Universal one-way quantum computation

with two-dimensional AKLT states

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GGI Florence,
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What is the computational power of quantum states?

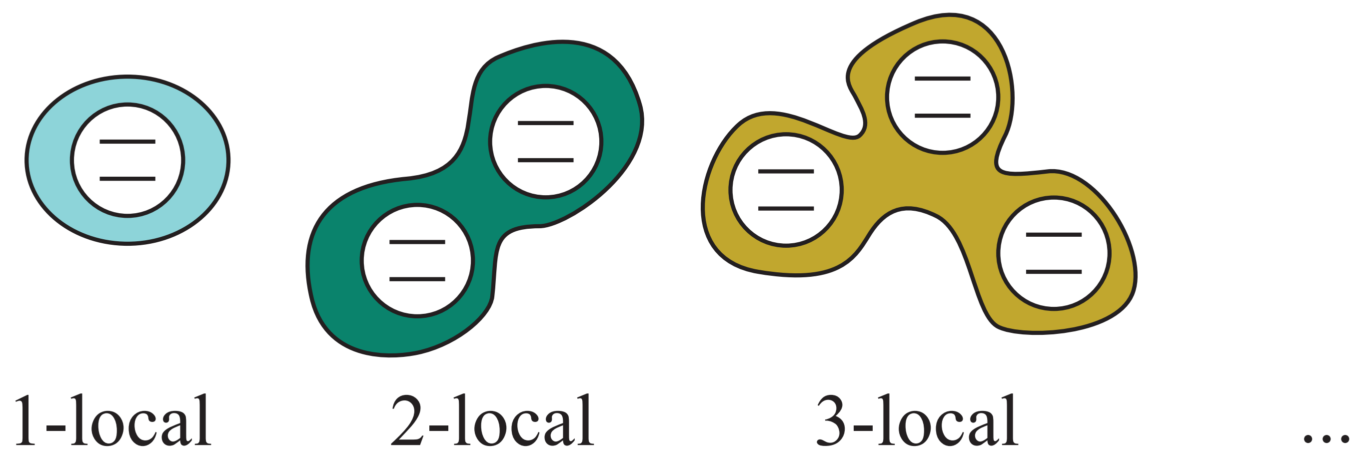
$$|\psi\rangle$$

How to extract the computational result?

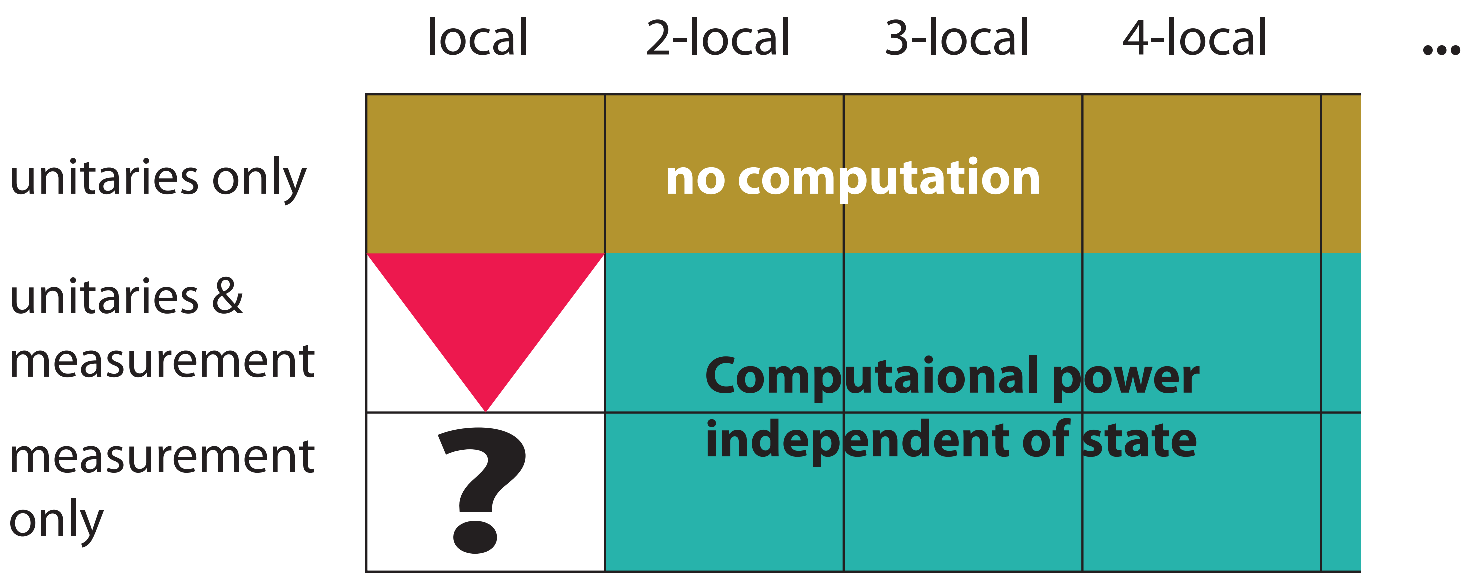
Allowed operations:

- Unitary gates and measurement
- Classical communication and *simple* processing

Organizing principle: Unitaries and measurements are n-local



How to extract the computational result?



?: *Is this computation universal?*

I.e.: Can anything computable by a quantum computer be computed in this fashion, with polynomial overhead?

