Quantum spin models in 1D and 2D
From the WZW model

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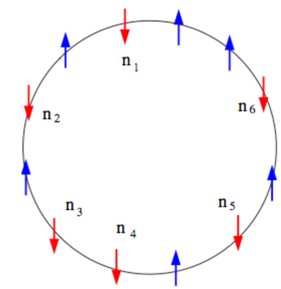
A bit of history (Gebhard-Vollhardt 1987)

Fermi state of a chain with N sites at half filling



$$|FS\rangle = \Pi_{|k|<k_F} c^*_{k\uparrow} c^*_{k\downarrow} |0\rangle \qquad k_F = \frac{\pi}{2}$$

Eliminated the states doubly occupied (Gutzwiller projection)

$$|\psi_G\rangle \propto P_G |FS\rangle = \Pi_i (1 - n_{i\uparrow} n_{i\downarrow}) |FS\rangle$$

Spin-spin correlator in the limit N -> infty (Gebhard-Vollhardt)

$$\langle S_n^a S_0^b \rangle = (-1)^n \delta_{ab} \frac{Si(\pi n)}{4\pi n} \approx \delta_{ab} \left[(-1)^n \frac{1}{8n} - \frac{1}{4\pi^2 n^2} \right] \quad (n >> 1)$$

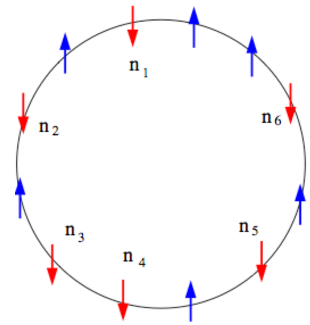
Spin-spin correlator in the AF Heisenberg model

$$\langle S_n^a S_0^b \rangle \approx \delta_{ab} \left[(-1)^n \frac{c\sqrt{\log n}}{n} - \frac{1}{4\pi^2 n^2} \right], \quad (n \to \infty)$$

The Gutzwiller state has only spin degrees of freedom and can be mapped into a hardcore boson state

$$|\uparrow\rangle \leftrightarrow |0\rangle \qquad \textit{empty}$$

$$|\downarrow\rangle \leftrightarrow a^{*}\,|0\rangle \quad \textit{occupied}$$



$$|\psi_G\rangle = \sum_{n_1,\ldots,n_{N/2}} \psi(n_1,\ldots,n_{N/2})\; a^{*}_{n_1}\ldots a^{*}_{n_{N/2}}|0\rangle$$

n_i : position of the i-boson (i.e. spin down)

$$\psi(n_1,\ldots,n_{N/2}) = \prod_i z_{n_i} \prod_{i<j} (z_{n_i} - z_{n_j})^2, \qquad z_n = e^{2\pi i n/N}$$

1D version of a bosonic Laughlin state at $\nu = \frac{1}{2}$

This state is a spin singlet

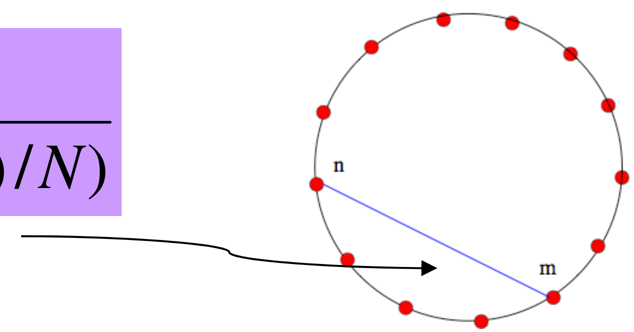
Haldane-Shastry Hamiltonian (1988)

$|\psi_G\rangle$ Is the exact ground state of the Hamiltonian

$$H = -2 \sum_{n<m} \frac{z_n \, z_m}{(z_n - z_m)^2} \vec{S}_n \cdot \vec{S}_m$$

AFH model with exchange couplings inversely proportionally to the chord distance

$$H = \frac{1}{2} \sum_{n<m} \frac{\vec{S}_n \cdot \vec{S}_m}{\sin^2(\pi(n-m)/N)}$$



Long range version of the AFH with NN couplings

Properties of the HS model

- Elementary excitations: spinons (spin 1/2 with fractional statistics)

- Degenerate spectrum described by the Yangian symmetry

- Closely related to the Calogero-Sutherland model

- Low energy physics described by the WZW SU(2)@k=1

- The HS model is at the fixed point of the RG while the AFH is a marginal irrelevant perturbation which gives rise to log corrections

Haldane, Bernard, Pasquier, Talstra, Schoutens, Ludwig,…

- Overlap between the HS state and the Bethe state

$$\left| \langle HS | Bethe \rangle_N \right| \approx 0.99, \quad N = 20 \ sites$$

- Truncate the HS Hamiltonian to NN and NNN couplings

$$H = \sum_n J_1 \vec{S}_n \cdot \vec{S}_{n+1} + J_2 \vec{S}_n \cdot \vec{S}_{n+2}$$

with $J_2 / J_1 = 1/4 = 0.25$

The J1-J2 model is critical with no log corrections at

$$J_2 / J_1 \cong 0.2411$$

Kalmeyer-Laughlin wave function (1987)

Bosonic Laughlin wave function on the square lattice at $\nu = \frac{1}{2}$

$$\psi_{KL}(z_1, \ldots z_{N/2}) = \prod_n \chi(z_n) \prod_{n<m} (z_n - z_m)^2 e^{-\sum_q |z_q|^2/4}$$

Where the z's are the position of hard core bosons $z_a = n_a + i\, m_a$

$$|\uparrow\rangle \leftrightarrow |0\rangle \qquad empty$$

$$|\downarrow\rangle \leftrightarrow a^* \, |0\rangle \quad occupied$$

The KL state is the wave function of a spin chiral liquid

Is there a H for which the KL state is the GS? -> Parent Hamiltonian

The Haldane-Shastry (1D) and the Kalmeyer-Laughlin (2D) states have a common origin

