GGI workshop, April 2012

Real-Space RG for dynamics of random spin chains and many-body localization

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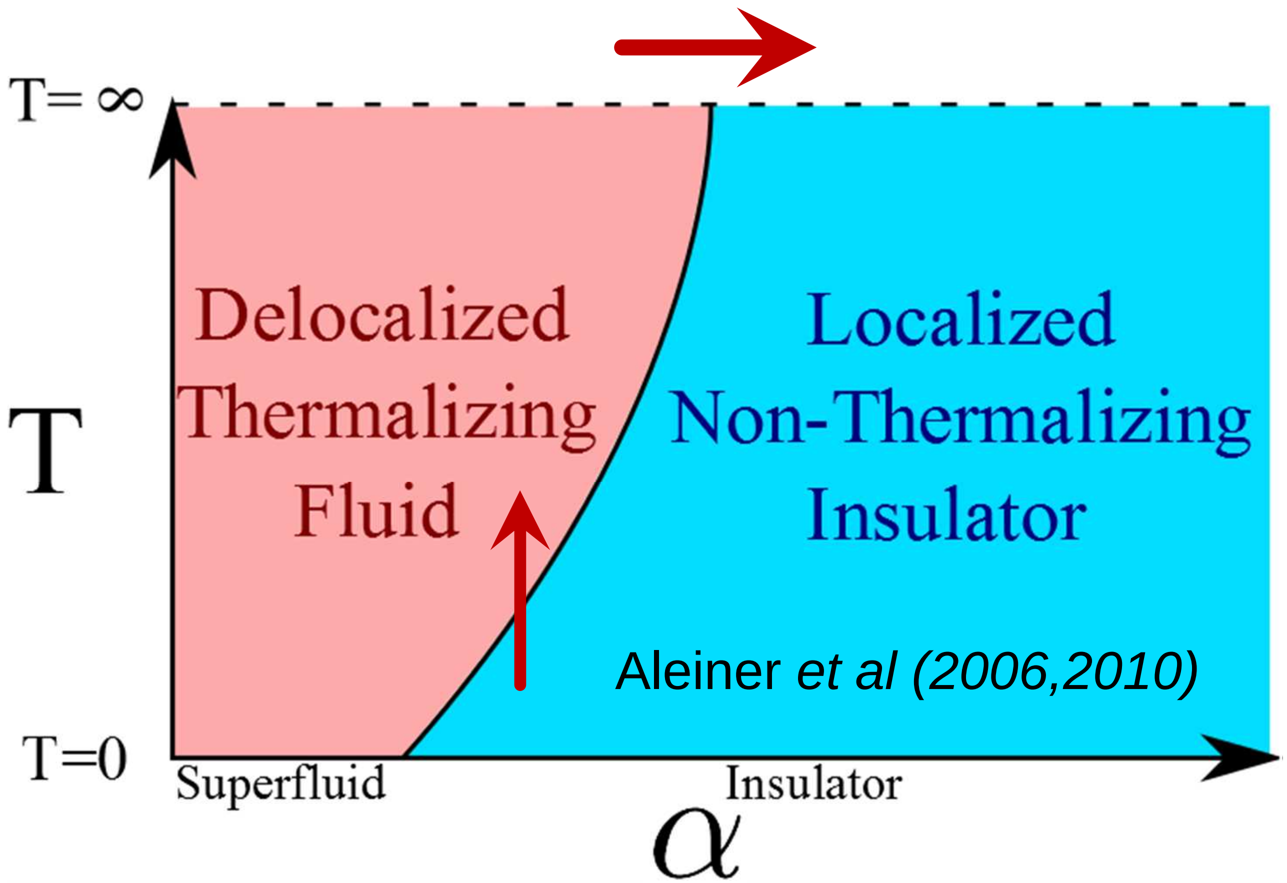
In collaboration with: Ronen Vosk

Many Body Localization



If the model has bounded spectrum, one can attempt to drive the transition at inifinite temperature
Oganesyan and Huse *(2007), Pal and Huse (2010)*

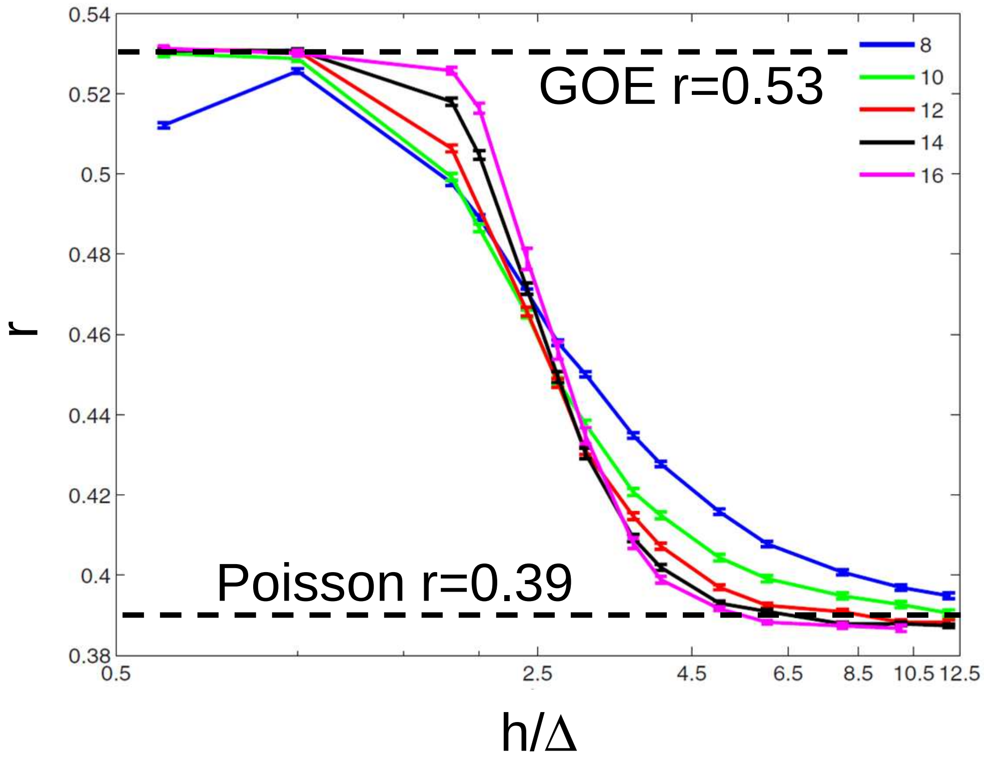
Disordered Spin Chains

A. Pal and D. Huse, Physical Review B **82**, 1 (2010)

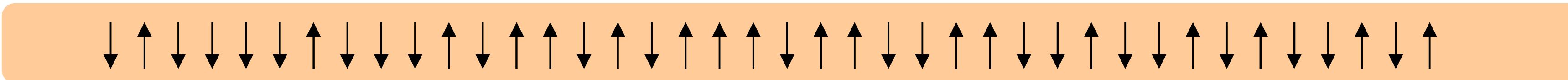
$$H = \frac{1}{2}\sum_{ij}\left(S_i^+ S_j^- + \mathrm{H.c}\right) + \Delta \sum_{ij} S_i^z S_j^z + \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

