Entanglement entropy: hints from the two intervals case

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based on

P. Calabrese, J. Cardy and E.T.; [0905.2069] (JSTAT)

P. Calabrese, J. Cardy and E.T.; [1011.5482] (JSTAT)

E.T.; [1011.0166] (JHEP)

A. Allais and E.T.; [1110.1607] (JHEP)

GGI, Florence, April 2012

Plan of the talk

- Introduction (definitions and replica trick)
- Twist fields
- Entanglement entropy of one interval in CFT
- $N = 2$ intervals:
  - $\mathrm{Tr}\rho_A^n$ for free compactified boson (Luttinger liquid) and Ising model
  - checks against existing numerical data
- Short intervals expansion for N intervals for a CFT in 2 dim.
  - small x expansion (free compactified boson and Ising model)
- Holographic entanglement entropy
  - Mutual information in Vaidya spaces (black hole formation)
  - Strong subadditivity and null energy condition
- Conclusions and open problems

Entanglement entropy: definition

- Quantum system ($\mathcal{H}$) in the ground state $|\Psi\rangle$
  Density matrix $\rho = |\Psi\rangle\langle\Psi| \quad \Longrightarrow \quad \mathrm{Tr}\rho^n = 1$

- Two observers: each one measures only a subset of a complete set of cummuting observables $\quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- A's reduced density matrix $\boxed{\rho_A = \mathrm{Tr}_B \rho}$

- Entanglement entropy $\equiv$ Von Neumann entropy of ρ_A

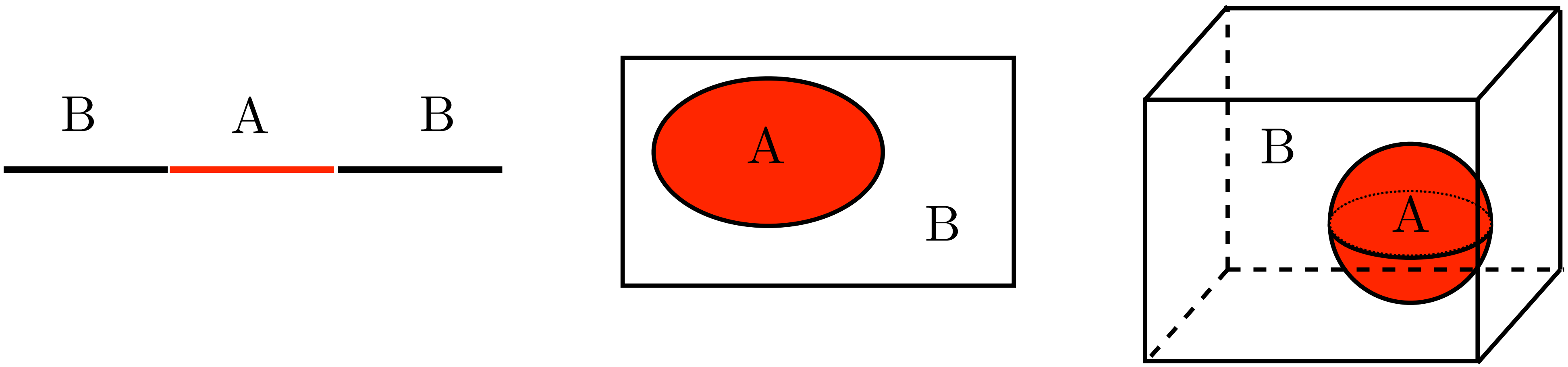
$$\boxed{S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A)}$$

  It measures the amount of information shared by A and B

- $S_A = S_B$ if ρ corresponds to a pure state $\quad \Longrightarrow \quad S_A$ is not extensive

Geometric entropy: area law

- Assume that A and B correspond to a spatial bipartition of the system



- In d spatial dimensions

$$S_A \propto \frac{\text{Area}(\partial A)}{a^{d-1}}$$

[Bombelli, Koul, Lee, Sorkin, PRD (1986)]
[Srednicki, PRL (1993)]

In $1+1$ dimensional CFT at $T=0$

$$S_A = \frac{c}{3} \log \frac{\ell}{a}$$

[Holzey, Larsen, Wilczek NPB (1994)]

Replica trick

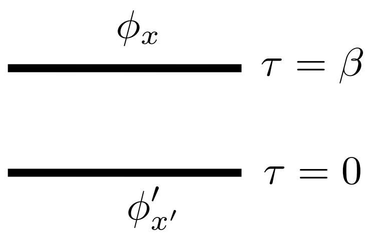
$$\boxed{S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\, \rho_A^n}$$

[Holzhey, Larsen, Wilczek, NPB (1994)]
[Calabrese, Cardy, JSTAT (2004)]

- Renyi entropies

$$\boxed{S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n}$$

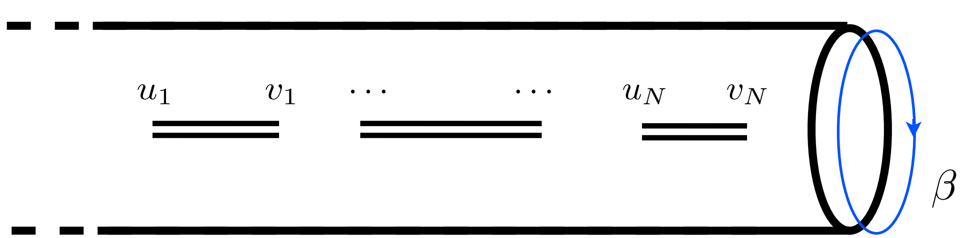
- Density matrix ρ in a thermal state at temperature $T = 1/\beta$



$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) = Z^{-1} \int [d\phi(y,\tau)] \prod_{x'} \delta(\phi(y,0) - \phi'_{x'}) \prod_{x} \delta(\phi(y,\beta) - \phi_x)\, e^{-S_E}$$

$Z = \mathrm{Tr}\, e^{-\beta H}$. The trace sews together the edges at $\tau = 0$ and $\tau = \beta$ providing a cylinder with circumference of length β.

- $\rho_A = \mathrm{Tr}_B \rho$



$A = (u_1, v_1) \cup \cdots \cup (u_N, v_N)$

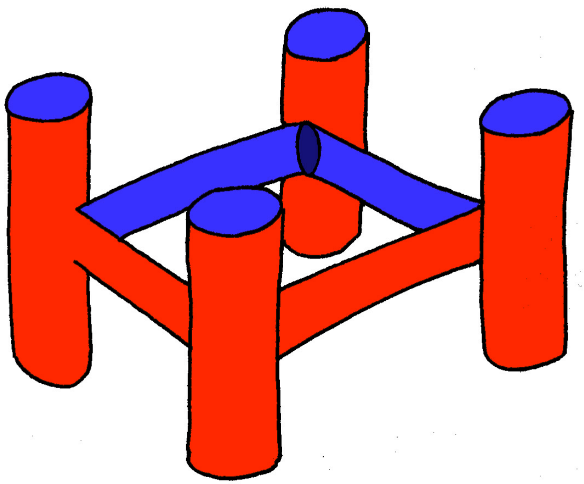
The trace Tr_B sews together only the points $\notin A$.
Open cuts are left along the disjoint intervals (u_j, v_j).

Replica trick and Riemann surfaces

- n copies of the cylinder above
  sewed together cyclically along the cuts

$$\boxed{S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}}$$

$$\text{“}\, \rho_A^{ij}\, \rho_A^{jk}\, \rho_A^{kl}\, \rho_A^{li} \,\text{”} \; =$$



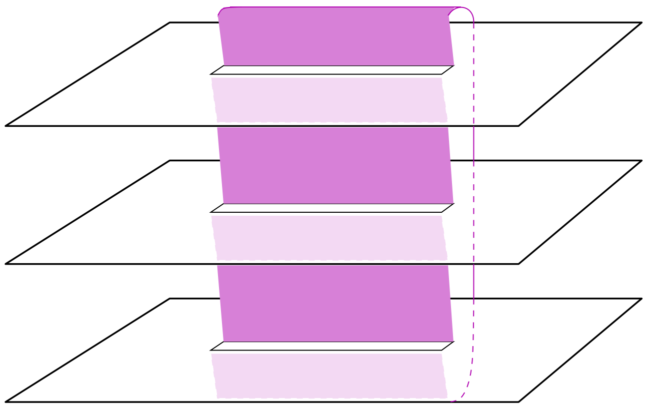
- $\mathrm{Tr}\, \rho_A^n$ as a partition function
  on the n sheeted Riemann surface $\mathcal{R}_{n,N}$

$$Z_{\mathcal{R}_{n,N}} = \int_{\mathcal{C}_{u_j,v_j}} [d\varphi_1 \cdots d\varphi_n]_{\mathbf{C}} \exp\left[-\int_{\mathbf{C}} dz d\bar{z} \left(\mathcal{L}[\varphi_1](z,\bar{z}) + \ldots + \mathcal{L}[\varphi_n](z,\bar{z}) \right) \right]$$

$$\mathcal{C}_{u_j,v_j} : \quad \varphi_i(x, 0^+) = \varphi_{i+1}(x, 0^-)$$

$$x \in \bigcup_{j=1}^{N} [u_j, v_j] \qquad\qquad i = 1, \ldots, n$$

$\mathcal{R}_{3,1}$



[Cardy, Castro-Alvaredo, Doyon, JSP (2007)]

Twist fields

- Global symmetry

$$\sigma \; : \; i \mapsto i+1 \bmod n \qquad \sigma^{-1} \; : \; i+1 \mapsto i \bmod n$$

$$\int dxdy\, \mathcal{L}[\sigma\varphi](x,y) = \int dxdy\, \mathcal{L}[\varphi](x,y)$$

