Geometric Phase Contribution to Quantum Nonequilibrium Many-Body Dynamics

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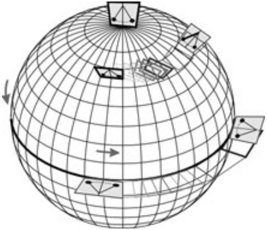
PRL 108, 080404 (2012) with A. Polkovnikov

Outline

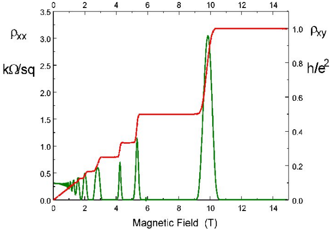
Geometry - Dynamics

Foucault Pendulum

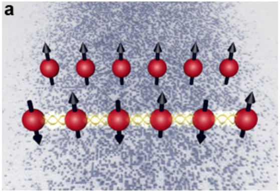


Berry Phase

Quantum Hall Effect



Haldane Gap



General Setting

Closed quantum non-equilibrium many-body system

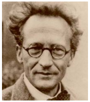
$$\hat{\mathrm{H}}(\boldsymbol{\lambda}(t)), \quad \boldsymbol{\lambda}(t) = (\lambda^1(t), \ldots, \lambda^p(t)) \in \mathcal{M}$$

Instantaneous basis

$$\hat{\mathrm{H}}(\boldsymbol{\lambda}(t))|n(\boldsymbol{\lambda}(t))\rangle = \epsilon_n(\boldsymbol{\lambda}(t))|n(\boldsymbol{\lambda}(t))\rangle$$

State of the system

$$|\psi(\boldsymbol{\lambda}(t))\rangle = \sum_n a_n(\boldsymbol{\lambda}(t))|n(\boldsymbol{\lambda}(t))\rangle$$



$$i\partial_t|\psi(t)\rangle = \hat{\mathrm{H}}(t)|\psi(t)\rangle$$

$$\Longrightarrow \; i\partial_t a_n + i\sum_{m\neq n}\langle n|\partial_t|m\rangle a_m + i\langle n|\partial_t|n\rangle a_n = \epsilon_n a_n$$

$$\boxed{\text{Density of excitations:} \quad n_{\text{ex}} = \frac{1}{L^d}\sum_{n\neq 0}|a_n|^2}$$

Quantum Geometric Tensor

$$i\partial_t a_n + i\sum_{m\neq n}\langle n|\partial_t|m\rangle a_m + i\langle n|\partial_t|n\rangle a_n = \epsilon_n a_n$$

$$t \mapsto \boldsymbol{\lambda}(t) \qquad \frac{\partial}{\partial\lambda^\mu} =: \partial_\mu$$

$$\sum_{\mu=1}^{p}\left[i\partial_\mu a_n + i\sum_{m\neq n}\langle n|\partial_\mu|m\rangle a_m + i\langle n|\partial_\mu|n\rangle a_n\right]\dot{\lambda}^\mu = \epsilon_n a_n$$

$$\boxed{Q_{\mu\nu}(\boldsymbol{\lambda}, n) := \partial_\mu\left(\langle n|\right)\partial_\nu|n\rangle - \partial_\mu\left(\langle n|\right)|n\rangle\langle n|\partial_\nu|n\rangle, \quad \mu,\nu = 1,\dots,p.}$$

J.P. Provost, G. Valle, (*Commun. Math. Phys.* 76, 289 (1980)).

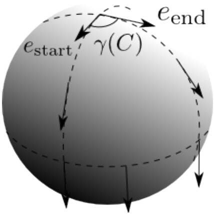
- $\mathrm{Re}Q_{\mu\nu} =: \mathsf{g}_{\mu\nu}$ Riemannian metric tensor
- $2\,\mathrm{Im}Q_{\mu\nu} =: F_{\mu\nu}$ Berry curvature

Quantum Geometric Tensor

- $\mathrm{Re}Q_{\mu\nu} =: \mathsf{g}_{\mu\nu}$ Riemannian metric tensor: $ds^2 = \mathsf{g}_{\mu\nu} d\lambda^\mu d\lambda^\nu$, which is related to the quantum fidelity

$$\mathcal{F}(\boldsymbol{\lambda}, \boldsymbol{\lambda} + d\boldsymbol{\lambda}) := |\langle 0(\boldsymbol{\lambda}) | 0(\boldsymbol{\lambda} + d\boldsymbol{\lambda}) \rangle| = 1 - \frac{1}{2} ds^2 + \dots$$

