Edge states at spin quantum Hall transitions

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Workshop at GGI, Florence:
New quantum states of matter in and out of equilibrium
24/04/2012

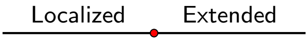
Based on
BJS: arXiv:1101.4361, BGJOS: arXiv:1109.4866

Outline

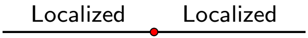
1. Motivations
   - Anderson transitions: IQHE, SQHE
   - Edge states
2. The model
   - Quantum network models
   - Quantum-classical localization
3. Solution
   - Superspins and σ-models
   - Universality and critical exponents
4. Conductance and numerics

Anderson transitions

- Disordered non-interacting electrons
  - Symmetry classification [Dyson '62, Altland,Zirnbauer '97]
    - 10 classes (WD, Chiral, BdG)
    - E. g. : IQHE in class A, $e^{it\mathcal{H}} \in \mathrm{U}(N)$
  - Escaping localization in 2d [Evers,Mirlin '08]
    - MIT

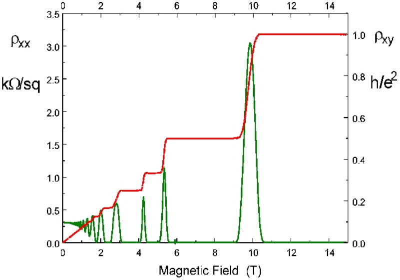


    - Topological phases



Integer Quantum Hall Effect

- 2DEG at high B, low T
- Plateaus $\rho_{xy} = \frac{h}{e^2 \nu}$
  [Von Klitzing Nobel '85]



1. Disorder

2. Chiral edge states

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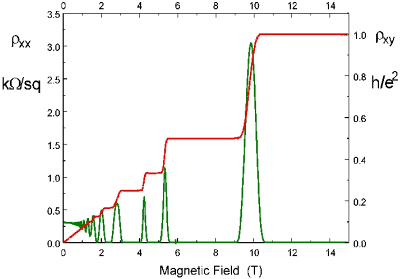

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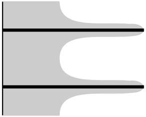

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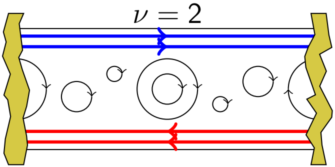


Spin quantum Hall effect (SQHE)

- $d + id$ disordered superconductors [Senthil,Marston,Fisher '99]
- $e^{it\mathcal{H}} \in \mathrm{Sp}(2N)$, class C
- Topological superconductor in 2d: $\sigma^{\mathrm{spin}} \in 2\mathbb{Z}$
- $\neq$ QSH ($\in \mathrm{AII}$, $T^2 = -1$, [Kane,Mele '05])
- Some aspects exactly solvable!

Low-energy field theories

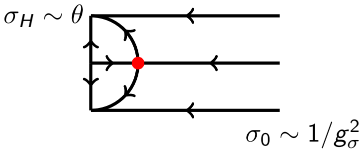
- Non linear σ-model [Wegner '79, Efetov '83]:

$$S = \frac{1}{2g_\sigma^2}\int \mathrm{d}^2 z\, \partial_\mu^\dagger Z_\alpha^\dagger \partial_\mu Z_\alpha$$

- IQHE criticality: topological term [Pruisken '84]

$$S_{\mathrm{top}} = i\theta N[Z]$$

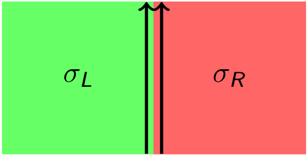
- At $\theta = \pi (\mathrm{mod}\ 2\pi)$, $g_\sigma = O(1)$, LogCFT $c = 0$ (?)



- Bulk exponents, ξ, etc. indep. of θ (if $\theta = \pi(\mathrm{mod}\ 2\pi)$)

Edge states and the θ-angle

- Bulk-boundary



$$\#(\text{edge states}) = \sigma_R - \sigma_L$$

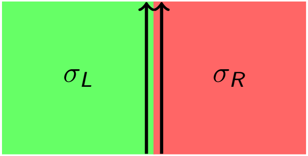
- Edge states as θ increased
  - 1D QED: quark-antiquark screen $F \propto \theta$ [Coleman '75, Affleck '85]
  - Boundary properties dep. on exact value θ [Xiong,Read,Stone '97]