- The twist fields implement this global symmetry

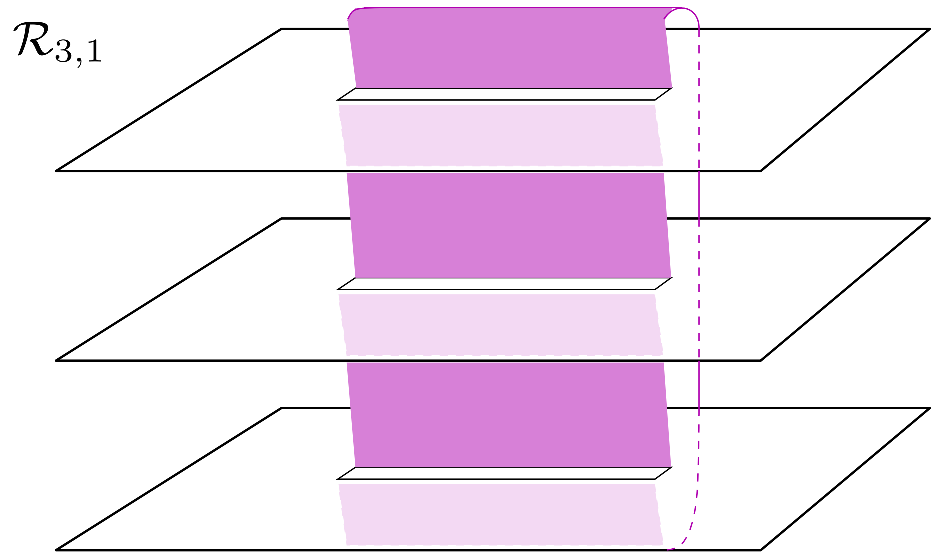
$$\mathcal{T}_n \equiv \mathcal{T}_\sigma \qquad \tilde{\mathcal{T}}_n \equiv \mathcal{T}_{\sigma^{-1}}$$

$$Z_{\mathcal{R}_{n,N}} = \langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\cdots\mathcal{T}_n(u_N,0)\tilde{\mathcal{T}}_n(v_N,0)\rangle_{\mathcal{L}^{(n)},\mathbf{C}}$$

$$\mathcal{T}_n = \prod_{k=0}^{n-1} \mathcal{T}_{n,k} \qquad \tilde{\mathcal{T}}_n = \prod_{k=0}^{n-1} \tilde{\mathcal{T}}_{n,k}$$

$$Z_{\mathcal{R}_{n,N}} = \prod_{k=0}^{n-1} \langle \mathcal{T}_{n,k}(u_1,0)\tilde{\mathcal{T}}_{n,k}(v_1,0)\cdots\mathcal{T}_{n,k}(u_N,0)\tilde{\mathcal{T}}_{n,k}(v_N,0)\rangle_{\mathcal{L}^{(n)},\mathbf{C}}$$

Boundary conditions and twist fields



- Boundary conditions:

$$\varphi_j(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \varphi_{j-1}(z, \bar{z})$$

- Linear combinations of basic fields which diagonalize the twist

$$\boxed{\tilde{\varphi}_k \equiv \sum_{j=1}^{n} e^{2\pi i \frac{k}{n} j} \varphi_j} \qquad k = 0, 1, \dots, n-1$$

$$\tilde{\varphi}_k(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = e^{2\pi i \frac{k}{n}} \tilde{\varphi}_k(z, \bar{z}) = \theta_k \tilde{\varphi}_k(z, \bar{z}) \qquad \theta_k \equiv e^{2\pi i \frac{k}{n}}$$

- Branch-point twist field $\mathcal{T}_{n,k}$ in the origin

[Dixon, Friedan, Martinec, Shenker, NPB (1987)] [Zamolodchikov, NPB (1987)]

Entanglement of a single interval

- Two-point function of twist fields for a free complex boson φ

[Dixon, Friedan, Martinec, Shenker, NPB (1987)]

$$\langle \mathcal{T}_{k,n}(u)\tilde{\mathcal{T}}_{k,n}(v)\rangle \propto \frac{1}{|u-v|^{4\Delta_{k/n}}}$$

$$\boxed{\Delta_{\frac{k}{n}} = \bar{\Delta}_{\frac{k}{n}} = \frac{1}{2}\,\frac{k}{n}\left(1-\frac{k}{n}\right)}$$

- Partition function on $\mathcal{R}_{n,1}$

[Calabrese, Cardy, JSTAT (2004)]

[Ryu, Takayanagi JHEP (2006)]

$$Z_{\mathcal{R}_{n,1}} = \prod_{k=0}^{n-1} Z_{k,n} = \prod_{k=0}^{n-1} \langle \mathcal{T}_{k,n}(u)\tilde{\mathcal{T}}_{k,n}(v)\rangle = \frac{c_n}{|u-v|^{\frac{1}{3}\left(n-\frac{1}{n}\right)}}$$

Entanglement entropy of a single interval for the free real boson

$c = 1$

$$\boxed{S_A = -\,\partial_n \mathrm{Tr}\rho_A^n\Big|_{n=1} = \frac{1}{3}\log\frac{\ell}{a} + c_1'}$$

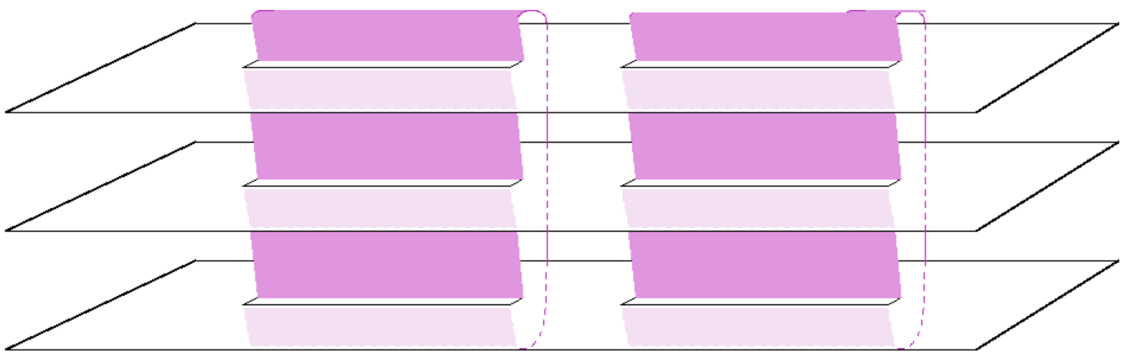
[Holzhey, Larsen, Wilczek, NPB (1994)]

Entanglement of two disjoint intervals

[Calabrese, Cardy and E.T.; JSTAT (2009)]

- $A = A_1 \cup A_2 = [u_1, v_1] \cup [u_2, v_2] \quad \Longrightarrow \quad \mathcal{R}_{n,2}$

e.g.: $\mathcal{R}_{3,2}$



- Four-point function of twist fields for a free, real, compactified boson φ

$$\mathrm{Tr}\rho_A^n \equiv Z_{\mathcal{R}_{n,2}} = \underbrace{c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)}}_{Z^W_{\mathcal{R}_{n,2}}} \mathcal{F}_n(x)$$

$$x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)}$$

Computation

- Compactification condition

$$\varphi_j(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \varphi_{j-1}(z,\bar{z}) + R(m_{j,1} + i m_{j,2}) \qquad m_j \in \mathbf{Z} + i\mathbf{Z}$$

$$\tilde{\varphi}_k(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \theta_k \tilde{\varphi}_k(z,\bar{z}) + \boxed{R\sum_{j=1}^{n} \theta_k^j m_j} \longrightarrow \xi \in R\Lambda_{\frac{k}{n}} \qquad \theta_k \equiv e^{2\pi i \frac{k}{n}}$$

- Partition function on $\mathcal{R}_{n,2}$ from the four-point function of twist fields

$$\boxed{Z_{\mathcal{R}_{n,2}} = \sum_{m\in\mathbf{Z}^{2n}} \prod_{k=0}^{n-1} Z^{\text{qu}}_{k,n} Z^{\text{cl}}_{k,n}} \longrightarrow$$ [Dixon, Friedan, Martinec, Shenker, NPB (1987)]

$$\mathcal{F}_n(x) = \sum_{m\,\in\,\mathbf{Z}^{2n}} \prod_{k\,=\,0}^{n-1} \frac{\text{const}}{\beta_{k/n}\left[F_{k/n}(x)\right]^2} \exp\left\{-\frac{2g\pi\sin\left(\pi\frac{k}{n}\right)}{n}\left[|\xi_1|^2\beta_{k/n} + \frac{|\xi_2|^2}{\beta_{k/n}}\right]\right\}$$

$$\beta_y \equiv \frac{F_y(1-x)}{F_y(x)} \qquad F_y(x) \equiv {}_2F_1(y, 1-y; 1; x)$$

- Z^{cl} does not contribute in the decompactification limit

Main result (I)

- Regularize the sum (non trivial step!)

- Riemann-Siegel theta function

Γ is a $G \times G$ symmetric matrix with positive imaginary part

$$\Theta(z|\Gamma) \equiv \sum_{m \in \mathbf{Z}^G} \exp\left[\, i\pi\, m^{\mathrm{t}} \cdot \Gamma \cdot m + 2\pi i\, m^{\mathrm{t}} \cdot z \,\right] \qquad z \in \mathbb{C}^G$$

- Final result

$$\beta_y \equiv \frac{F_y(1-x)}{F_y(x)} \qquad F_y(x) \equiv {}_2F_1(y, 1-y; 1; x)$$

$$\boxed{\eta \equiv g R^2}$$

$$\boxed{\mathcal{F}_n(x) = \text{const} \frac{\left[\Theta\left(0|\eta\Gamma\right)\Theta\left(0|\eta\widetilde{\Gamma}\right)\right]^2}{\prod_{k=1}^{n-1} F_{k/n}(x) F_{k/n}(1-x)}}$$

$$\Gamma_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \beta_{k/n} \cos\left[2\pi \frac{k}{n}(r-s)\right] \qquad \widetilde{\Gamma}_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \frac{1}{\beta_{k/n}} \cos\left[2\pi \frac{k}{n}(r-s)\right]$$

$$r, s = 1, \dots, n-1$$

Main result (II)

$$\boxed{\Gamma_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \beta_{k/n} \cos\left[2\pi \frac{k}{n}(r-s)\right]}$$

$$\widetilde{\Gamma}_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \frac{1}{\beta_{k/n}} \cos\left[2\pi \frac{k}{n}(r-s)\right]$$

$r, s = 1, \ldots, n-1$

$$\beta_y \equiv \frac{{}_2F_1(y, 1-y; 1; 1-x)}{{}_2F_1(y, 1-y; 1; x)}$$

$$\boxed{\eta \equiv gR^2}$$

$$\mathcal{F}_n(x) = \text{const} \frac{[\Theta(0|\eta\Gamma)\,\Theta(0|\eta\widetilde{\Gamma})]^2}{\prod_{k=1}^{n-1} F_{k/n}(x) F_{k/n}(1-x)}$$

nasty
n dependence

→ $\mathcal{F}_n(x)$ is invariant under $x \leftrightarrow 1-x$ $\quad (S_A = S_B)$