SU(2)@k=1

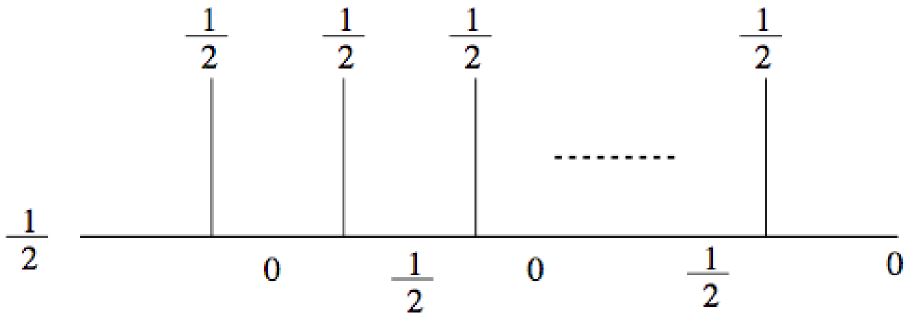
The WZW model $SU(2)_{k=1}$

CFT with c =1 and two primary fiels $\phi_{0,}\ \phi_{1/2}, \quad h_0 = 0, h_{1/2} = 1/4$

Fusion rule: $\phi_{1/2} \times \phi_{1/2} = \phi_0$

Unique conformal block

(N even)



The Haldane-Shastry state: $z_n = e^{2\pi i n/N}$

The Kalmeyer-Laughlin state $z = n + i\, m$

Bosonization

Spin ½ primary field $\phi_{s_n}(z_n) = \chi_n \, e^{i s \varphi(z)/\sqrt{2}}, \quad s_n = \pm 1$

$\chi_n = 1 \, (n : even), \quad \chi_n = e^{i\pi(s_n - 1)/2} \, (n : odd)$ Marshall sign rule

$$\left\langle \phi_{s_1}(z_1) \ldots \phi_{s_N}(z_N) \right\rangle = \delta_{\sum_i s_i, 0} \prod_{n:odd} e^{i\pi(s_n - 1)/2} \prod_{i<j} (z_i - z_j)^{s_i s_j / 2}$$

Map:
Spin-> hard boson

$s_n = 1 : \quad |\uparrow\rangle \leftrightarrow |0\rangle \qquad q_n = 0$

$s_n = -1 : |\downarrow\rangle \leftrightarrow a_n^* |0\rangle \quad q_n = 1$

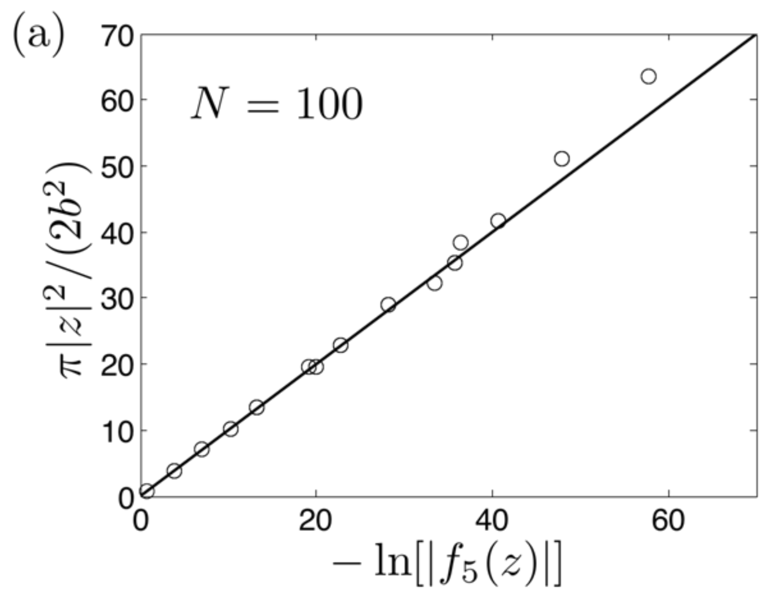
$s_i = 1 - 2 q_i$

Choosing $z_n = e^{2\pi i n / N}$ we recover the HS state

$$\left\langle \phi_{s_1}(z_1) \ldots \phi_{s_N}(z_N) \right\rangle = \prod_i z_{n_i} \prod_{i<j} (z_{n_i} - z_{n_j})^2$$

In 2D $$\psi_{WZW}(z_1,\ldots z_{N/2}) \propto \prod_n \chi(z_n) \prod_{n<m} (z_n - z_m)^2 \prod_{n=1}^{M} f_M(z_n)$$

where $$f_M(z_n) = z_n \prod_{m(\neq n)}^{N} \left(1 - \frac{z_n}{z_m}\right)^{-1} \to e^{-|z_n|^2/4} \; (M = N/2 \to \infty)$$



The KL state is approached asymptotically

Questions:

- can one derive the parent Hamiltonians for the HS and KL states using purely CFT methods?

- can one generalize these states to higher spin?

Hamiltonians from null vectors

SU(2)@k=1

In the spin ½ module at k=1 there is the null vector

$$\chi_{3/2,\,3/2} = J^{+}_{-1}\,\phi_{1/2,\,1/2}$$ (Kac, Gepner, Witten)

This is the heighest weight vector of a spin 3/2 multiplet.
The whole multiplet is given by

$$\chi_{a,\,m}(z) = \sum K^{\,a,m}_{\,b,m'}\, J^{b}_{-1}\, \phi_{1/2,\,m'}(z), \quad a = 1,2,3, \quad m = \pm 1/2$$

$$K^{a,m}_{b,m'} = \frac{2}{3}\left(\delta_{a,b}\,\delta_{m,m'} - i\sum_{c}\varepsilon_{abc}\, t^{c}_{m,m'}\right), \quad t^{a} = \frac{\sigma^{a}}{2}$$

Clebsh-Gordan coefficients for $\frac{1}{2}\otimes 1 \rightarrow \frac{3}{2}$

Decoupling eqs for these null vectors imply that the conformal blocks

$$\psi_{m_1,\ldots,m_N}(z_1 \ldots z_N) = \left\langle \phi_{m_1}(z_1) \ldots \phi_{m_N}(z_N) \right\rangle$$

