Critical scaling in a trap

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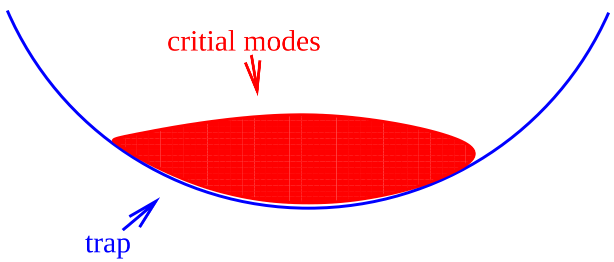
Physical systems are generally inhomogeneneous, homogeneous systems are often an ideal limit of experimental conditions.

General issue: How quantum and thermal critical behaviors develop in the presence of external space-dependent fields

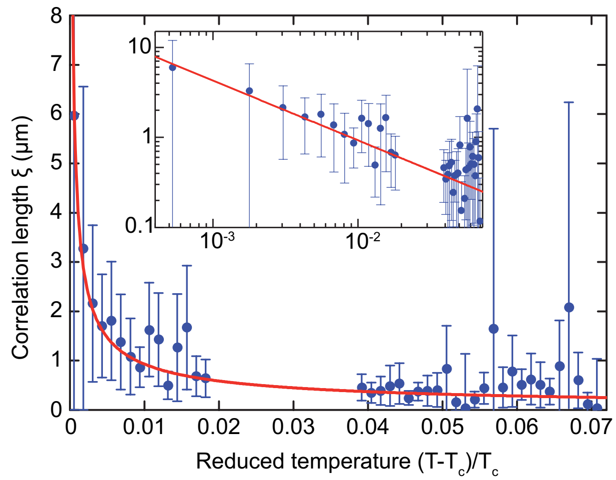
Ex.: interacting particles trapped within a limited region of space by an external potential, such as in experiments of



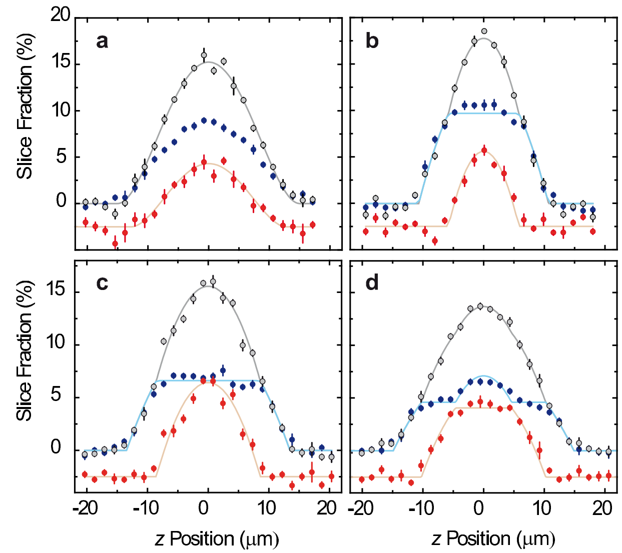
Bose-Einstein condensation in diluted atomic vapors and of cold atoms in optical lattices → interplay between quantum and statistical behaviors

Trap effects at thermal and quantum transitions are discussed in the framework of the trap-size scaling (M.Campostrini, EV, PRL 102,240601,2009; PRA 81,023606,2010)

Finite-T transition related to the **Bose-Einstein condensation in interacting gases**, experiments show an increasing correlation length compatible with a continuous transition (Donner, etal, Science 2007). Moreover, experimental evidences of the Kosterlitz-Thouless transition in 2D (e.g., Hung etal, Nature 2010)



Quantum Mott insulator to superfluid transitions and different Mott phases (where the density is independent of μ) have been observed in many experiments with ultracold atomic gases loaded in optical lattices (e.g., Fölling etal 2006, Inguscio etal 2009)



A common feature is a **confining potential**, which can be varied to achieve different spatial geometries, allowing also to effectively reduce the spatial dims

A classical example:

The lattice gas model in a confining field $V(r) = (|\vec{r}|/l)^p$,

$$\mathcal{H}_{\rm Lgas} = -4J \sum_{\langle ij \rangle} \rho_i \rho_j - \mu \sum_i \rho_i + \sum_i 2V(r_i)\rho_i,$$

where $\rho_i = 0, 1$ whether the site is empty or occupied.

Far from the origin $\langle \rho_x \rangle \to 0$ (as $\langle \rho_x \rangle \sim e^{-2V(x)}$), thus particles are trapped.

It can be exactly mapped to a standard Ising model:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j + h \sum_i s_i - \sum_i V(r_i) s_i, \qquad s_i = 1 - 2\rho_i, \qquad h = 2qJ + \mu/2$$

In the absence of the trap, liquid-gas transition and Ising critical behavior with a diverging length scale, at $T = T_c$ and $\mu = \mu_c = -4qJ$ ($h = h_c = 0$).

No diverging length scale in the presence of the confining potential

How is the critical behavior distorted by the trap, and recovered in the limit $l \to \infty$?

A quantum example:

Atomic gases loaded in optical lattices are generally described by the Bose-Hubbard (BH) model with a confining potential

$$H_{\rm BH} = -\frac{J}{2}\sum_{\langle ij\rangle}(b_i^\dagger b_j + b_j^\dagger b_i) + \sum_i [(\mu + V(r_i))n_i + U n_i(n_i - 1)],$$

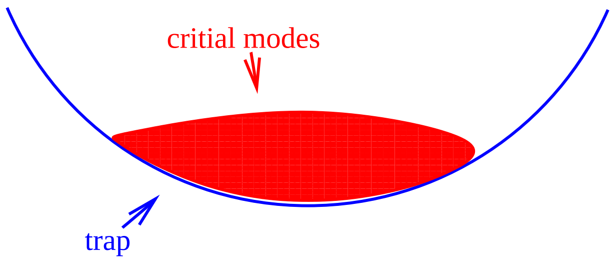
where $n_i = b_i^\dagger b_i$, $V(r) = v^p r^p$, and the trap length scale $l \equiv J^{1/p}/v$

The trapping potential strongly affects the critical behavior at the Mott transitions and within the superfluid phases: correlation functions are not expected to develop a diverging length scale.