- $2\,\mathrm{Im}Q_{\mu\nu} =: F_{\mu\nu}$ Berry curvature: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A_\mu(\boldsymbol{\lambda}, n) := i\langle n | \partial_\mu | n \rangle$ is the Berry connection (connection 1-form)



$$\boxed{\gamma(\mathcal{C}, n) = \int_\Sigma F_{\mu\nu} d\lambda^\mu \wedge d\lambda^\nu = \int_{\mathcal{C}=\partial\Sigma} A_\mu d\lambda^\mu}$$

Competition: Dynamical and Geometrical Phase

$$i\partial_t a_n + i\sum_{m\neq n}\langle n|\partial_t|m\rangle a_m + i\langle n|\partial_t|n\rangle a_n = \epsilon_n a_n$$

Gauge transformation: $a_n = \tilde{a}_n \exp\left[\int_{t_i}^{t} d\tau(-i\epsilon_n(\tau) + iA_\tau(|n\rangle))\right]$

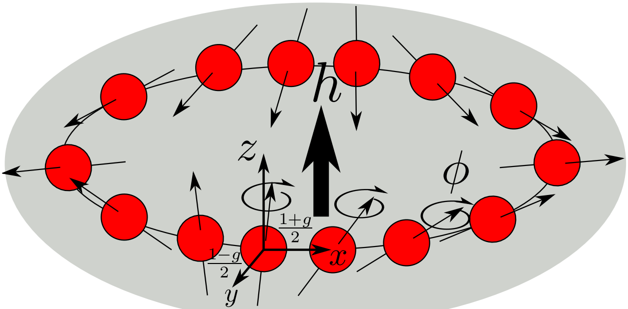
$$\boxed{\partial_t \tilde{a}_n = -\sum_{m\neq n}\langle n|\partial_t|m\rangle \exp\left[iE_{nm}(t) - i\Gamma_{nm}(t)\right]\tilde{a}_m}$$

$$E_{nm}(t) = \int_{t_i}^{t}\left[\epsilon_n(\tau) - \epsilon_m(\tau)\right] d\tau, \qquad \Gamma_{nm}(t) = \int_{t_i}^{t}\left[A_\tau(|n\rangle) - A_\tau(|m\rangle)\right] d\tau$$

The XY Spin Chain

$$\hat{\mathrm{H}}_0(g,h) = -\sum_{l=1}^{N}\left(\frac{1+g}{2}\hat{\sigma}_l^x\hat{\sigma}_{l+1}^x + \frac{1-g}{2}\hat{\sigma}_l^y\hat{\sigma}_{l+1}^y + h\hat{\sigma}_l^z\right)$$

$$\hat{\mathrm{H}}(g,h,\phi) = \hat{\mathrm{U}}(\phi)\,\hat{\mathrm{H}}_0(g,h)\,\hat{\mathrm{U}}^\dagger(\phi), \quad \hat{\mathrm{U}}(\phi(t)) = \exp\left(-i\sum_{l=1}^{N}\hat{\sigma}_l^z\frac{\phi(t)}{2}\right)$$

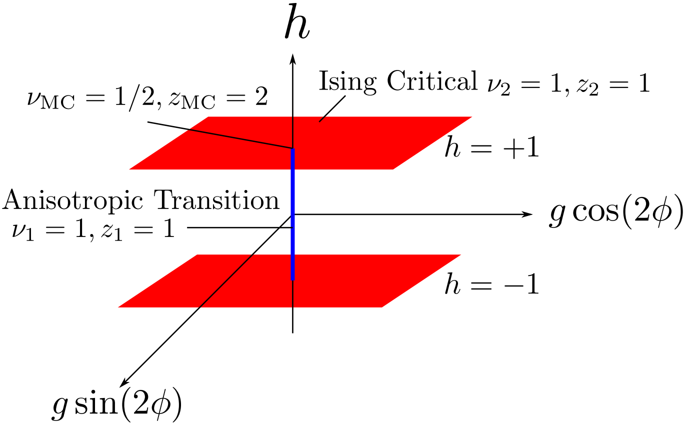


J.K. Pachos and C.M. Carollo, Phil. Trans. R. Soc. A15 364, (2006)

Criticalities of the XY Spin Chain

$$\hat{\mathrm{H}}(g, h, \phi(t)) = \sum_k \epsilon_k(g, h) \left(b_k^\dagger b_k + 1\right)$$

$$\epsilon_k(g, h) = \sqrt{(h - \cos k)^2 + g^2 \sin^2 k}$$



Circular Dynamics:

Start in the ground state $|\mathrm{GS}\rangle$ at $t_i = 0$, $\phi(t) = \omega t$, $\phi_f = \phi(t_f)$

$$\hat{\mathrm{H}}(g, h, \phi(t)) = -\sum_k \hat{\mathbf{c}}_k^\dagger \, \hat{\mathrm{H}}_k(g, h, \phi(t)) \, \hat{\mathbf{c}}_k$$

where

$$\hat{\mathrm{H}}_k = \begin{pmatrix} (h - \cos k) & i\, g \sin k \, e^{-2i\phi(t)} \\ -i\, g \sin k \, e^{2i\phi(t)} & -(h - \cos k) \end{pmatrix}, \quad \text{and} \quad \hat{\mathbf{c}}_k^\dagger = (c_{-k}, c_k^\dagger)$$

$$k = \frac{2\pi\, n}{N}, \quad n = \pm 1, \pm 2, \ldots, \pm \frac{N}{2}$$

$$\boxed{|\Psi(t)\rangle = \bigotimes_k |\psi(t)\rangle_k, \qquad |\psi(t)\rangle_k = a_{\mathrm{gs},k}(t)|\mathrm{gs}\rangle_k + a_{\mathrm{es},k}(t)|\mathrm{es}\rangle_k}$$

$$\Longrightarrow \qquad p_{\mathrm{ex},k} = |a_{\mathrm{es},k}(\phi_f)|^2$$

Circular Dynamics:

$$n_{\text{ex}}(\omega, \phi_f) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} p_{\text{ex},k} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \mathsf{g}_{\phi\phi}(|\text{gs}\rangle_k) \frac{\sin^2\left[\frac{1}{2}\Omega_k(\omega)\phi_f\right]}{\left[\frac{1}{2}\Omega_k(\omega)\right]^2}$$

$$\Omega_k(\omega) := \sqrt{\left[\frac{\epsilon_{\text{gs},k} - \epsilon_{\text{es},k}}{\omega} - (A_\phi(|\text{gs}\rangle_k) - A_\phi(|\text{es}\rangle_k))\right]^2 + 4\mathsf{g}_{\phi\phi}(|\text{gs}\rangle_k)}$$

- Slow driving $\omega \sim 0$:

$$\Omega_k(\omega) = \frac{\epsilon_{\text{gs},k} - \epsilon_{\text{es},k}}{\omega} - (A_\phi(|\text{gs}\rangle_k) - A_\phi(|\text{es}\rangle_k)) + \mathcal{O}(\omega)$$

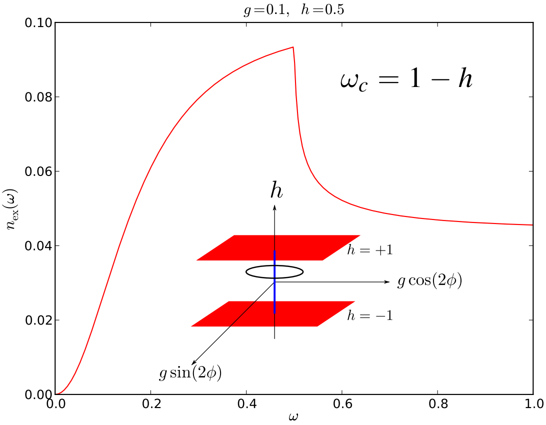
- Fast driving $\omega \to \infty$:

$$\Omega_k(\omega) = 2 - \frac{1}{2}\frac{\epsilon_{\text{gs},k} - \epsilon_{\text{es},k}}{\omega}(A_\phi(|\text{gs}\rangle_k) - A_\phi(|\text{es}\rangle_k)) + \mathcal{O}(\frac{1}{\omega^2})$$

Circular Dynamics: Limit of large ϕ_f

$$\boxed{n_{\rm ex} \sim \int_{-\pi}^{\pi} \frac{dk}{2\pi} \mathsf{g}_{\phi\phi}(|\mathrm{gs}\rangle_k)\frac{\omega^2}{\epsilon_k^2}, \quad \omega \ll \Delta\epsilon}$$

$$\boxed{n_{\rm ex} \sim \mathsf{g}_{\phi\phi}(|\mathrm{GS}\rangle)\frac{1}{2}, \quad \omega \gg \Delta\epsilon}$$