= interacting fermions: $$H = \frac{1}{2}\sum_{ij}\left(a_i^\dagger a_j + \mathrm{H.c.}\right) + \sum_i h_i n_i + \Delta \sum_{ij} n_i n_j$$

Ratio of adjacent energy gaps from exact diagonalization of 16 sites:



Thermalization and dynamics of entanglement entropy



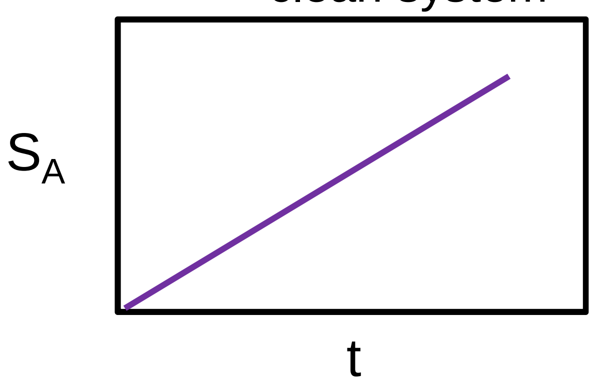
$$e^{-iHt}\,|\,\Psi_0\,\rangle \qquad H=\frac{1}{2}\sum_{ij}\left(S_i^+S_j^-+\mathrm{H.c}\right)+\Delta\sum_{ij}S_i^zS_j^z+\sum_i h_iS_i^z$$

$$h_i\in[-h,h]$$

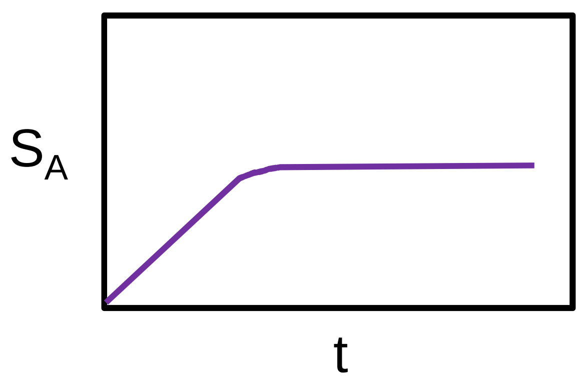
ρ_A ρ_B

Von-Neuman entropy generated in the dynamics: $S_A(t)=-Tr\left[\rho_A(t)\ln\rho_A(t)\right]$

clean system



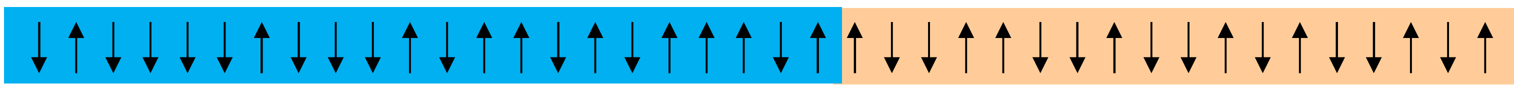
localized system (?)



$S_{saturation} \sim \xi_{localization}$

Bounded entanglement allows efficient numerics (using DMRG).
Approach transition from the localized side?

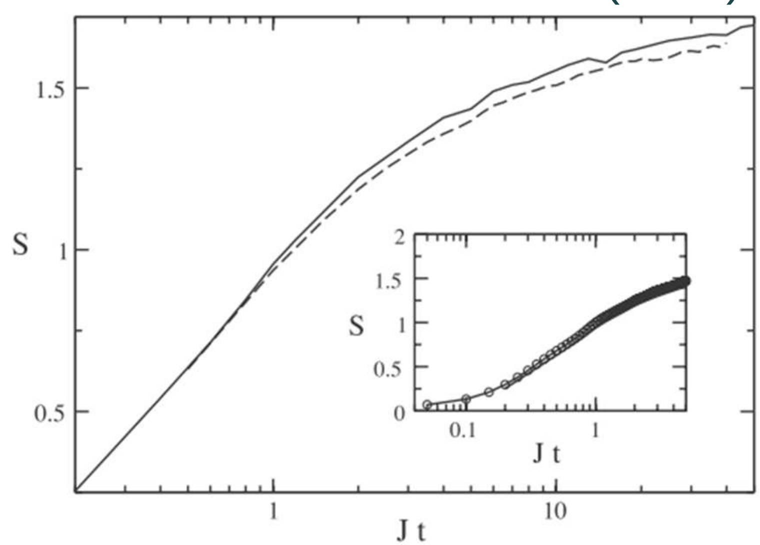
Entanglement dynamics: numerics



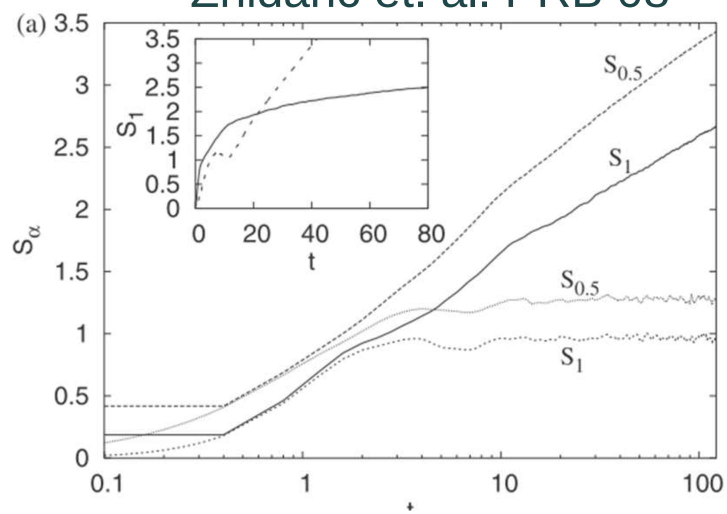
$$H = \frac{1}{2}\sum_{ij}\left(S_i^+ S_j^- + \text{H.c}\right) + J_z \sum_{ij} S_i^z S_j^z + \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

S~ log(t) growth seen in the interacting disordered model.