Outline

1. Quantum computation by measurement

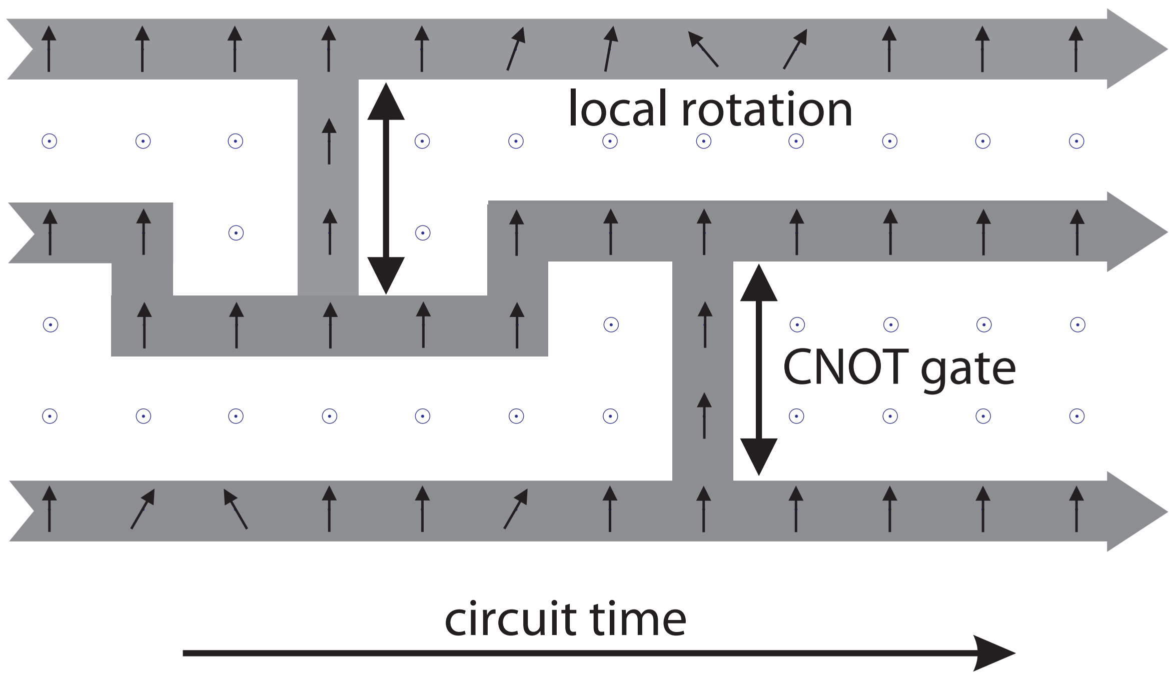
2. The role of entanglement

3. Quantum computation and statistical mechanics

4. The 2D AKLT state is computationally universal

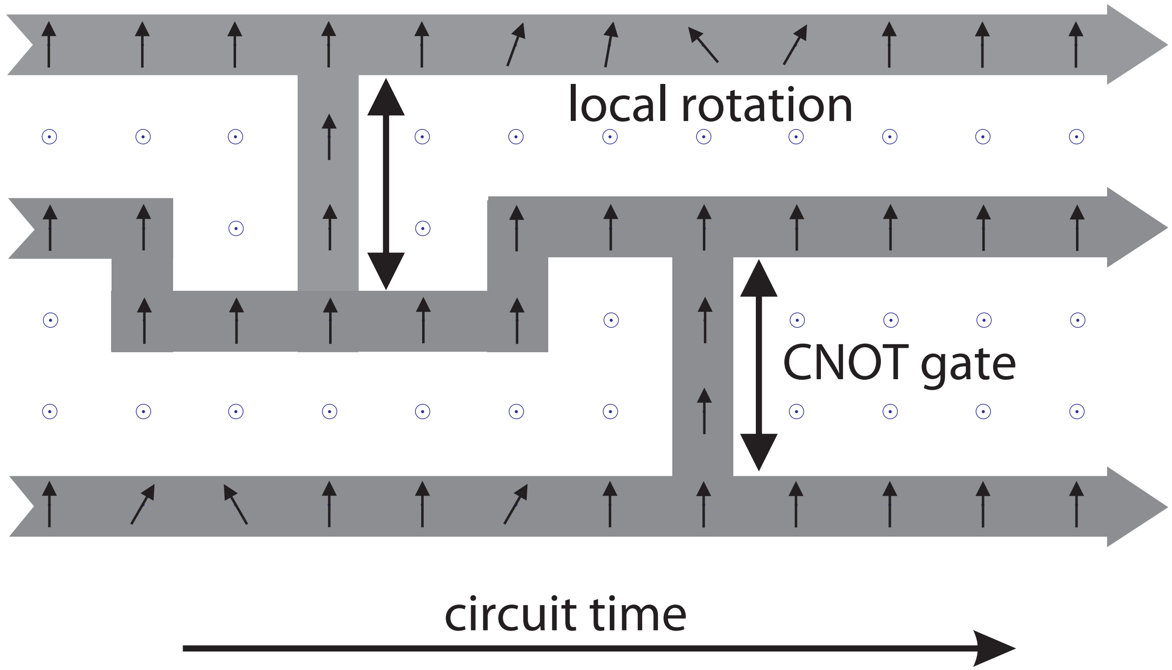
5. Discussion

Quantum computation by measurement



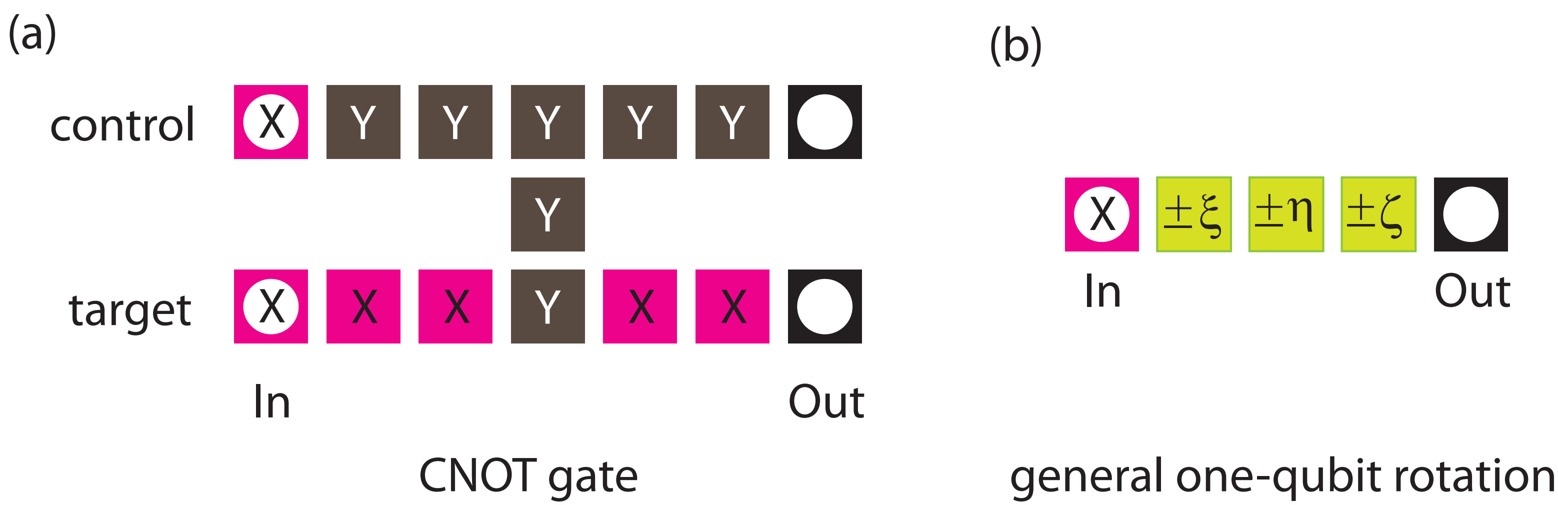
- Use an entangled quantum state $|\Psi\rangle$ as computational resource.
- Information written onto $|\Psi\rangle$, processed and read out by one-qubit measurements only.

Quantum computation by measurement



- *Are there universal resource states $|\Psi\rangle$?* – Yes: cluster states.
- *Can universal states $|\Psi\rangle$ be ground states of simple Hamiltonians?* — Yes: 2D AKLT states.

Simulating a universal gate set



Measurement-based realization of a universal set of gates, for the cluster state.

Cluster states are universal

- Cluster states $|\Phi\rangle_{\mathcal{C}}$ can be created from a product state via unitary evolution under the Ising Hamiltonian [1].

$$|\Phi\rangle_{\mathcal{C}} = e^{i\pi/4 \sum_{\langle i,j \rangle} \sigma_z^{(i)} \sigma_z^{(j)}} \bigotimes_k |+\rangle_k.$$

- Cluster states do not arise as ground states of two-body Hamiltonians [2].

[1] Raussendorf and Briegel, PRL 86, 910 (2001);
[2] M. Nielsen, Math. Phys. 57, 147 (2006).

But ... ?

Individual measurement outcomes are random. How can one compute in a deterministic fashion?

A: Measurement outcomes are random *but correlated*. Only the correlations represent output bits.

The support of the quantum correlations $K(|\Psi\rangle)$ used for the computation is of the order of the system size

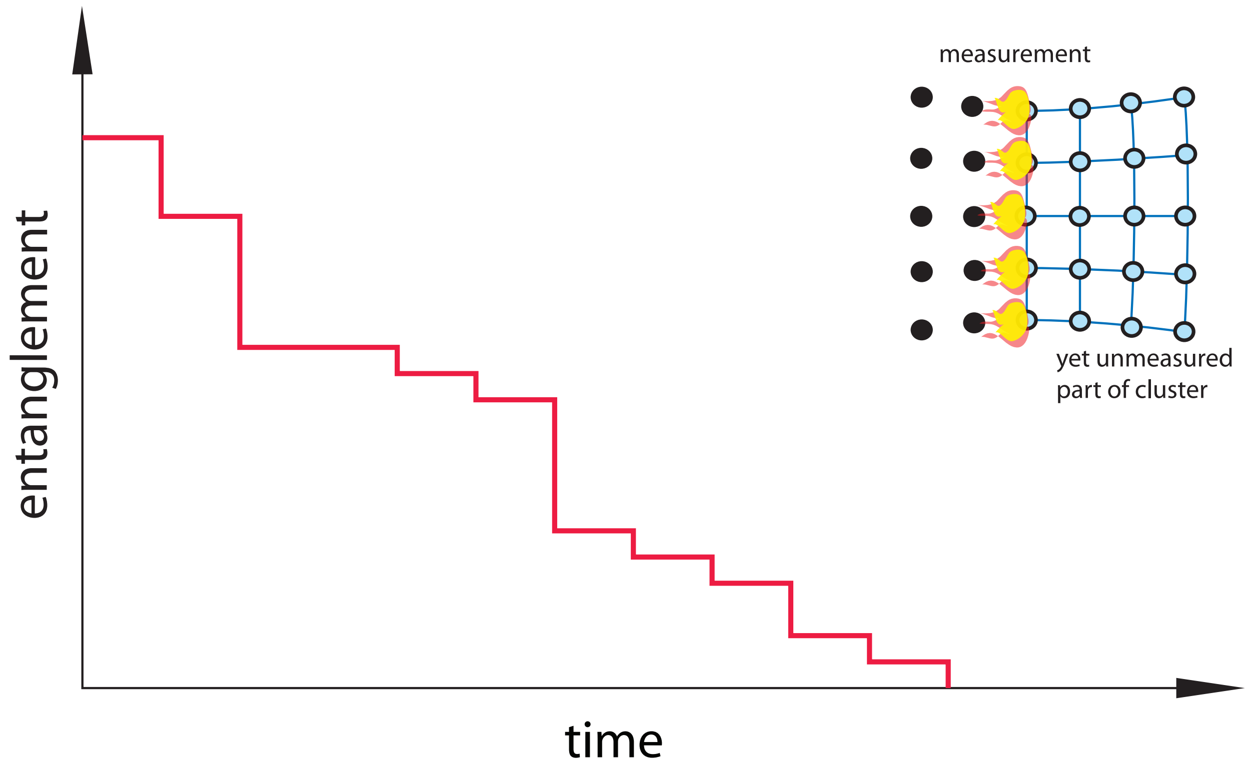
$$|K(|\Psi\rangle)| \sim \text{supp}(|\Psi\rangle)$$

- In addition: Measurement bases need to be adjusted to measurement outcomes obtained at other qubits.

$\Rightarrow$ Leads to *temporal order*.

The role of entanglement

Entanglement as a resource



- Burn entanglement to obtain a computational result.
- Resource state $|\Psi\rangle$ can be used only once.

Entanglement as a resource

Theorem (Van den Nest et *al.*, '06) The operational cost O of classically simulating measurement-based quantum computation on an n-qubit state $|\Psi\rangle$ is at most

$$O = \text{Poly}(n)\,\exp(\chi),$$

where χ is the *entanglement width* of $|\Psi\rangle$.

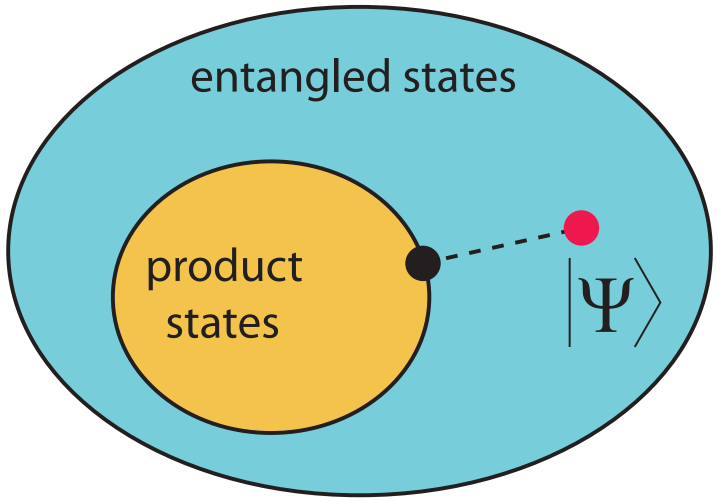
Remark: χ is an *entanglement measure*.

Message: *Substantial entanglement is necessary for a speedup.*

MBQC • entanglement theory • tensor networks • graph theory

But is entanglement *sufficient* ?

Consider *geometric entanglement* $\mathcal{E}(|\Psi\rangle)$ [Wei & Goldbart '03]:

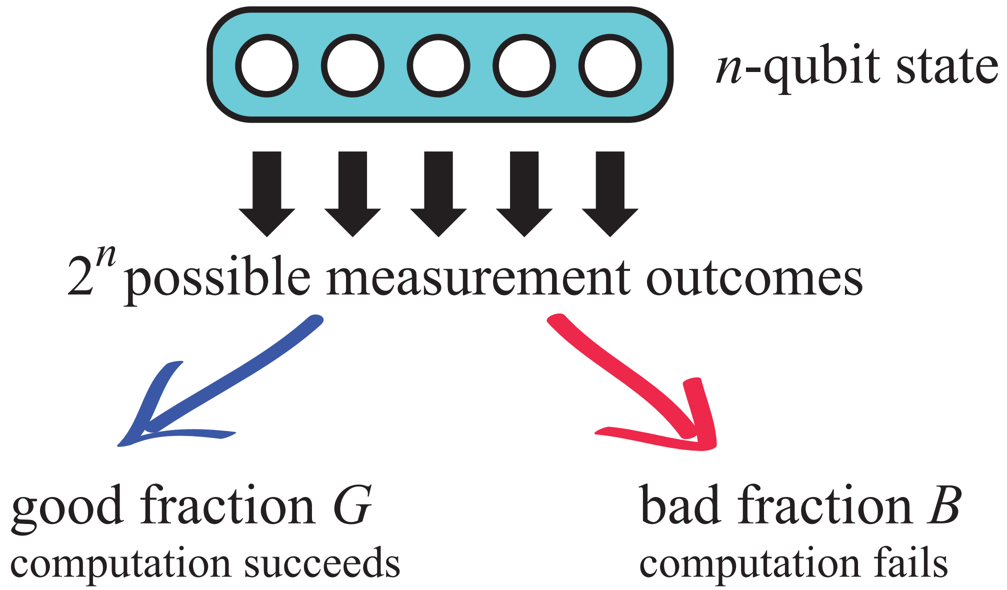


$$\mathcal{E}(|\Psi\rangle) := -\log_2 \left(\max_{\text{prod. states } |\phi\rangle} |\langle\phi|\Psi\rangle|^2 \right) \qquad (1)$$