A theoretical description of the critical correlations in trapped systems is important for experimental investigations.

In a trap, correlations do not develop a diverging length scale.

The critical behavior of the homogeneous system is observed around the middle of the trap only when $\xi \ll l_{\rm trap}$



If $\xi \gtrsim l_{\rm trap}$, it gets distorted by the trap, although it may still show universal effects controlled by the universality class of the transition of the unconfined system.

These universal effects are described by the trap-size scaling theory, resembling the finite-size scaling theory in critical phenomena, but characterized by a further nontrivial *trap critical exponent* θ, which describes how the critical length scale ξ depends on the trap size at criticality, i.e., $\xi \sim l_{\rm trap}^{\theta}$ (the above naive relations in magenta thus become respectively $\xi \ll l_{\rm trap}^{\theta}$ and $\xi \gtrsim l_{\rm trap}^{\theta}$)

Plan of the rest of the talk:

- Trap-size scaling at *thermal* transitions

  - Lattice gas model, • Finite-T transitions related to the formation of BEC in interacting gases

- TSS at $T = 0$ quantum transitions in D-dim quantum systems (described by $(D + z)$-dim QFT's)

  - The XY chain in the presence of a space-dependent transverse field, as a laboratory model

  - The Bose-Hubbard (BH) model, describing cold bosonic atoms in optical lattices

- list of related issues and further developments

Scaling law of homogeneous systems

$$\mathcal{F}_{\text{sing}}(u_1, u_2, \ldots, u_k, \ldots) = b^{-d}\mathcal{F}_{\text{sing}}(b^{y_1}u_1, b^{y_2}u_2, \ldots, b^{y_k}u_k, \ldots)$$

u_k are nonlinear scaling fields (analytic functions of the model parameters)

In a standard continuous transition: **two relevant scaling fields** $u_t \sim t = T/T_c - 1$ (with $y_t = 1/\nu$) and $u_h \sim h$ (external field, with $y_h = (d-2+\eta)/2$), and irrelevant u_i ($i \geq 3$) with $y_i < 0$.

When $u_t, t \to 0$ and $u_h, h \to 0$

$$\mathcal{F}_{\text{sing}} \approx \xi^{-d}\left[f(h\xi^{y_h}) + \xi^{-\omega} f_\omega(h\xi^{y_h}) + \ldots\right], \qquad \xi \sim t^{-\nu}$$

$O(\xi^{-\omega})$ arises from the leading irrelevant u_3, and $\omega = -y_3$.

Finite-size scaling in a finite system

$$\mathcal{F}_{\text{sing}}(u_1, u_2, \ldots, L) = b^{-d}\mathcal{F}_{\text{sing}}(b^{y_1}u_1, b^{y_2}u_2, \ldots, L/b)$$

thus $\mathcal{F}_{\text{sing}}(u_t, u_h) = L^{-d}\mathcal{F}_{\text{sing}}(L^{y_t}u_t, L^{y_h}u_h)$

Trap-size scaling (**TSS**) in the presence of the confining potential

$$V(r) = v^p |\vec{r}|^p, \qquad l \equiv 1/v \quad \text{is the trap size}$$

Ex.: $\mathcal{H}_{\rm Lgas} = -4J \sum_{\langle ij \rangle} \rho_i \rho_j - \mu \sum_i \rho_i + \sum_i 2V(r_i)\rho_i$ with $\rho_i = 0, 1$

Scaling Ansatz to allow for the confining potential:

$$\mathcal{F}(u_t, u_h, u_v, x) = b^{-d} \mathcal{F}(u_t b^{y_t}, u_h b^{y_h}, u_v b^{y_v}, x/b)$$

where $y_t = 1/\nu$, $y_h = (d + 2 - \eta)/2$, while y_v must be determined.

Then, fixing $u_v b^{y_v} = 1$, and defining the trap exponent $\theta \equiv 1/y_v$,

$$\text{TSS}: \qquad \mathcal{F} = l^{-\theta d} \mathcal{F}(u_t l^{\theta y_t}, u_h l^{\theta y_h}, x l^{-\theta})$$

resembling FSS: $\mathcal{F}_{\rm sing}(u_t, u_h) = L^{-d} \mathcal{F}_{\rm sing}(u_t L^{y_t}, u_h L^{y_h})$, with $L \to l^\theta$

Critical dynamics by adding a time dependence, through the scaling variable $t l^{-z\theta}$

Finite-size effects by adding the L dependence, through $L l^{-\theta}$ (de Queiroz, dos Santos, Stinchcombe, PRE 81,051122,2010)

The correlation length ξ around the middle of the trap, or any generic length scale associated with the critical modes, behaves as

$$\xi = l^{\theta} \mathcal{X}(t l^{\theta/\nu}), \qquad \mathcal{X}(y) \sim y^{-\nu} \text{ for } y \to 0$$

The trap induces a critical length scale $\xi \sim l^{\theta}$ at $t = 0$.

A generic quantity S is expected to asymptotically behave as

$$S = l^{-\theta y_s} f_s(t l^{\theta/\nu}, x l^{-\theta}) = l^{-\theta y_s} \bar{f}_s(\xi l^{-\theta}, x l^{-\theta})$$

The hard-wall limit, $p \to \infty$ of $V(r) = (|\vec{r}|/l)^p$, $\longrightarrow$ homogeneous system of size $L = 2l$ with open boundary conditions.