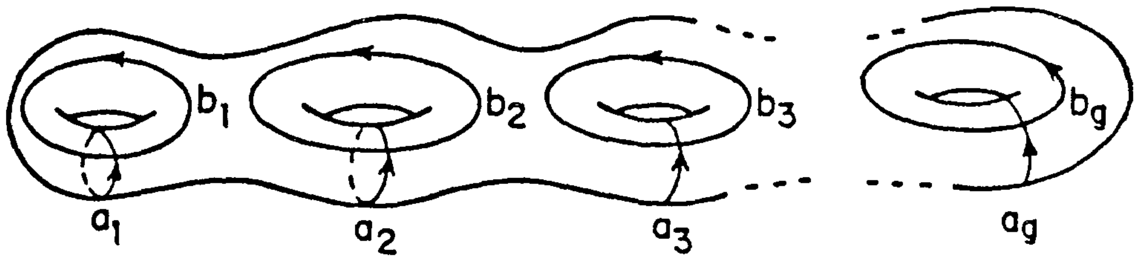
- Fix the constant s.t. $\mathcal{F}_n(0) = 1$
- Riemann-Siegel theta function manipulations
- Final result

$$\boxed{\mathcal{F}_n(x) = \left[\frac{\Theta(0|\eta\Gamma)\,\Theta(0|\Gamma/\eta)}{\Theta(0|\Gamma)^2}\right]^2}$$

complex
boson

→ $\mathcal{F}_n(x)$ is invariant under $\eta \leftrightarrow 1/\eta$

Higher genus Riemann surfaces



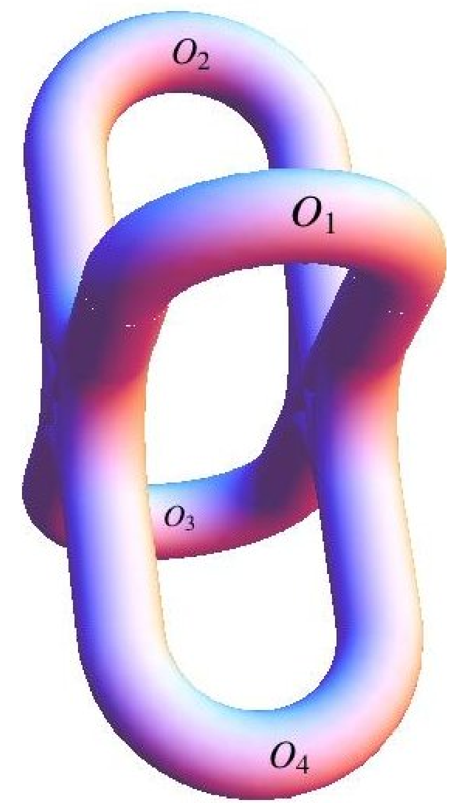
$$\oint_{a_i} \omega_j = \delta_{ij} \qquad \oint_{b_i} \omega_j = \tau_{ij}$$

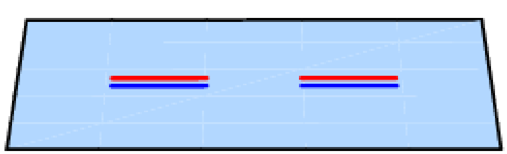
$$Z \;=\; Z^{\mathrm{qu}} Z^{\mathrm{cl}}$$

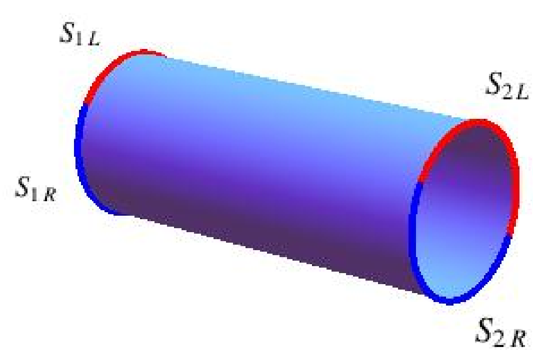
- $$Z^{\mathrm{cl}}(R) \;=\; \sum_{(p,\bar{p})\,\in\, G_R^g} \exp\Big[\, i\pi(p\cdot\tau\cdot p - \bar{p}\cdot\bar{\tau}\cdot\bar{p})\,\Big]$$

  $$G_R \;=\; \left\{ (p,\bar{p}) = \left(\frac{n}{R}+\frac{mR}{2}\,,\,\frac{n}{R}-\frac{mR}{2}\right);\, n,m\in\mathbb{Z} \right\}$$

- Γ is the period matrix of the
  n sheeted Riemann surface $(g = n-1)$
  (from replication)







decompactification regime: analytic continuation

- different compactification radii

$$\mathcal{F}_n(x) = \left[\frac{\Theta(0|\eta_1\Gamma)\,\Theta(0|\Gamma/\eta_1)}{\Theta(0|\Gamma)^2}\right]\left[\frac{\Theta(0|\eta_2\Gamma)\,\Theta(0|\Gamma/\eta_2)}{\Theta(0|\Gamma)^2}\right]$$

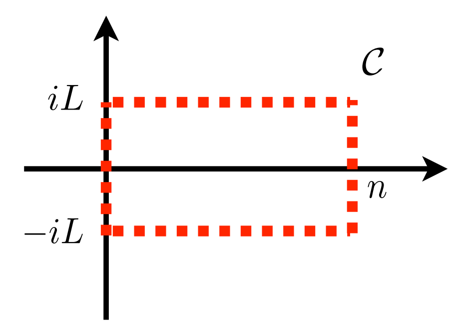
- decompactification regime: large η (recall the symmetry $\eta \leftrightarrow 1/\eta$)

$$\mathcal{F}_n(x) = \frac{\eta^{n-1}}{\prod_{k=1}^{n-1} F_{k/n}(x) F_{k/n}(1-x)}$$

- Useful representation:

$$D_n(x) = \sum_{k=1}^{n-1} \log F_{k/n}(x) = \int_{\mathcal{C}} \frac{dz}{2\pi i}\pi \cot(\pi z) \log F_{z/n}(x)$$

$$D_1'(x) \equiv -\left.\frac{\partial D_n(x)}{\partial n}\right|_{n=1} = \int_{-i\infty}^{i\infty} \frac{dz}{i}\,\frac{\pi z}{\sin^2 \pi z} \log F_z(x)$$



- **Mutual information**

$$I_{A_1:A_2} \equiv S_{A_1} + S_{A_2} - S_{A_1\cup A_2}$$

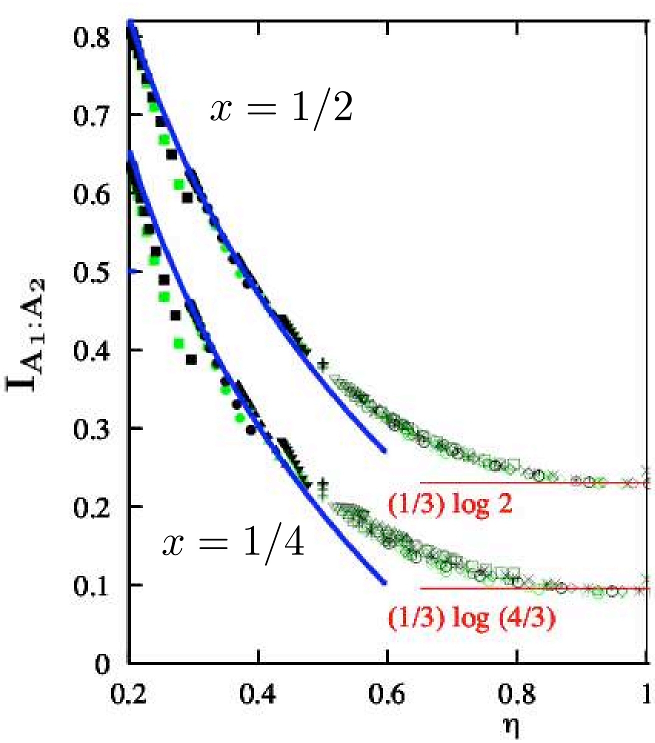
$$I_{A_1:A_2}(\eta \ll 1) - I^W_{A_1:A_2} \simeq -\frac{1}{2}\ln \eta + \frac{D_1'(x) + D_1'(1-x)}{2}$$

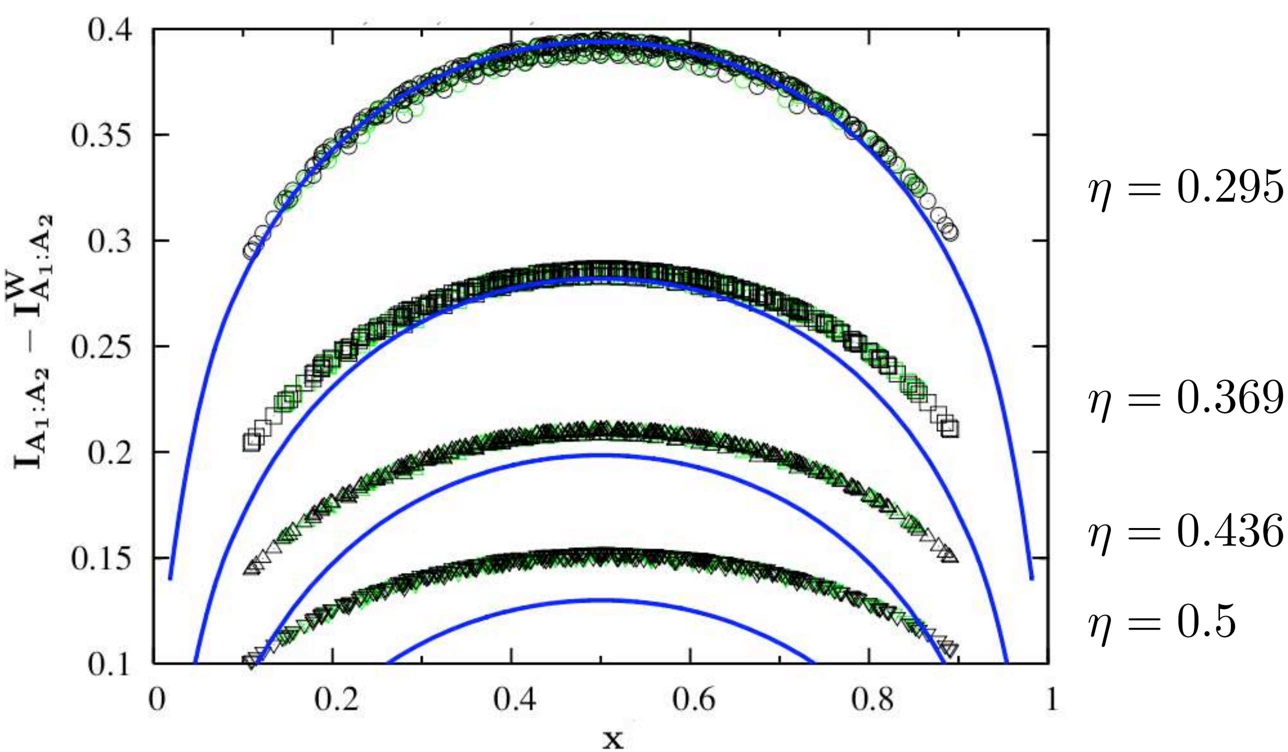
Comparison with the numerical data

- Exact diagonalization of the XXZ spin chain in a magnetic field (up to $L = 30$) [Furukawa, Pasquier, Shiraishi, PRL (2009)]

$$H \equiv \sum_{j=1}^{L} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta\, S_j^z S_{j+1}^z - h\, S_j^z \right)$$

$$\left. \begin{array}{l} \Delta \in (-1, 1] \\ h = 0 \end{array} \right\} \eta = 1 - \frac{1}{\pi} \arccos \Delta$$





confirms the formula

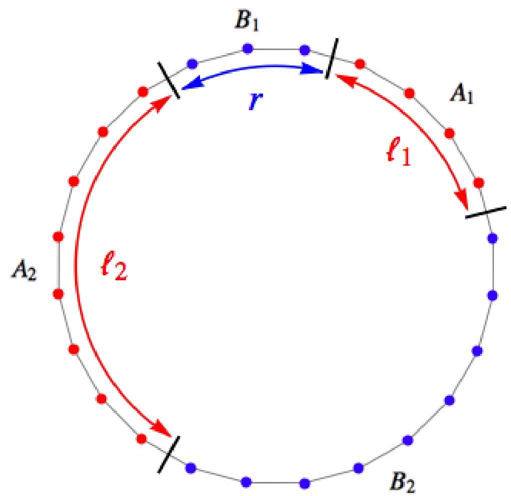
$$I_{A_1:A_2}(\eta \ll 1) - I^W_{A_1:A_2} \simeq -\frac{1}{2} \ln \eta + \frac{D_1'(x) + D_1'(1-x)}{2}$$

