Entanglement Spectrum and Boundary Theories (in quantum magnets) using PEPS

Didier Poilblanc

Laboratoire de Physique Théorique, Toulouse
Entanglement concepts & tools for studying many-body systems

Some motivation (and warm-up): entanglement spectra of Heisenberg ladder [DP, PRL 105, 077202 (2010)]

“Holographic mapping”: Boundary Hamiltonians from PEPS wavefunctions

Application to 2d AKLT wavefunctions: connection between bulk and boundary

Application to 2d topological states: dimer and SU(2)-RVB wavefunctions
* Ignacio Cirac  
  (Max-Planck Garching)

* Norbert Schuch  
  (Cal. Tech.  
  -> RWTH Aachen)

* Frank Vertraete  
  (Univ. Vienna)

* David Perez-Garcia  
  (Univ. Madrid)

- Phys. Rev. B 83, 245134 (2011)  [1,2,3]
- arXiv:1202.0947  [1,2,4]
- arXiv:1203.4816  [1,2,4]
Exotic states of matter

* no broken symmetry
* no local order
* GS degeneracy depends on topology of space

Topological order  X. G. Wen

Exemple: (topological) spin liquid

Beyond the “order parameter paradigm”: correlations “missed” by two-point correlation functions can be detected by entanglement measures
Edge states in (topological) FQH systems

Li & Haldane
PRL 2008

Lauchli et al., 2009

Also topological insulators, etc...

crutial role of edges!
Reduced density matrix

\[ |\Psi\rangle \in \mathcal{E}_A \otimes \mathcal{E}_B \quad \rho = |\Psi\rangle\langle\Psi| \quad \text{projector} \]

**Definition:**
\[ \rho_A = \sum_j \langle j|_B (|\Psi\rangle\langle\Psi|) |j\rangle_B = \text{Tr}_B \rho \]

**SINGLET:**
\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B) \]
\[ \rho_A = \frac{1}{2}(|\uparrow\rangle_A \langle \uparrow|_A + |\downarrow\rangle_A \langle \downarrow|_A) \]

**In general:**
\[ \rho_A = \sum_i \lambda_i^2 |i\rangle_A \langle i|_A \quad \text{“mixed” ensemble} \]
Entanglement Entropy

A quantitative measure: 

Reduced density matrix \( \rho_A = \text{Tr}_B |\Psi\rangle\langle\Psi| \)

\( S_{\text{entanglement}} = -\text{Tr}\{\rho_A \ln \rho_A\} \) (Von Neumann)

\( S_{\text{entanglement}} \propto \xi L^{d-1} \) “area” law

\( d=2: \propto L \) (perimeter)

More complex if critical or \( d=1 \) ...

Rewrite $\rho_A$ as thermal density matrix

$$\rho(T) = \frac{1}{Z} \exp(-\beta H) = \sum_{\alpha} \exp(-\beta e_{\alpha}) |\alpha\rangle\langle\alpha|$$

$$\beta = 1/T \quad \text{inverse temperature}$$

$$\rho_A = \sum_i \lambda_i^2 |i\rangle_A \langle i|_A$$

rewrite the weights as: $$\lambda_i = \exp(-\xi_i/2)$$

Entanglement spectrum: $\{\xi_i\}$

$$\rho_A = \exp(-\hat{\xi})$$
"Haldane" Conjecture:

Precise correspondence between the entanglement spectrum of a FQH system partitioned into two sub-systems linked by some "edge" and the true sub-system spectrum

Questions:
- More generally, can the ES always be connected to the true edge spectrum?
- Is there any edge property that reflect bulk properties?
A simple example: the 2-leg antiferromagnetic spin “ladder”

Entanglement spectrum

D.P., PRL 105, 077202 (2010)
A precise characterization of the “boundary hamiltonien” is in fact possible!

\[ \rho_A = \exp(-H_b) \]

\[ H_b = A_0 N_v + \sum_{r,k} A_r \mathbf{S}_k \cdot \mathbf{S}_{k+r} + R \hat{X} \]

Exponential decay!
- Extend to long cylinders with $N_h$ legs? $N_h \to \infty$?
- Get a simple physical description of the degrees of freedom of $H_b$
Tensor Network approaches

Matrix Product States (1D):

\[ |\Psi\rangle = \sum_{I} c_I |i_1, i_2, ..., i_{N_h}\rangle \]
\[ i_k = -S, -S+1, ..., S-1, S \]

\[ \begin{array}{cc}
M_{\alpha_1, \alpha_2}^{i} & D \times D \text{ matrix} \\
\end{array} \]

\[ c_I = \sum_{\alpha} L_{\alpha_1}^{i_1} M_{\alpha_1, \alpha_2}^{i_2} \cdots M_{\alpha_{N_h-2}, \alpha_{N_h-1}}^{i_{N_h-1}} R_{\alpha_{N_h-1}}^{i_{N_h}} \]
\[ = L^{i_1} M^{i_2} \cdots M^{i_{N_h-1}} R^{i_{N_h}} \]

Equivalent to DMRG!!