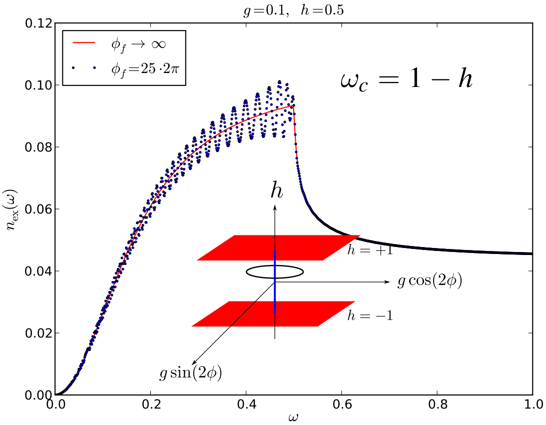


$$\epsilon_{k,\rm rot} = \sqrt{(h + \omega - \cos k)^2 + g^2 \sin^2 k}$$

Circular Dynamics: Limit of large ϕ_f

$$n_{\mathrm{ex}} \sim \int_{-\pi}^{\pi} \frac{dk}{2\pi} \mathsf{g}_{\phi\phi}(|\mathrm{gs}\rangle_k) \frac{\omega^2}{\epsilon_k^2}, \quad \omega \ll \Delta\epsilon$$

$$n_{\mathrm{ex}} \sim \mathsf{g}_{\phi\phi}(|\mathrm{GS}\rangle) \frac{1}{2}, \quad \omega \gg \Delta\epsilon$$



$$\epsilon_{k,\mathrm{rot}} = \sqrt{(h + \omega - \cos k)^2 + g^2 \sin^2 k}, \qquad h_c = 1 \to h_c = 1 - \omega$$

Summary

- How the geometric phase influences quantum many-body non-equilibrium dynamics.
- Signatures of quantum criticalities in the dependence of the density of excited quasi-particles on the driving velocity, "Dynamical quantum phase transition". Which allows to probe quantum phase transitions in an indirect way, without crossing the critical point.

- Outlook
  - Generalization to more complex systems.
  - Dicke model
    $\hat{\mathrm{H}}(\phi(t)) = \omega_0 \hat{\mathrm{J}}_z + \omega \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} + \frac{\lambda}{\sqrt{N}} \left(\hat{\mathrm{a}}^\dagger + \hat{\mathrm{a}}\right) \left(e^{i\phi(t)} \hat{\mathrm{J}}_+ + e^{-i\phi(t)} \hat{\mathrm{J}}_-\right),$
    $\lambda_c = \sqrt{\frac{\omega\omega_0}{2}} \longrightarrow \lambda_c = \sqrt{\frac{\omega(\omega_0 + \delta_\phi)}{2}}$

Rotated Dicke Model

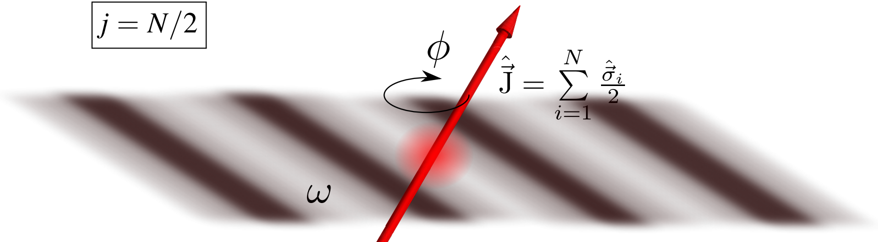
R.H. Dicke, Phys. Rev. 93, 99 (1954)

$$\hat{\mathrm{H}}_{0,\mathrm{D}} = \omega_0 \hat{\mathrm{J}}_z + \omega \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} + \frac{\lambda}{\sqrt{N}} \left(\hat{\mathrm{a}}^\dagger + \hat{\mathrm{a}}\right) \left(\hat{\mathrm{J}}_+ + \hat{\mathrm{J}}_-\right)$$

$$\hat{\mathrm{H}}_\mathrm{D}(\phi(t)) = \hat{\mathrm{U}}(\phi)\, \hat{\mathrm{H}}_{0,\mathrm{D}}\, \hat{\mathrm{U}}^\dagger(\phi), \quad \hat{\mathrm{U}}(\phi(t)) = \exp\left(-i\phi(t)\hat{\mathrm{J}}_z\right)$$

$$\boxed{\hat{\mathrm{H}}(\phi(t)) = \omega_0 \hat{\mathrm{J}}_z + \omega \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} + \frac{\lambda}{\sqrt{N}} \left(\hat{\mathrm{a}}^\dagger + \hat{\mathrm{a}}\right) \left(e^{i\phi(t)}\hat{\mathrm{J}}_+ + e^{-i\phi(t)}\hat{\mathrm{J}}_-\right)}$$

$$\boxed{j = N/2}$$



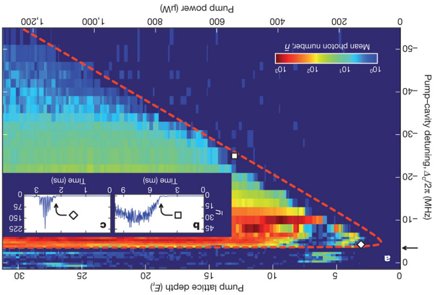
F. Plastina, G. Liberti and A. Carollo, Europhys. Lett., 76, 182188 (2006)