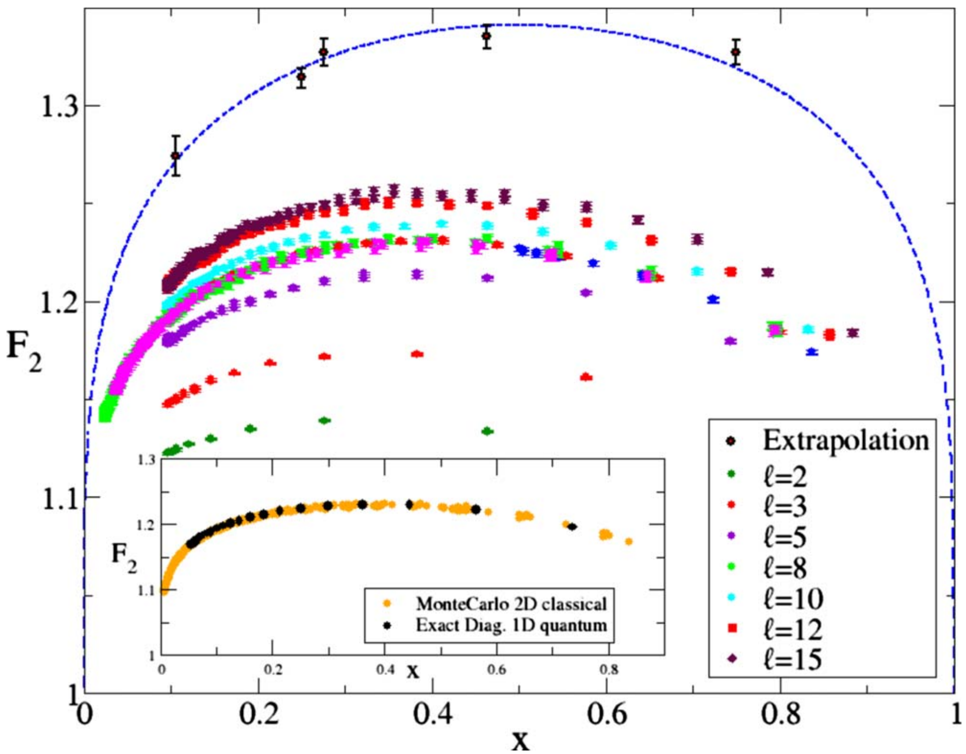
Ising model: 2 sheets

$$H_{XY} \equiv -\sum_{j=1}^{L}\left(\frac{1+\gamma}{4}\,\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{4}\,\sigma_j^y\sigma_{j+1}^y + \frac{h}{2}\,\sigma_j^z\right)$$

$$\gamma = \text{anisotropy} \qquad \begin{cases} 1 & \text{Ising model} \\ 0 & \text{XX model} \end{cases}$$

$$h = \text{magnetic field}$$





[Alba, Tagliacozzo and Calabrese; PRB (2010)]

$$\mathcal{F}_2(x) = \frac{1}{\sqrt{2}}\left\{\left[\frac{(1+\sqrt{x})(1+\sqrt{1-x})}{2}\right]^{1/2} + x^{1/4} + [x(1-x)]^{1/4} + (1-x)^{1/4}\right\}$$

Ising model

[Calabrese, Cardy and E.T.; JSTAT (2011)]

- Bosonization on higher genus Riemann surfaces

$$\boxed{\mathcal{F}_n(x) = \frac{1}{2^{n-1}\Theta(0|\Gamma)} \sum_{\varepsilon,\delta} \left| \Theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(0|\Gamma) \right|}$$

$$\Gamma_{rs} \equiv \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi\frac{k}{n}\right) \beta_{k/n} \cos\left[2\pi\frac{k}{n}(r-s)\right] \qquad \beta_y \equiv \frac{F_y(1-x)}{F_y(x)} \qquad F_y(x) \equiv {}_2F_1(y, 1-y; 1; x)$$

$$r, s = 1, \ldots, n-1$$

- Riemann-Siegel theta function with characteristic

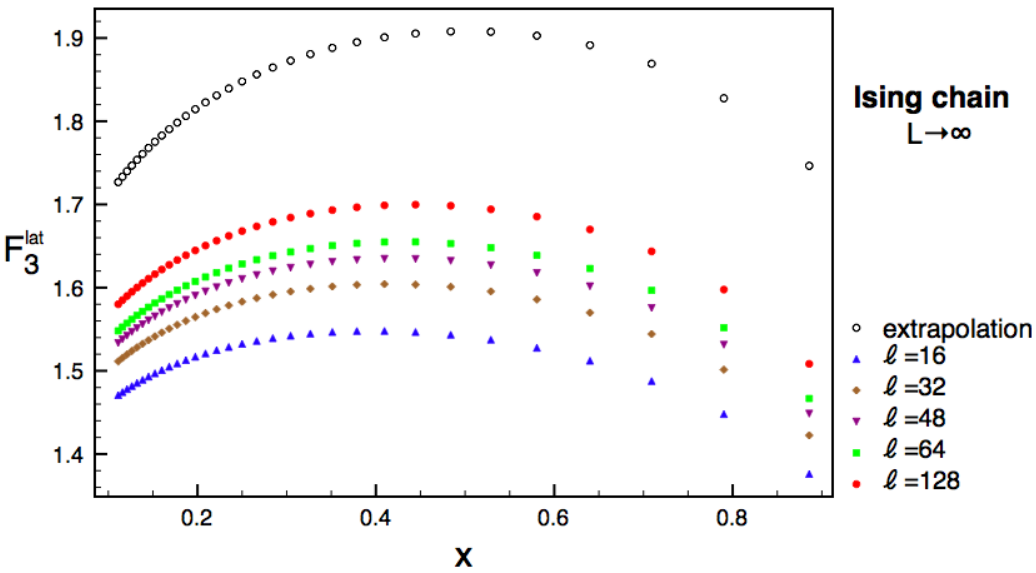
$$\Theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(z|\Gamma) \equiv \sum_{m \in \mathbf{Z}^G} \exp\left[\, i\pi\,(m+\varepsilon)^{\mathrm{t}} \cdot \Gamma \cdot (m+\varepsilon) + 2\pi i\,(m+\varepsilon)^{\mathrm{t}} \cdot (z+\delta) \,\right]$$

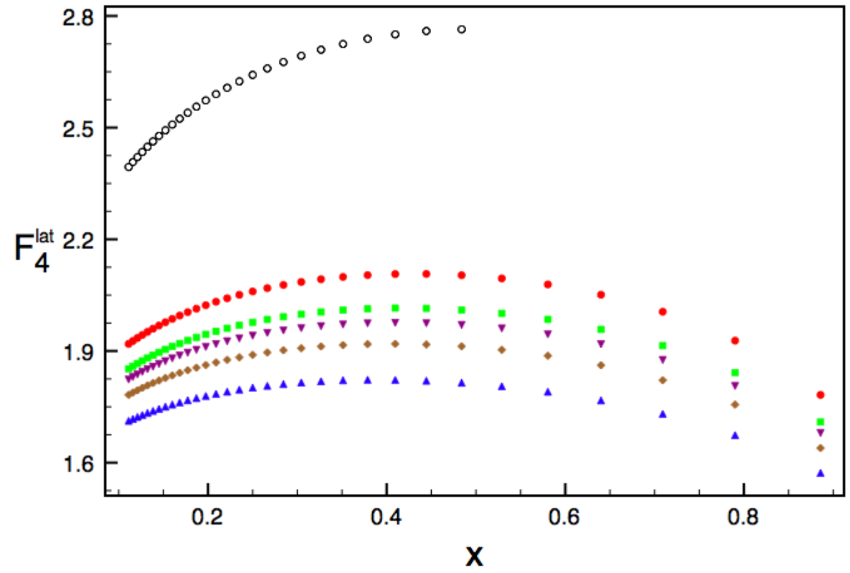
ε and δ are vectors with $n-1$ elements which are either 0 or $1/2$

- $\mathcal{F}_n(x)$ is invariant under $x \leftrightarrow 1-x$ $\quad (S_A = S_B)$

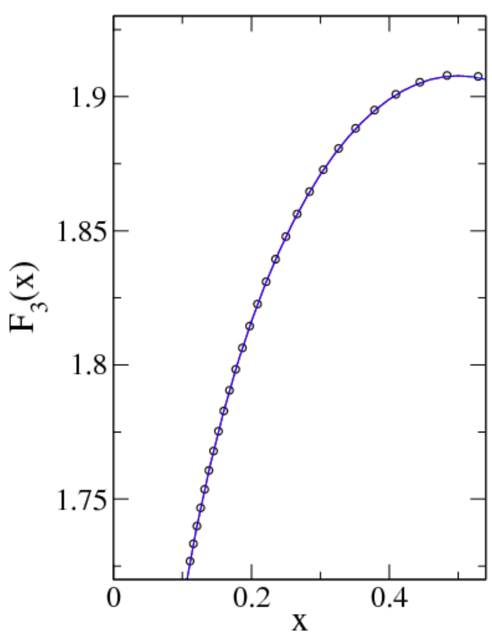
Ising model: 3,4, ... sheets

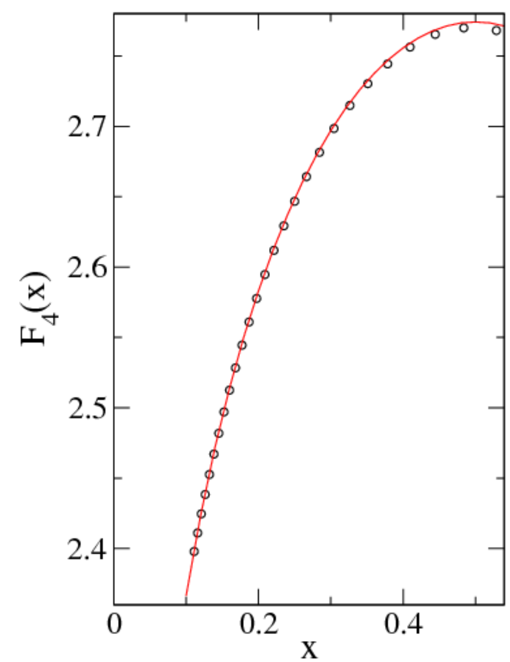
[Fagotti, Calabrese; JSTAT (2010)]