By probability conservation: If $|\Psi\rangle$ is an n-qubit state, then

$$\mathcal{E}(|\Psi\rangle) \leq n.$$

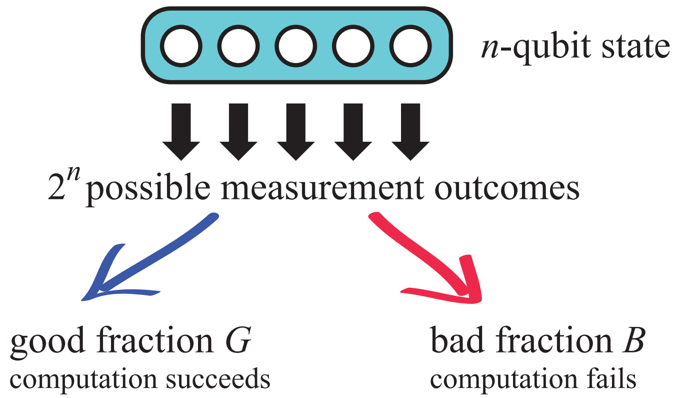
Too entangled to be useful



Assumptions:

- Entanglement is close to maximal: $\mathcal{E}(|\Psi\rangle) = n - \delta$
- Computation succeeds with high probability: $p_{\text{success}} = 1/2$.

Too entangled to be useful



Assumptions:

1. Entanglement: $\mathcal{E}=n-\delta$

2. Success probability: 1/2

Q: What is the fraction of "good" outcomes $|G|/2^n$?

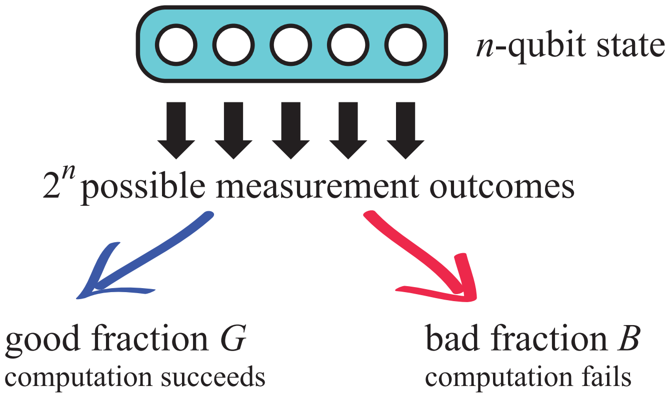
For all post-measurement states $|\phi(\mathbf{s})\rangle$:

$$|\langle\phi(\mathbf{s})|\Psi\rangle|^2 \leq 2^{-\mathcal{E}(|\Psi\rangle)} = 2^{-n+\delta}.$$

Hence,

$$p_{\text{success}} = \frac{1}{2} = \sum_{\mathbf{s}\in G} |\langle\phi(\mathbf{s})|\Psi\rangle|^2 \leq |G|\cdot 2^{-n+\delta}.$$

Too entangled to be useful



Assumptions:

1. Entanglement: $\mathcal{E}=n-\delta$
2. Success probability: 1/2

Q: What is the fraction of "good" outcomes $|G|/2^n$ *?*

Answer:

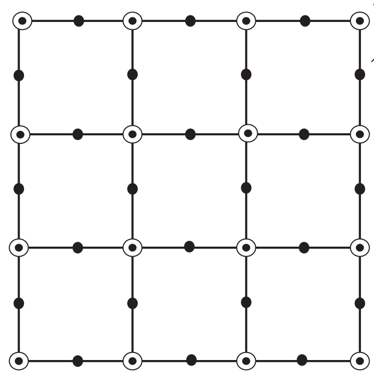
$$\boxed{\frac{|G|}{2^n} \geq 2^{-1-\delta}.}$$

- This MBQC can be efficiently simulated by coin flipping.
  No quantum speedup!

[Gross D, Flammia ST, Eisert J. 2009. Phys. Rev. Lett. 102:190501]

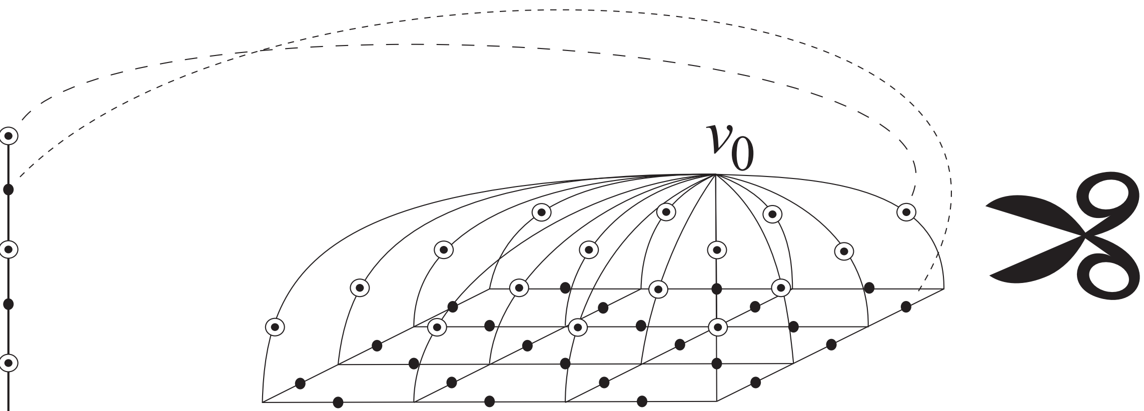
MBQC and Statistical Mechanics

MBQC and the Ising model



2D cluster state

universal

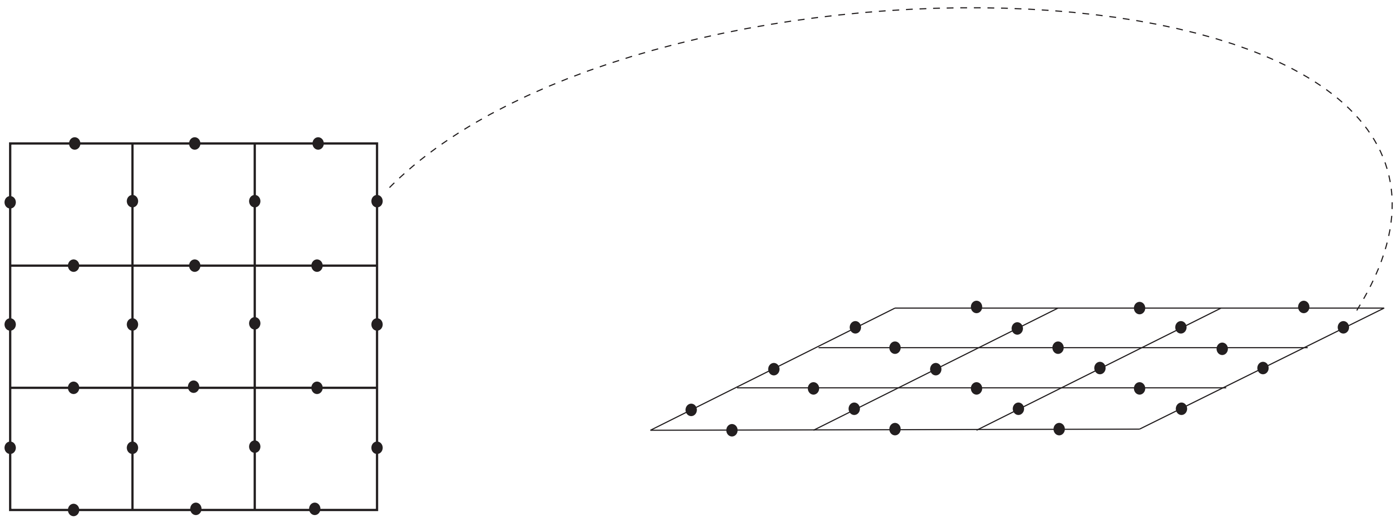


corresp. Ising interaction graph

partition function hard

$$\langle \phi_{\text{loc}} | \Psi \rangle \sim Z_{\text{Ising}}(\{J_e/kT\}) \quad (2)$$

MBQC and the Ising model



Kitaev surface code state

Not useful for MBQC

corresp. Ising interaction graph

partition function easy to compute

$$\langle \phi_{\mathrm{loc}} | \Psi \rangle \sim Z_{\mathrm{Ising}}(\{J_e/kT\})$$

2D AKLT states are universal resources

Why ground states?

- Experiment:

  Create computational resource states by cooling.

- Theory:

  Make contact with condensed matter physics.

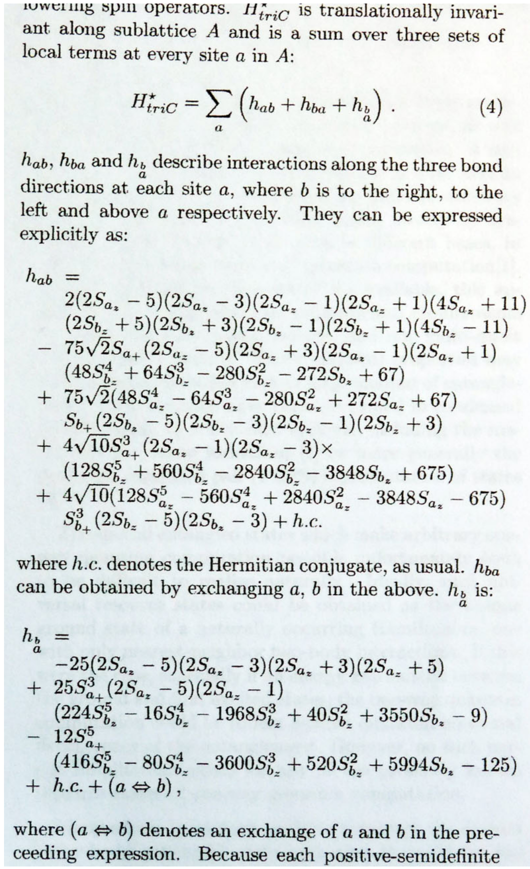
  Connections already made for 1D AKLT states:

  * Stability by symmetry [1],
  * Renormalization group [2],
  * Holographic principle [1].

[1] A. Miyake, Phys. Rev. Lett. 105, 040501 (2010).