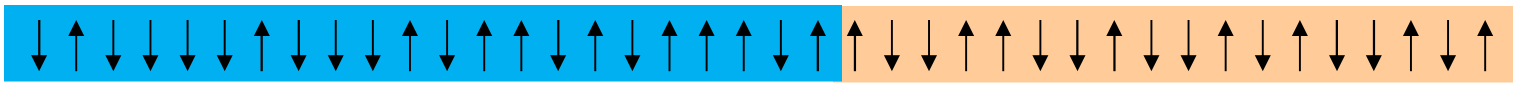
De Chiara et. al. J. Stat. Mech (2006)



Znidaric et. al. PRB 08



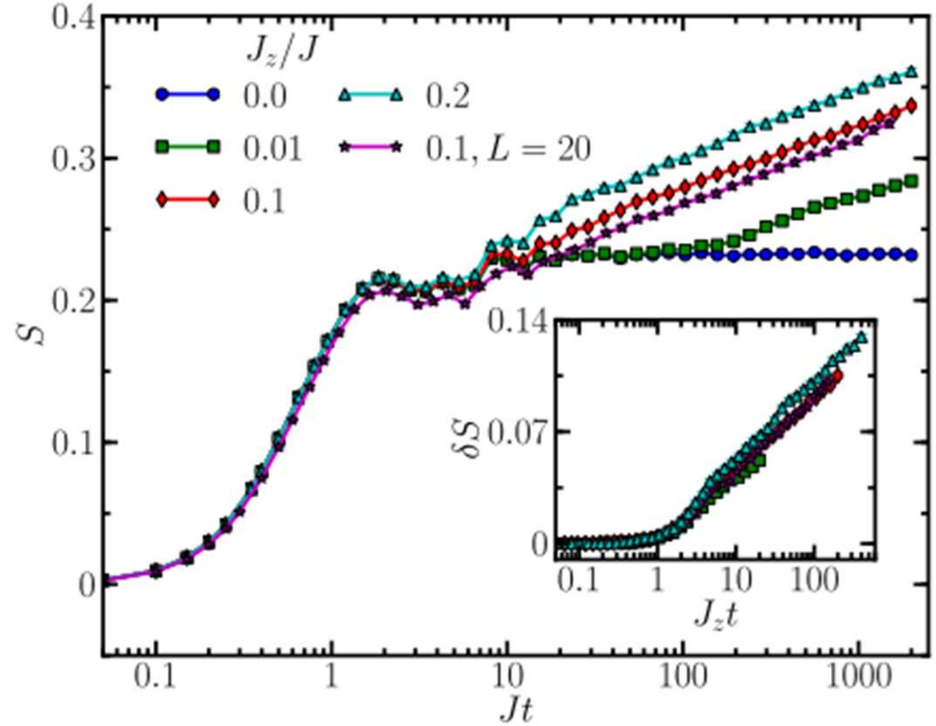
Entanglement dynamics: numerics (preliminary)



$$H = \frac{1}{2}\sum_{ij}\left(S_i^+ S_j^- + \mathrm{H.c}\right) + J_z \sum_{ij} S_i^z S_j^z + \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

S~ log(t) growth seen in the interacting disordered model.

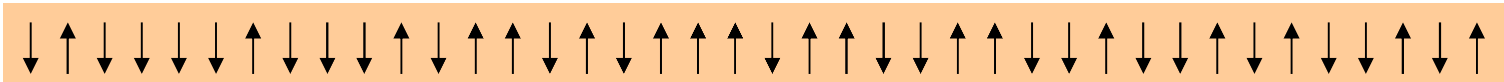
Bardarson et. al. arXiv:1202.5532



Outline

- Real space RG for quantum time evolution in strong disorder.
  Basic idea and scheme
- Application: random spin chain quenched from AFM state.
  Flow to Infinite randomness fixed point
- Criterion for the Many-body localization transition.
- Evolution of entanglement entropy and number fluctuations.
  Non-thermalization and asymptotic GGE
- Generalization to generic initial state and generic disorder
  (Preliminary)

Real space RG for the dynamics, general scheme



Model: $$H = \frac{1}{2}\sum_{ij} J_{ij}\left(S_i^+ S_j^- + S_i^- S_j^+ + 2\Delta_i S_i^z S_j^z\right)$$

Basic idea:

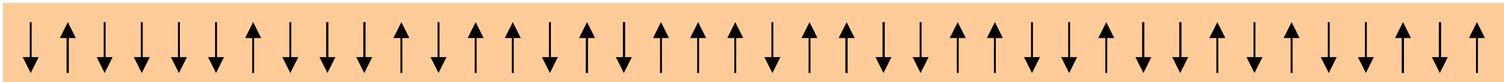
Large local separation of scales (disorder)

solve the local fast time evolution exactly

Compute effect on rest of chain perturbatively

Relation to but somewhat different philosophy than RSRG that targets the ground state (Dasgupta & Ma 1980, D. Fisher 1994, …)

Real space RG for the dynamics, general scheme



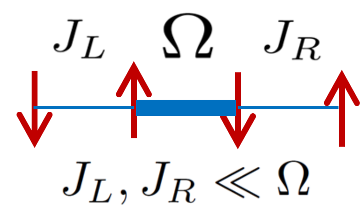
<u>Model:</u>

$$H = \frac{1}{2}\sum_{ij} J_{ij}\left(S_i^+ S_j^- + S_i^- S_j^+ + 2\Delta_i S_i^z S_j^z\right)$$

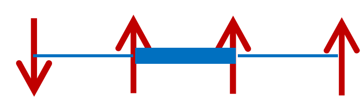
<u>Scheme:</u>

1. Successively eliminate pairs of spins coupled by the largest J's.

   These pairs perform rapid oscillations (frequency Ω) if initially anti-aligned.