[Calabrese, Cardy and E.T.; JSTAT (2011)]





Short intervals expansion

- $\mathrm{Tr}\rho_A^n$ when the lengths ℓ_p of the intervals are small
  w.r.t. to other characteristc lengths of the system

$$\frac{Z_n(\{I_p\})}{Z_1^n} = \langle \prod_{p=1}^{N} I_p \rangle_{\mathcal{R}^n} \qquad\qquad |I_p\rangle = \sum_{\{k_j\}} C_{\{k_j\}} \otimes_j |\phi_{k_j}\rangle_j$$

- $C_{\{k_j\}}$ for a given interval are independent
  of the location and lengths of the other intervals
  $\Longrightarrow$ find them considering the simplest case
  of a single interval $I = (-\ell/2, \ell/2)$ on the infinite line

$$C_{\{k_j\}} = c_n \ell^{-(c/6)(n-1/n)+\sum_j(\Delta_{k_j}+\bar{\Delta}_{k_j})} d_{\{k_j\}}$$

$$n^{-\sum_j(\Delta_{k_j}+\bar{\Delta}_{k_j})} \langle \prod_{j=1}^{n} \phi_{k_j}\big(e^{2\pi i j/n}\big) \rangle_{\mathbf{C}}$$

Short intervals expansion: two intervals case

- $A = [u_1, v_1] \cup [u_2, v_2]$ and r is the distance between the centers

$$\mathrm{Tr}\,\rho_A^n = \sum_{\{k_{1_j}\}} \sum_{\{k_{2_j}\}} C_{\{k_{1_j}\}}(\ell_1) C_{\{k_{2_j}\}}(\ell_2) \prod_{j=1}^{n} \langle \phi_{k_{1_j}}(r) \phi_{k_{2_j}}(0) \rangle_{\mathbf{C}}$$

$$\boxed{= c_n^2 (\ell_1 \ell_2)^{-c/6(n-1/n)} \sum_{\{k_j\}} \left(\frac{\ell_1 \ell_2}{n^2 r^2} \right)^{\sum_j (\Delta_j + \bar{\Delta}_j)} \langle \prod_{j=1}^{n} \phi_{k_j} \big(e^{2\pi i j/n} \big) \rangle_{\mathbf{C}}^2}$$

- This becomes an expansion in the cross ratio x
and the leading contribution comes from the two point function

$$s_k(n) = \sum_{0 \le j_1 < j_2 \le n-1} \frac{e^{4\pi i (j_1 + j_2) s_k / n}}{|\sin(\pi(j_2 - j_1)/n)|^{4x_k}} \qquad \begin{aligned} x_k &= \Delta_k + \bar{\Delta}_k \\ s_k &= \Delta_k - \bar{\Delta}_k \end{aligned}$$

- MAIN MESSAGE:
The short length expansion of $\mathrm{Tr}\rho_A^n$ for two intervals
provides a series of powers whose terms encode *all* the data of the CFT
(conformal dimensions and OPE coefficients)

Two intervals: small x expansions, first order

$$\boxed{\mathcal{F}_n(x) = 1 + \left(\frac{x}{4n^2}\right)^{\alpha} s_2(n) + \left(\frac{x}{4n^2}\right)^{2\alpha} s_4(n) + \dots}$$

- → compactified boson $\qquad \alpha = \min[\eta, 1/\eta] \qquad \mathcal{N} = 2$
- → Ising $\qquad \alpha = 1/4 \qquad \mathcal{N} = 1$

The first order comes from the two point function

$$s_2(n) = \mathcal{N}\,\frac{n}{2}\sum_{j=1}^{n-1}\frac{1}{\left[\sin\left(\pi\frac{j}{n}\right)\right]^{2\alpha}}$$

Analytic continuation of the first order

$$s_2'(1) = \mathcal{N}\,\frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{4\Gamma\left(\alpha+\frac{3}{2}\right)}$$

Two intervals: small x expansions, second order

- The second order in x comes from the four point function

$$s_4(n) = 2 \sum_{0\leq j_1<j_2<j_3<j_4\leq n-1} \left[Q_0^{2\alpha} + Q_1^{2\alpha} + Q_2^{2\alpha}\right]$$ compactified boson

$$s_4(n) = \sum_{0\leq j_1<j_2<j_3<j_4\leq n-1} Q_0^{1/2}$$ Ising model

$$Q_0 = \frac{\sin(\pi j_{42}/n)\sin(\pi j_{31}/n)}{\sin(\pi j_{21}/n)\sin(\pi j_{43}/n)\sin(\pi j_{41}/n)\sin(\pi j_{32}/n)} \qquad j_{kl} = j_k - j_l$$

$$Q_1 = \frac{\sin(\pi j_{41}/n)\sin(\pi j_{32}/n)}{\sin(\pi j_{21}/n)\sin(\pi j_{43}/n)\sin(\pi j_{42}/n)\sin(\pi j_{31}/n)}$$

$$Q_2 = \frac{\sin(\pi j_{21}/n)\sin(\pi j_{43}/n)}{\sin(\pi j_{42}/n)\sin(\pi j_{31}/n)\sin(\pi j_{41}/n)\sin(\pi j_{32}/n)}$$

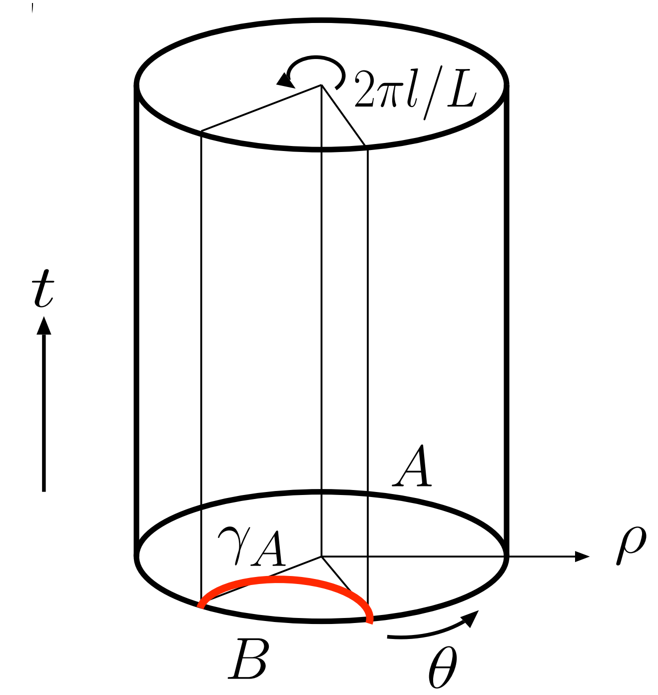
Agreement with the short length expansion discussed for N intervals

Holographic entanglement entropy

AdS_{d+2}/CFT_{d+1} correspondence

[Ryu, Takayanagi, PRL, JHEP (2006)]

- Prescription: in regularized AdS_{d+2}
  - Find the *minimal area* surface γ_A s.t. $\partial\gamma_A = \partial A$
  - $$\boxed{S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N^{(d+2)}}}$$



- $d=1$ formula $S_A = (c/3)\log(l/a)$ and the area law

$$S_A \propto \frac{\mathrm{Area}(\partial A)}{a^{d-1}}$$

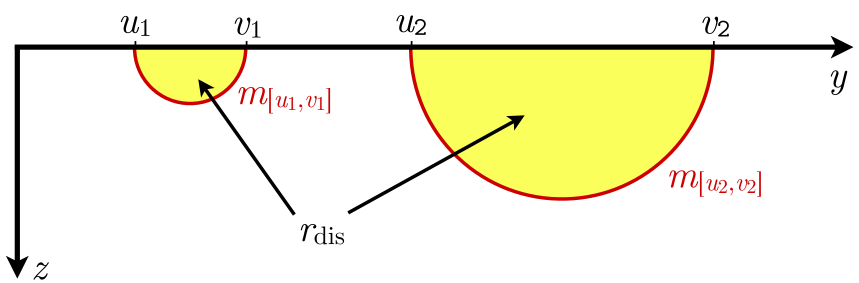
are recovered.

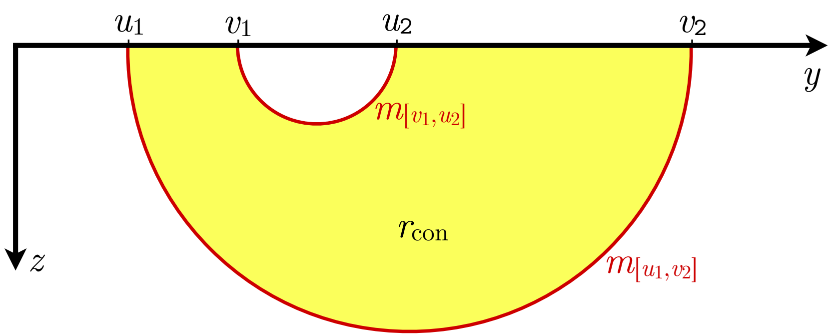
[Bombelli, Koul, Lee, Sorkin, PRD (1986)]
[Srednicki, PRL (1993)]

Transition in the holographic mutual information

- The holographic prescription predicts a transition for the mutual information

[Headrick, PRD (2010)]

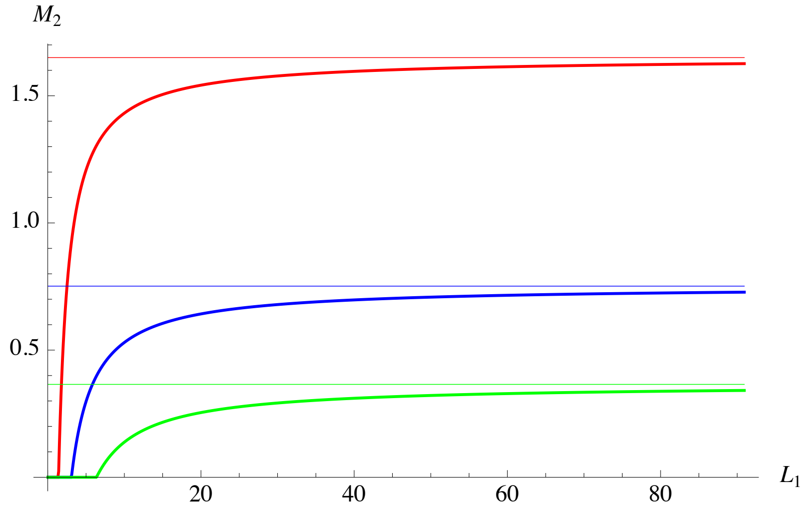


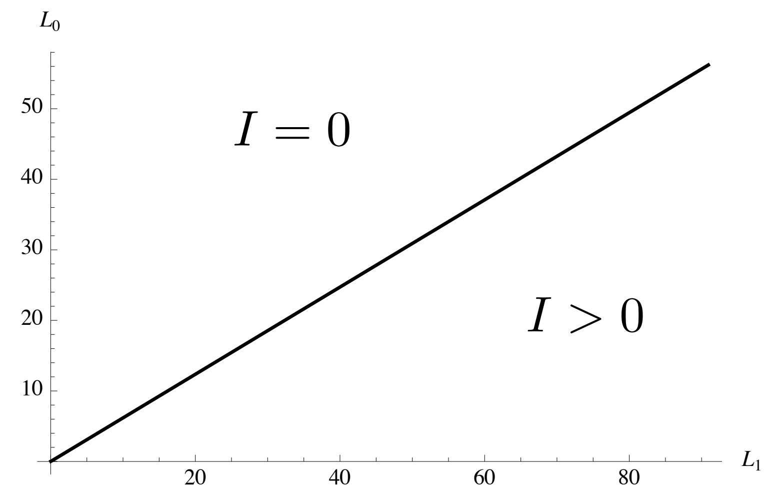


- AdS_{d+2}

[E.T., JHEP (2011)]

$$S_d(L_1, L_2; L_0) \equiv \min\Big[\underbrace{\tilde{A}_d(L_1) + \tilde{A}_d(L_2)}_{\text{disconnected surface}}\,;\,\underbrace{\tilde{A}_d(L_0) + \tilde{A}_d(L_1 + L_0 + L_2)}_{\text{connected surface}}\Big]$$