[2] S.D. Bartlett et al., Phys. Rev. Lett. 105, 110502 (2010).

2D: A known universal 2-body ground state



lowering spin operators. $H^{\star}_{triC}$ is translationally invariant along sublattice A and is a sum over three sets of local terms at every site a in A:

$$H^{\star}_{triC} = \sum_{a} \left(h_{ab} + h_{ba} + h_{\substack{b\\a}} \right) . \qquad (4)$$

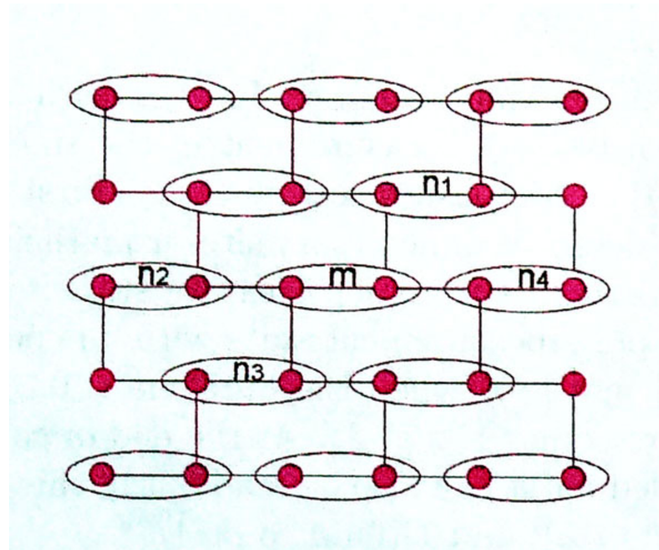
h_{ab}, h_{ba} and $h_{\substack{b\\a}}$ describe interactions along the three bond directions at each site a, where b is to the right, to the left and above a respectively. They can be expressed explicitly as:

$$\begin{aligned} h_{ab} \;=&\\ & 2(2S_{a_z}-5)(2S_{a_z}-3)(2S_{a_z}-1)(2S_{a_z}+1)(4S_{a_z}+11)\\ & (2S_{b_z}+5)(2S_{b_z}+3)(2S_{b_z}-1)(2S_{b_z}+1)(4S_{b_z}-11)\\ - & \; 75\sqrt{2}S_{a_+}(2S_{a_z}-5)(2S_{a_z}+3)(2S_{a_z}-1)(2S_{a_z}+1)\\ & (48S_{b_z}^4+64S_{b_z}^3-280S_{b_z}^2-272S_{b_z}+67)\\ + & \; 75\sqrt{2}(48S_{a_z}^4-64S_{a_z}^3-280S_{a_z}^2+272S_{a_z}+67)\\ & S_{b_+}(2S_{b_z}-5)(2S_{b_z}-3)(2S_{b_z}-1)(2S_{b_z}+3)\\ + & \; 4\sqrt{10}S_{a_+}^3(2S_{a_z}-1)(2S_{a_z}-3)\times\\ & (128S_{b_z}^5+560S_{b_z}^4-2840S_{b_z}^2-3848S_{b_z}+675)\\ + & \; 4\sqrt{10}(128S_{a_z}^5-560S_{a_z}^4+2840S_{a_z}^2-3848S_{a_z}-675)\\ & S_{b_+}^3(2S_{b_z}-5)(2S_{b_z}-3)+h.c. \end{aligned}$$

where $h.c.$ denotes the Hermitian conjugate, as usual. h_{ba} can be obtained by exchanging a, b in the above. $h_{\substack{b\\a}}$ is:

$$\begin{aligned} h_{\substack{b\\a}} \;=&\\ & -25(2S_{a_z}-5)(2S_{a_z}-3)(2S_{a_z}+3)(2S_{a_z}+5)\\ + & \; 25S_{a_+}^3(2S_{a_z}-5)(2S_{a_z}-1)\\ & (224S_{b_z}^5-16S_{b_z}^4-1968S_{b_z}^3+40S_{b_z}^2+3550S_{b_z}-9)\\ - & \; 12S_{a_+}^5\\ & (416S_{b_z}^5-80S_{b_z}^4-3600S_{b_z}^3+520S_{b_z}^2+5994S_{b_z}-125)\\ + & \; h.c.+(a\Leftrightarrow b)\,, \end{aligned}$$

where $(a \Leftrightarrow b)$ denotes an exchange of a and b in the preceeding expression. Because each positive-semidefinite

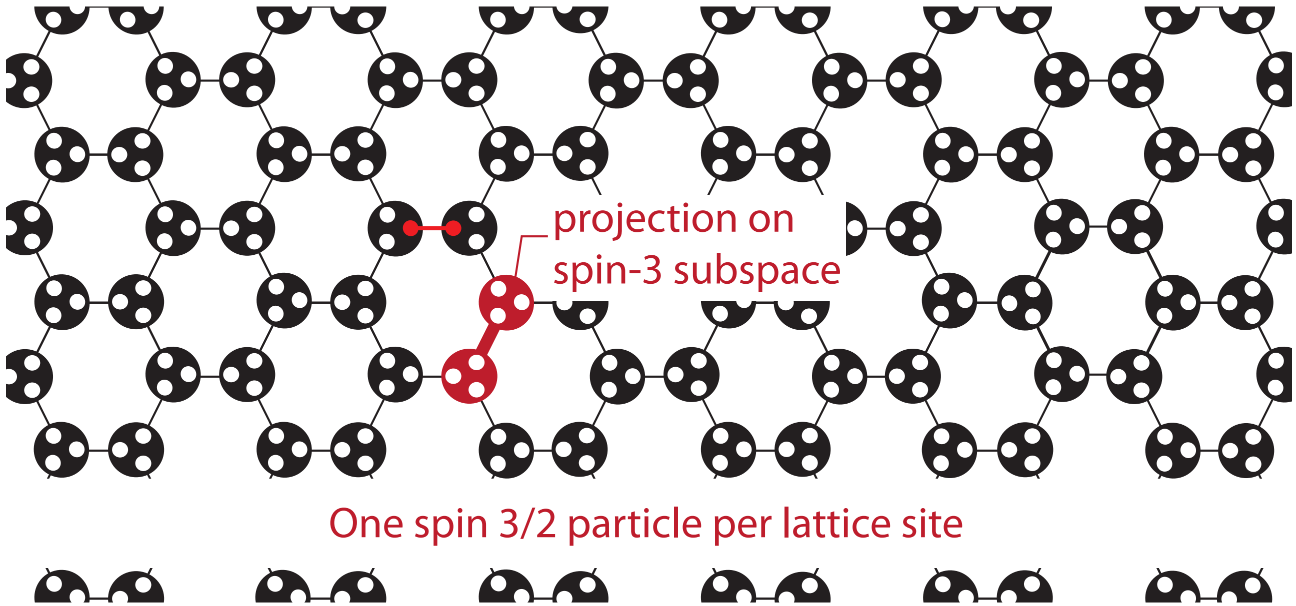


- Universal resource
- 2-body ground state
- Spin 5/2, honeycomb

Xie Chen et al., Phys. Rev. Lett. 102, 220501 (2009).

Also see [for different construction]: J. Cai *et al.*, arXiv:1004.1907

The AKLT state in two dimensions

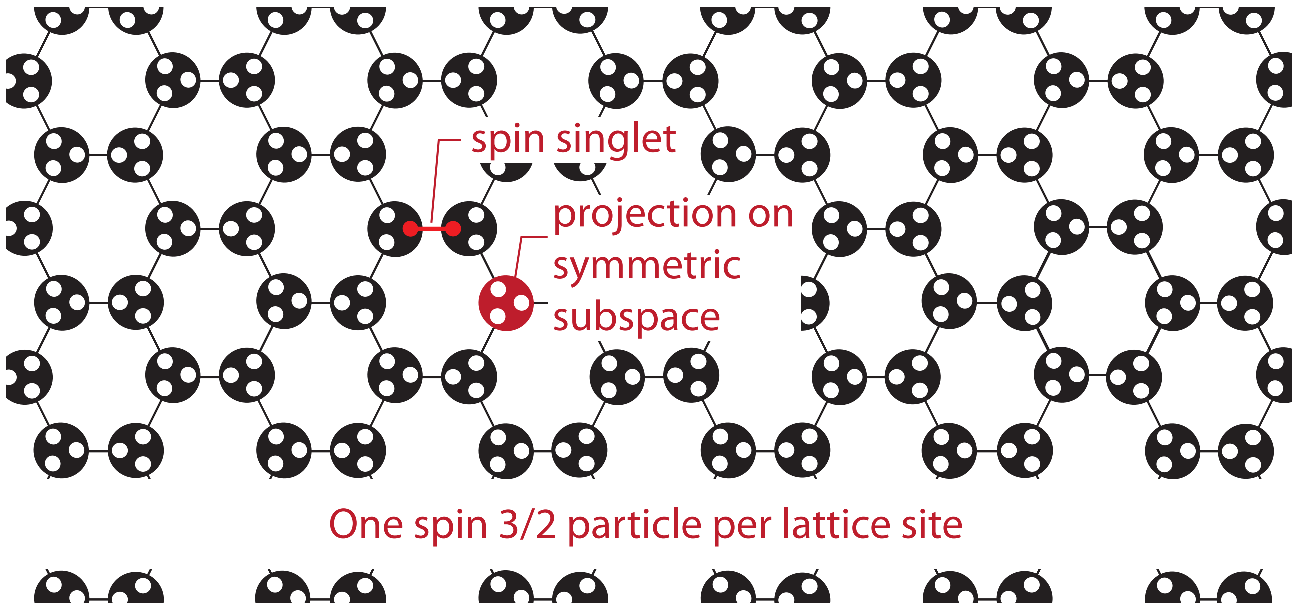


It is the ground state of the Hamiltonian

$$H = J \sum_{e \in E(\mathcal{L})} P_{3,e}. \quad (3)$$

$P_{3,e}$ is the projector on the spin-3 subspace of six spins 1/2.