Satisfy the algebraic equations (use Ward identities)

$$C_{i,a}\left(z_1,\ldots z_N\right)\psi(z_1 \ldots z_N) = 0, \quad i = 1,\ldots N, \quad a = 1,2,3$$

$$C_{i,a}\left(z_1,\ldots z_n\right) = \sum_{j(\neq i)}^{n} w_{ij}(t_j^a + i\varepsilon_{abc}\, t_i^b\, t_j^c), \quad w_{ij} = \frac{z_i + z_j}{z_i - z_j}$$

$$H_i = \sum_a C_{i,a}^* \, C_{i,a}, \quad H_i \psi = 0, \quad i = 1,\ldots, N$$

Parent Hamiltonian $$H = \sum_i H_i, \quad H\, \psi = 0$$

This construction works for generic z's

If $\quad |z_n| = 1 \quad$ 1D-model

$$H_{HS}\,\psi = E_0\,\psi$$

$$H_{HS} = -\sum_{n\neq m}\left(\frac{z_n z_m}{(z_n - z_m)^2} + \frac{1}{12} w_{n,m}(c_n - c_m)\right)\vec{S}_n \cdot \vec{S}_m$$

$$w_{n,m} = \frac{z_n + z_m}{z_n - z_m}, \quad c_n = \sum_{m(\neq n)} w_{n,m}, \quad E_0 = \frac{1}{16}\sum_{n\neq m} w_{n,m}^2 - \frac{N(N+1)}{16}$$

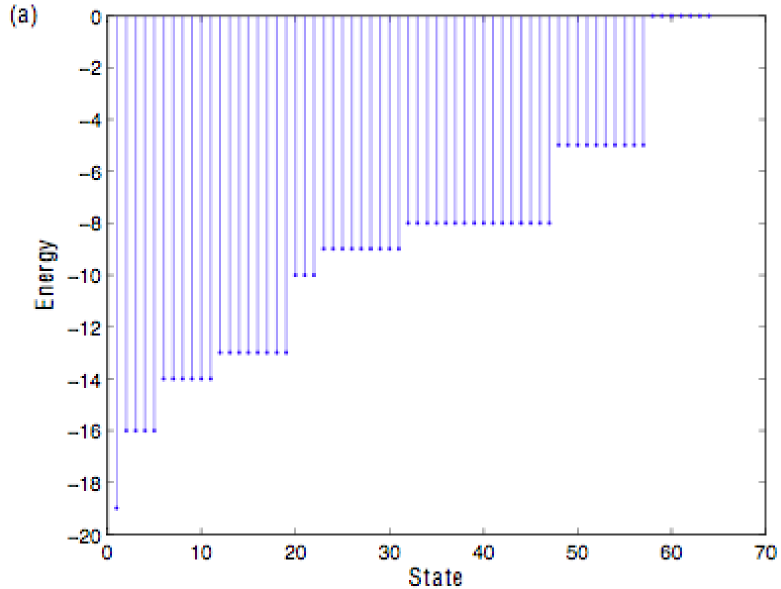
Uniform case $\quad z_n = e^{2\pi i n/N} \to c_n = 0 \quad$ recover the HS Hamiltonian

Yangian symmetry

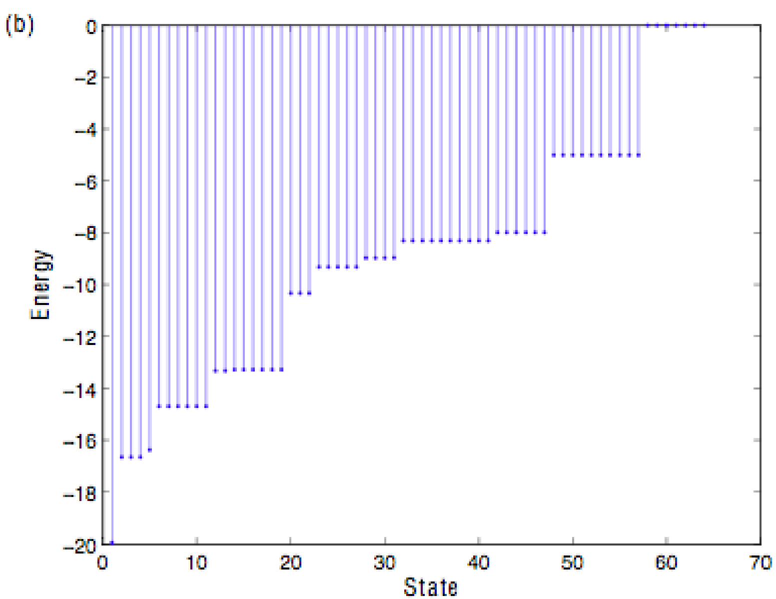
$$\left[H, \vec{S}\right] = \left[H, \vec{\Lambda}\right] = 0, \quad \left[\vec{S}, \vec{\Lambda}\right] \neq 0, \quad \vec{S} = \sum_i \overline{S}_i, \quad \vec{\Lambda} = \sum_{i,j} w_{ij}\, \vec{S}_i \times \vec{S}_j$$