Aim of the talk

Edge states for SQHE

Edge states and the θ-angle

- Bulk-boundary



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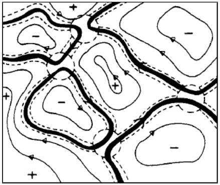
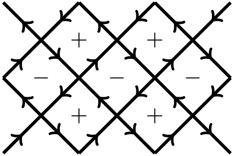
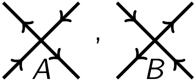
Network models: quantum percolation

- Chalker-Coddington model [Chalker,Coddington '88]:



$\longrightarrow$



- 

: $$S = \begin{pmatrix} \sqrt{1-t^2} & t \\ -t & \sqrt{1-t^2} \end{pmatrix}$$

$\Rightarrow$ $\mathcal{U}_{e,e'} = S_{e',e} U_e, \; U_e \in U(1)$

Edge states in CC model

- Extreme limits

$t = 1$: Insulator $\qquad t_c = \frac{1}{\sqrt{2}} \qquad t = 0$: QH state $\nu = 1$

- Higher plateaus: chiral extra edge channels [BGJOS '12]:

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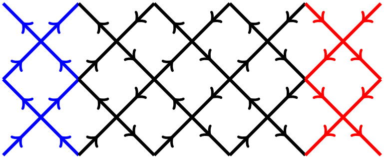
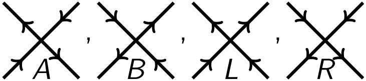


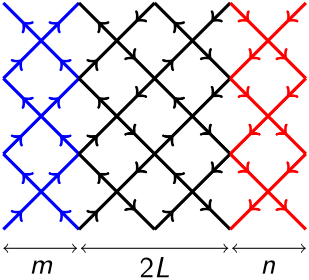
Network model SQHE with edge channels

- $U_e \in SU(2)$

- 

$: \quad S_x = \mathbf{1} \otimes \begin{pmatrix} \sqrt{1-t_x^2} & t_x \\ -t_x & \sqrt{1-t_x^2} \end{pmatrix}$

- $\mathcal{L} = m, \mathcal{R} = n$:



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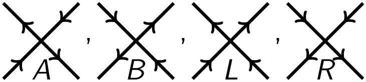
- 

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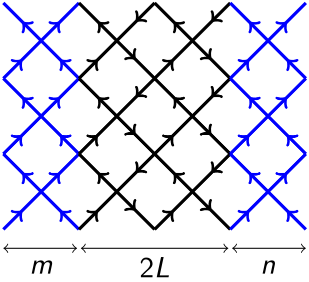

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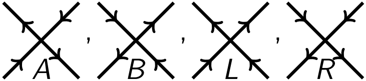
- 

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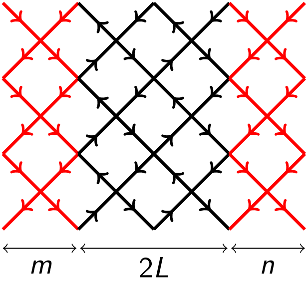

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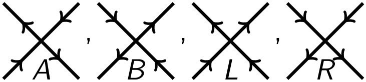
- 

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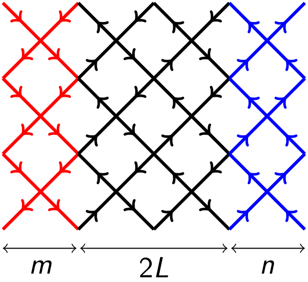


Disorder average I

[Gruzberg,Read,Ludwig '99,Beamond,Cardy,Chalker '02,Mirlin,Evers,Mildenberger '03,Cardy '04]

- $G(e, e', z) = \langle e| \left(1 - z\mathcal{U}\right)^{-1} |e'\rangle = \sum_{\gamma(e,e')} \cdots z U_{e_j} s_j \cdots$
- SUSY path integral
  - $x_\sigma(e), \eta_\sigma(e), \sigma = \uparrow, \downarrow$
  - Lattice action: $W[x, \eta] = zx^*(e')\mathcal{U}_{e',e}x(e) + z\eta^*(e')\mathcal{U}_{e',e}\eta(e)$
  - $\langle \bullet \rangle = \int D\mu(x, \eta) \bullet \exp(W[x, \eta])$

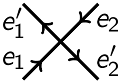
$\Rightarrow$ $G(e, e', z) = \langle x(e)x^*(e') \rangle$

Disorder average II

1. $\int \mathrm{d}U =$ projects $\mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^{1|1})$ onto SU(2)-inv. :

   Truncation: $\Psi = \left(1, \frac{1}{\sqrt{2}}(x_\uparrow \eta_\downarrow - x_\downarrow \eta_\uparrow), \eta_\uparrow \eta_\downarrow\right)$