The AKLT state in two dimensions

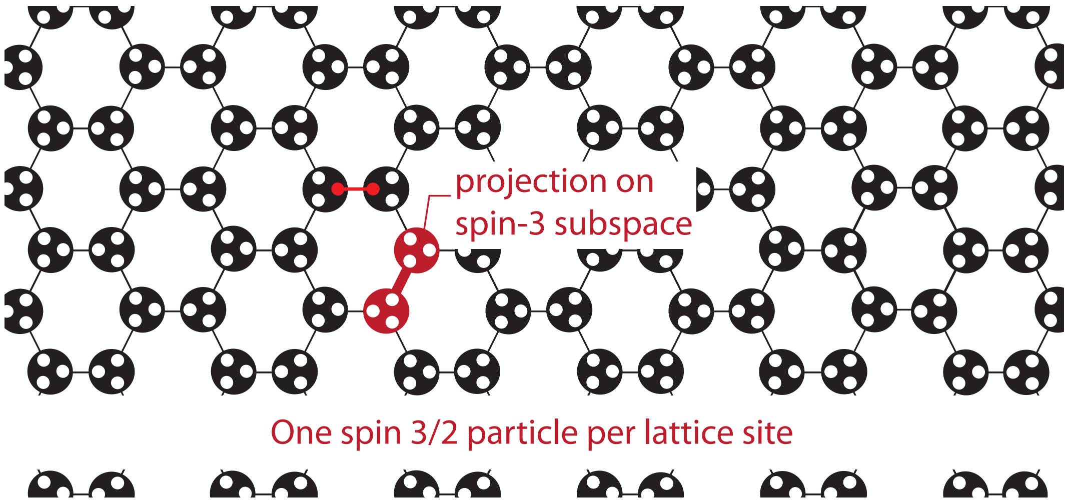


The AKLT-state on a honeycomb lattice $\mathcal{L}$ is

$$|\mathrm{AKLT}\rangle_{\mathcal{L}} = \left(\bigotimes_{a=V(\mathcal{L})} P_{3/2,a} \right) \bigotimes_{e \in E(\mathcal{L})} |s=0\rangle_e. \qquad (4)$$

$P_{3/2,a}$ is the projector on the spin-3/2 subspace of three spin 1/2.

Our result



The AKLT state on the honeycomb lattice is universal for quantum computation by local measurements.

PRL **107**, 070501 (2011)
Also see: A. Miyake, Ann. Phys. 326:165671 (2011)

Away from the AKLT point

Deformed AKLT-type Hamiltonian

$$H(a) = \sum_{\langle i,j \rangle} D(a)_i \otimes D(a)_j P_{3,(i,j)} D(a)_i \otimes D(a)_j,$$

where $D(a) = \text{diag}(\sqrt{3}/a, 1, 1, \sqrt{3}/a)$. (AKLT: $a = \sqrt{3}$)

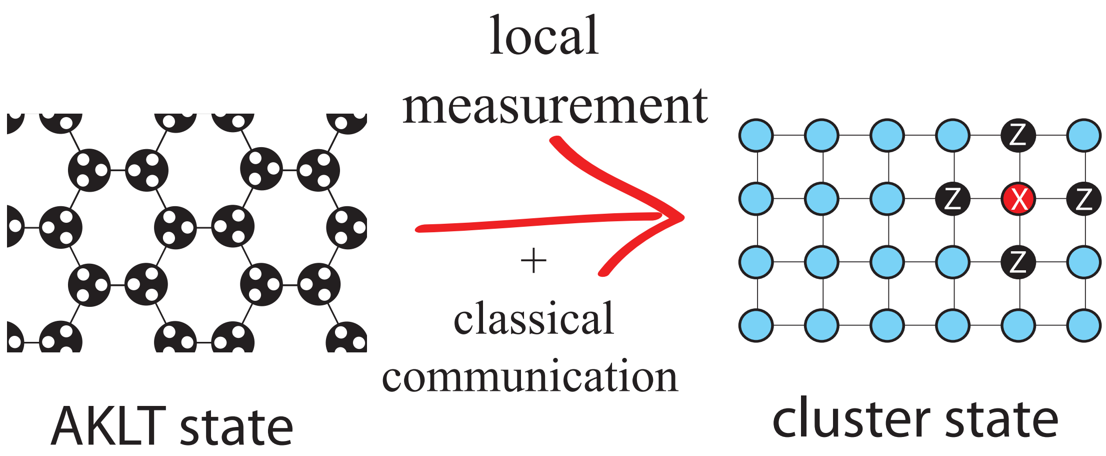
- Known: Phase transition to Neel order at $a = 6.46$ [Niggemann *et al.* '97]
- Phase transition in computational power from universal to non-universal at the same point!

[A. Darmawan *et al.*, New. J. Phys. **14**, 013023 (2012)]

Proof outline:

The 2D AKLT is universal for MBQC

Strategy – reduction to cluster state



Steps:

1. Devise a symmetry breaking measurement.
2. Map to percolation. Universality is a connectivity problem!
3. Monte-Carlo shows: System is in the percolating phase.

Step 1: Symmetry-breaking measurement

Apply to every lattice site a POVM with the three elements

$$
\begin{aligned}
F_z &= \sqrt{\frac{2}{3}}\frac{I_{12}+Z_1Z_2}{2}\frac{I_{23}+Z_2Z_3}{2} = \sqrt{\frac{2}{3}}(|000\rangle\langle 000| + |111\rangle\langle 111|),\\
F_x &= \sqrt{\frac{2}{3}}\frac{I_{12}+X_1X_2}{2}\frac{I_{23}+X_2X_3}{2} = \sqrt{\frac{2}{3}}(|+++\rangle\langle +++| + |---\rangle\langle ---|),\\
F_y &= \sqrt{\frac{2}{3}}\frac{I_{12}+Y_1Y_2}{2}\frac{I_{23}+Y_2Y_3}{2} = \sqrt{\frac{2}{3}}(|iii\rangle\langle iii| + |-i,-i,-i\rangle\langle -i,-i,-i|).
\end{aligned}
$$

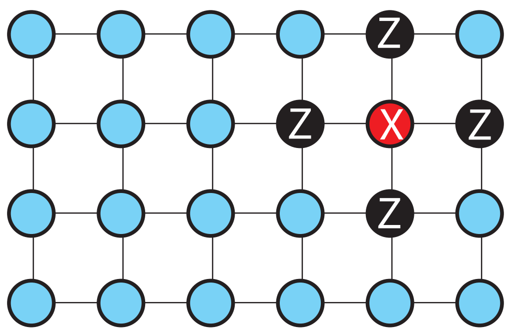
$\{1,2,3\}$: qubit locations at given lattice site, $|\pm\rangle = \frac{|0\rangle\pm|1\rangle}{\sqrt{2}}$, $|\pm i\rangle = \frac{|0\rangle\pm i|1\rangle}{\sqrt{2}}$.

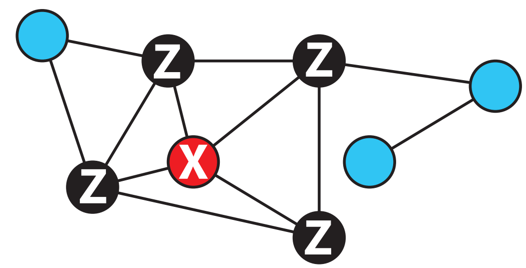
- This is indeed a POVM, $F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I_{\text{sym}}$.
- Post-POVM state: $|\Psi(\mathcal{A})\rangle = \bigotimes_{v\in V(\mathcal{L})} F_{v,a}\,|\text{AKLT}\rangle_{\mathcal{L}}$.

Claim: $|\Psi(\mathcal{A})\rangle$ is an encoded graph state.

What is that & what's the graph?

Cluster states and graph states





A graph state $|G\rangle$ corresponding to a graph G is the single common eigenstate of the stabilizer operators $\{K_a\}$,

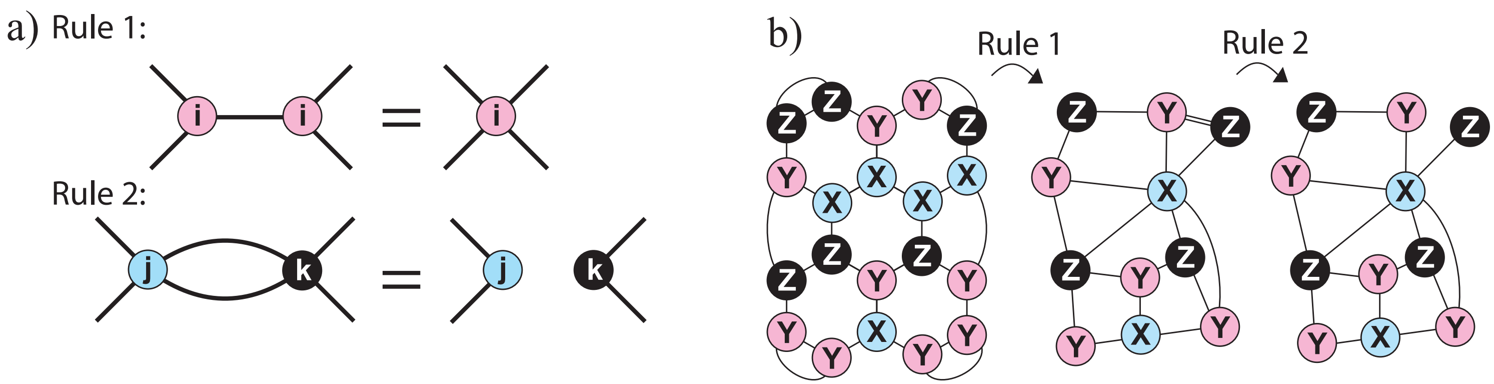
$$K_a|G\rangle = |G\rangle, \ \forall a \in V(G),$$

with

$$K_a = X_a \bigotimes_{b|(a,b)\in E(G)} Z_b, \quad \forall a \in V(G). \tag{5}$$

Therein, $E(G)$ is the set of edges of G.