   Or remain stuck if initially parallel

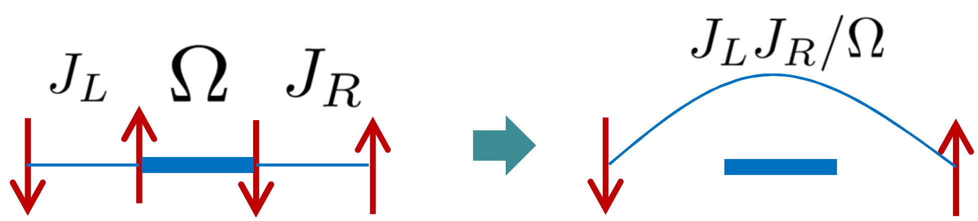


2. Compute feedback on the dynamics of the rest of the chain.

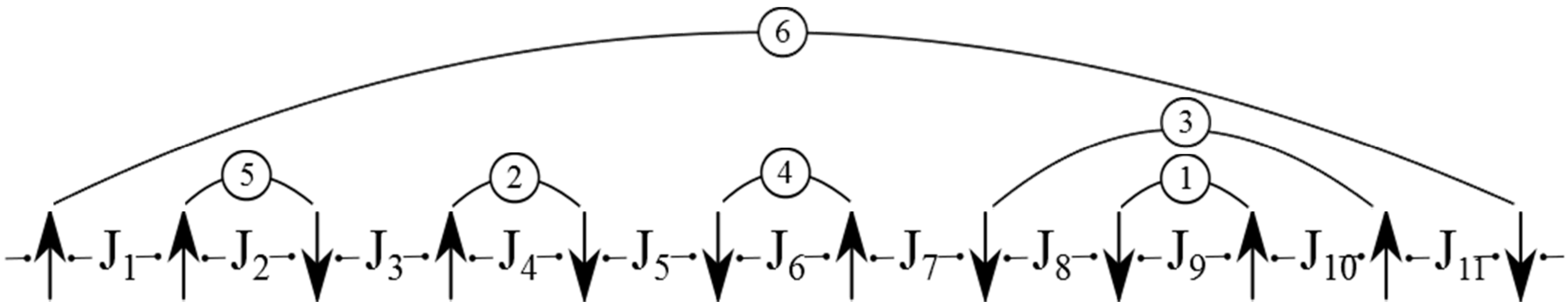
We obtain the full dynamics by solving only isolated pairs of spins at a time.
<u>The price:</u> renormalization of the coupling constants with time

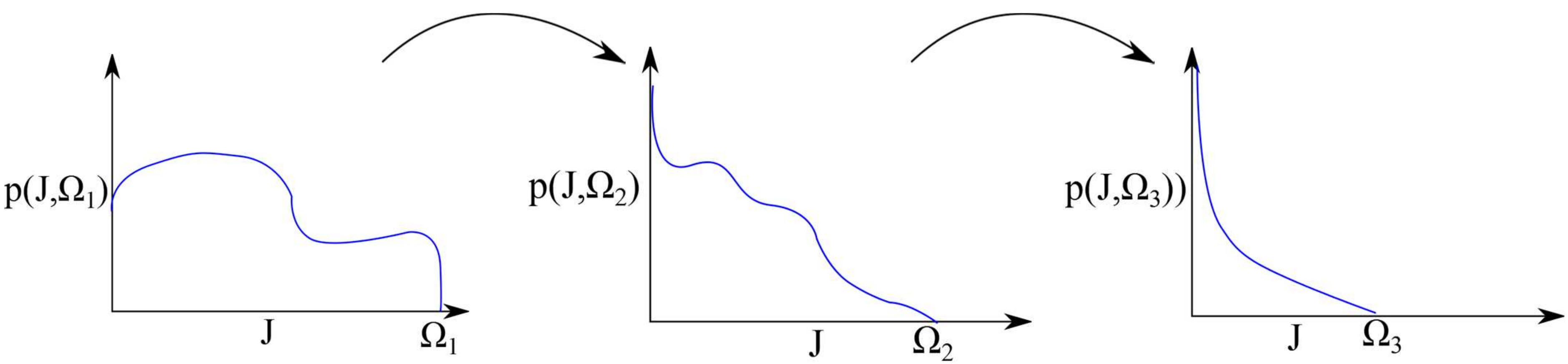
For classical dynamics: Fisher et. al. PRL 99, Polkovnikov and Mathey PRA 2009

Real space RG for dynamics, general scheme



To compute the evolution up to time $t \approx \Omega^{-1}$, successively eliminate all pairs precessing at frequencies Ω_0 down to Ω.





The RG decimation step: formal

Transform to interaction picture w.r.t the strong bond and compute the time dependent density matrix in second order perturbation theory starting from the non entangled initial state:

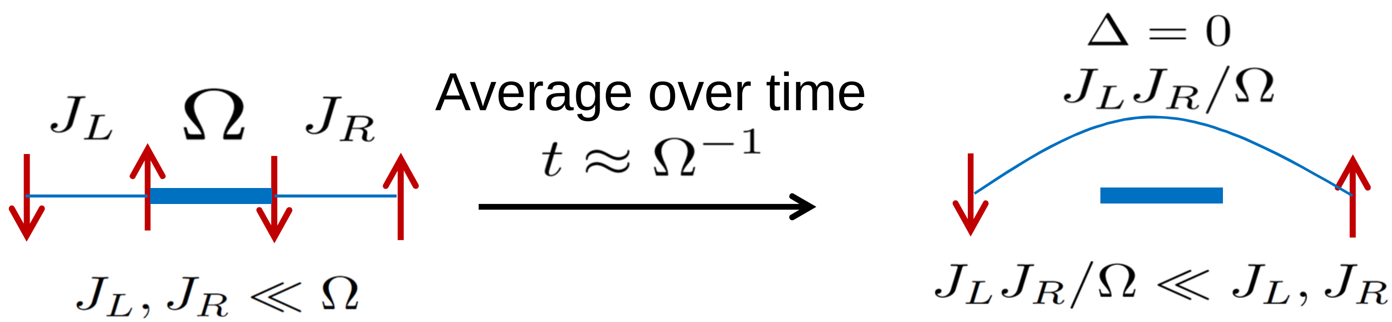
$$U_I = 1 - \frac{i}{\hbar}\int_0^t dt_1 e^{\frac{i}{\hbar}H_0 t_1}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_1}$$
$$-\frac{1}{\hbar^2}\int_0^t dt_1 \int_0^{t_1} dt_2 e^{\frac{i}{\hbar}H_0 t_1}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_1} e^{\frac{i}{\hbar}H_0 t_2}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_2}$$

Average over fast oscillations and
Equate term by term:

$$\rho(t) = U_I^\dagger \rho_0 U_I \;= e^{iH_{eff}t}\rho_0 e^{-iH_{eff}t}$$

H_{eff} depends on H and on initial state !