Criticalities of the Dicke Model

Solved in the TDL ($N \to \infty$) using Holstein-Primakoff transformation and Displacement of the bosonic modes: C. Emary and T. Brandes, Phys. Rev. E 67, 066203 (2003)

$$\boxed{\lambda_c = \frac{1}{2}\sqrt{\omega\omega_0}}$$

K. Baumann et al., Nature 464, 1301-1306 (29 April 2010)

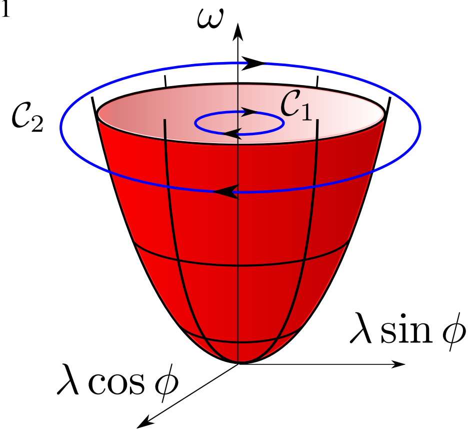


Criticalities of the Dicke Model

Solved in the TDL ($N \to \infty$) using Holstein-Primakoff transformation of SU(2) and Displacement of the bosonic modes:
C. Emary and T. Brandes, Phys. Rev. E 67, 066203 (2003)

$$\boxed{\lambda_c = \frac{1}{2}\sqrt{\omega\omega_0}}$$

$\omega_0 = 1$



Coherent States

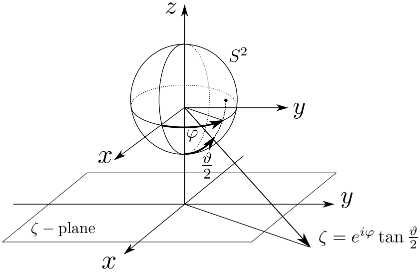
Spin (atomic) coherent states

$$|\zeta(t)\rangle := \exp\left(\zeta(t)\hat{J}_+ - \zeta^*(t)\hat{J}_-\right)|j,-j\rangle, \qquad \zeta(t) \in \mathbb{C}$$

where $|j,-j\rangle$ is the "minimal-weight" eigenstate: $\hat{J}_z|j,-j\rangle = -j|j,-j\rangle$.

$$\langle\zeta|\hat{\mathbf{J}}|\zeta\rangle = j\begin{pmatrix}\sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta\end{pmatrix}$$

$$\zeta = e^{i\varphi}\tan\frac{\vartheta}{2}$$



$\{|\zeta\rangle\}$ over-complete set!

$$\hat{1} = \int d^2\zeta \frac{2j+1}{\pi(1+\zeta\zeta^*)^2}|\zeta\rangle\langle\zeta|$$

Coherent States

Field (oscillator) coherent states

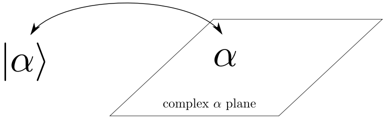
$$|\alpha(t)\rangle := \exp\left(\alpha(t)\hat{a}^\dagger - \alpha^*(t)\hat{a}\right)|0\rangle, \qquad \alpha(t) \in \mathbb{C}$$

where $|0\rangle$ is the vacuum state $\hat{a}|0\rangle = 0$. $\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$$

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

$$\hat{D}(\beta)|\alpha\rangle = e^{i\mathrm{Im}(\alpha\beta^*)}|\alpha+\beta\rangle$$



$\{|\alpha\rangle\}$ over-complete set!

$$\hat{1} = \int d^2\alpha \frac{1}{\pi}|\alpha\rangle\langle\alpha|$$

Time-Dependent Mean-Field Approach

$$\boxed{\mathcal{S}\left[\alpha(t), \zeta(t)\right] = \int_{t_i}^{t_f} dt \left[\langle \alpha\zeta | i\partial_t - \hat{\mathrm{H}}_{\mathrm{D}}(\phi(t)) | \zeta\alpha \rangle \right]}$$

$$\delta\mathcal{S}\left[\alpha(t), \zeta(t)\right] = 0$$

$$\zeta(t) = \frac{q_1(t) + ip_1(t)}{\sqrt{4j - (q_1^2(t) + p_1^2(t))}}$$

$$\alpha(t) = \frac{1}{\sqrt{2}}(q_2(t) + ip_2(t))$$

$$\mathcal{H}_{\mathrm{cl}} := \langle \alpha\zeta | \hat{\mathrm{H}}_{\mathrm{D}}(\phi(t)) | \zeta\alpha \rangle$$