Romer and Ostlund (PRL, 1995)

D ~ m parameter controlling the DMRG truncation
Tensor Network for $d=2$ (and higher): Projected Entangled Paired States (PEPS)

"contract" product of tensors

\[ c_I = \sum_\Lambda L_{\Lambda_1}^{I_1} B_{\Lambda_1,\Lambda_2}^{I_2} \ldots B_{\Lambda_{N_h-2},\Lambda_{N_h-1}}^{I_{N_h-1}} R_{\Lambda_{N_h-1}}^{I_{N_h}} \]

\[ B_{\Lambda_{n-1},\Lambda_n}^{I_n} = \text{tr} \left[ \prod_{k=1}^{N_v} \hat{A}_{\alpha_{k,n-1},\alpha_{k,n}}^{i_{k,n}} \right] \]

\[ \Lambda_n = (\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{N_v,n}) \]

\[ I_n = (i_{1,n}, i_{2,n}, \ldots, i_{N_v,n}) \]
Holographic framework

Basic formula: $\rho_A = U \sigma_b^2 U^\dagger$

isometry: maps 2D onto 1D

$\sigma_b^2 = \exp(-H_b)$

Consequence: expect area law!
Boundary theories: main message

To what extend $H_b$ is a local Hamiltonian? Can we describe TOPOLOGICAL systems?

* gapped systems (AKLT):
  $H_b$ is short-range

* approaching a critical point
  (deformed AKLT):
  $H_b$ becomes long-range

* for topological GS (Kitaev toric code, dimer wf, su(2)-RVB):
  => $H_b$ non-local
Application to AKLT ladders
(Affleck-Lieb-Kennedy-Tasaki)

\[ S_i = \frac{z_i}{2} \]

\[ H_{AKLT} = \sum_{\langle ij \rangle} P_{S_i+S_j} \]

\[ N_h = 2 \]

PEPS representation

D=2!

# of legs \( N_h \) from 2 to \( \infty \)
Entanglement spectra of AKL ladders/cylinders

(b) AKLT 16x2
(c) AKLT 16x8

same perimeter = 16 sites
low energy c=1 CFT

2 legs
8 legs
Extrapolate to infinite AKLT cylinders

\[ A(r) \sim \exp \left( -\frac{r}{\xi_b} \right) \]

Short-range boundary Hamiltonians

\[ H_b = A_0 N_v + \sum_{r,k} A_r S_k \cdot S_{k+r} + R^X \]
Critical point:

Deformed AKLT model

\[ A_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}^m = \langle s_m | Q(\Delta) | \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \]

\[ Q_n(\Delta) = e^{-8\Delta S_{z,n}^2} \]

breaks SU(2) down to U(1)

\[ A(r) \sim \exp(-r/\xi_b) \]
I. INTRODUCTION

In order to detect topological order, a common setup consists of dividing the system into two regions named \(-\) and \(\tilde{c}\) and compute the reduced density matrix \(\text{RDM}\) in the GS of the \(-\) subsystem. In particular, the entanglement entropy \(\text{EE}\) defined as the Von Neumann entropy of the RDM

\[
S_{VN} = -\ln \rho_A
\]

contains an extensive term – proportional to the length of the boundary – and a universal subleading constant, the topological EE characterizing the topological nature. In addition, \(-\ln \rho_A\) can be seen as a dimensionless Hamiltonian \(\tilde{H}\) which provides even more information on the system. First, its spectrum, the so-called entanglement spectrum \(\text{ES}\), has been conjectured to have a deep correspondence with the actual boundary spectrum. This remarkable property was first established in fractional quantum Hall states – the ES was shown to reproduce faithfully the spectrum of edge states – and later on, in quantum spin systems. Furthermore, beyond its spectrum, the nature of \(\tilde{H}\) itself is directly linked to the property of the bulk.

PEPS offer a natural formulation of the relation between bulk and boundary. In Ref. an explicit isometry was constructed which maps the Hamiltonian \(\tilde{H}\) onto another one \(\tilde{\tilde{H}}\) acting on the space of auxiliary spins living at the boundary of region \(-\), while keeping the spectrum. Furthermore, for various two-dimensional \(B\) models displaying quantum phase transitions, like a deformed KLT or an Ising type model, it was found that a gapped bulk phase with local order corresponds to a boundary Hamiltonian with local interactions, whereas critical behavior in the bulk is reflected on a diverging interaction length of \(\tilde{\tilde{H}}\).