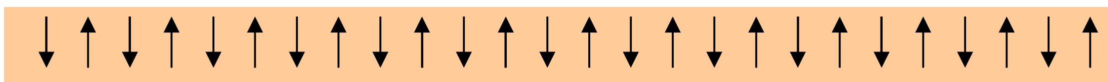
$$\rho_0 = \left|\,\psi_0^p\right\rangle\left\langle\psi_0^p\,\right|\rho_0^R$$



The RG decimation steps: anti-parallel pair

$$H = \frac{1}{2}\sum_{ij} J_{ij}\left(S_i^+ S_j^- + S_i^- S_j^+ + 2\Delta_i S_i^z S_j^z\right)$$

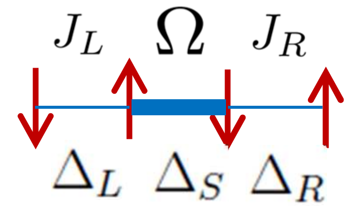
This is all we will ever need if the initial state is a Neel state



<u>Simplest case Δ=0 :</u>

$$H_{eff} = \frac{J_L J_R}{2\Omega}(S_L^+ S_R^- + \mathrm{H.c.})$$

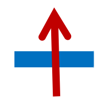
The RG decimation step: anti-parallel pair

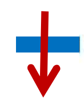


<u>For $\Delta > 0$</u> need to keep track of a new spin on the strong bond

$$H_{\text{eff}} = \frac{J_L J_R}{2\Omega(1-\Delta_S^2)}\left(S_L^+ S_R^- + S_L^- S_R^+\right)$$

$$+\frac{\Delta_S J_L J_R}{2\Omega(1-\Delta_S^2)}\left[S_L^+ S_R^- + S_L^- S_R^+ - \frac{\Delta_L \Delta_R}{\Delta_S}(1-\Delta_S^2) S_L^z S_R^z\right] S_n^z$$

 $= |\, s\,\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

 $= |\, t_0\,\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

The new spin initially points along x or –x therefore the evolution is a superposition of the dynamics given an up-spin on the bond and the dynamics with a down-spin:

$$|\,\psi(t)\,\rangle = e^{-\frac{i}{\hbar}H_{\text{eff}}t}\frac{1}{\sqrt{2}}\Big(|\uparrow_n\rangle\,|\psi_0^R\rangle \pm |\downarrow_n\rangle\,|\psi_0^R\rangle\Big)$$

$$H_{\text{eff}} \approx \frac{J_L J_R}{2\Omega}\left(1 \pm \frac{\Delta_S}{2}\right)\left(S_L^+ S_R^- + S_L^- S_R^+ \mp 2\frac{\Delta_L \Delta_R}{4} S_L^z S_R^z\right)$$

This generates entanglement between decimated bond and the nearby spins after a time

$$t_{\text{ent}} = \frac{2\Omega}{J_L J_R \Delta_S}$$

But no effect on subsequent renormalization of coupling constants!

$\tilde{J} \approx J_L J_R/\Omega$ $\quad |\tilde{\Delta}| \approx |\Delta_L||\Delta_R|/4$

Flow of distributions for initial Neel state

↓↑↓↑↓↑↓↑↓↑↓↑↓↑↓↑↓↑↓↑↓↑↓↑↓↑

<u>Scaling variables:</u>

$$\Gamma = \ln(\Omega_0/\Omega) = \ln(\Omega_0 t)$$

$$\zeta = \ln(\Omega/J) \qquad \beta = -\ln|\Delta|$$

<u>RG rules:</u>

$$\zeta_L + \zeta_R \to \tilde{\zeta}$$

$$\beta_L + \beta_R - \ln 4 \to \tilde{\beta}$$

<u>Flow equations for distributions:</u>

$$\frac{\partial \rho(\zeta)}{\partial \Gamma} = \frac{\partial \rho(\zeta)}{\partial \zeta} + \rho(0) \int_0^\infty d\zeta_L d\zeta_R \delta(\zeta - \zeta_L - \zeta_R)\rho(\zeta_L)\rho(\zeta_R)$$

$$\frac{\partial f(\beta)}{\partial \Gamma} = \rho(0) \int_0^\infty d\beta_L d\beta_R \delta(\beta - \beta_L - \beta_R + \ln 4) f(\beta_L) f(\beta_R) - f(\beta)\rho(0)$$

Solution of the flow equations

$$\rho(\zeta,\Gamma) = \alpha(\Gamma)e^{-\alpha(\Gamma)\zeta}$$

$$f(\beta,\Gamma) = \eta(\Gamma)e^{-\eta(\Gamma)\beta}$$

$\Rightarrow$

$$\frac{d\alpha}{d\Gamma} = -\alpha^2$$

$$\frac{d\eta}{d\Gamma} = -\alpha\eta$$

$\Rightarrow$

$$\alpha(\Gamma) = \frac{1}{\Gamma + \alpha_0^{-1}}$$

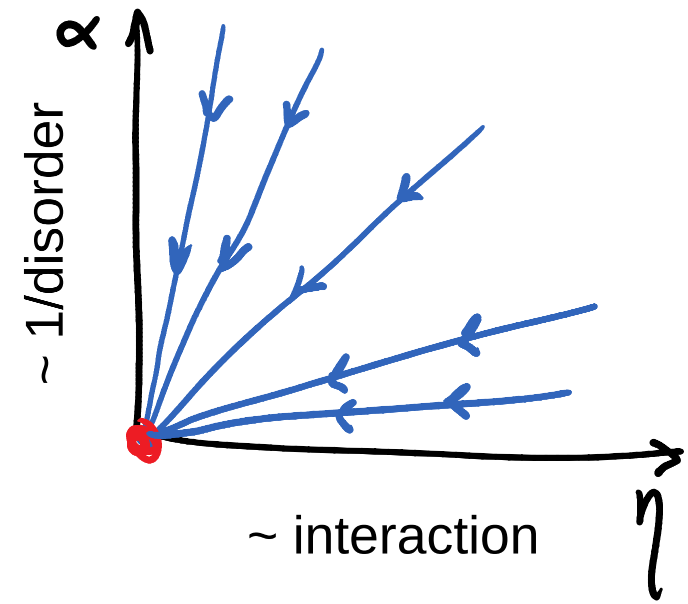
$$\eta(\Gamma) = \frac{1}{\Gamma + \eta_0^{-1}}$$

Or in the original variables:

$$P(J) = \frac{\alpha}{\Omega}(J/\Omega)^{\alpha-1} \qquad F(\Delta) = \eta|\Delta|^{\eta-1}$$

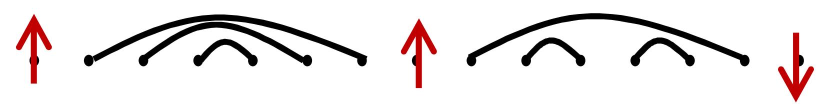
Flow to an infinite randomness fixed point!