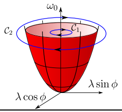
$$\frac{\partial\mathcal{H}_{\mathrm{cl}}}{\partial q_i} = -\dot{p}_i, \qquad \frac{\partial\mathcal{H}_{\mathrm{cl}}}{\partial p_i} = \dot{q}_i$$

Time-Dependent Mean-Field Approach

$$
\begin{aligned}
\dot{q}_1 &= \omega_0 p_1 - 2\lambda \frac{\left(\cos\phi(t) q_1 + \sin\phi(t) p_1\right) p_1 q_2}{\sqrt{4j\left(4j - \left(q_1^2 + p_1^2\right)\right)}} + \\
&+ 2\lambda \sqrt{\frac{4j - \left(q_1^2 + p_1^2\right)}{4j}} \sin\phi(t) q_2 \\
\dot{p}_1 &= -\omega_0 q_1 + 2\lambda \frac{q_1 \left(\cos\phi(t) q_1 + \sin\phi(t) p_1\right) q_2}{\sqrt{4j\left(4j - \left(q_1^2 + p_1^2\right)\right)}} + \\
&- 2\lambda \sqrt{\frac{4j - \left(q_1^2 + p_1^2\right)}{4j}} \cos\phi(t) q_2 \\
\dot{q}_2 &= \omega p_2 \\
\dot{p}_2 &= \omega q_2 + 2\lambda \sqrt{\frac{4j - \left(q_1^2 + p_1^2\right)}{4j}} \left(\cos\phi(t) q_1 + \sin\phi(t) p_1\right)
\end{aligned}
$$

Fixed Points

$$\phi(t) = \delta_\phi\, t$$



$$q_{1,c_1} = 0, p_{1,c_1} = 0, q_{2,c_1} = 0, p_{2,c_1} = 0, \quad \text{stable if} \quad \lambda < \frac{\sqrt{\omega(\omega_0 + \delta_\phi)}}{2}$$

$$q_{1,c_{2,3}} = \mp\sqrt{2j\left(1 - \frac{\omega(\omega_0+\delta_\phi)}{4\lambda^2}\right)}\cos\phi(t)$$

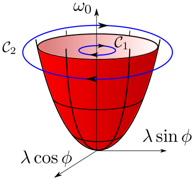
$$p_{1,c_{2,3}} = \mp\sqrt{2j\left(1 - \frac{\omega(\omega_0+\delta_\phi)}{4\lambda^2}\right)}\sin\phi(t)$$

$$q_{2,c_{2,3}} = \frac{2\lambda}{\omega}\sqrt{j\left(1 - \left(\frac{\omega(\omega_0+\delta_\phi)}{4\lambda^2}\right)^2\right)}$$

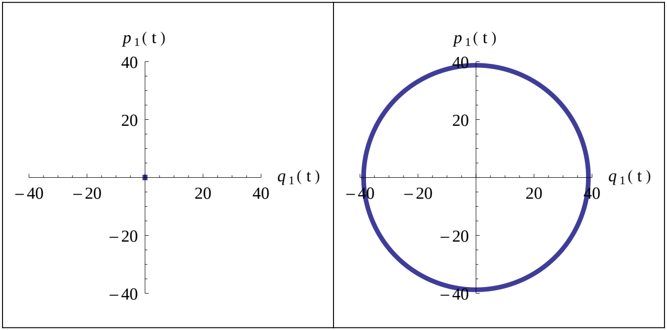
$$p_{2,c_{2,3}} = 0$$

$$\text{real and stable if} \quad \lambda > \frac{\sqrt{\omega(\omega_0 + \delta_\phi)}}{2}$$

Fixed Point - Circles



$$\phi(t) = \delta_\phi\, t$$



$$\lambda < \lambda_{c,(\mathrm{ROT})} = \frac{1}{2}\sqrt{\omega(\omega_0 + \delta_\phi)} \qquad\qquad \lambda > \lambda_{c,(\mathrm{ROT})}$$