Standard finite size scaling for $p \to \infty$, thus $\lim_{p\to\infty} \theta = 1$ (in FSS the RG dimension of the size L is $y_L = -1$, since $\xi \sim L$ at T_c)

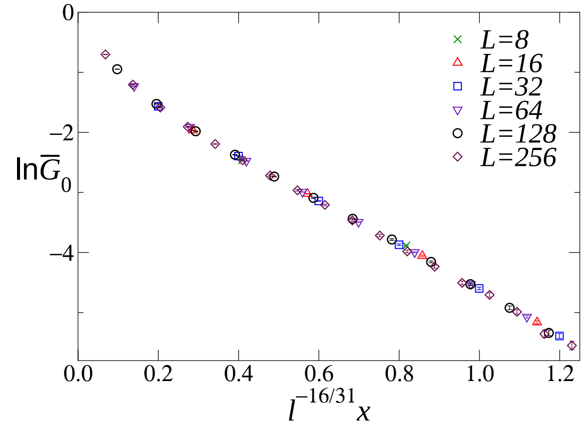
The trap exponent θ can be computed by analyzing the RG properties of the corresponding perturbation at the critical point.

In the lattice gas model, $\mathcal{H}_{\rm Lgas} = -4J\sum_{\langle ij\rangle}\rho_i\rho_j - \mu\sum_i \rho_i + \sum_i 2V(r_i)\rho_i$, the trapping potential is coupled to the order parameter, thus $P_V = \int d^dx\, V(x)\phi(x)$ to $H_{\phi^4} = \int d^dx \left[(\partial_\mu\phi)^2 + r\phi^2 + u\phi^4\right]$.

Using scaling relations ($y_V = p/\theta - p = d - y_\phi$, $y_\phi = (d-2+\eta)/2$) $\rightarrow$
$\theta = 2p/(d+2-\eta+2p)$, $p = 2$: $\theta = 16/31,\ 0.4462,\ 2/5$ in 2,3 and 4D.

$$G_0(x) \equiv \langle\rho_0\rho_x\rangle - \langle\rho_0\rangle\langle\rho_x\rangle = l^{-2\theta y_\phi} f_g(tl^{\theta/\nu}, xl^{-\theta})$$

Results of MC simulations: $\overline{G}_0(x) \equiv l^{4/31}G_0(x)$ vs $xl^{-16/31}$ at T_c



Relaxational dynamics: time scale diverging as $\tau \sim l^{\theta z}$ where z is the dynamic exponent; confirmed by MC simulations with $z = 2.170(6)$

Finite-T transitions in interacting Bose gases and BEC

The condensate wave function $\Psi(x)$ provides the $U(1)$ symm complex order parameter of the transition, thus expected to belong to

the XY universality class: $H_{XY} = \int d^d x \, (|\partial_\mu \Psi|^2 + r|\Psi|^2 + u|\Psi|^4)$

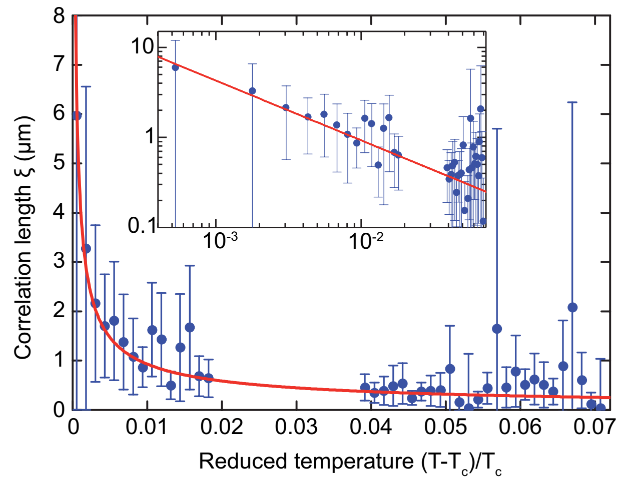
$\nu = 0.6717(1)$, $\eta = 0.0387(1)$ in 3D, shared with the superfluid transition in ^{4}He, superconductor transitions, transition in easy-plane magnets, etc...

No real BEC in 2D, but a finite-T Kosterlitz-Thouless transition with an exponential behavior of ξ, formally $\nu \to \infty$, to a quasi-long range order phase with one-body correlation functions decaying algebraically

In a harmonic trap, the confining potential $V(x) = v^2 x^2 \equiv (x/l)^2$ is coupled to the particle density, giving rise to $P_V = \int d^3 x \, v^2 |x|^2 |\Psi(x)|^2$.

By scaling arguments: $\theta = 1/y_v = 2\nu/(1+2\nu)$, thus $\theta = 0.57327(4)$ in 3D $(\nu = 0.6717(1))$ and $\theta = 1$ in 2D $(\nu = \infty)$, for comparison, $\theta = 1/2$ for a Gaussian theory $(\nu = 1/2)$

Experimental results for a trapped Bose gas at BEC (Donner, etal, Science 2007) showed an increasing correlation length, leading to the estimate $\nu = 0.67(13)$ by fitting to $\xi \sim t^{-\nu}$ (to be compared with $\nu_{XY} = 0.6717(1)$).



Trap effects are negligible when $\xi \ll c l^\theta$, but relevant when $\xi \approx l^\theta$.

However, exp results are not sufficiently precise to show trap effects, (analogously to the experimental evidences of the KT transition in 2D)

Experiments may probe TSS, by varying the trapping potential and matching the trap-size dependence of the TSS Ansatz, analogously to experiments probing FSS behavior in ^{4}He at the superfluid transition.

One may exploit TSS, using it to infer the critical exponents from the data, analogously to FSS techniques to determine the critical parameters.

$T = 0$ transitions driven by quantum fluctuations: **quantum critical behavior** with a peculiar interplay between quantum and thermal fluctuations at low T.

Nonanaliticity of the ground-state energy, where the gap Δ vanishes

Continuous QPT $\longrightarrow$ diverging length scale ξ, and scaling properties.