Like the "Random singlet" phase of spin chains
(Dasgupta&Ma 80, Baht&Lee 82, Fisher 94)
Here oscillating pairs play the role of singlets



Relation between frequency (or time) scale and length scale:

$$L(\Gamma) = (\alpha_0\Gamma + 1)^2 \approx \alpha_0^2 \ln^2(\Omega_0 t)$$



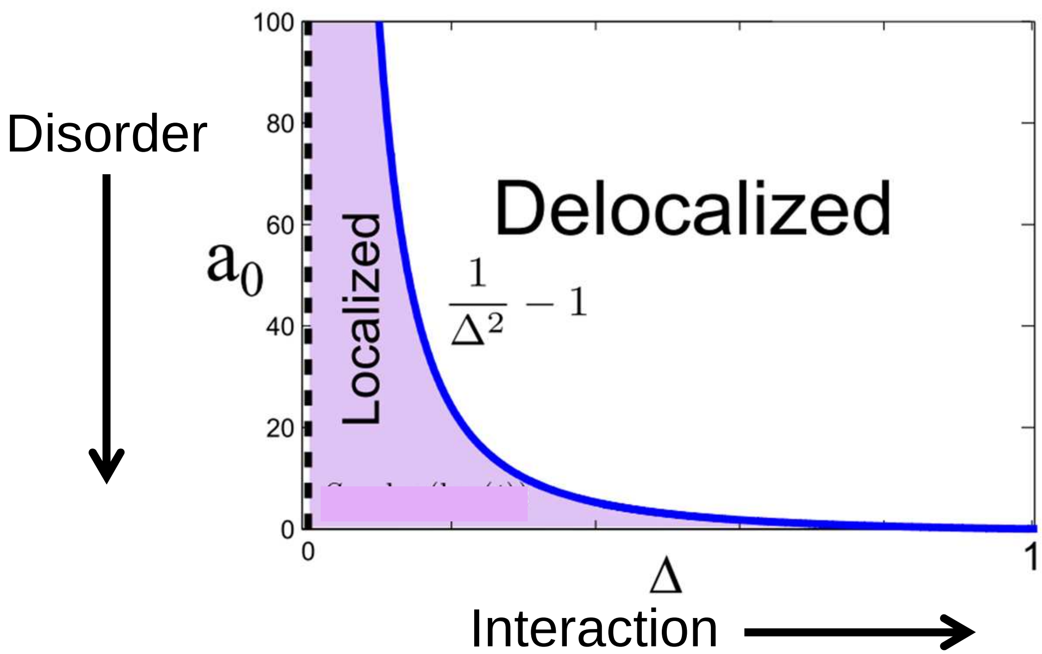
(Distance between remaining spins at that scale)

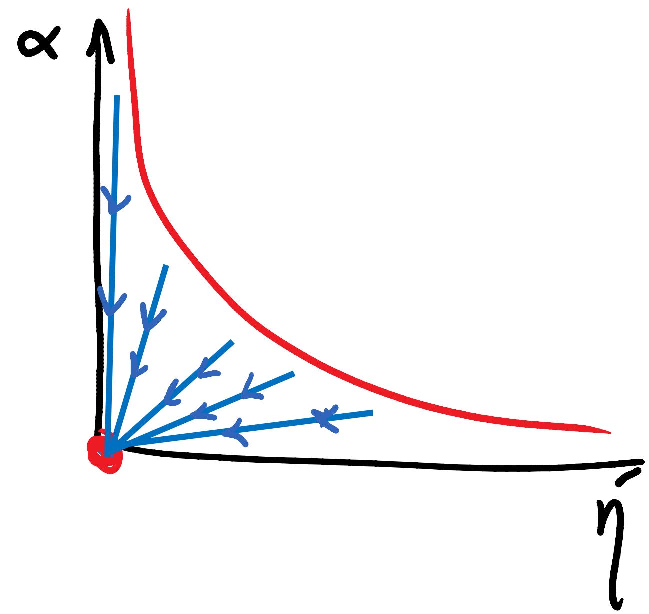
Phase diagram - Extent of the localized state

A criterion for initial conditions that lead to the localized fixed point can be found from the RG rule: $\tilde{J} = \dfrac{J_L J_R}{\Omega(1-\Delta_S^2)}$

In order to flow to increasing randomness the typical J must decrease in the process. Therefore demand:

$$\frac{J_{typ}^2}{\Omega(1-\bar{\Delta}^2)} < J_{typ} \quad \Rightarrow \quad \alpha < \frac{1}{\Delta^2} - 1$$

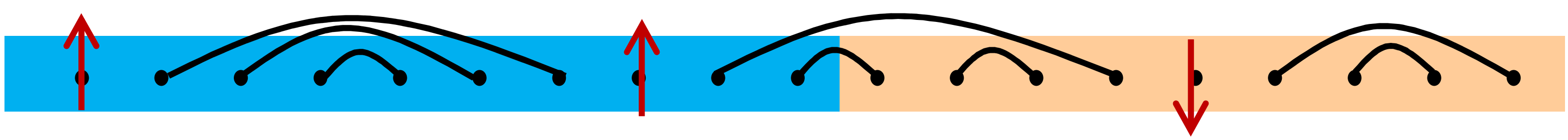




Evolution of the entanglement entropy

<u>Simplest case $\Delta=0$ ("non interacting"):</u> Only intra-pair entanglement

Compute entanglement entropy by counting the number of decimated bonds that cut the interface.
Each decimated bond crossing the interface contributes ~log2.
(As in the ground state of random singlet phase – Refael & Moore PRL 2004)

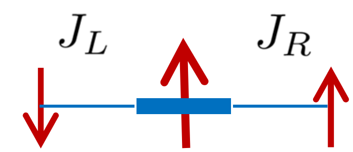


$$S_{ent} \sim \int_0^{\Gamma} \alpha(\Gamma')d\Gamma' = \ln(\Gamma + \alpha_0^{-1}) = \ln\left(\ln(\Omega_0 t) + \alpha_0^{-1}\right)$$

Evolution of the entanglement entropy

<u>Δ>0 Neel initial state :</u>

A bond eliminated at t_1 builds entanglement with neighbors only at a later time $t=t_1+t_{ent}$.



$$t_{\mathrm{ent}} = \frac{2\Omega}{J_L J_R \Delta_S}$$

The interaction generates entanglement only after a delay time from the start of time evolution

$$t_{\mathrm{delay}} \approx \frac{2\Omega_0}{J_0^2 \Delta_0} = \left(\frac{2\Omega_0}{J_0}\right)\frac{1}{J_0^z}$$

Evolution of the entanglement entropy

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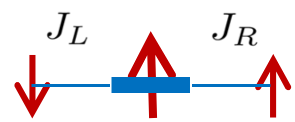
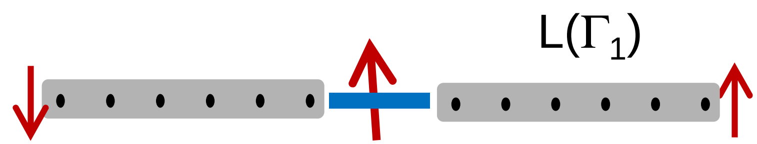
$$t_{\text{delay}} \approx \frac{2\Omega_0}{J_0^2 \Delta_0} = \left(\frac{2\Omega_0}{J_0}\right) \frac{1}{J_0^z}$$

How much entanglement is generated?