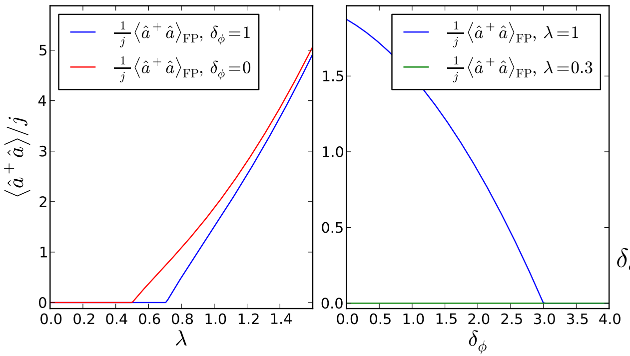
Mean Photon Number

$$\phi(t) = \delta_\phi \, t$$

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \psi(t_f) | \hat{a}^\dagger \hat{a} | \psi(t_f) \rangle = |\alpha_{c_2}|^2 =$$

$$= \begin{cases} \frac{1}{2}\left(\frac{2\lambda}{\omega}\right)^2 \left[1 - \left(\frac{\omega(\omega_0+\delta_\phi)}{4\lambda^2}\right)^2\right] & \text{if } \lambda \geq \frac{1}{2}\sqrt{\omega(\omega_0+\delta_\phi)} \\ 0 \quad \text{if } \lambda < \frac{1}{2}\sqrt{\omega(\omega_0+\delta_\phi)} \end{cases}$$

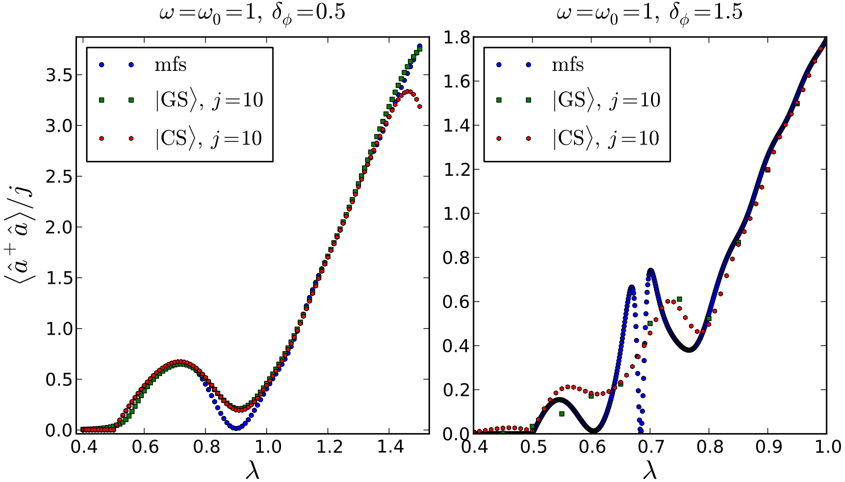
$\omega = 1, \quad \omega_0 = 1$



$$\delta_{\phi,c} = \frac{4\lambda^2}{\omega} - \omega_0$$

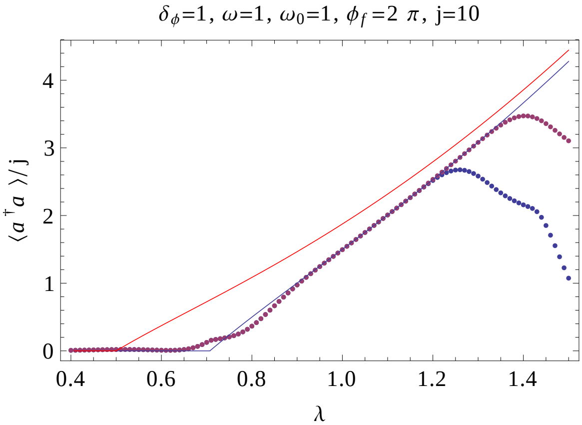
Check validity of TDMF equations

$$\phi(t) = \delta_\phi\, t, \quad \phi_f = 2\pi, \qquad n_{\max} = 50$$



Check validity of TDMF equations

$$\phi(t) = \delta_\phi\, t, \quad \phi_f = 2\pi, \qquad n_{\max} = 50, 40, \qquad |\psi(0)\rangle = |\alpha_{c_2}\rangle|\zeta_{c_2}\rangle$$

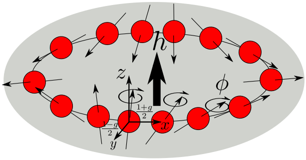


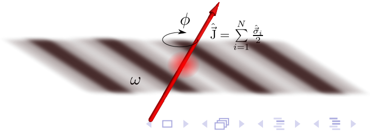
Summary

- The geometric phase

$$\boxed{\gamma = \int_{\mathcal{C}} i \langle n|\partial_j|n\rangle \, d\lambda^j}$$

can induces “dynamical quantum phase transition” in different rotationally driven quantum systems. Which allows one to probe quantum phase transitions in an indirect way, without crossing the critical point.