2. Dilute-dense mapping:



:

$$\prod_i \Psi_{\alpha_i'}(e_i') \left[\delta_{\alpha_1,\alpha_1'}\delta_{\alpha_2,\alpha_2'} S_{11} S_{22} - \delta_{\alpha_1,\alpha_2'}\delta_{\alpha_2,\alpha_1'} S_{12} S_{21}\right] \prod_j \Psi^*_{\alpha_j}(e_j)$$

- Example: $\Psi = 1$: $(1 - t^2) + t^2 = 1$

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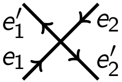

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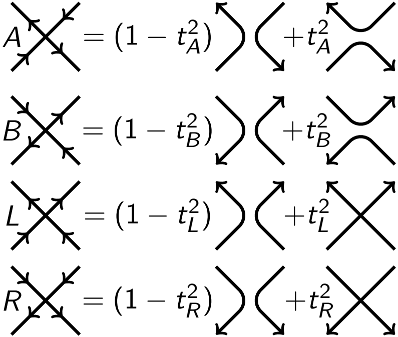
New geometrical model

- Decomposition:



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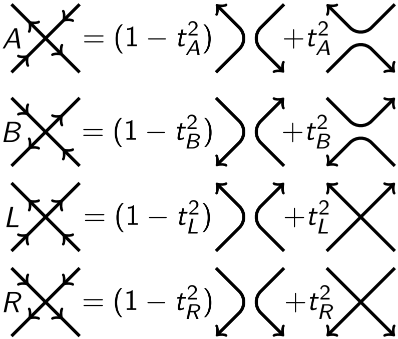

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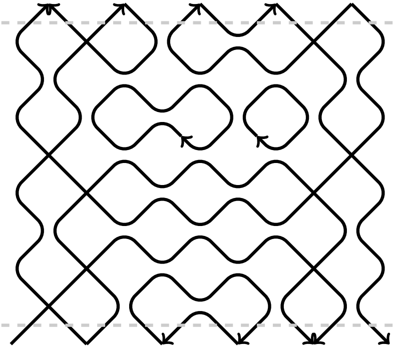
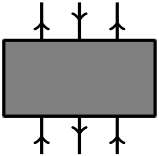


Conductance

- Landauer: PCC $g = \mathrm{Tr}\, \mathbf{t}\mathbf{t}^\dagger$, $\mathbf{t} = \langle e^{\mathrm{out}}| \, (1-\mathcal{U})^{-1} \, |e^{\mathrm{in}}\rangle$
- No replicas:
$\overline{g} = 2\langle \eta_\downarrow(e^{\mathrm{out}})\eta_\uparrow(e^{\mathrm{out}})\eta^*_\uparrow(e^{\mathrm{in}})\eta^*_\downarrow(e^{\mathrm{in}})\rangle = 2P(e^{\mathrm{in}}, e^{\mathrm{out}})$

$\Rightarrow$  $\overline{g} = 2\sum_{e\in C^{\mathrm{in}}}\sum_{e'\in C^{\mathrm{out}}} P(e,e')$

$\Rightarrow$ Loops $\equiv$ transport

What next:

Solve loop model, then go back to SQH.

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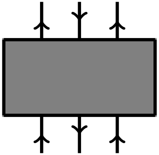
$\Rightarrow$  $\overline{g} = 2 \sum_{e \in C^{\text{in}}} \sum_{e' \in C^{\text{out}}} P(e, e')$

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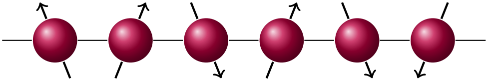
Edge states in superspin chains



- $\color{red}{\mathfrak{sl}(2|1)}$-irreps. $\downarrow \equiv V, \uparrow \equiv V^\star$ dim$=3$, sdim$=1$
- $$\mathcal{H}^{\mathcal{L},\mathcal{R}} = \begin{cases} V^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes (V^\star)^{\otimes n} & (m;n) \\ V^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes V^{\otimes n} & (m;-n) \\ (V^\star)^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes (V^\star)^{\otimes n} & (-m;n) \\ (V^\star)^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes V^{\otimes n} & (-m;-n). \end{cases}$$

Hamiltonian

- $P_i^{\mathfrak{sl}}$: $V_i \otimes V_{i+1}$ $(V_i^\star \otimes V_{i+1}^\star)$
- $E_i^{\mathfrak{sl}}$: $V_i \otimes V_{i+1}^\star$ $(V_i^\star \otimes V_{i+1})$

$$\Rightarrow H = -u \sum P_i^{\mathfrak{sl}} - \sum E_i^{\mathfrak{sl}} - v \sum P_i^{\mathfrak{sl}}$$

bulk

boundary

- Indefinite inner product, Jordan cells, ... [Read,Saleur '07]