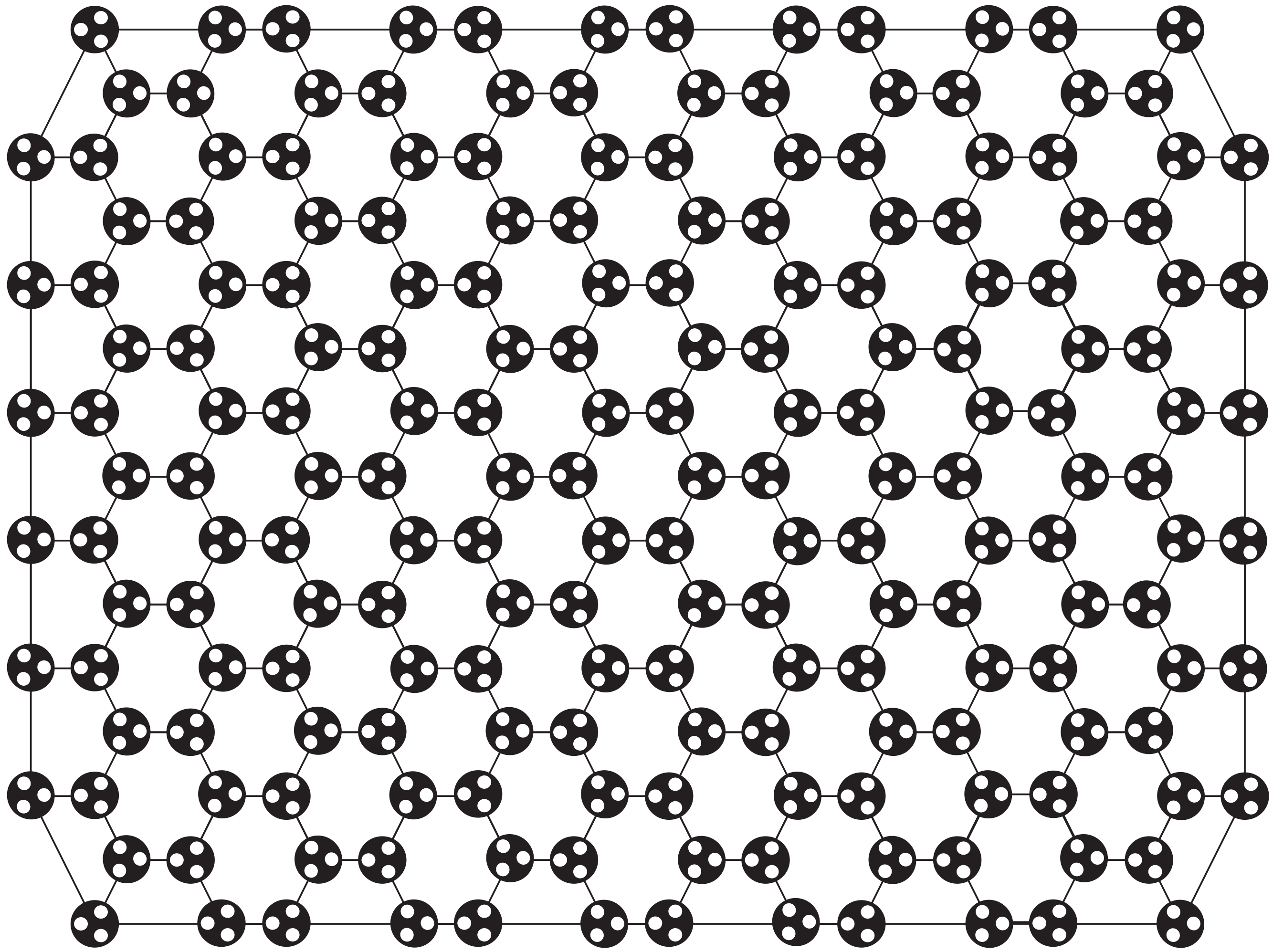
Step2: Rules for mapping to percolating graph



R1 (Edge contraction): Contract all edges $e \in E(\mathcal{L})$ that connect sites with the same POVM outcome.

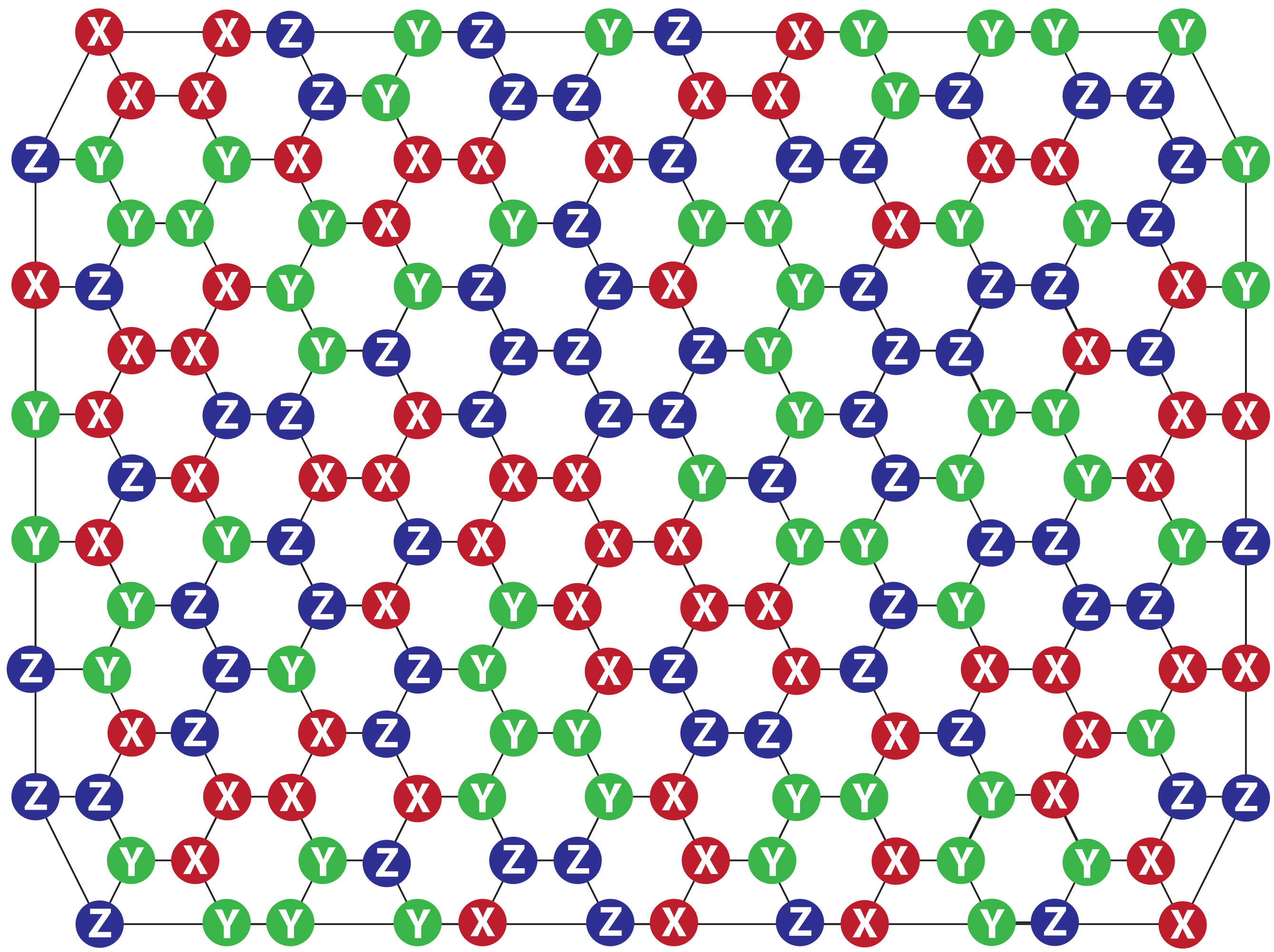
R2 (Mod-2 edge deletion): In the resultant multi-graph, delete all edges of even multiplicity and convert all edges of odd multiplicity into conventional edges of multiplicity 1.

Illustration of the graph rules



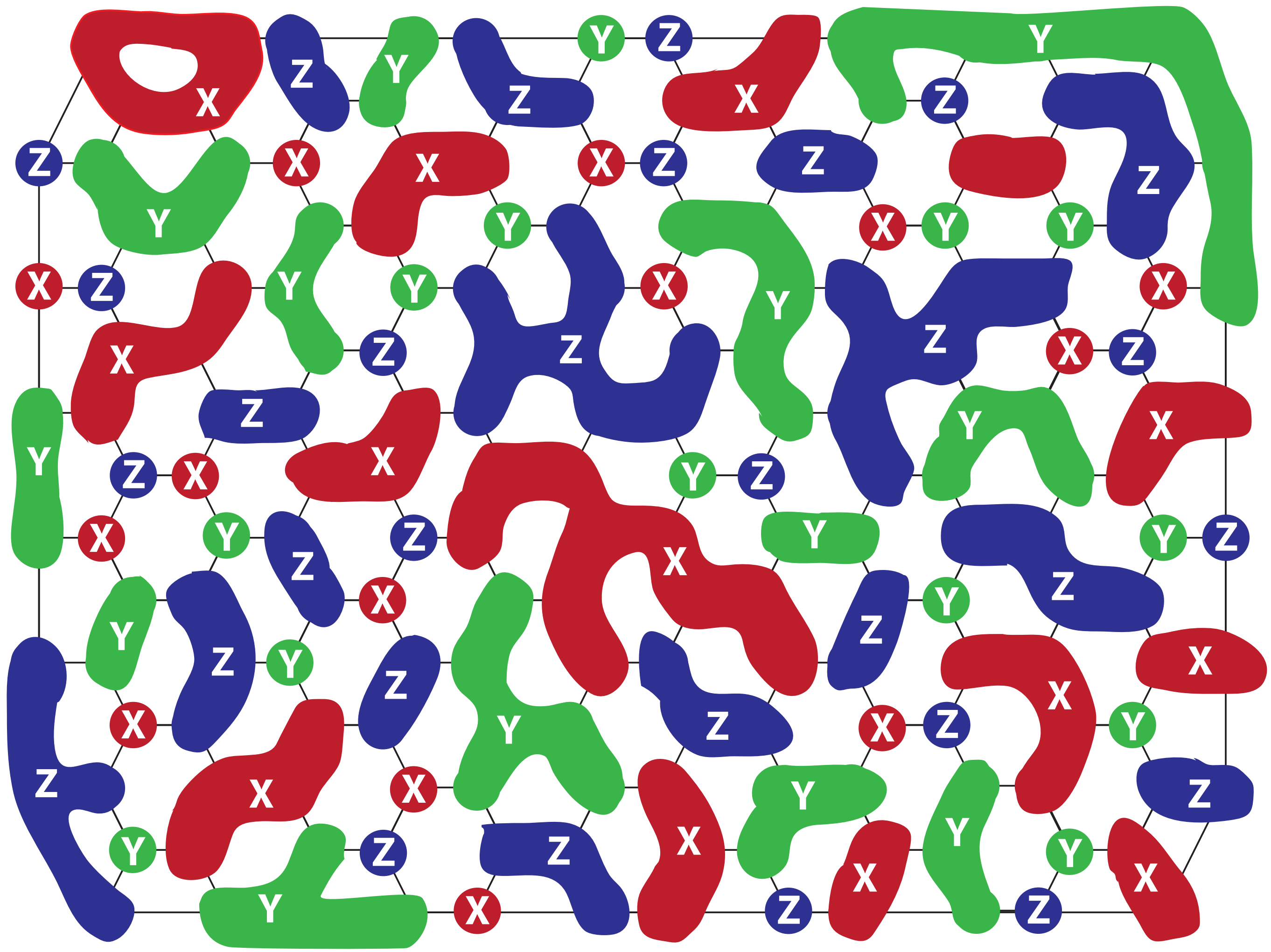
Take an AKLT state on a honeycomb lattice ...