Non uniform case -> Yangian is broken-> less degeneracy

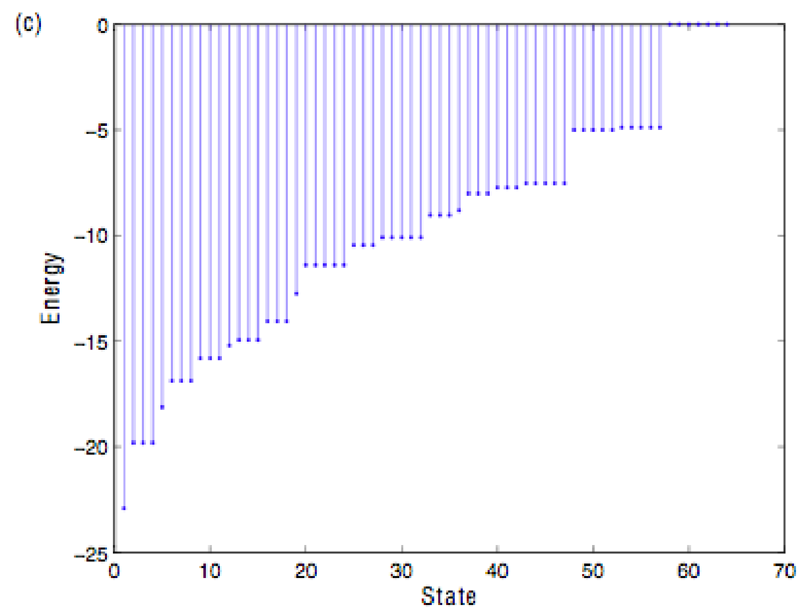
Uniform $z_n = e^{2\pi i n/N}$



Dimer $z_n = e^{2\pi i(n+\delta(-1)^n)/N}$



Random $z_n = e^{2\pi i(n+\phi_n)/N}$



Decoupling equations -> eqs. for spin correlators

$$w_{ij}\left\langle t_i^a\, t_j^a \right\rangle + \sum_{k(\neq i,j)} w_{ik}\left\langle t_j^a\, t_k^a \right\rangle + \frac{3}{4} w_{ij} = 0, \quad i \neq j$$

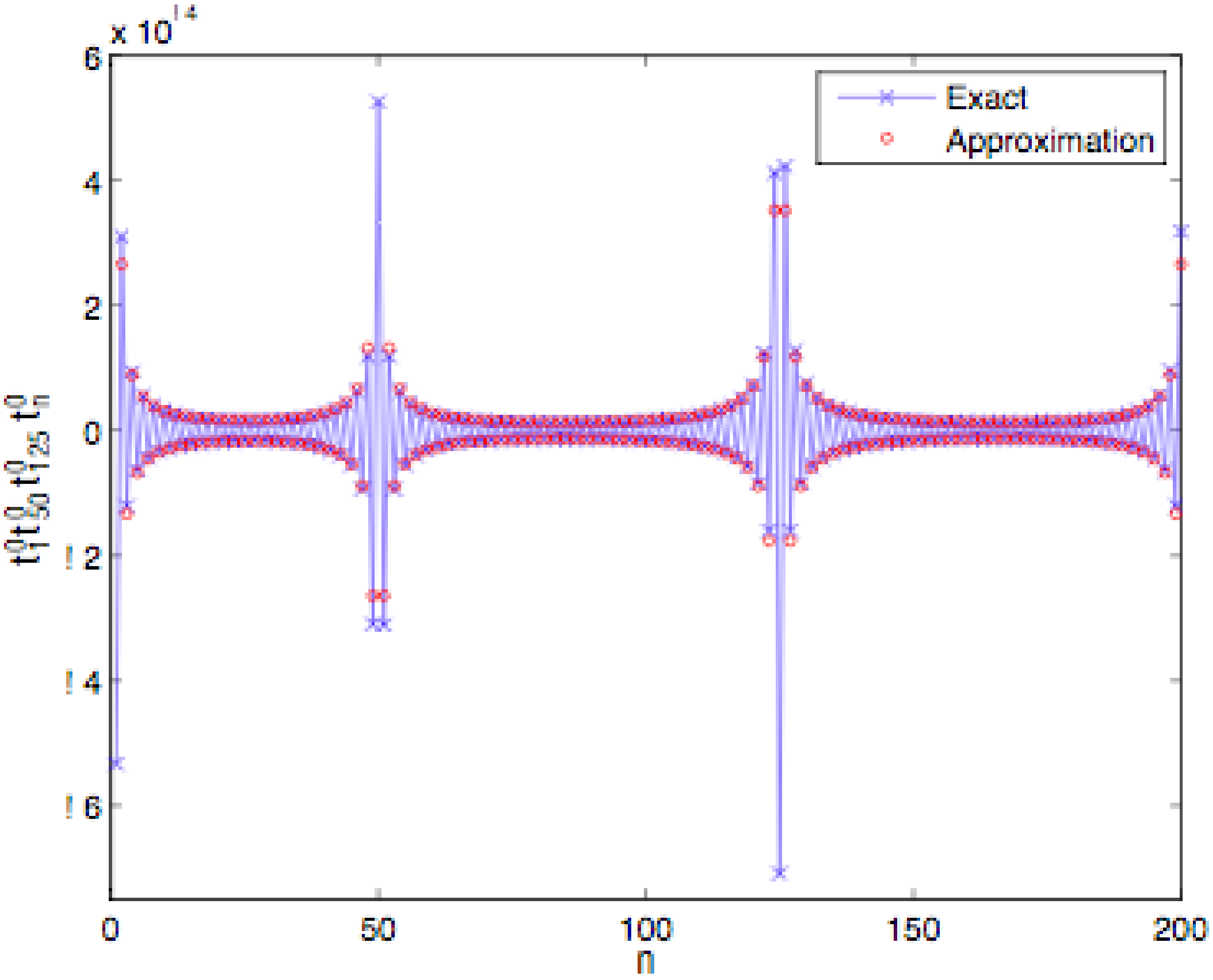
In the uniform case, N -> infinite (Gebhard-Vollhardt 1987)

$$\left\langle t_n^a\, t_0^b \right\rangle = (-1)^n \delta_{ab} \frac{Si(\pi n)}{4\pi n}, \qquad Si(z) = \int_0^z dt \frac{\sin t}{t}$$

Exact formula for finite N

$$\left\langle t_n^a\, t_0^b \right\rangle = (-1)^n \delta_{ab} \frac{(-1)^n}{4N\sin(\pi n/N)} \sum_{m=1}^{N/2} \frac{\sin(2\pi n(m-1/2)/N)}{m-1/2}$$

Four point spin correlator $\left\langle t_1^0\, t_{50}^0\, t_{125}^0\, t_n^0 \right\rangle \quad n = 1, \cdots, 200$



Excited states (uniform model)

They are known exactly in terms of quasimomenta

$$S^z = \frac{N}{2} - M, \quad \{m_j\}_{j=1}^{M} = \{1, 2, \ldots, N-1\}, \quad m_{j+1} > m_j + 1$$

Energy and momenta $\quad E = \sum_j m_j (m_j - N), \quad P = \sum_j 2\pi m_j / N$