Bulk topological σ-models

- Continuum limit periodic chain [Gruzberg,Read,Ludwig '99, Read,Saleur '01]:
  - Target: $\mathbb{CP}^{1|1} = \frac{\mathrm{U}(2|1)}{\mathrm{U}(1)\times\mathrm{U}(1|1)}$, $\pi_2 = \mathbb{Z}$
  - $S = \frac{1}{2g_\sigma^2}\int \mathrm{d}^2 z\, D_\mu^\dagger Z_\alpha^\dagger D_\mu Z_\alpha - \frac{i\theta}{2\pi}\int \mathrm{d}^2 z\, \epsilon^{\mu\nu}\partial_\mu a_\nu$

- At $\theta = \pi (\mathrm{mod}\ 2\pi)$, $g_\sigma = O(1)$, percolation

$\Rightarrow$ Bulk exponents (n-hulls), Loc. length ($\nu = 4/3$)

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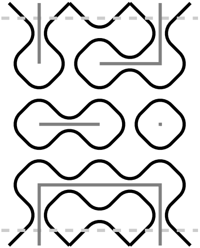

Edge states as conformal boundaries

- Symmetric CBC [Candu,Mitev,Quella,Saleur,Schomerus '10]

$$(\partial_y + i a_y) Z_\alpha = \Theta_1 g_\sigma^2 (\partial_x + i a_x) Z_\alpha \ ,$$

$$(\partial_y - i a_y) Z_\alpha^\dagger = -\Theta_1 g_\sigma^2 (\partial_x - i a_x) Z_\alpha^\dagger$$

$\Theta_1 = (2\mathcal{L} + \theta/\pi)$, $\Theta_2 = (2\mathcal{R} + \theta/\pi)$

$\Rightarrow$ Dep. on exact value of θ:

$$\theta \to \theta + 2\pi p \Longleftrightarrow (V \otimes V^\star)^{\otimes L} \to V^{\otimes p} \otimes (V \otimes V^\star)^{\otimes L} \otimes (V^\star)^{\otimes p}$$

- $\mathcal{L}, \mathcal{R}$ monopole charges

Remarks on conformal boundary conditions

- Disorder leads to non unitary, non rational (Log) CFT
- Trivial in unitary, rational case. E.g. $O(3)$ model:
  - Bulk CFT: $SU(2)_1$ WZW, spin chain: XXX
  - Only two $SU(2)$ CBC ($P_i \sim E_i$).
- Situation is far richer in our case.

Relation blob-edge states

- $\tilde{H} := -u \frac{1}{(m+1)!} \sum_{\sigma \in \mathfrak{S}_L} \sigma - \sum E_i - v \frac{1}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_R} \sigma$

$\Rightarrow$ When $L \to \infty$: $H \simeq \tilde{H}$

$\Rightarrow$ Effective boundary loops

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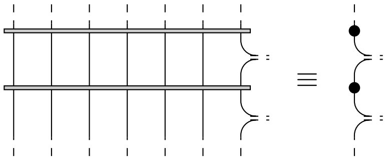
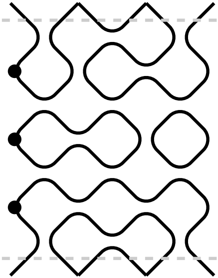


Boundary loop models

- Blob algebra [Martin,Saleur '94]
  -  $= \delta$
  - $h_{r(\delta),r(\delta)+2j}$ [Jacobsen,Saleur '06]
    1. Irrational
    2. Indep. of λ
- Two-bdry [Dubail,Jacobsen,Saleur'09]



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Leading exponents $h^{m,n}(k)$ in sector k

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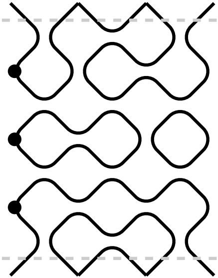

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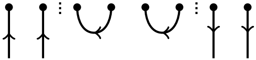
- E. g. : 