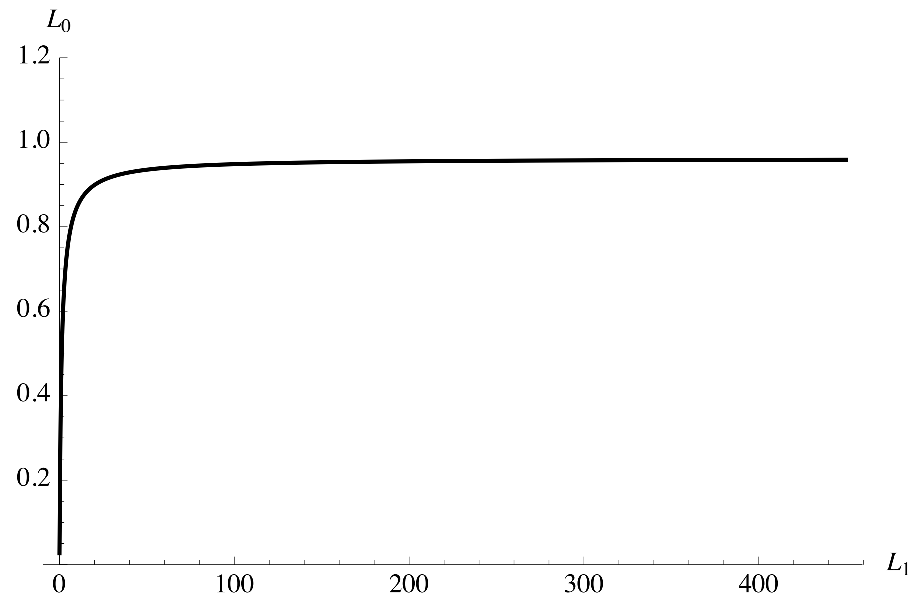


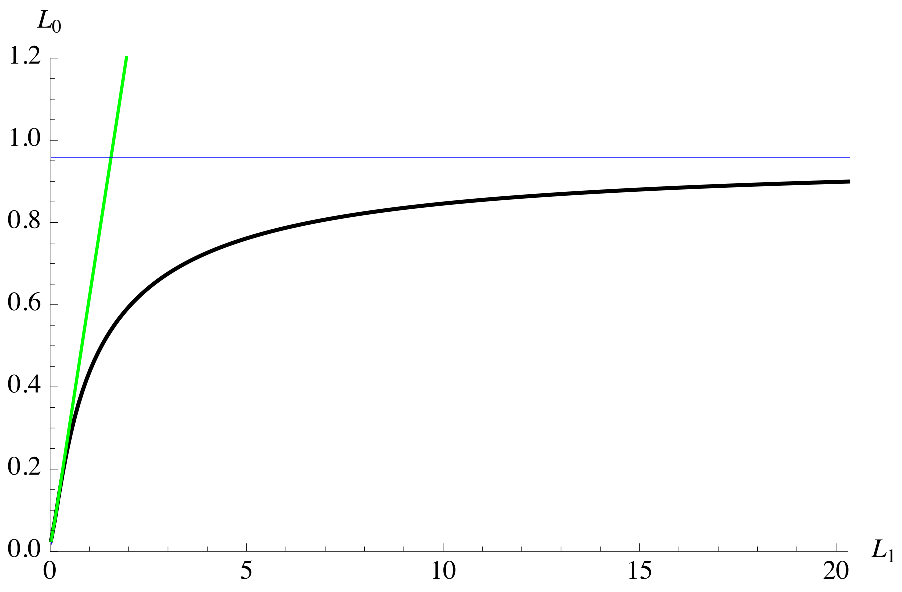
Holographic mutual information: charged black hole

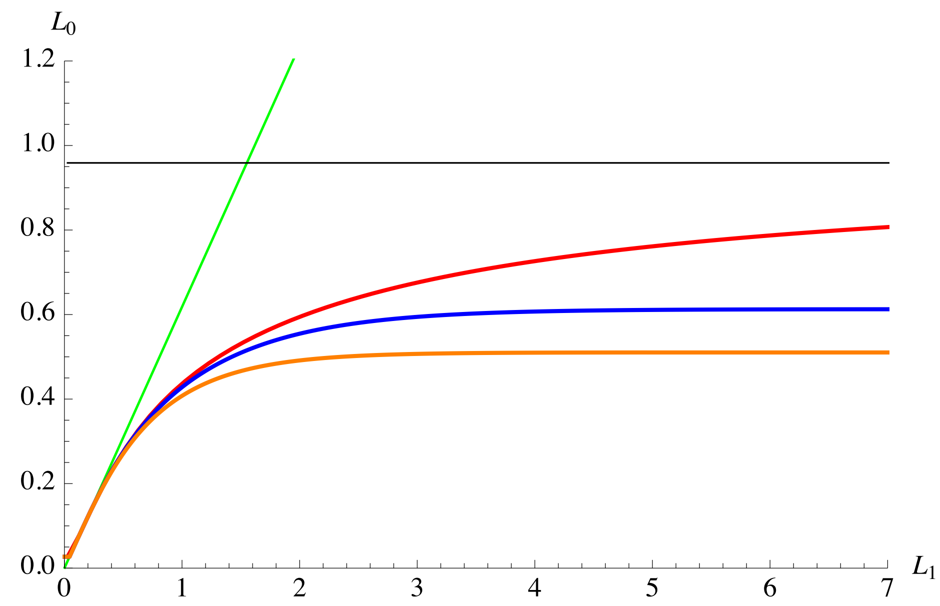
- $$\frac{ds^2}{R^2} = \frac{-f\,dt^2 + d\vec{x}^2}{z^2} + \frac{dz^2}{f\,z^2}$$

  $$f = 1 + Q^2\left(\frac{z}{R^2}\right)^{2d} - M\left(\frac{z}{R^2}\right)^{d+1}$$

- Transition curve for the mutual information when $L_1 = L_2$







Vaidya metrics

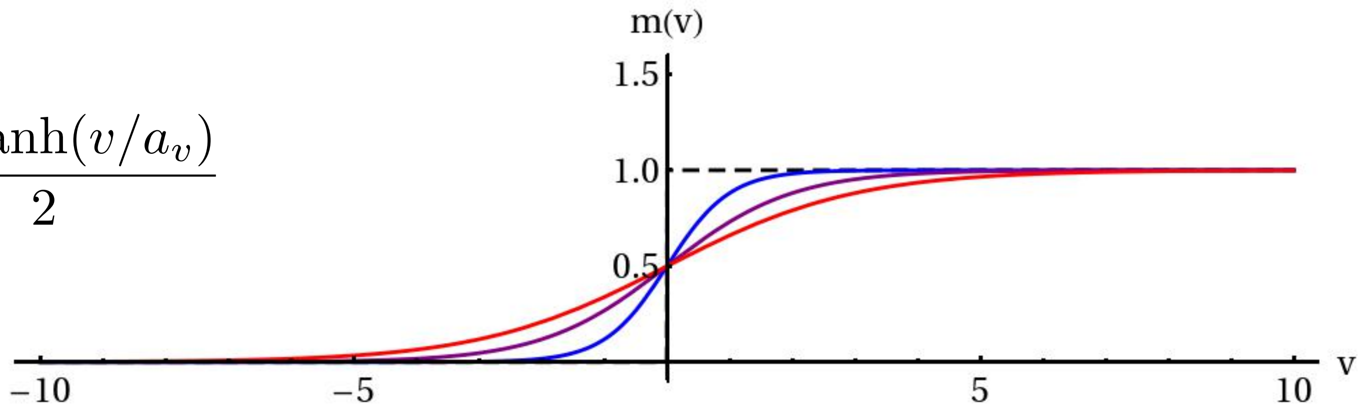
- $d+1$ dimensional Vaidya metrics

$$ds^2 = \frac{l^2}{z^2}\Big[-\big(1-m(v)z^d\big)dv^2 - 2dzdv + d\vec{x}^2\Big]$$

formation of a black hole through the collapse of null dust

$$G_{\mu\nu} - \frac{d(d+1)}{2l^2}g_{\mu\nu} = T_{\mu\nu} \qquad\qquad T_{vv} = \frac{d-1}{2}z^{d-1}\partial_v m(v)$$

$$m(v) = M\,\frac{1+\tanh(v/a_v)}{2}$$



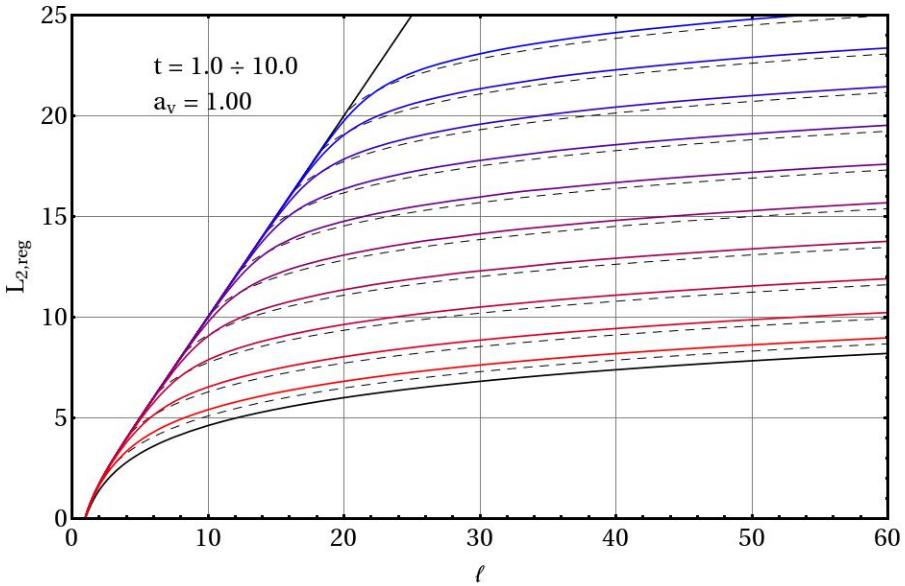
- Null energy condition $T_{\mu\nu}N^\mu N^\nu \geqslant 0$ for Vaidya metrics is $\partial_v m(v) \geqslant 0$