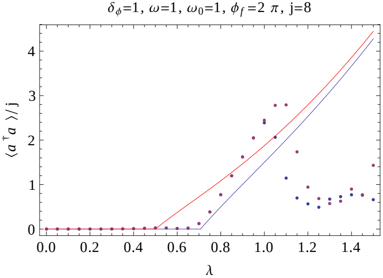


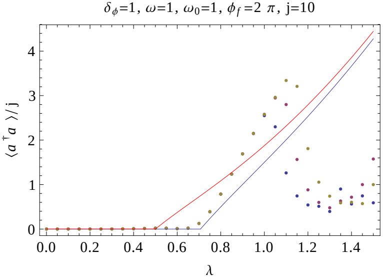
Thank you!

$$\hat{\mathrm{H}}(\phi(t)) = \omega_0 \hat{\mathrm{J}}_z + \omega \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} + \frac{\lambda}{\sqrt{N}} \left(\hat{\mathrm{a}}^\dagger + \hat{\mathrm{a}}\right) \left(e^{i\phi(t)} \hat{\mathrm{J}}_+ + e^{-i\phi(t)} \hat{\mathrm{J}}_-\right)$$

$$\langle \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} \rangle = \langle \psi(t_f) | \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} | \psi(t_f) \rangle = \langle 0 | \langle j, -j | \hat{\mathrm{U}}^\dagger(t_f) \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} \hat{\mathrm{U}}(t_f) | j, -j \rangle | 0 \rangle$$

$$\phi(t) = \delta_\phi \, t, \quad t_f = \frac{2\pi}{\delta_\phi},$$



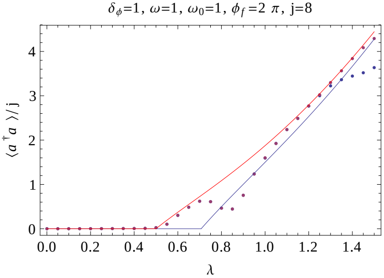


$$\frac{\alpha}{j} = \frac{1}{2} \left(\frac{2\lambda}{\omega}\right)^2 \left[1 - \left(\frac{\omega(\omega_0 + \delta_\phi)}{4\lambda^2}\right)^2\right]$$

$$\hat{\mathrm{H}}(\phi(t)) = \omega_0 \hat{\mathrm{J}}_z + \omega \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} + \frac{\lambda}{\sqrt{N}} \left(\hat{\mathrm{a}}^\dagger + \hat{\mathrm{a}} \right) \left(e^{i\phi(t)} \hat{\mathrm{J}}_+ + e^{-i\phi(t)} \hat{\mathrm{J}}_- \right)$$

$$\langle \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} \rangle = \langle \psi(t_f) | \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} | \psi(t_f) \rangle = \langle \mathrm{GS}_T | \hat{\mathrm{U}}^\dagger(t_f) \hat{\mathrm{a}}^\dagger \hat{\mathrm{a}} \hat{\mathrm{U}}(t_f) | \mathrm{GS}_T \rangle$$

$$\phi(t) = \delta_\phi \, t, \quad t_f = \frac{2\pi}{\delta_\phi},$$



$$\frac{\alpha}{j} = \frac{1}{2} \left(\frac{2\lambda}{\omega} \right)^2 \left[1 - \left(\frac{\omega(\omega_0 + \delta_\phi)}{4\lambda^2} \right)^2 \right]$$

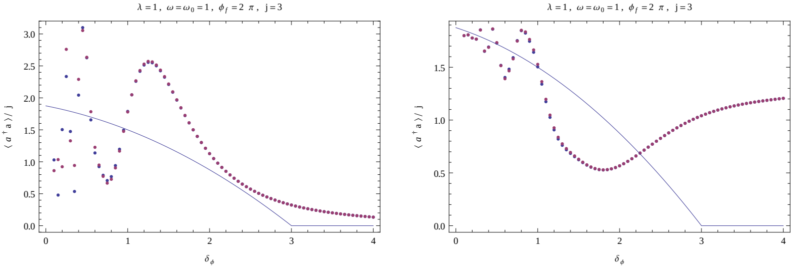


Figure: (left) start in $|j, -j\rangle|0\rangle$ (right) start in $|\mathrm{GS}_T\rangle$

Thank you!