The wave functions of excited states are also chiral correlators

$$\left\langle \phi_{m_1}(z_1) \ldots \phi_{m_N}(z_N) \right\rangle \to E_0 = -(N^3 + 2N)/12 \to \quad S = 0$$

$$\left\langle \phi_{m_\infty}(\infty)\, \phi_{m_1}(z_1) \ldots \phi_{m_N}(z_N)\, \phi_{m_0}(0) \right\rangle \to E_{exc} = N/2 \to \quad S = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

$$\left\langle \phi_{m_1}(z_1) \ldots \phi_{m_N}(z_N)\, J^a(0) \right\rangle \to E_{exc} = N - 1 \to \quad S = 1$$

$$\left\langle J^a(\infty)\, \phi_{m_1}(z_1) \ldots \phi_{m_N}(z_N) \right\rangle \to E_{exc} = N - 1 \to \quad S = 1$$

$$\left\langle J^a(\infty)\, \phi_{m_1}(z_1) \ldots \phi_{m_N}(z_N)\, J^b(0) \right\rangle \to E_{exc} = 2(N-1) \to \quad S = 1 \otimes 1 = 0 \oplus 1 \oplus 2$$

Generalization to 2D

Take the z's generic in the complex,

$$H_i = \frac{1}{2}\sum_{j(\neq i)} |w_{ij}|^2 + \frac{2}{3}\sum_{j(\neq i)} |w_{ij}|^2\, t_i^a\, t_j^a + \frac{2}{3}\sum_{j\neq k(\neq i)} w_{ij}^* w_{ik}\, t_j^a\, t_k^a$$

$$-\frac{2i}{3}\sum_{j\neq k(\neq i)} w_{ij}^* w_{ik}\, \varepsilon^{abc}\, t_i^a\, t_j^b\, t_k^c,$$

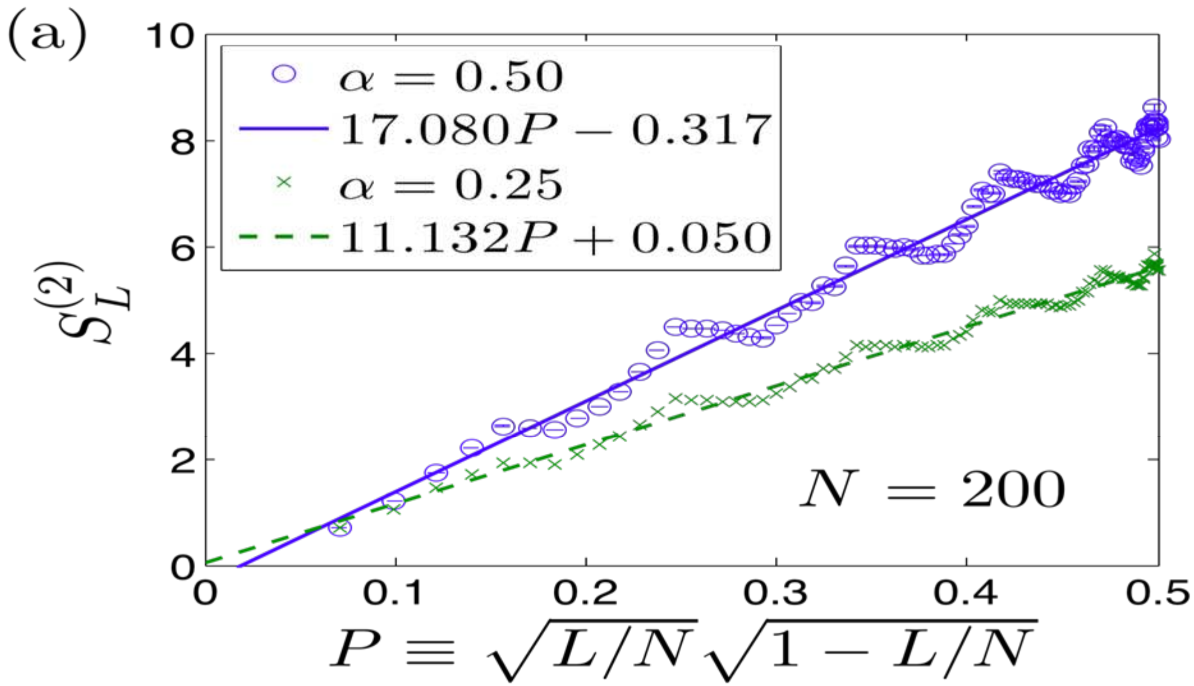
$$w_{ij} = \frac{g(z_i)}{z_i - z_j} + h(z_i)$$ g(z), h(z): generic

The three body term breaks time reversal (in 1D it was absent)

If $z_i = n_i + i\, m_i$ this is the Parent H for the "KL state"

Greiter et al. have constructed H's for the large N limit

Area law on the sphere



Topological entropy: $-\gamma = 0.341 \pm 0.057 \approx -\frac{1}{2}\log 2 = -0.34$

Same as the bosonic Laughlin states at $\nu = \frac{1}{2}$

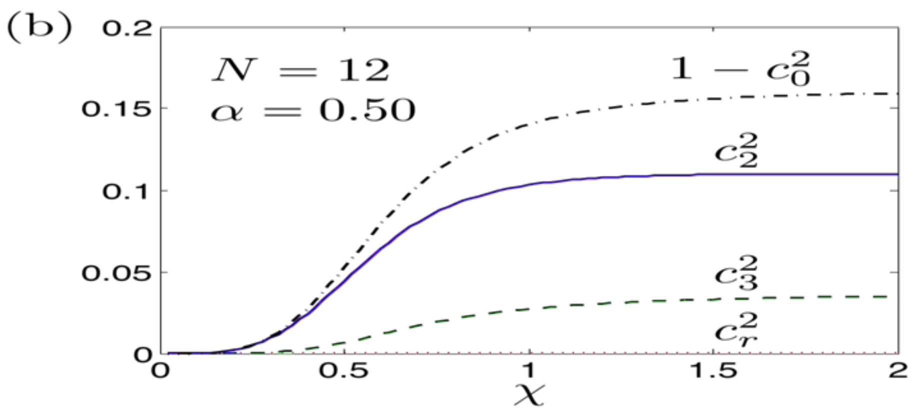
Two leg ladder and entanglement Hamiltonian

$$z_{nA} = e^{(2in+\chi)\pi/Q}, \quad z_{nB} = e^{(2in-\chi)\pi/Q}, \quad n = 1,\ldots Q \qquad \chi = \text{Inter leg distance}$$