- The formula for the holographic entanglement entropy has been proposed

[Hubeny, Rangamani, Takayanagi, JHEP (2007)]

Holographic entanglement entropy for Vaidya metrics

[Abajo-Arrastia, Aparicio, Lopez, JHEP (2010)]



$$S_A\Big|_{T>0} = \frac{c}{3}\log\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi\ell}{\beta}\right)\right] \qquad \text{BTZ}$$

time evolution

$$S_A\Big|_{T=0} = \frac{c}{3}\log\left(\frac{\ell}{\epsilon}\right) \qquad AdS_3$$

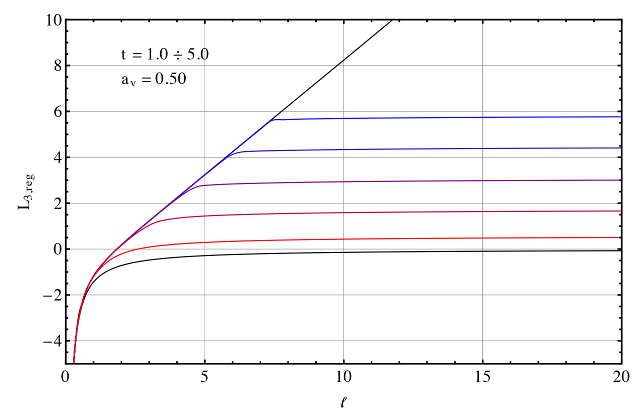
- The problem has been addressed analytically in the *thin shell limit*

  $a_v \to 0 \qquad \Longrightarrow \qquad m(v) = M\delta(v)$

  [Balasubramanian et al., PRL, PRD (2011)]

- Similar behavior in $3+1$ bulk dimensions

  [A. Allais and E.T., JHEP (2012)]

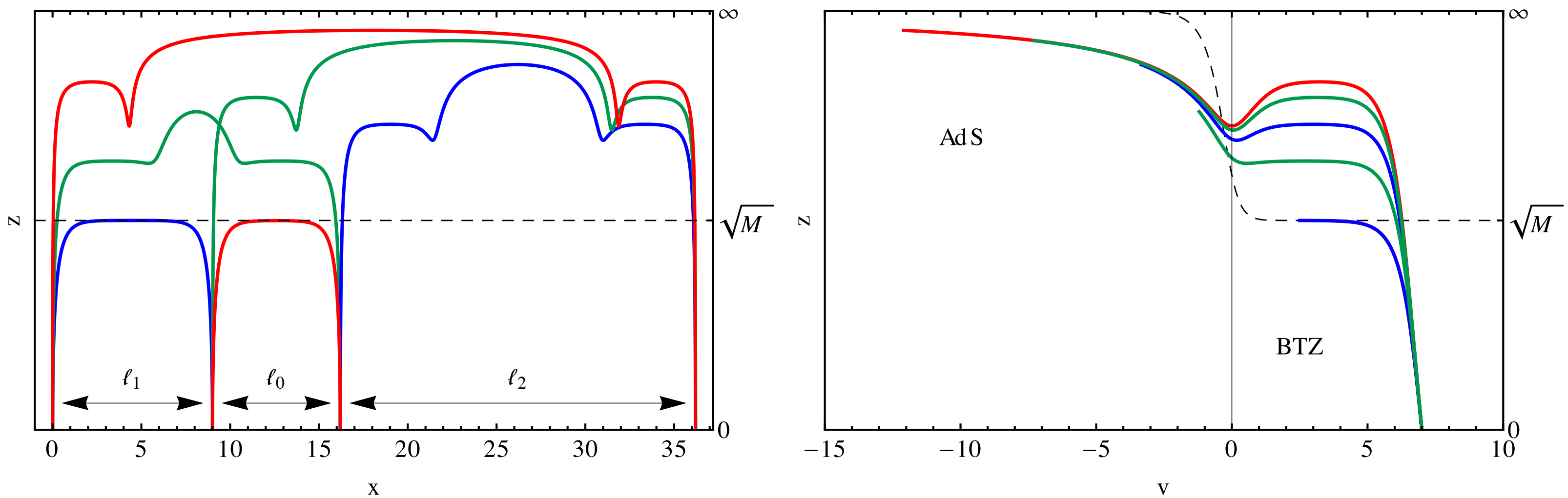


- Related to the holographic description of *global quenches* in CFT ?

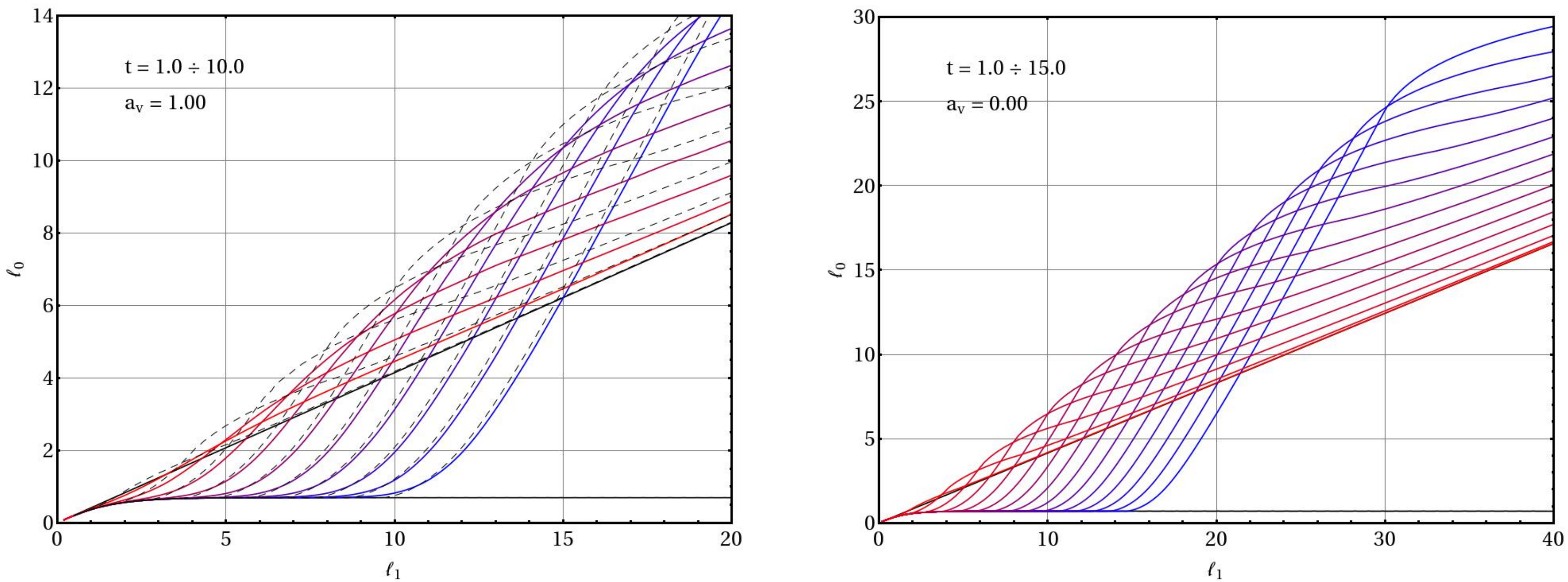
Holographic mutual information for Vaidya metrics (I)

- Geodesics configuration
  (connected, disconnected and mixed)

[Balasubramanian et al., PRD (2011)]
[A. Allais and E.T., JHEP (2012)]

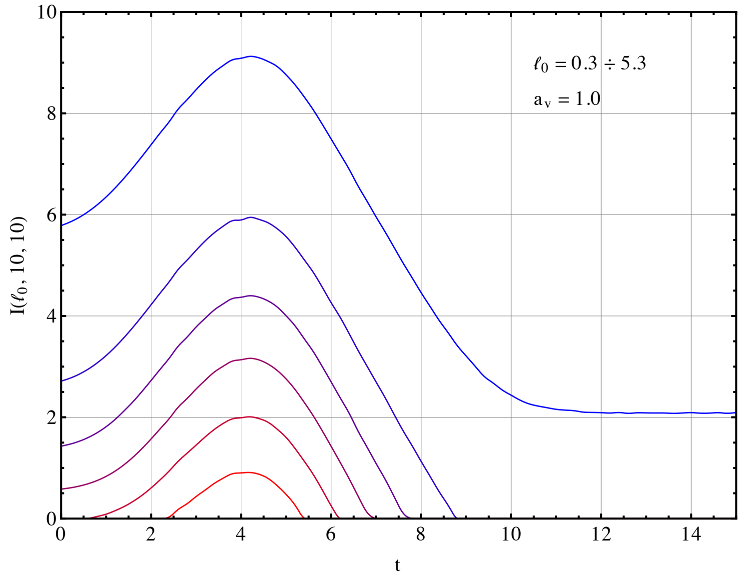


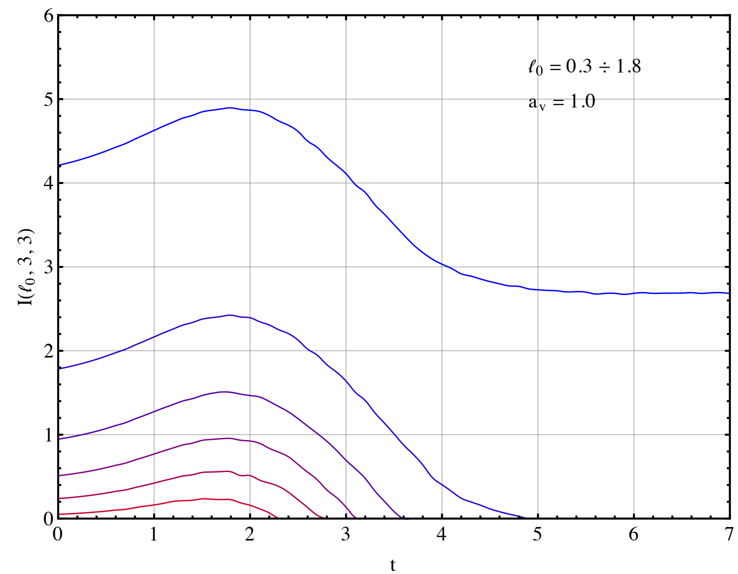
- Transition point in the configuration space