Example: **the Ising chain in a transverse field**

$$H_{\rm Is} = -J \sum_i \sigma_i^x \sigma_{i+1}^x - \mu \sigma_i^z$$

$\mu = 0 \longrightarrow$ two degenerate ground states $\prod_i | \rightarrow_i \rangle$ and $\prod_i | \leftarrow_i \rangle$

$\mu \to \infty \longrightarrow$ GS=$\prod_i | \uparrow_i \rangle$, breaking Z_2

These phases extend to finite μ, quantum transition at $\mu_c/J = 1$, between quantum paramagnetic and ordered phases

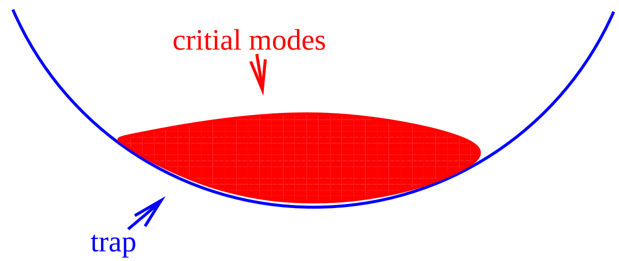
2D Ising quantum critical behavior with $\Delta \sim \xi^{-1} \sim |\mu - \mu_c|$

A QPT is generally characterized by a relevant parameter μ, with RG dimensions $y_\mu \equiv 1/\nu$, and dynamic exponent z:

$$\xi \sim |\bar{\mu}|^{-\nu}, \qquad \Delta \sim |\bar{\mu}|^{z\nu} \sim \xi^{-z}, \qquad \bar{\mu} = \mu - \mu_c$$

Scaling law of the free energy $F(\mu, T) = b^{-(d+z)} F(\bar{\mu} b^{1/\nu}, T b^z)$

A trapping potential significantly changes the phenomenology of QPT: correlations are not expected to develop a diverging length scale.



TSS to describe how critical correlations develop in large traps.

Scaling Ansatz in the presence of the trap $V(r) = v^p r^p \equiv (r/l)^p$:

$$F(\mu, T, v, x) = b^{-(d+z)} F(\bar{\mu} b^{y_\mu}, T b^z, v b^{y_v}, x/b)$$

$F = l^{-\theta(d+z)} \mathcal{F}(\bar{\mu} l^{\theta/\nu}, T l^{\theta z}, x l^{-\theta})$ where $\nu \equiv 1/y_\mu$ and $\theta \equiv 1/y_v$

For example, TSS of the gap and the length scale:

$$\Delta = l^{-\theta z} \mathcal{D}(\bar{\mu} l^{\theta/\nu}), \qquad \mathcal{D}(y) \sim y^{z\nu} \ \text{ for } \ y \to 0$$

$$\xi = l^{\theta} \mathcal{X}(\bar{\mu} l^{\theta/\nu}, T l^{\theta z}), \qquad \mathcal{X}(y, 0) \sim y^{-\nu} \ \text{ for } \ y \to 0$$

implying a critical length scale scaling as $\xi \sim l^{\theta}$ at $\bar{\mu} = 0$.

The *trap exponent* θ depends on the universality class of the QPT, and the way the potential is coupled to the system.

θ can be computed using RG scaling arguments

The hard-wall limit $p \to \infty$ is equivalent to confining a homogeneous system in a box of size $L = 2l$ with open boundary conditions, thus $\theta \to 1$

TSS provides a general framework for quantum critical behaviors in confined systems.

The quantum XY chain in a transverse field is a standard theoretical laboratory for issues related to quantum transitions.

A space-dependent transverse field gives rise to an inhomogeneity analogous to a trapping potential in particle systems

$$H_{\rm XY} = -\sum_i \frac{1}{2}[(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y] - \mu\sigma_i^z - V(x_i)\sigma_i^z,$$

where $0 < \gamma \leq 1$, $V(x) = v^p|x|^p \equiv (|x|/l)^p$

Map into **spinless fermions** by a Jordan-Wigner transformation:
$\sigma_i^x = \Pi_{j<i}(1-2c_j^\dagger c_j)(c_i^\dagger + c_i)$, $\sigma_i^y = i\Pi_{j<i}(1-2c_j^\dagger c_j)(c_i^\dagger - c_i)$, $\sigma_i^z = 1 - 2c_i^\dagger c_i$

$$H = \sum[c_i^\dagger A_{ij} c_j + \frac{1}{2}(c_i^\dagger B_{ij} c_j^\dagger + {\rm h.c.})], \qquad A_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1} + 2Q(x_i)\delta_{ij},$$
$$B_{ij} = -\gamma\,(\delta_{i+1,j} - \delta_{i,j+1})\,, \quad Q(x) = \bar\mu + V(x), \quad \bar\mu \equiv \mu - 1.$$

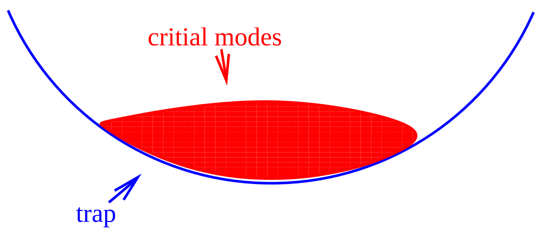
μ plays the role of *chemical potential* for the *c-particles*, and $V(x)$ acts as a *trap*

There are experimental realizations of Ising chains, such as the insulators $CsCoBr_3$, $CoNb_2O_6$, etc.., in a magnetic field.