Illustration of the graph rules



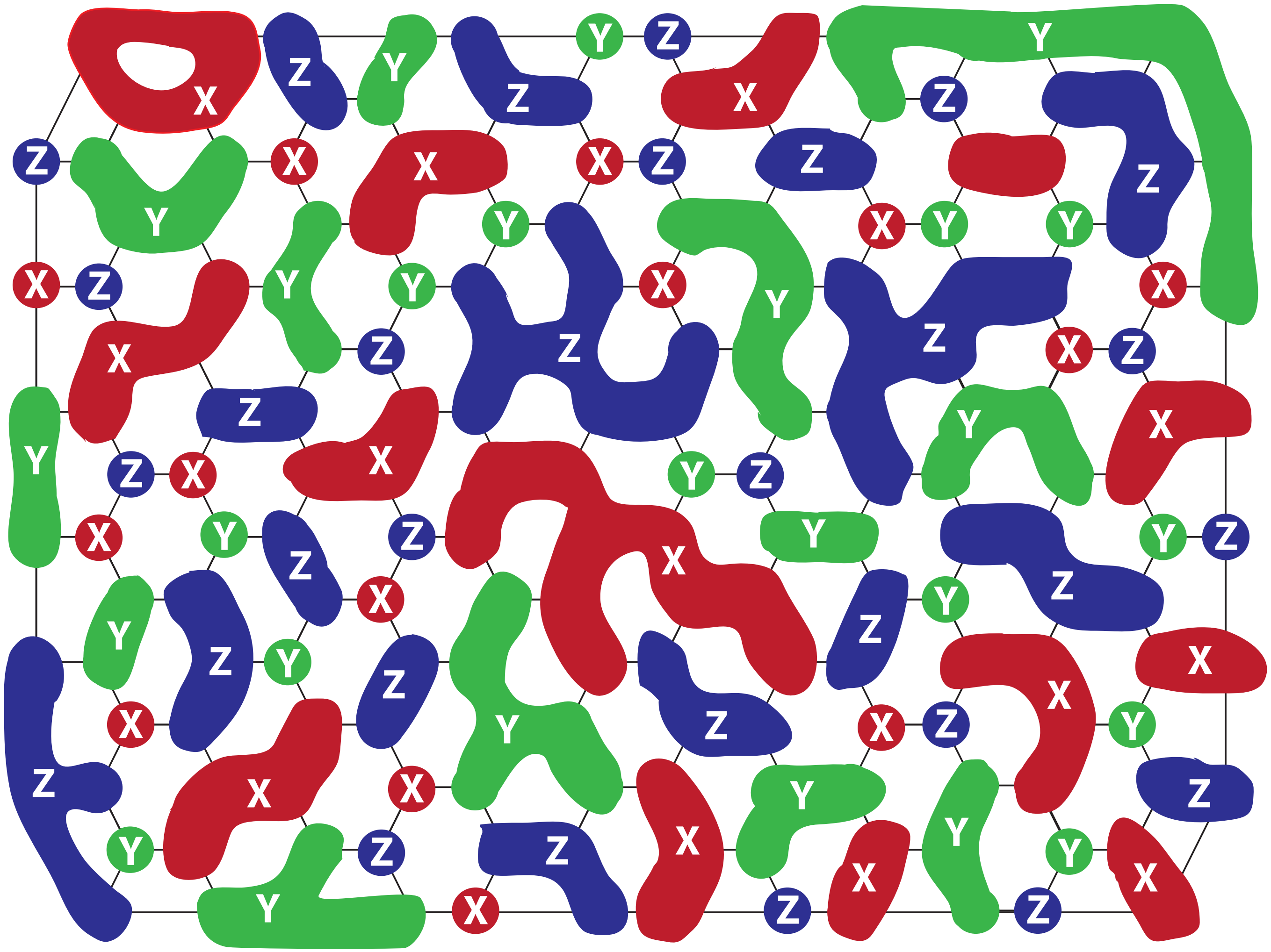
... and apply the 3-outcome POVM (5).

Illustration of the graph rules



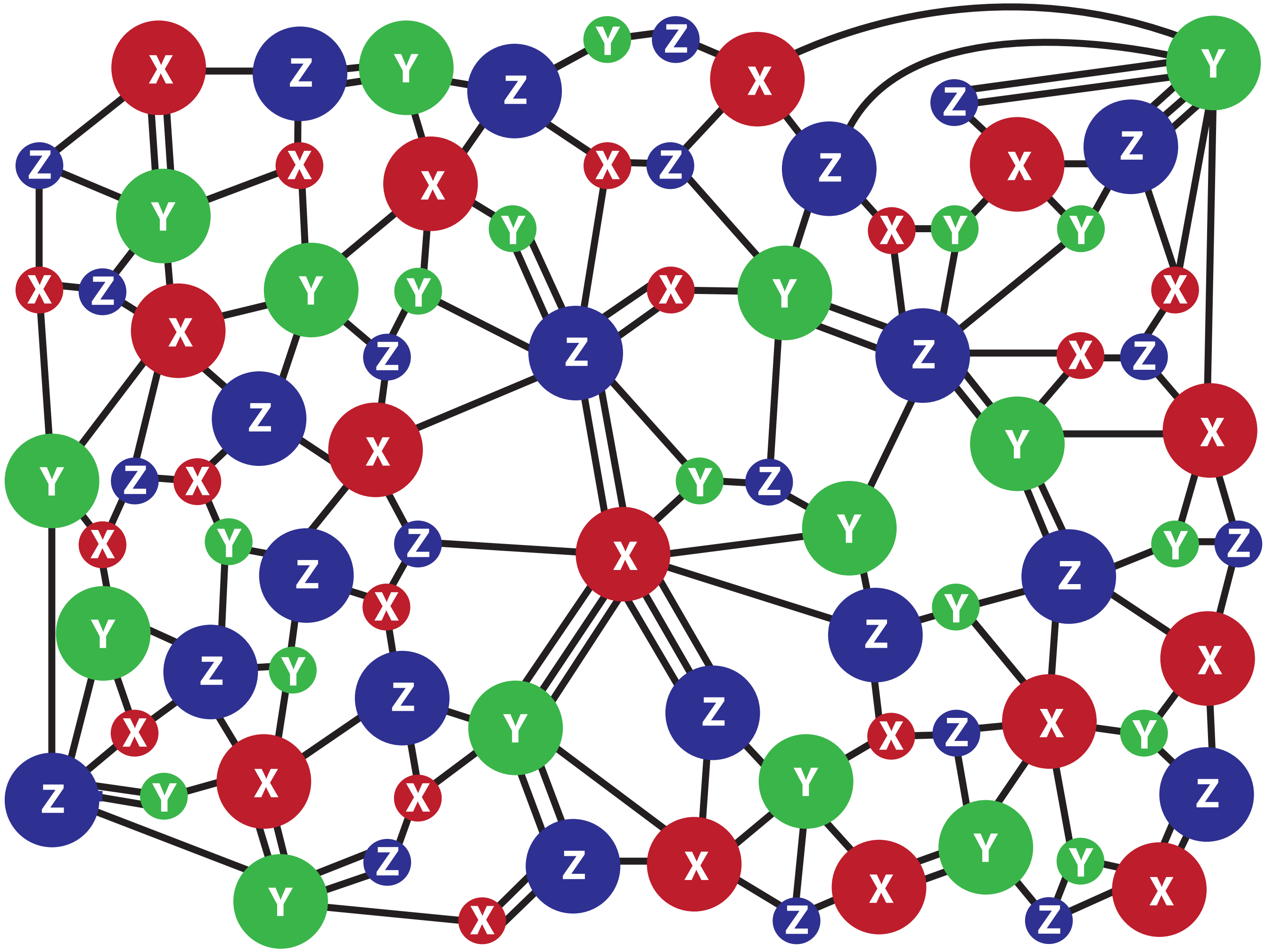
Neighboring sites with same outcome form a *domain*.

Illustration of the graph rules



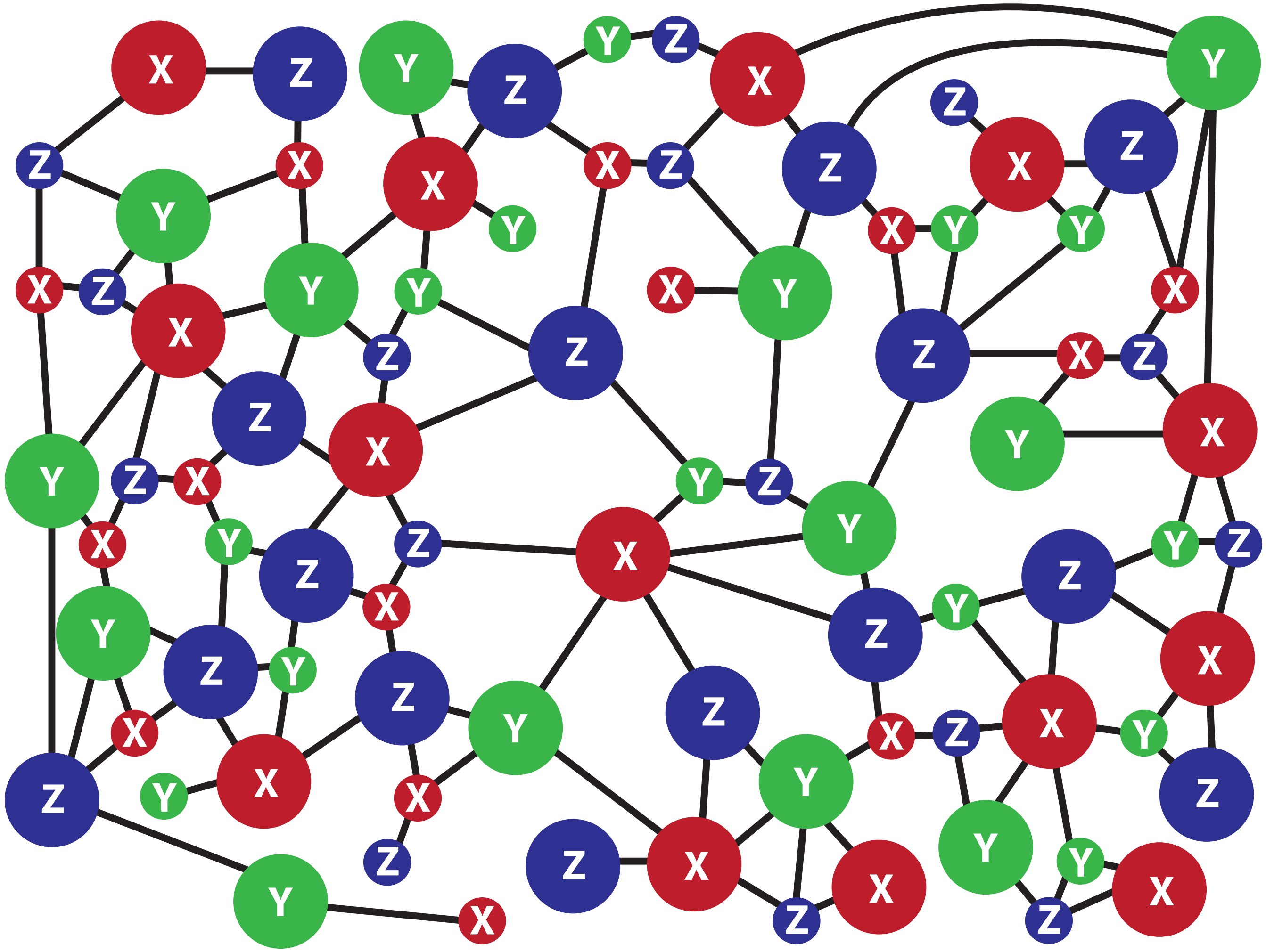
R1: Contract the domains. One encoded qubit per domain.