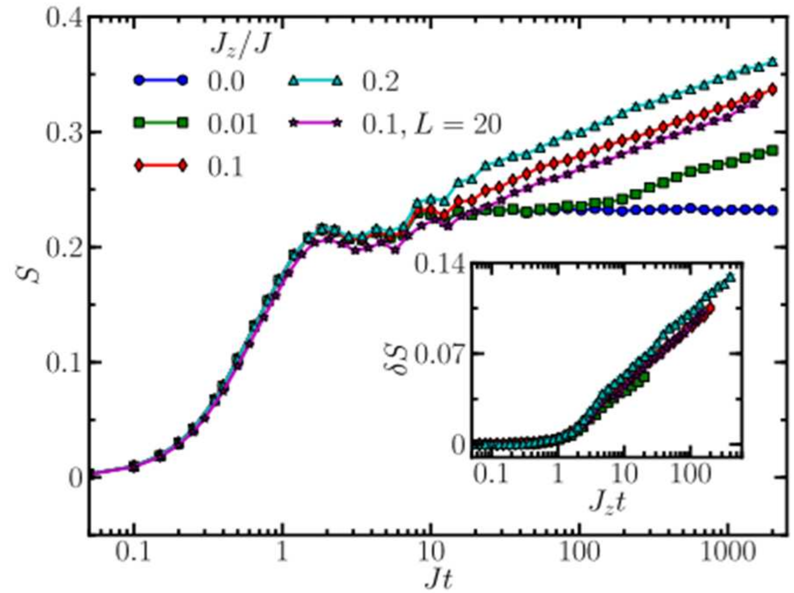
Recall there is a decimated region of length $L(\Gamma_1)$ between spins at Γ_1



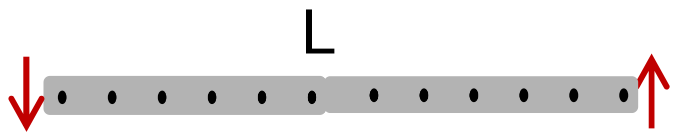
Reasonable to assume that by the time $t=t_1+t_{ent}$ that these spins become entangled the decimated pairs between them are also entangled with each other.

$$S(\Gamma) = L(\Gamma_1)\,\Theta(\Gamma - \Gamma_{\text{delay}})$$

$$S(t) \approx \frac{\alpha_0^2}{3 - \ln \Delta_0}\Theta(t - t_{\text{delay}}) \ln^2(t/t_{\text{delay}})$$



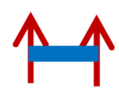
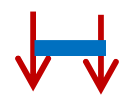
Saturation of entanglement entropy in a finite system



Saturation time: $$t(L) = \Omega_0^{-1} e^{\Gamma(L)} \approx \Omega_0^{-1} e^{\sqrt{L}/\alpha_0}$$

Entropy saturates to an extensive value: S(L) ~ L

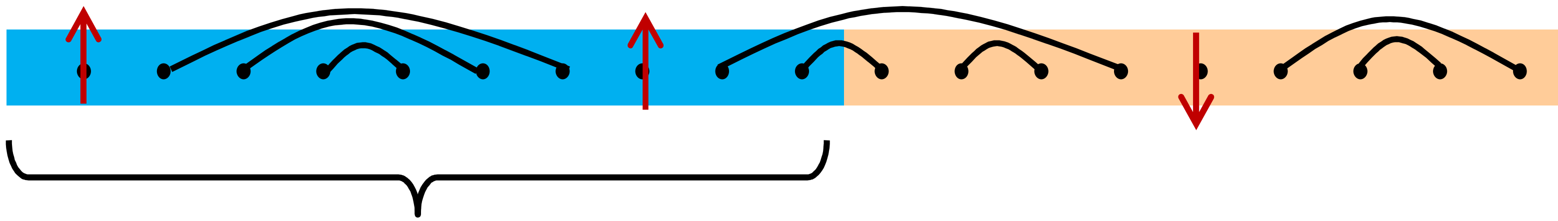
What is the prefactor?
Is it the expected thermalized value S(L)= L ln2 ?
(T is infinite for the given initial state)

No! in every decimated pair the states  and 
are not populated therefore S(L)<(L/2)ln2

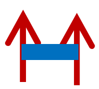
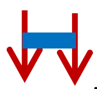
No thermalization!

(Later we relate this to an infinite set of effective constants of motion)

Evolution of particle number fluctuations



$$\langle \delta N^2(t) \rangle = ?$$

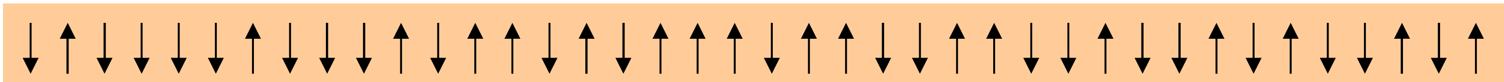
Since the  and  states of decimated pairs are not populated, only pairs that intersect the interface contribute to $\langle \delta N^2(t) \rangle$

$$\langle \delta N^2(t) \rangle = \int_0^{\Gamma(t)} d\Gamma' \alpha(\Gamma') = \ln\left(\ln(\Omega_0 t) + \alpha_0^{-1}\right)$$

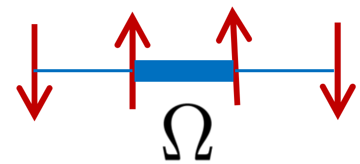
Much slower than the growth of the ent. Entropy $S(t) \sim \ln^2 t$
And independent of interaction!

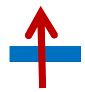
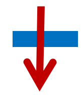
Saturates to a non-extensive value in a finite system: $\langle \delta N^2(\infty) \rangle \sim \ln L$

Generalize to random initial state



RG rule for a strong bond connecting parallel spins:



Keep track of a new spin:  $= |\uparrow\uparrow\rangle$  $= |\downarrow\downarrow\rangle$

$$H_{eff} \approx -\frac{J_L J_R}{2\Omega(1-\Delta_S^2)}(S_L^+ S_R^- + \mathrm{H.c.}) + J_L \Delta_L S_L^z S_n^z + J_R S_R^z S_n^z$$
$$-\frac{J_L J_R \Delta_S}{2\Omega(1-\Delta_S^2)}\left(S_L^+ S_R^+ S_n^- + S_L^- S_R^- S_n^+\right)$$

Generates slow switching between the $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ states.