In the absence of the trap, quantum transition at $\bar{\mu} \equiv \mu - 1 = 0$ in the 2D Ising universality class, separating a quantum paramagnetic phase for $\bar{\mu} > 0$ from a quantum ferromagnetic phase for $\bar{\mu} < 0$.

$$\xi \sim |\bar{\mu}|^{-\nu}, \quad \nu = 1/y_\mu = 1; \qquad \qquad \Delta \sim \xi^{-z}, \quad z = 1$$

In the presence of the confining potential $V(x) = v^p x^p$ ($l \equiv 1/v$ is the trap size), the critical behavior can be observed around the center of the trap in the large-l limit.



Analyzing the RG dim of the corresponding perturbation

$$P_V = \int d^d x \, dt \, V(x) \phi(x)^2 \quad \longrightarrow \quad \theta \equiv 1/y_v = p/(p+1)$$

Using the relations $y_V = p y_v - p$, $y_{\phi^2} = d + z - y_\mu$, $y_V + y_{\phi^2} = d + z$, $p y_v - p = y_\mu$, and the value $z = 1$ and $y_\mu = 1$

TSS can be proved in the XY chain model:

The Hamiltonian can be solved by exact numerical diagonalization, even in the presence of the trapping potential:

- New fermi variables $\eta_k = g_{ki} c_i^\dagger + h_{ki} c_i$ so that $H = \sum_k \omega_k \eta_k^\dagger \eta_k$, $(\omega_k \geq 0)$
- Introduce $\phi_{ki} = g_{ki} + h_{ki}$ and $\psi_{ki} = g_{ki} - h_{ki}$ satisfying the equations $(A + B)\phi_k = \omega_k \psi_k$ and $(A - B)\psi_k = \omega_k \phi_k$
- Solution by solving $(A - B)(A + B)\phi_k = \omega_k^2 \phi_k$,
- The continuum limit, by rewriting the discrete differences in terms of derivatives, has a nontrivial TSS limit: by rescaling

$$x = \gamma^{1/(1+p)} l^{p/(1+p)} X, \quad \bar{\mu} = \gamma^{p/(1+p)} l^{-p/(1+p)} \mu_r, \quad \omega_k = 2\gamma^{p/(1+p)} l^{-p/(1+p)} \Omega_k,$$

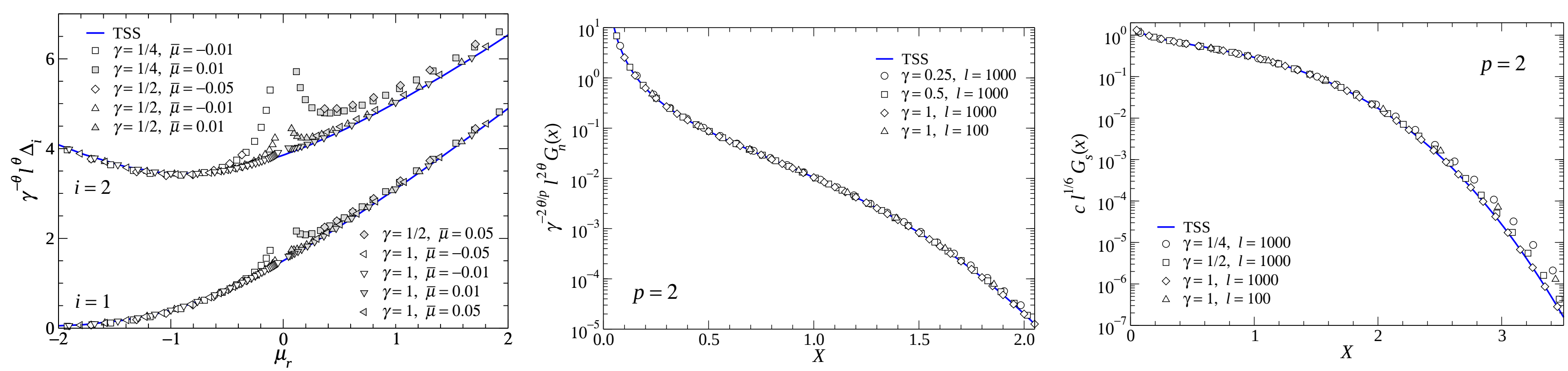
- Keeping only the leading terms in the large-l limit, Schrödinger-like eq

$$(\mu_r + X^p - \partial_X)(\mu_r + X^p + \partial_X)\phi_k(X) = \Omega_k^2 \phi_k(X)$$

Thus, $\theta = p/(p+1)$ **in agreement with RG**. Next-to-leading terms in the large-trap limit give rise to $O(l^{-\theta})$ scaling corrections.

TSS can be analytically derived in the XY chain model, arriving at a continuum Schrödinger-like equation for the lowest states. The asymptotic trap-size dependence confirms the RG scaling arguments

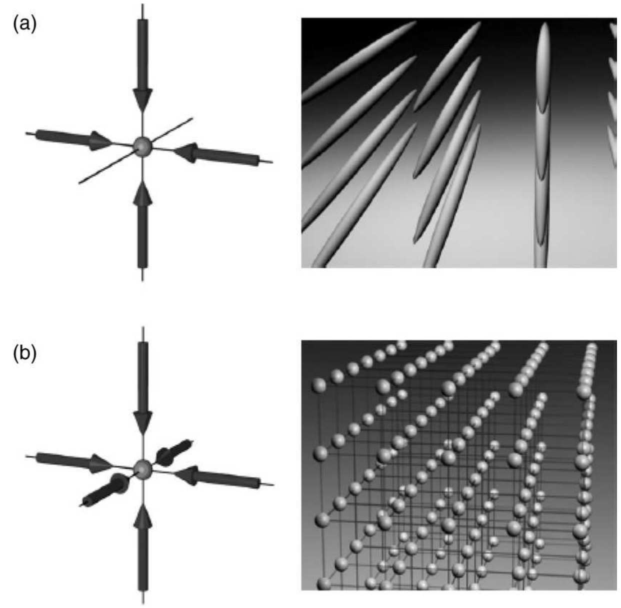
- any low-energy scale behaves as $\Delta \approx \gamma^{\theta} l^{-\theta} \mathcal{D}(\mu_r)$ where $\mu_r \equiv \gamma^{-\theta} l^{\theta} \bar{\mu}$
- particle-density correlator: $G_n(x) \equiv \langle n_0 n_x \rangle_c \approx \gamma^{2\theta/p} l^{-2\theta} \mathcal{G}_n(X)$
- two-point function: $G_s(x) \equiv \langle \sigma_0^x \sigma_x^x \rangle = a_s l^{-\theta\eta} \mathcal{G}_s(X)$
- its second moment correlation length: $\xi = a_\xi \gamma^{\theta/p} l^{\theta} [1 + O(l^{-\theta})]$