Density matrix for leg A $\qquad \rho_A = Tr_B \left|\psi\right\rangle\left\langle\psi\right| = e^{-H_A}$

Entanglement Hamiltonian $\qquad H_A = c_0 + c_2 H_2 + c_3 H_3 + c_r H_r$

$$H_2 = -2\sum_{n<m} \frac{z_n z_m}{z_{nm}^2} \vec{S}_n \cdot \vec{S}_m, \quad H_3 = -i \sum_{n<m} \frac{z_n z_m z_p}{z_{nm} z_{np} z_{mp}} \vec{S}_n \cdot (\vec{S}_m \times \vec{S}_p)$$



Higher spin generalizations of the HS model

SU(2)@k=2

Primary fields: $\phi_0,\ \phi_{1/2},\ \phi_1$

Fusion rules $\phi_{1/2}\times\phi_{1/2}=\phi_0+\phi_1,\quad \phi_1\times\phi_1=\phi_0,\quad \phi_1\times\phi_{1/2}=\phi_{1/2}$

For spin 1 there is only one conformal block

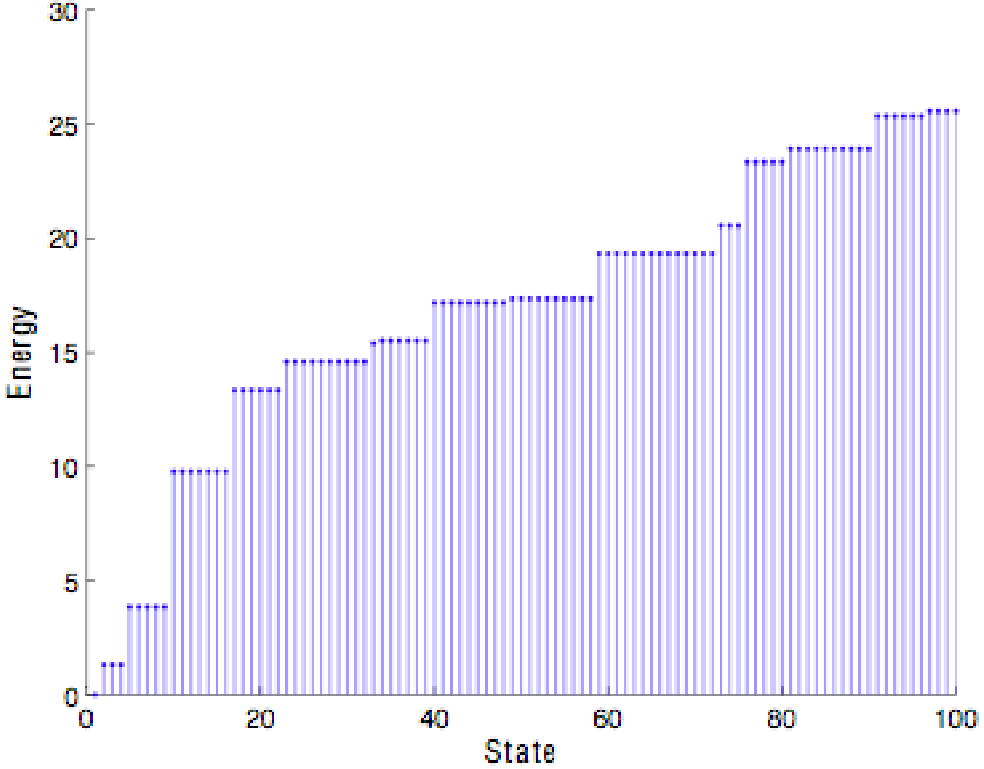
$$\psi_{s_1\ldots s_N}=\left\langle \phi_{s_1}(z_1)\cdots\phi_{s_N}(z_N)\right\rangle,\qquad s_i=0,\pm 1$$

Using the null vectors method one finds the parent Hamiltonian

$$H=-\frac{4}{3}\sum_{i\neq j}w_{ij}^2-\frac{1}{3}\sum_{i\neq j}\left(w_{ij}^2+2\sum_{k(\neq i,j)}w_{ki}\,w_{kj}\right)t_i^a\,t_j^a+\frac{1}{6}\sum_{i\neq j}w_{ij}^2\left(t_i^a\,t_j^a\right)^2+\frac{1}{6}\sum_{i\neq j\neq k}w_{ij}\,w_{ik}t_i^a\,t_j^a\,t_i^b\,t_k^b$$

(See also Greiter et al and Paredes)

Spectrum in the uniform case



There are not accidental degeneracies-> No Yangian symmetry

SU(2)@k=2 = Boson + Ising (c= 3/2 = 1+ 1/2)

Primary spin 1 fields (h=1/2)