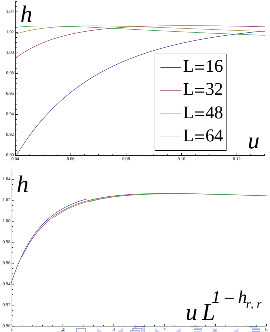
Critical exponents

k	$\#(\text{legs})$	$h^{m,n}(k)$
0	$m-n$	$h_{r_0,r_0}=0$
1	$m-n+2$	h_{r_1,r_1}
$\vdots$	$\vdots$	$\vdots$
n	$n+m$	h_{r_n,r_n}
$n+1$	$n+m+2$	$h_{1,3}$
$\vdots$	$\vdots$	$\vdots$
$n+j$	$n+m+2j$	$h_{1,1+2j}$
$\vdots$	$\vdots$	$\vdots$

$$r_k=\frac{6}{\pi}\arccos\left(\frac{\sqrt{3}}{2}\sqrt{\frac{(n+1-k)(m+1+k)}{(m+1)(n+1)}}\right)$$

$$h_{r,s}=\frac{\left((3r-2s)^2-1\right)}{24}$$

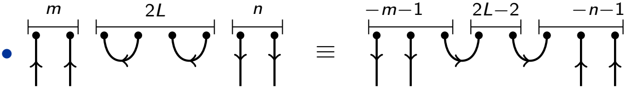
- Indep. of m, n for $\#(\text{legs}) > n+m$
- Irrational
- Indep. of couplings: boundary RG flow



Symmetries and other cases

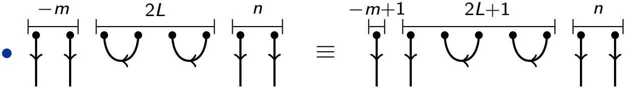
1. Left $\leftrightarrow$ Right: $h^{\mathcal{L},\mathcal{R}}(k) = h^{\mathcal{R},\mathcal{L}}(k)$

2. 

- 

$\Rightarrow$ $h^{-m,-n}(k) = h^{m-1,n-1}(k)$

3. $\mathcal{L}\cdot\mathcal{R} < 0$, $\#(\text{legs}) \geq |\mathcal{L}| + |\mathcal{R}|$

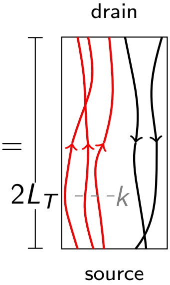
- 

$\Rightarrow$ $h^{-m,n}(k) = h^{m,-n}(k) = h_{1,2+2k} = \frac{k(2k+1)}{3}$

Critical conductance in a strip

- Bottom-top $\bar{g}^{\mathcal{L},\mathcal{R}} = 2\max(0, \mathcal{L} - \mathcal{R}) + 2\sum_{k=1}^{\infty} kP(k, L_T/L)$

$$P(k, L_T/L) =$$



$$= \langle D | T^{2L_T} | S \rangle \underset{\frac{L_T}{L} \to \infty}{\approx} e^{-\pi h^{m,n}(k) \frac{L_T}{L}}$$

- In quasi 1D geometry:

$$\bar{g}^{\mathcal{L},\mathcal{R}} \sim 2\max(0, \mathcal{L} - \mathcal{R}) + Ae^{-\pi h^{\mathcal{L},\mathcal{R}}(1) \frac{L_T}{L}}$$

Numerics for network model

- $g^{\mathcal{L},\mathcal{R}}$ from transfer matrices
  - Fit $\bar{g}^{\mathcal{L},\mathcal{R}} \sim g_\infty + Ce^{-\lambda \frac{L_T}{L}}$
  - Typically $L_T/L \in [2, 40]$, disorder $\mathcal{O}(10^5) \sim \mathcal{O}(10^6)$

$\Rightarrow$ Confirmed $h^{\mathcal{L},\mathcal{R}}(1)$

$\Rightarrow$ Verified indep. on bdry couplings (even random)

$\mathcal{L},\mathcal{R}$	numerics $h^{\mathcal{L},\mathcal{R}}(1)$	analytical $h^{\mathcal{L},\mathcal{R}}(1)$
$0, 0$	0.3333(12)	$1/3$
$0, 1$	0.3330(7)	$1/3$
$0, 10$	0.3325(24)	
$1, 1$	0.03775(25)	0.037720
$2, 2$	0.01600(2)	0.015906
$1, 2$	0.0520(25)	0.052083
$2, 4$	0.02954(7)	0.029589
$-2, -2$	0.0377(4)	0.037720
$-3, -2$	0.0522(2)	0.052083
$-1, 0$	0.999(9)	1
$-2, 0$	0.999(3)	
$-2, 1$	0.998(3)	

Conclusions and Outlooks

- Mapping SQH extra edge channels to classical loop model
- Exact critical exponents of boundary CFT
- Verified predictions of decay conductance
- Outlooks
  - Fractal properties of wave function near extra edges [Mirlin,Evers,Mildenberger '03]
  - Exact conductance [Cardy '00]
  - Edge states of wires in other AZ classes