- Bipartite entanglement entropy: $\lim_{L\to\infty} S_{\rm vN}(L/2;L) \approx (c/6)\ln l^{\theta}$, instead of $S_{\rm vN}(L/2;L) \approx (c/6)\ln L$ for homogeneous systems

Ultracold atomic gases in optical lattices (arrays of microscopic potentials induced by ac Stark effects of interfering laser beams, which constrain the atoms at the sites of a lattice)

Experiments are set up with a harmonic trap, inducing an effective external potential $V(r) = v^2 r^2$

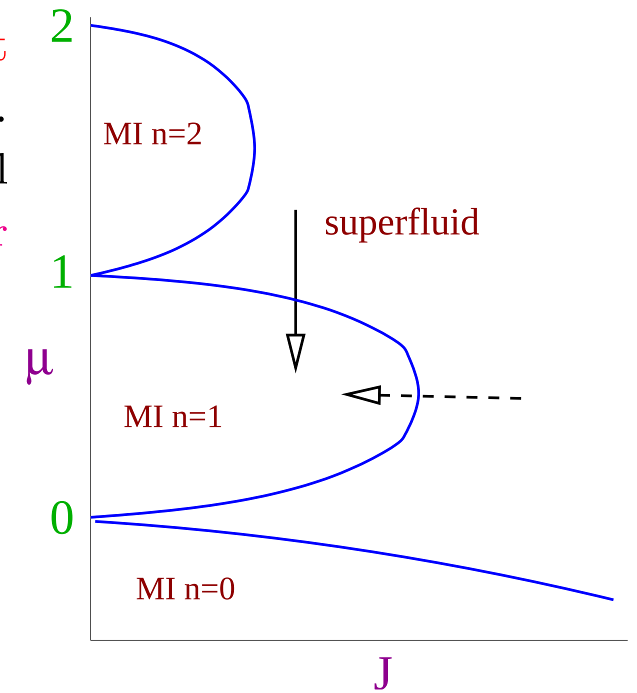


Boson systems described by **the Bose-Hubbard model** ($[b_i, b_j^\dagger] = \delta_{ij}$, $n_i \equiv b_i^\dagger b_i$) (D. Jaksch etal, 1998)

$$H_{\rm BH} = -\frac{J}{2} \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) + \sum_i U n_i (n_i - 1) + \mu \sum_i n_i + \sum_i V(r_i) n_i$$

The experimental capability of varying the confining potential allows to vary the spatial geometry, achieving also quasi-1D geometries.

The BH model presents Mott insulators ($\partial\langle n_i\rangle/\partial\mu = 0$) and superfluid phases. At the transitions driven by μ (ex. along the full arrow), *nonrelativistic* bosonic field theory (Fisher etal, 1989)



$$Z = \int [D\phi] \exp(-\int_0^{1/T} dt\, d^dx\, \mathcal{L}_c),$$

$$\mathcal{L}_c = \phi^* \partial_t \phi + \frac{1}{2m}|\nabla\phi|^2 + r|\phi|^2 + u|\phi|^4,$$

where $r \sim \mu - \mu_c$.

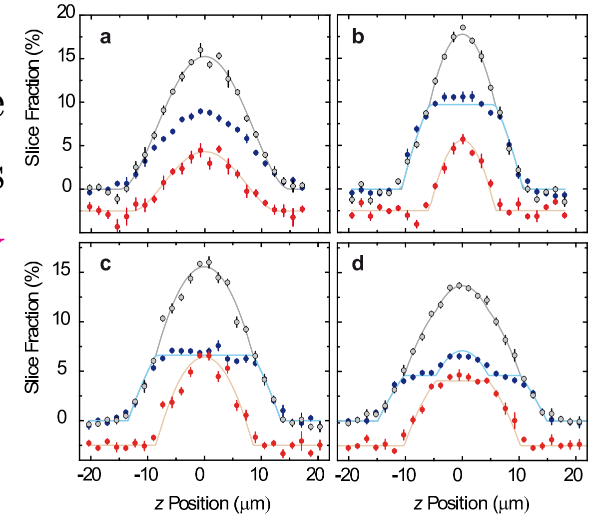
The upper critical dimension is $d_c = 2 \longrightarrow$ mean field for $d > 2$.

For $d = 2$ the FT is free (apart from logs), thus $z = 2, \quad y_\mu = 2$.

The 1D critical theory is equivalent to a free field theory of nonrelativistic spinless fermions, thus $z = 2, \quad y_\mu = 2$.

The special transitions at fixed integer density (along the dashed arrow) $\in$ the $d+1$ XY universality class (*relativistic FT*), thus $z = 1, \quad y_\mu = 1/\nu_{\rm XY}$.