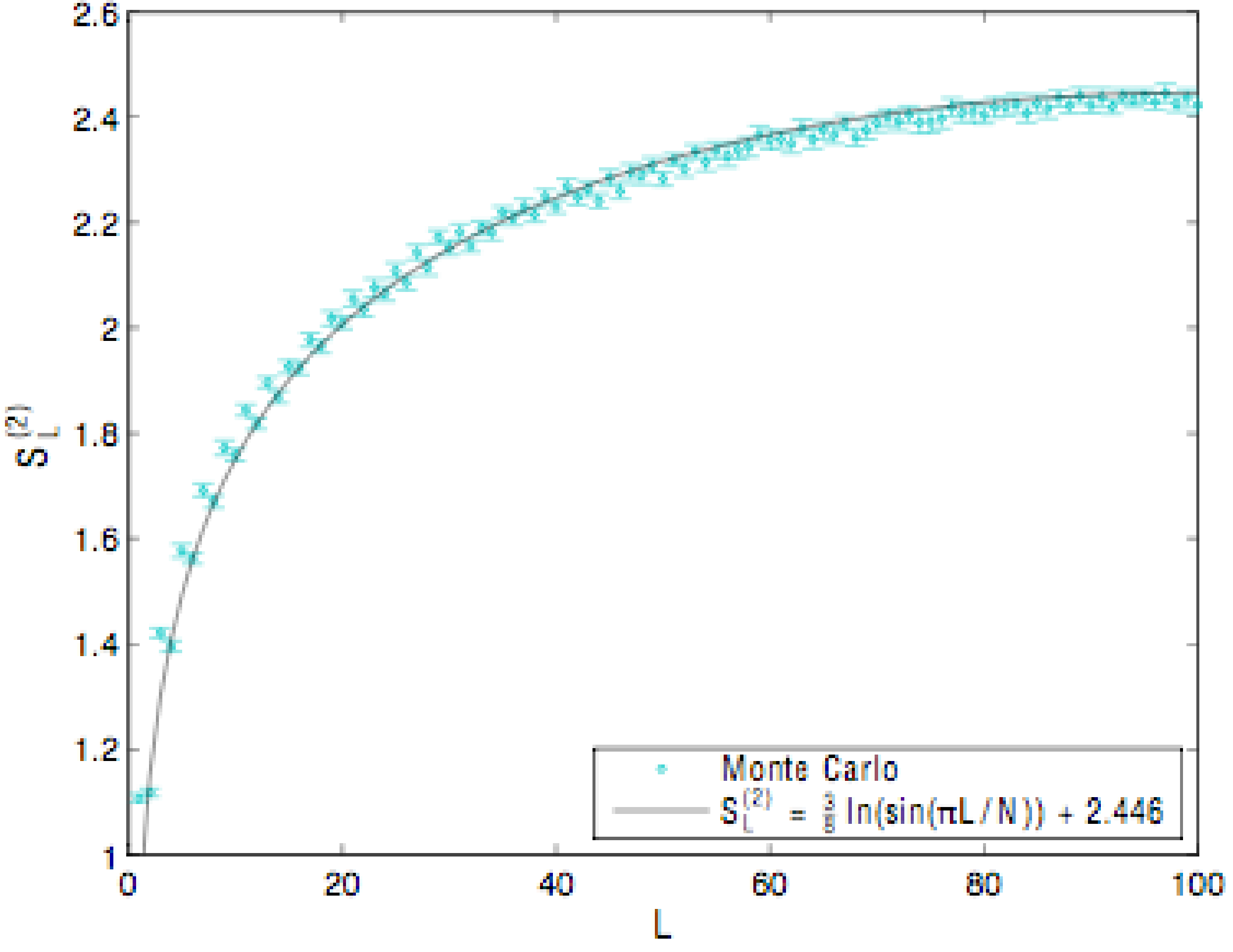
$$\phi_{\pm 1}(z_j) = e^{\pm i\varphi(z_j)}, \quad \phi_0(z_j) = (-1)^j \chi(z_j)$$ Majorana fermion

$$\psi_{s_1 \cdots s_N} = (-1)^{\sum_{i:\,odd} s_i} \prod_{i<j} \left(z_i - z_j\right)^{s_i s_j} \; Pf_0 \frac{1}{z_i^0 - z_j^0}, \quad \sum_i s_i = 0, \quad N : even$$

In the uniform case we expect the low energy spectrum of this model to be described by SU(2)@k=2 model

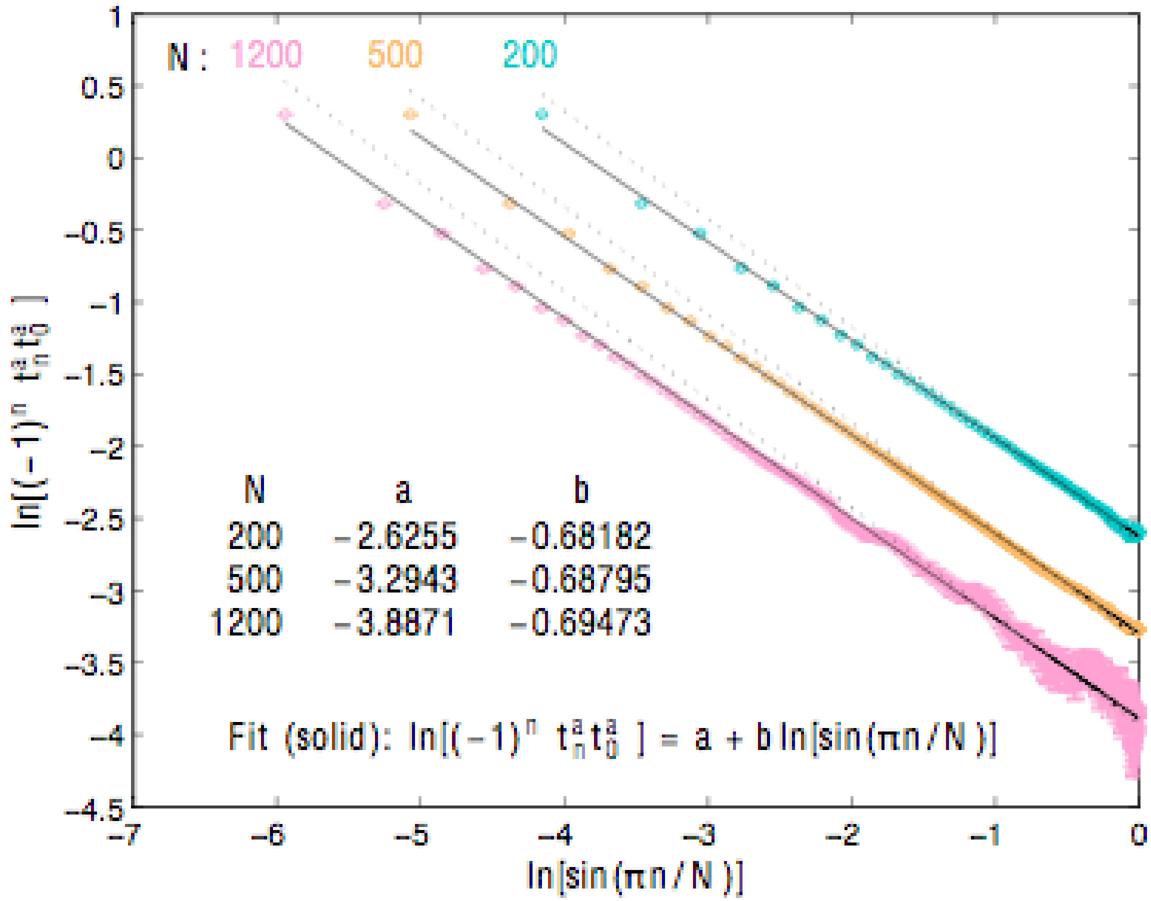
$$SU(2)@k=2 \to c=\frac{3}{2}$$

Renyi-2 entropy $S_L = -\log Tr\,\rho_L^2$



Spin-spin correlator

CFT prediction $$\left\langle t_n^a\, t_0^a \right\rangle \approx (-1)^n \left(\sin \frac{\pi n}{N} \right)^b, \quad b = -\frac{3}{4}$$