Illustration of the graph rules



R2: Reduce multiple edges. Odd - keep, even - remove.

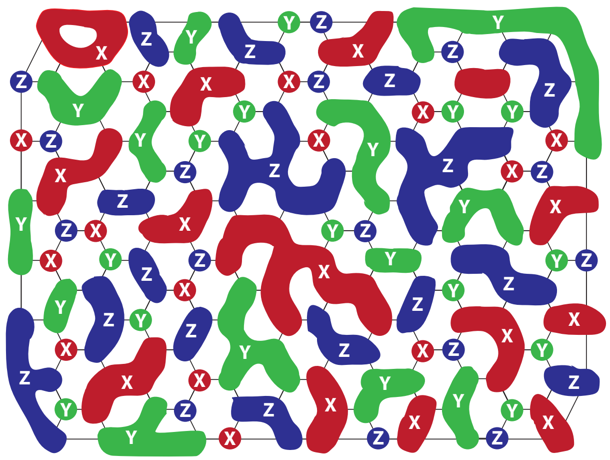
Illustration of the graph rules



Resulting graph defines an encoded graph state.

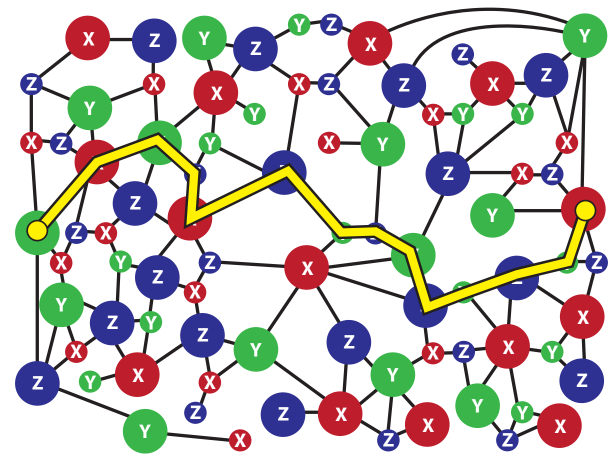
The percolation problem

- Require two properties of typical resulting graphs:



Domains are microscopic.

⇒ **Percolation**

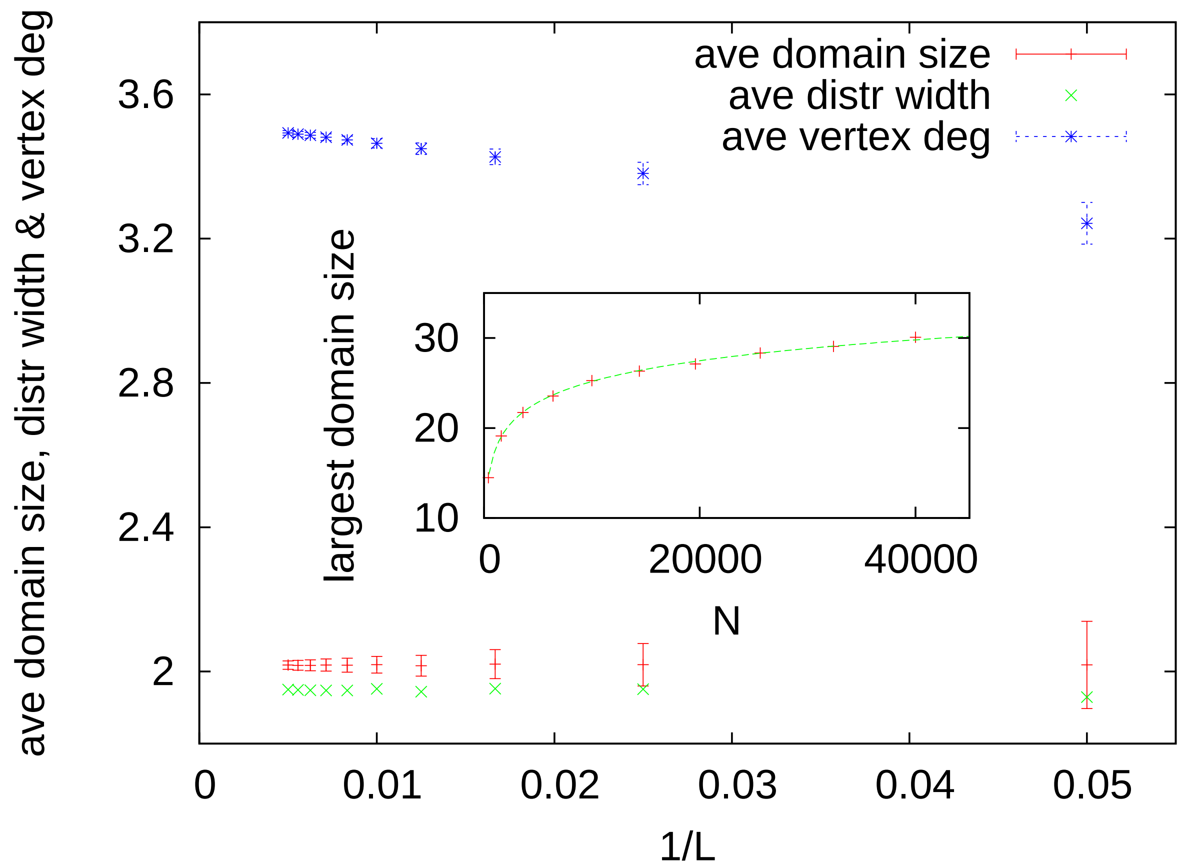


Left-right path always exists.

⇒ **Supercritical phase**

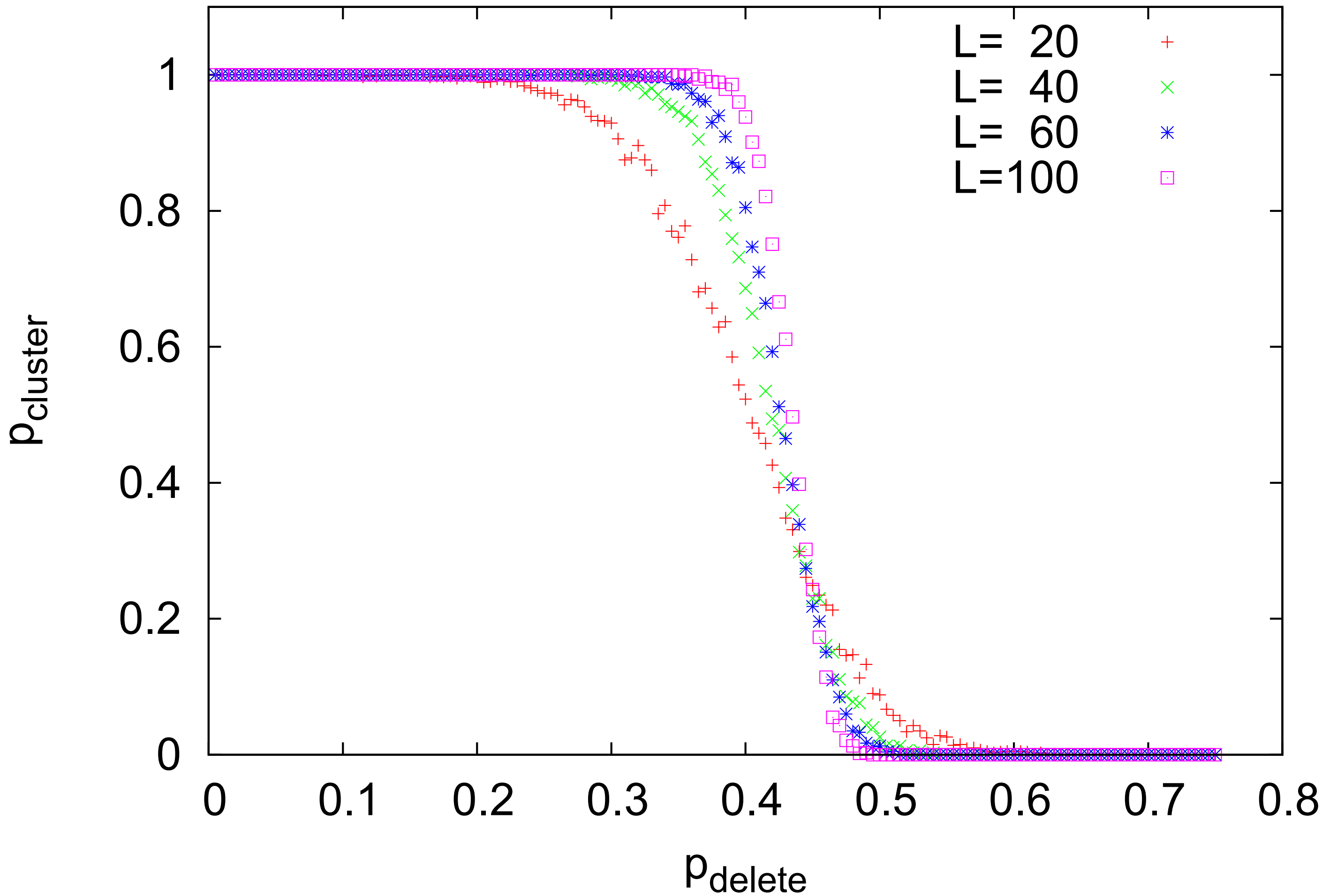
- To show:
  * If 2 properties obeyed then reduction to 2D cluster possible.
  * Typical resulting graph states have these properties.

Step 3: Monte Carlo



domains are small • average vertex degree $d = 3.5 \gg 2$

Step 3: Monte Carlo



Traversing path always exists • stable against edge deletion

Conclusion

- We have shown that the AKLT state on a honeycomb lattice is a universal resource for measurement-based QC.

- The AKLT state is the ground state of a simple, highly symmetric Hamiltonian with only 2-body interactions.

- Universality extends beyond the AKLT point.

What does condensed matter physics have to say about computational universality?