II. RVB WAVEFUNCTIONS ON CYLINDERS

Here, to investigate the boundary Hamiltonian \(\tilde{\tilde{H}}\) of the RV\(\tilde{\tau}\) wavefunction, we consider cylinders of length \(N_h\) and circumference \(N_v\) as depicted in Fig. 1. By partitioning the cylinder into two half-cylinders playing the role of the two \(-\) and \(\tilde{c}\) subsystems defined above, reveals two edges \(L\) and \(R\) along the cut, as shown in Fig. 1. Ultimately, we will take the limit of infinite cylinders, i.e., we shall see later, for a topological state, the boundary Hamiltonian \(\tilde{\tilde{H}}\) depends on the choice of the boundaries \(B_L\) and \(B_R\). Open boundary conditions \(\text{OBC}\) on the cylinder ends are obtained by setting the outgoing virtual indices to "l" as shown in Fig. 2. Arbitrary boundary conditions can be realized as in Fig. 2.

FIG. 2 Cylinder geometry for computing the RDM. Equal-weight superposition of hardcore dimers [see Eqn 1] have simple representations in terms of PEPS. The bipartition generates two \(L\) and \(R\) edges along the cut. Various fixed boundary conditions \(B_L\) and \(B_R\) can be chosen on the cylinder ends by fixing the boundary virtual variables. OBC are defined by setting all boundary indices to "r" as in Fig. 2. Arbitrary boundary conditions can be defined physically by freezing [with a local magnetic field] some spins at the boundaries translating in the PEPS language by setting the boundary indices to \(\downarrow\) spin or \(\uparrow\) spin.

Topological spin liquids

RVB = equal-weight superposition of NN singlet coverings

D=3 PEPS

Short-range spin-spin correlations
RVB on the kagome lattice
Evidence for Z2 liquid from recent numerics:
Yan, Huse & White, Science 2011

PEPS:

map on a square lattice
(but no reflection symmetry)
Some properties of RVB wavefunctions

* Square lattice: algebraic dimer-dimer correlations
  Albuquerque & Alet, PRB 2010

* Kagome lattice: short-range dimer-dimer correlations
  Misguich et al., PRL 2002
  Moessner & Sondhi, PRL 2001

Disconnected topological sectors in the space of dimer lattice coverings

E.g. on a cylinder:
* square lattice: \( \mathbb{Z}_2 \) sectors
* kagome lattice: 2 sectors
Structure of the Boundary Hamiltonian

Topological sectors translate into CONSERVATION LAWS of “transfer matrix”
[i.e. on (parity) of # of “2” on each row]

\[ \sigma_b^2 = \exp(-\tilde{H}_b) \]

\[ \tilde{H}_b = H_1 + \beta_\infty(1^{\otimes N_v} - \mathcal{P}) \]

\[ \beta_\infty \to \infty \]

supported by the non-zero eigenvalue sector of the RDM

\[ H_1 = H_{\text{local}} \mathcal{P} \]

projector characterizing the sectors
Different sectors can be obtained by choosing different boundary conditions

\[ P = P_{\text{even}} \]

(kagome)

\[ P = P_{\text{odd}} \]

NB: for the square lattice, \( N_v \) sectors/projectors
Non-locality of Boundary Hamiltonian (acting on the edge)

local operator (on the edge):
$D \times D$ matrix $\Rightarrow$ basis of $D^2$ operators

\[ O_{\text{edge}} = c_0 N_v + \sum_{\lambda,i} c_{\lambda} \hat{x}^i_{\lambda} + \sum_{\lambda,\mu,r,i} d_{\lambda \mu}(r) \hat{x}^i_{\lambda} \hat{x}^{i+r}_{\mu} \]
\[ + \sum_{\lambda,\mu,\nu,r,r',i} e_{\lambda \mu \nu}(r, r') \hat{x}^i_{\lambda} \hat{x}^{i+r}_{\mu} \hat{x}^{i+r'}_{\nu} + \ldots, \]

$H_1 = H_{\text{local}} \mathcal{P}$

$H_{\text{local}}$ is an extended t-J model!
Topological entropy

# of states on the edge contributing to even sector:

$$
\# = \frac{3^{N_v}}{2}
$$

$$
S_{VN} \sim - \ln \#
$$

$$
S_{VN} = S_0 + AN_v
$$
Interpolation between orthogonal dimer and RVB states can be mapped onto Kitaev’s toric code. ES spectrum reflects the edge spectrum (conjecture).
Conclusion and outlook

* natural mapping between bulk and boundary
  ➔ properties of bulk reflected in the property of the boundary Hamiltonian
  ➔ property of the bulk can be “read off” the property of boundary Hamiltonian

* applied also to **topological** states
  ➔ tool to identify spin liquids in microscopic models

* extensions to chiral SL, fermions, (non-Abelian) anyons or gauge models ...