Suggest existence of log corrections (Narajan and Shastry 2004)

Take again SU(2)@k=2

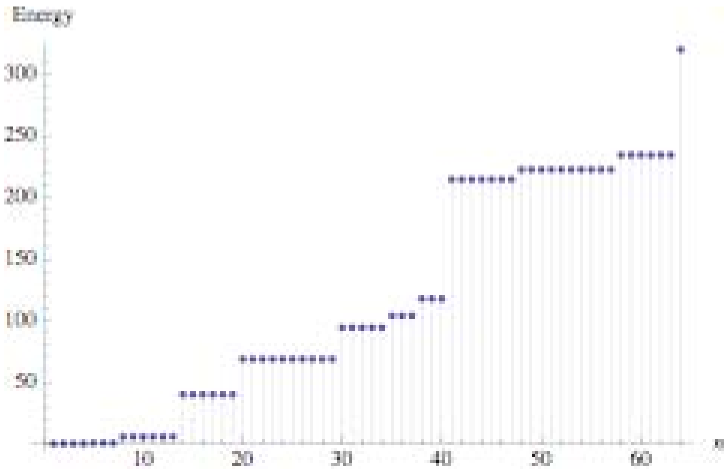
Fusion rule of spin 1/2 field $$\phi_{1/2} \times \phi_{1/2} = \phi_0 + \phi_1$$

Number of chiral correlators of N spin 1/2 fields

$$\left\langle \phi_{1/2,m_1}(z_1) \ldots \phi_{1/2,m_N}(z_N) \right\rangle_p \quad p = 1, 2 \ldots , 2^{N/2-1}$$

Now the GS is NOT unique but degenerate !!

Example N = 6 -> 4 GS



The spin Hamiltonian contains 4 body terms

Mixing spin 1/2 and spin 1 for SU(2)@k=2

$$\left\langle \phi_{1/2} \cdots^{N_{1/2}} \cdots \phi_{1/2}\, \phi_1 \cdots^{N_1} \cdots \phi_1 \right\rangle_p \quad \text{->} \quad 2^{\frac{1}{2}N_{1/2}-1}$$

The degeneracy only depends on the number of spin 1/2 fields

SU(2)@2 = Boson + Ising
c =3/2 = 1 + 1/2

Spin 1 field $\phi_{1,\pm 1}(z) = e^{\pm i\varphi(z)}, \quad \phi_{1,0}(z) = \chi(z)$

Spin 1/2 field $\phi_{1/2,\pm 1/2}(z) = \sigma(z)\, e^{\pm i\varphi(z)/2}$

$\chi(z)$ is the Majorana field and $\sigma(z)$ is the spin field of the Ising model

Ising fusion rules $\chi \times \chi = id, \quad \chi \times \sigma = \sigma, \quad \sigma \times \sigma = id + \chi$

Moore-Read wave function for FQHE @5/2 (1992)

CFT = boson (c=1) + Ising (c=1/2)

Electron operator $$\psi_e(z) = \chi(z)\, e^{i\sqrt{2}\,\varphi(z)}$$

Ground state wave function

$$\left\langle \psi_e(z_1)\ldots\psi_e(z_N)\right\rangle = \prod_{i<j}(z_i - z_j)^2 \left\langle \chi(z_1)\ldots\chi(z_N)\right\rangle$$

$$\left\langle \chi(z_1)\ldots\chi(z_N)\right\rangle = Pfaffian\,\frac{1}{z_i - z_j} = \sqrt{\det\frac{1}{z_i - z_j}}$$

Quasihole operator $$\psi_{qh}(z) = \sigma(z)\, e^{\frac{i}{2\sqrt{2}}\varphi(z)}$$

$$\left\langle \psi_{qh} \cdots^{N_{qh}} \cdots \psi_{qh}\, \psi_e \cdots^{N_e} \cdots \psi_e \right\rangle_p$$ -> Degeneracy $2^{\frac{1}{2}N_{qh}-1}$

Fusion rules of Ising

An analogy via CFT

FQHE ←	CFT →	Spin Models
Electron	χ field	spin 1
Quasihole	σ field	spin 1/2
Braiding of quasiholes	Monodromy of correlators	Adiabatic change of H

In the FQHE braiding is possible because electrons live effectively in 2 dimensions

To have "braiding" for the spin systems we needs 2D

The SU(2)@k=2 in 2D is the spin analogue of the Moore-Read state

In the FQHE the z's are the positions of the electrons or quasiholes

In the spin models the z's parametrize the couplings of the Hamiltonian. They are not real positions of the spins.

Braiding amounts to change these couplings is a certain way.

One can in principle do topological quantum computation in these spin systems.

But one has first to show that Holonomy = Monodromy

This problem has been solved for the Moore-Read state (Bonderson, Gurarie, Nayak, 2010)

Conclusions

- Using WZW we have proposed wave functions for spin systems which are analogue of FQH wave functions

- Generalization of the Haldane-Shastry model in several directions
  1) non uniform
  2) higher spin
  3) degenerate ground states
  4) 1D -> 2D

Prospects

- Physics of the generalized HS models
- WZW's with other Lie groups and supergroups, other chiral algebras
- Relation with the CFT approach to the Calogero-Sutherland model
- Excited states
- Topological Quantum Computation with HS models

THANK YOU

Grazie Mille