factors.
- Now it is proved that this ambiguity permits to match the Möbius form of BFKL kernel and the BC kernel and to construct the quasi-conformal NLO kernel.