Experimental and theoretical results have shown the coexistence of Mott insulator and superfluid regions in trapped systems, but the critical behavior can only be observed in the large trap-size limit



Within the TSS framework:

$$F(\mu, T, l, x) = l^{-\theta(d+z)}\mathcal{F}(\bar{\mu}l^{\theta/\nu}, Tl^{\theta z}, xl^{-\theta}), \quad \langle O\rangle(\mu, l, x) \sim l^{-y_o\theta}\mathcal{O}(\bar{\mu}l^{\theta/\nu}, xl^{-\theta})$$

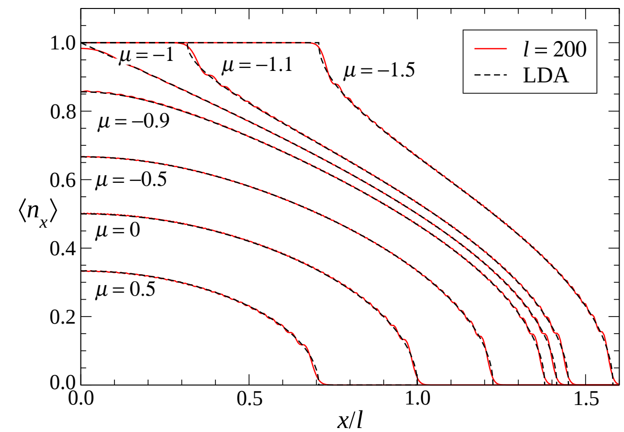
for several physically interesting observables, such as particle density and its correlators, one-particle density matrix, entanglement, etc...

The trap exponent θ can be determined by an analysis of the corresponding RG perturbation, $P_V = \int d^dx\, dt\, V(x)|\phi(x)|^2$, obtaining $\theta = p/(p+y_\mu)$. By replacing the corresponding value of y_μ, this relation yields the value of θ for each specific transition. $\theta = p/(p+2)$ at the μ-driven transitions.

$$H_{\rm BH} = -\frac{J}{2}\sum_{\langle ij\rangle}(b_i^\dagger b_j + b_j^\dagger b_i) + \sum_i U n_i(n_i - 1) + \mu \sum_i n_i + \sum_i V(r_i) n_i$$

- *Thermodynamic limit* at fixed μ: $N, l \to \infty$ keeping N/l^d fixed

$\langle n_x \rangle$ of the 1D hard-core BH model approaches its LDA in the large-l limit, i.e., the value of the particle density of the homogeneous system at $\mu_{\rm eff}(x) \equiv \mu + (x/l)^p$

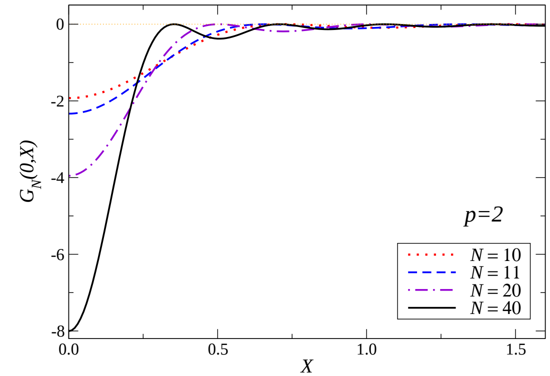


Corrections are suppressed by powers of l, and present a nontrivial scaling: $\langle n_x \rangle = \rho_{\rm lda}(x/l) + l^{-\theta}\mathcal{D}(x/l^\theta, Tl)$. This feature is likely shared by other particle systems, finite U or Hubbard models

- Another interesting TSS regime is that at fixed particle number $N = \sum_i \langle n_i \rangle$, corresponding to the low-density regime $N/l^d \to 0$.

Results for 1D BH models using various approaches:

• Analytical results: TSS in the dilute limit (universal within trapped boson gases with short-range interactions)



• Numerical results by exact diagonalization in the superfluid phase and at the $n = 1$ Mott transition at $T = 0$, showing a modulated TSS at $T = 0$, essentially due to level crossings of the lowest states at finite trap size, $\Delta = l^{-2\theta} A_\Delta(\phi)[1 + O(l^{-\theta/2})]$, $\langle n_0 \rangle = 1 - l^{-\theta} D_0(\phi)[1 + O(l^{-\theta/2})]$

• Numerical results by DMRG to check universality when adding interaction terms in the Hamiltonian.

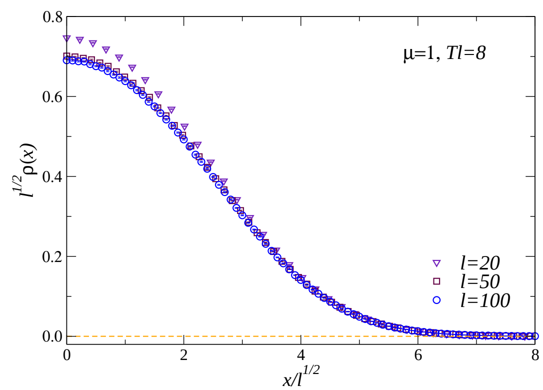
• Numerical results by quantum-MC at finite temperature, showing TSS, while the $T = 0$ modulation phenomenon gets averaged out.

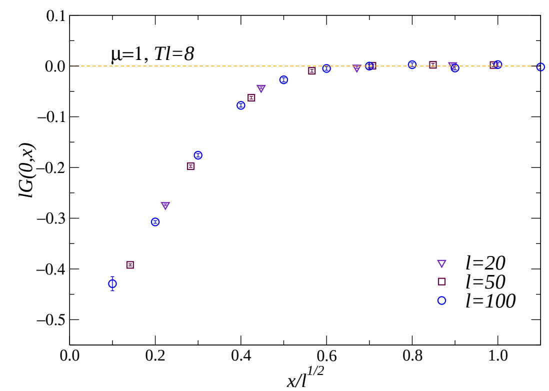
At the Mott transitions:

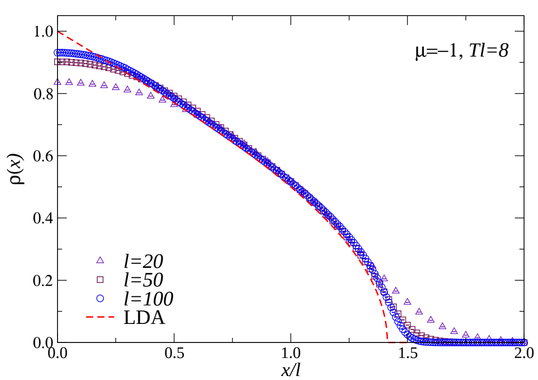
$\langle n_x \rangle = \rho_{\rm lda}(x/l) + l^{-\theta}\mathcal{D}(x/l^\theta, Tl)$ and $\langle n_x n_y \rangle_c = l^{-2\theta}\mathcal{G}(x/l^\theta, y/l^\theta, Tl^{2\theta})$

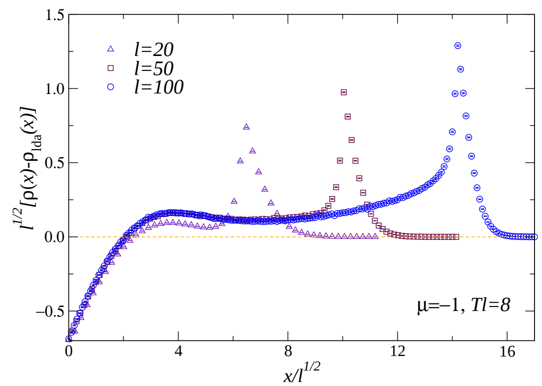
G. Ceccarelli, C. Torrero, EV, PRA 2012)

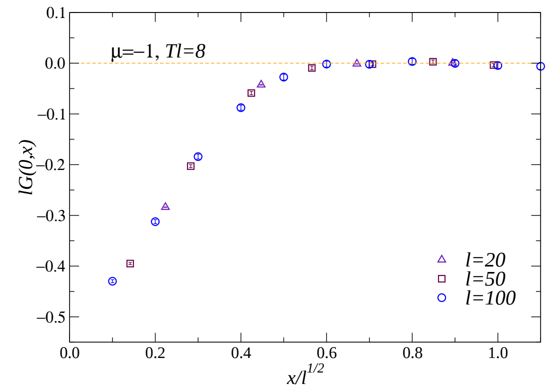
Numerical results by QMC at finite T showing TSS











Entanglement in 1D trapped particle systems

Let us divide the system into two parts A and B: $- - A - - - - B - -$

Entanglement quantified by: $S_{\rm vN} = -\ln {\rm Tr}[\rho_A \ln \rho_A]$ where $\rho_A = {\rm Tr}_B \rho_{\rm GS}$ is the reduced density matrix of the subsystem A.

Quantum critical behaviors described by 2D conformal field theories show logarithmically divergent entanglement entropies.

Dividing the chain in two parts of length l_A and $L - l_A$, CFT predicts $S_{vN}(l_A; L) \approx (c/6)\,[\ln L + \ln \sin(\pi l_A/L) + b]$, where c is the central charge (Calabrese, Cardy, 2004)

The trap destroys conformal invariance. What is the scaling behavior of the entanglement entropies in the TSS limit?

In the presence of the trap of size l, the dependence on L disappears when $L \to \infty$, and the half-space entanglement behaves as $S_{\rm vN}(L/2; L) \approx (c/6)\,(\ln \xi_e + b)$, with $\xi_e = a_e l^\theta$ an entanglement length.
Results for the XY and XX chains (CV, JSTAT P08020, 2010)

CONCLUSIONS: TSS provides a theoretical framework to describe thermal and quantum critical behaviors in confined particle systems.

- Finite-T transitions: $\mathcal{F} = l^{-\theta d}\mathcal{F}(u_t l^{\theta y_t}, u_h l^{\theta y_h}, x l^{-\theta})$, with $\xi \sim l^\theta$ at T_c

θ by scaling arguments: it depends on the universality class, the power law $V(x) = (x/l)^p$, the way it is coupled to the critical modes.

  - Static and dynamics • Lattice gas models • QLRO of 2D systems
  - finite-T transitions of interacting Bose gases with BEC

- At quantum transitions, $\mathcal{F}(\mu, T, l, x) = l^{-\theta(d+z)}\mathcal{F}(\bar{\mu} l^{\theta/\nu}, T l^{z\theta}, x l^{-\theta})$

  - the quantum XY chain in a space-dependent transverse field
  - the Bose-Hubbard model which describes cold atomic gases confined in optical lattices.

Further studies ...

- TSS of the unitary off-equilibrium quantum dynamics, e.g., in the presence of a time-dependent confining potential (Campostrini, V, PRA 82 063636 2010; Collura, Karevski, PRL 104, 200601, 2010)
- TSS of critical dynamics (Costagliola, EV, JSTAT 2011 L08001)
- Finite-size scaling effects in TSS (de Queiroz, dos Santos, Stinchcombe, PRE 81,051122,2010)
- TSS of bipartite entanglement entropies in 1D XY and BH chains (Campostrini, V, JSTAT P08020, 2010)
- TSS in 2D BH models by QMC (Ceccarelli, Torrero, arXiv:1203.2030)

Further related works:

- Quantum correlations and entanglement in Fermi gases trapped within limited space regions, of any dimensions:

- Systematic framework to compute them based on the $N \times N$ *overlap* matrix $\mathbb{A}_{nm} = \int_A d^d z \, \psi_n^*(z)\psi_m(z)$ where $\psi_n(x)$ (Calabrese, Mintchev, EV, PRL 107, 020601, 2011).

- Relations between particle fluctuations and entanglement entropy of an extended region A in noninteracting Fermi gases,
vN entropy/particle variance $\approx \pi^2/3$ for any subsystem A in any dimension, (Calabrese, Mintchev, EV, EPL 98, 20003, 2011), even in the presence of a space-dependent confining potential (V, arXiv:1204.2155).

- Quantum dynamics and entanglement of Fermi gases released from a trap (V, arXiv:1204.3371).