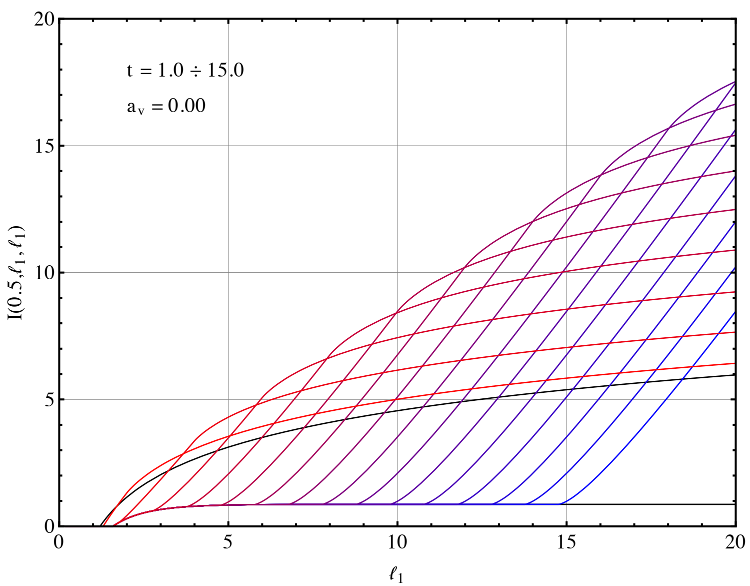
Holographic mutual information for Vaidya metrics (II)

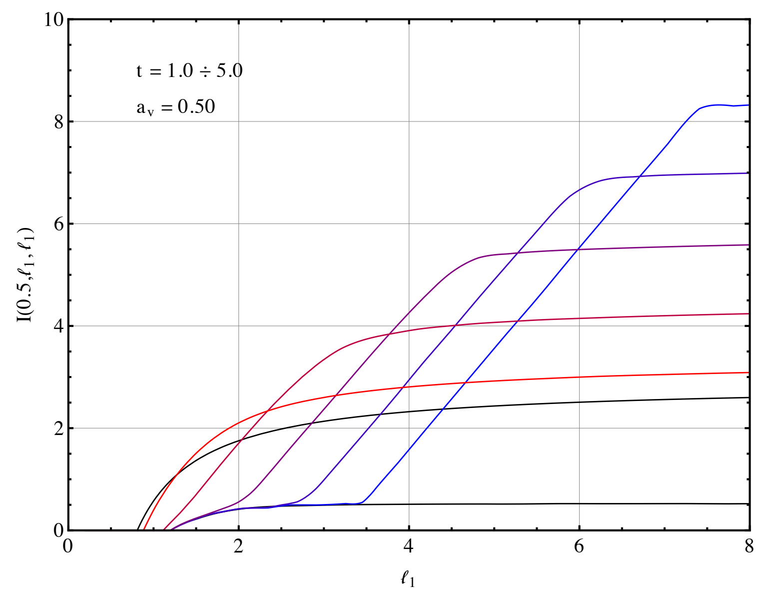
- Dependence on the boundary time t ($d = 2$ and $d = 3$)





- Dependence on $\ell_2 = \ell_1$ ($d = 2$ and $d = 3$)





Null energy condition and strong subadditivity

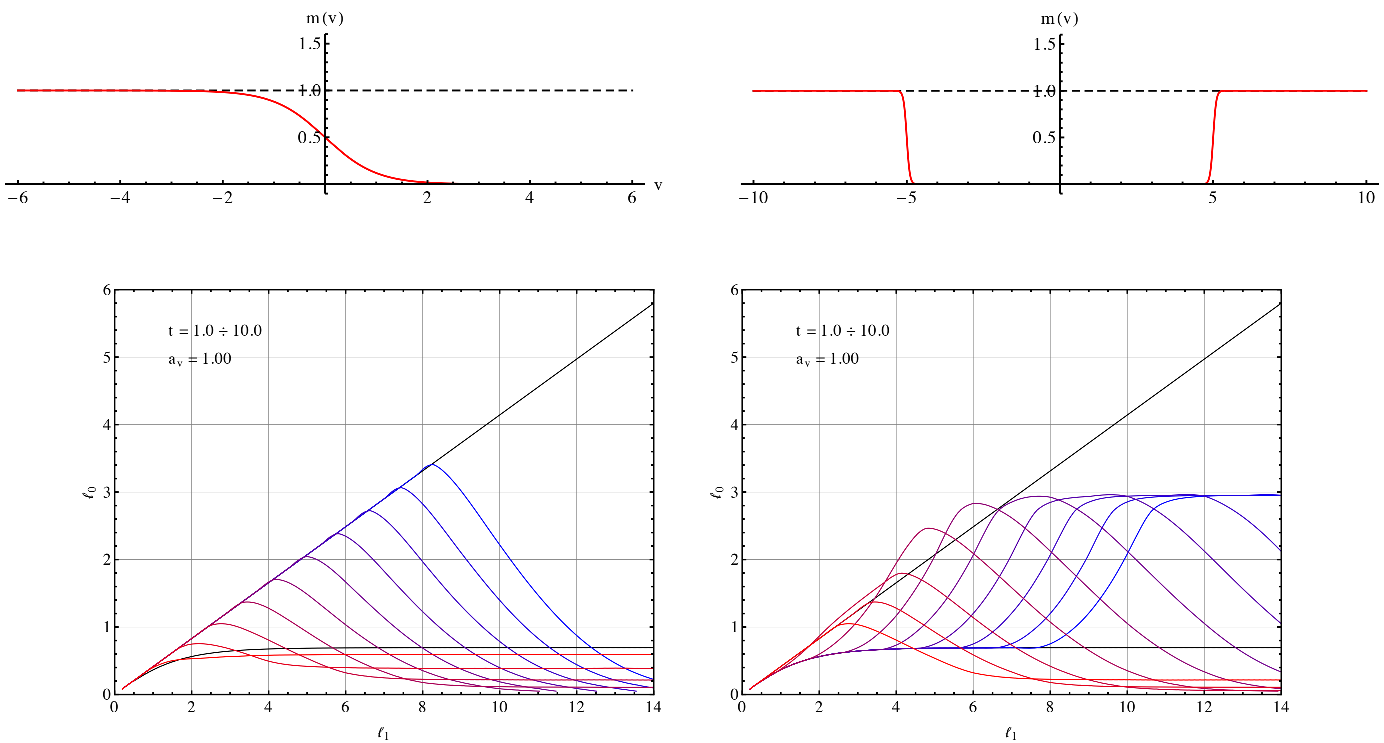
[A. Allais and E.T., JHEP (2012)]

- Subadditivity $\quad S_{A_1} + S_{A_2} \geqslant S_{A_2 \cup A_2} \qquad I(A_1, A_2) \geqslant 0$

  Strong subadditivity

  $S_{A_1 \cup A_2} + S_{A_2 \cup A_3} \geqslant S_{A_2} + S_{A_1 \cup A_2 \cup A_3} \qquad I(A_1, A_2 \cup A_3) \geqslant I(A_1, A_2)$

- For time independent backgrounds the holographic formula satisfies the strong subadditivity condition [Headrick, Takayanagi, PRD (2007)]

- 

NEC violated

↓

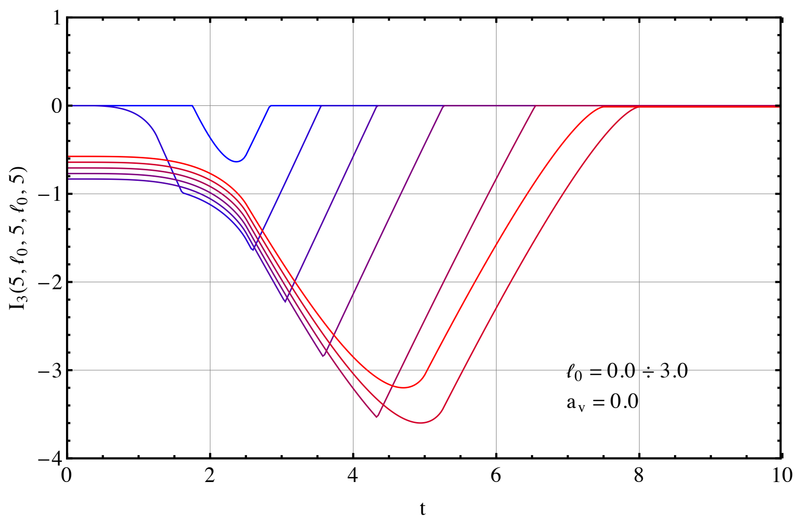
SSA violated

Holographic tripartite information

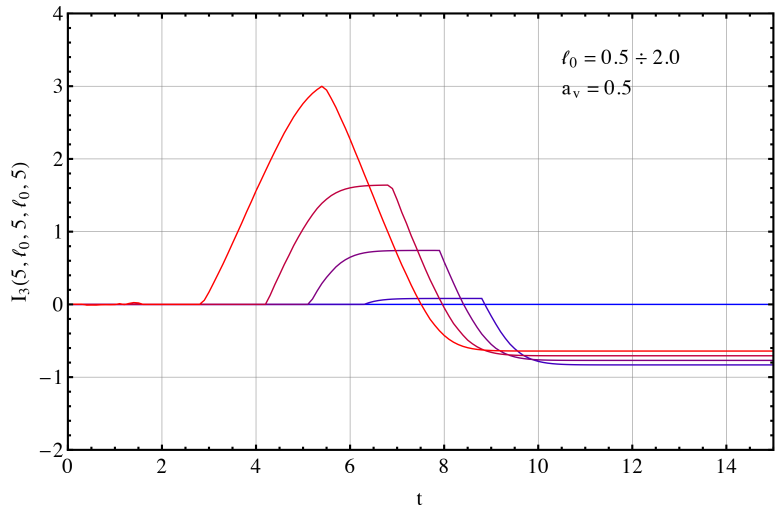
- $$I_3(A_1, A_2, A_3) \equiv S_{A_1} + S_{A_2} + S_{A_3} - S_{A_1 \cup A_2} - S_{A_1 \cup A_3} - S_{A_2 \cup A_3} + S_{A_1 \cup A_2 \cup A_3}$$

- For time independent backgrounds the holographic formula gives $I_3 < 0$

  [Hayden, Headrick, Maloney, 1107.2940]

- 

  Holographic I_3 for Vaidya metric ($d = 2$ in the thin shell limit)

- 

  A violation of the null energy condition leads to a violation of the monogamy

Conclusions

- Two intervals case:
  - $\mathrm{Tr}\rho_A^n$ for the **compactified boson** and the **Ising model**
  - Analytic continuation in some regimes (e.g. decompactification regime)

  Results checked against numerical data from spin chain analysis

- N intervals case:
  - Short intervals expansion

$$\boxed{\text{The } N \geqslant 2 \text{ intervals case includes all the data of the CFT}}$$

- Holographic mutual information for Vaidya spacetimes

- A violation of the null energy condition leads to a violation of the strong subadditivity and the monogamy condition

Open issues

- Analytical continuation for $n \to 1$ of $\mathcal{F}_n(x)$
- Presence of boundaries (quantum quenches), finite T
- $\mathbb{Z}_2$ orbifolded target space at generic radius, minimal models, etc.
- Interactions
- Generalization to $N > 2$ intervals
- Higher dimensions

- Holographic computation of the Renyi entropies found for small c
- Quantum quenches from the holographic point of view (more examples needed)
- Role of the null energy condition

Thank you!