But the $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ states of the pair are not populated.

The operators $S_1^z S_2^z$ of decimated pairs are asymptotic constants of motion if the system still flows to infinite randomness.

But the flow is complicated by the switching term (last term) and the generated interaction between the new spin and its neighbors.

RG flow for random initial state and random Zeeman fields neglecting resonances

$$\frac{\partial \rho(\zeta)}{\partial \Gamma} = \frac{\partial \rho(\zeta)}{\partial \zeta} + (\rho(0) + g(1)) \int_0^\infty d\zeta_L \int_0^\infty d\zeta_R \delta(\zeta - \zeta_L - \zeta_R) \rho(\zeta_L) \rho(\zeta_R) - \rho(\zeta) g(1)$$

$$\frac{\partial f(|\Delta|)}{\partial \Gamma} = p_1 \rho(0) \int_0^1 d|\Delta|_L \int_0^1 d|\Delta|_R \delta(|\Delta| - |\Delta|_L |\Delta|_R) f(|\Delta|_L) f(|\Delta|_R)$$

$$+ (p_0 \rho(0) + g(1, \Gamma))\, \delta(\Delta) - f(\Delta)\, (\rho(0) + g(1))$$

$$\frac{\partial g(\tilde{h})}{\partial \Gamma} = \tilde{h} \frac{\partial g(\tilde{h})}{\partial \tilde{h}} - g(\tilde{h})\, (g(1) - 1)\,. \qquad \tilde{h} = h/\Omega$$

Solved by the Ansatz:

Gradual condensation

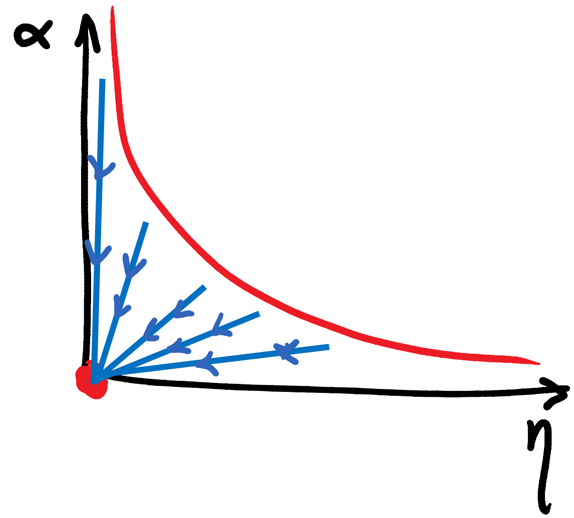
$$\rho(\zeta, \Gamma) = a(\Gamma) e^{-a(\Gamma)\zeta},$$

$$f(|\Delta|, \Gamma) = (1 - b_2(\Gamma))\, b_1(\Gamma) |\Delta|^{b_1(\Gamma) - 1} + b_2(\Gamma) \delta(|\Delta|)$$

$$g(\tilde{h}, \Gamma) = c \tilde{h}^{c-1}.$$

Flow to infinite rand. in J (peaked at small J) and large Local Zeeman fields

Summary



- Formulated RSRG for quantum spin chains
- Established fixed point depends both on H and on initial state!
- Infinite randomness fixed point for xxz chain with initial Neel state.
- Evolution of entanglement entropy: $S(\tilde{t}) \sim \Theta\left(\tilde{t}-1\right) \ln^2 \tilde{t}$

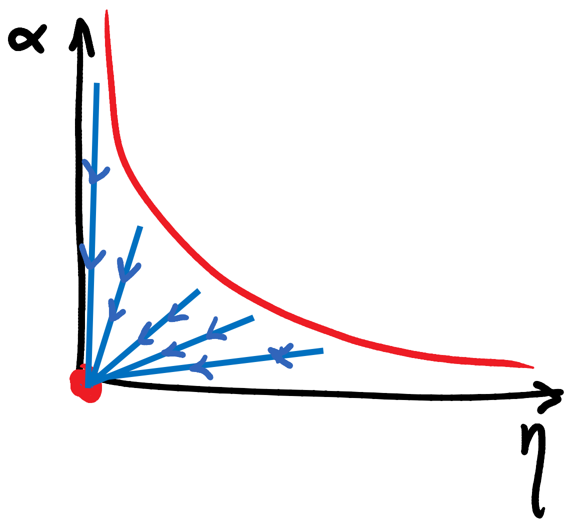
$$\tilde{t} = t/t_{delay} = J_z t(J/2\Omega_0)$$

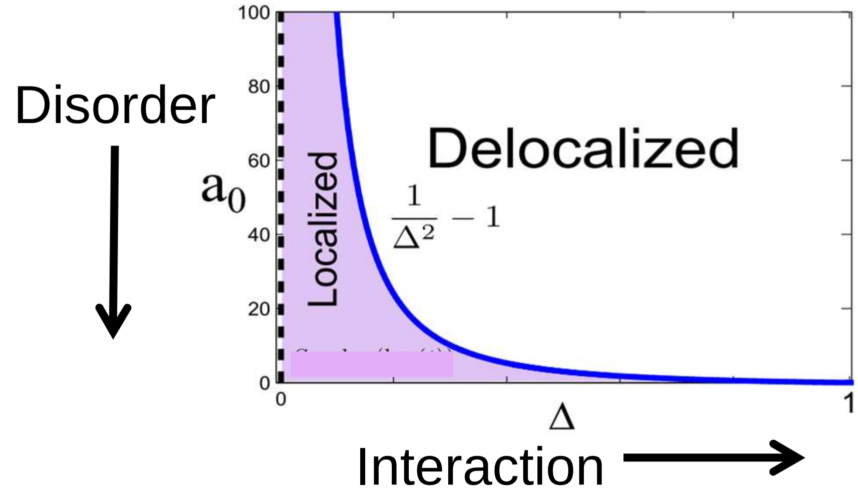
- Particle number fluctuations: $\langle \delta N^2(t) \rangle \sim \ln \ln(\Omega_0 t)$

- Non thermal steady state can be understood as Generalized Gibbs ensemble with the asymptotic conserved quantities: $(S_1^z S_2^z)_{\mathrm{pair}}$

Summary

- Conjectured criterion for the many-body localization transition from the basin of attraction of the infinite-randomness FP.





Outlook

- Nature of the steady state for generic initial conditions and possibly generic disorder (allow local Zeeman fields)
- Access the critical point which controls the many-body localization transition

Thank you!