CITY UNIVERSITY LONDON

Entanglement Entropy of Degenerate Ground States:
Permutation Symmetric States

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New quantum states of matter in and out of equilibrium

Galileo Galilei Institute for Theoretical Physics, Florence

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- In the context of high energy physics much of the motivation to study the entanglement entropy has come from its behaviour at quantum critical points [Holzhey, Larsen and Wilczek '94; Calabrese and Cardy '04]:

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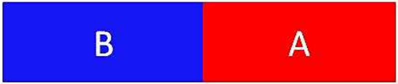
- In this talk I will provide evidence that the entropy can also reveal interesting information about states even when there is no critical point.
- Why bi-partite? It is unclear what would be a good measure of multi-partite entropy... So for now we stick to the following configuration and consider the entanglement between a region A and its complement $B = \bar{A}$



Overview of the talk

- Entanglement entropy

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- Large distance behaviour of the entropy
- Generalizations: Permutation Symmetric States

This talk is mainly based on the following publications:

O.A. Castro-Alvaredo and B. Doyon,
Phys. Rev. Lett. 108 120401 (2012); arXiv:1103.3247
J. Stat. Mech. 1102 P02001 (2011); arXiv:1011.4706

Bi-partite Entanglement Entropy

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$

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then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

and $|\Psi\rangle$ a ground state.

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- Other entropies may also be defined such as Rényi's

Other Entropies

$$S_A^{(n)} = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}$$

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$$\cdots S_{i-1} \otimes \underbrace{S_i \otimes S_{i+1} \otimes \quad \cdots \quad \otimes S_{i+L-1}}_{\text{A}} \otimes S_{i+L} \quad \cdots$$

Replica Trick

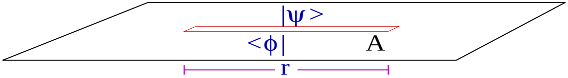
$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_A(\rho_A^n)$$

- For general QFTs the "replica trick" naturally leads to the notion of replica theories on multi-sheeted Riemann surfaces $\quad\Rightarrow\quad$ interpretation of $\mathrm{Tr}_A(\rho_A^n)$

Partition functions on multi-sheeted Riemann surfaces

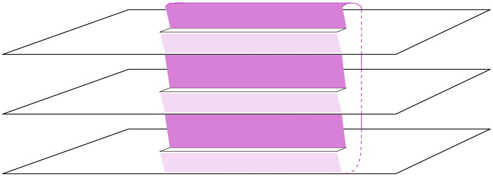
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04] ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n =$$



Twist Fields

- For general 1+1 dimensional QFT we have found [Cardy, OCA & Doyon '08] that the entropy may be expressed in terms of a two-point function of twist fields

$$\boxed{\mathrm{Tr}_A(\rho_A^n) \propto \langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle}$$

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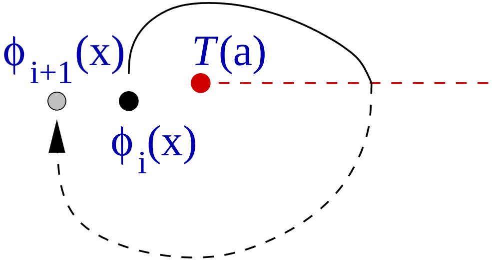
- In QFT the twist field generates cyclic permutations of the n-copies of a QFT in its replica version:



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YES!

- We may consider a replica spin chain theory, consisting of n non-interacting copies of some known model.

Local Cyclic Replica Permutation Operators

- We may consider a replica spin chain theory, consisting of n non-interacting copies of some known model.
- We define $\mathcal{T}_i$, the local cyclic replica permutation operator. It acts on site i and permutes its spin with that of different copies of the same site $\mathcal{T}_i|s_1 s_2 \ldots s_n\rangle = |s_2 \ldots s_n s_1\rangle$.

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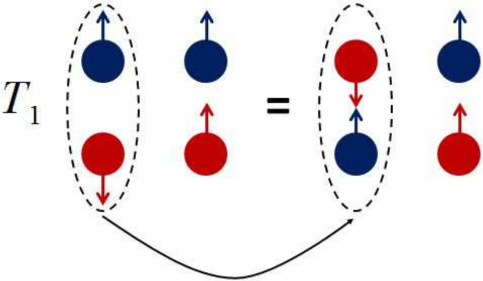

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- In this simple case, the operator $\mathcal{T}_1$ is nothing but the permutation operator. It may be written as

Cyclic Permutation Operator for n=N=2

$$\mathcal{T}_i = E^{11}_{1,i}E^{11}_{2,i} + E^{12}_{1,i}E^{21}_{2,i} + E^{21}_{1,i}E^{12}_{2,i} + E^{22}_{1,i}E^{22}_{2,i} \quad \text{with} \quad i = 1, 2,$$

with $(E^{\epsilon\epsilon'}_{\alpha,i})_{jk} = \delta_{\epsilon,j}\delta_{\epsilon',k} \quad \alpha = 1, \ldots, n \quad \text{and} \quad i = 1, \ldots, N$

Local Cyclic Replica Permutation Operators

- Similarly, by increasing the number of copies to $n=3$ and $n=4$ we find more involved expressions:

$\mathcal{T}_i$ for n=3

$$E_{1,i}^{11}E_{2,i}^{11}E_{3,i}^{11} + E_{1,i}^{12}E_{2,i}^{21}E_{3,i}^{22} + E_{1,i}^{21}E_{2,i}^{12}E_{3,i}^{11} + E_{1,i}^{22}E_{2,i}^{22}E_{3,i}^{22} +$$
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- We can then find a general expression:

General Cyclic Permutation Operator

$$\mathcal{T}_i=\mathrm{Tr}_{\mathrm{aux}}\left(\prod_{\alpha=1}^{n}T_{\alpha,i;\mathrm{aux}}\right)\quad \text{with}\quad T_{\alpha,i;\mathrm{aux}}=\begin{pmatrix} E_{\alpha,i}^{11} & E_{\alpha,i}^{21}\\ E_{\alpha,i}^{12} & E_{\alpha,i}^{22}\end{pmatrix}_{\mathrm{aux}}$$

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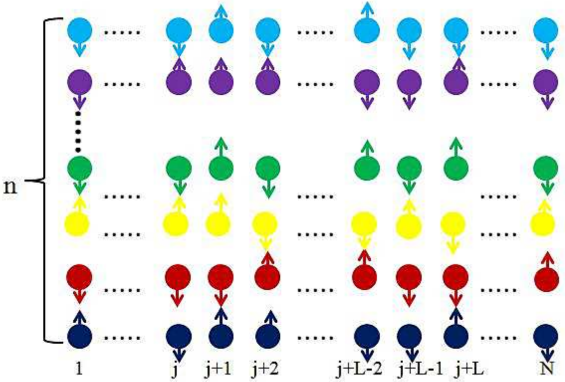

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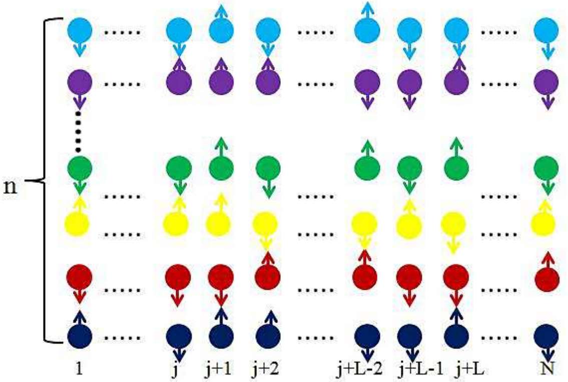

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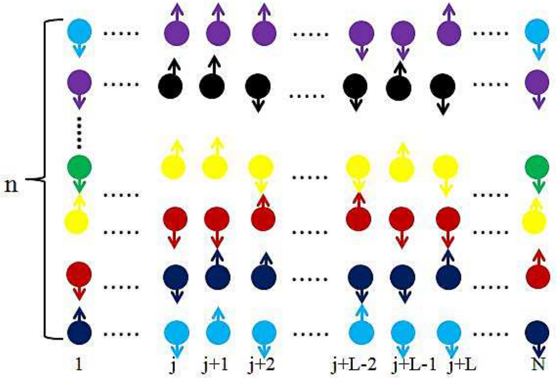

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General Formula for the Entanglement Entropy

- The Rényi entropy of such a "block" of spins would be given by:

$$S^{(n)}_{j,\ldots,j+L} = \frac{1}{1-n} \log\left(\frac{\langle\Psi|\mathcal{T}_j\mathcal{T}_{j+1}\cdots\mathcal{T}_{j+L-1}\mathcal{T}_{j+L}|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

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- And employing the replica trick we also know the von Neumann entropy to be:

$$S_{j,\ldots,j+L} = -\lim_{n \to 1} \frac{d}{dn} \left[\frac{\langle \Psi | \mathcal{T}_j \mathcal{T}_{j+1} \cdots \mathcal{T}_{j+L-1} \mathcal{T}_{j+L} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right]$$

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- In the scaling limit, when both $N, L \to \infty$, expectation values of products of cyclic permutation operators play the role of the two point function of twist fields in QFT.

Examples: the spin-$\frac{1}{2}$ XXZ chain

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Entanglement Entropy of Two Spins

$$\begin{aligned} S_{\{1,m+1\}} \quad &= \quad -\frac{(1-z(m)+4s(m))}{4}\log\left[\frac{(1-z(m)+4s(m))}{4}\right] \\ &\quad -\frac{(1-z(m)-4s(m))}{4}\log\left[\frac{(1-z(m)-4s(m))}{4}\right] \\ &\quad -\frac{(1+z(m))}{2}\log\left[\frac{1+z(m)}{4}\right]. \end{aligned}$$

with $\langle\sigma_1^z\sigma_{m+1}^z\rangle = z(m)$ and $\langle\sigma_1^+\sigma_{m+1}^-\rangle = s(m)$.

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- Computations become very involved whenever more than two spins are considered (e.g. very complex combinatorial problem).
- We therefore decided to look at a theory where correlation functions are so simple, that one may hope to find a general formula for the entropy.
- We obtained those correlation functions simply by taking the $\Delta \to -1^+$ limit of the correlation functions of the XXZ chain. This leads to

The Correlation Functions

$$\lim_{\Delta \to -1^+} \langle E_{j_1}^{\epsilon_1 \epsilon'_1} E_{j_2}^{\epsilon_2 \epsilon'_2} \cdots E_{j_m}^{\epsilon_m \epsilon'_m} \rangle = \frac{1}{2^m} \prod_{j \in B} (-1)^j$$

where B is the subset of sites at which either an operator E^{12} or an operator E^{21} sit.

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Rényi Entropy

$$S_m^{(n)} = -\frac{nm\log 2}{1-n} + \frac{1}{1-n}\log\left(\sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right)^n\right)$$

where m is the number of spins whose entropy is being computed (not necessarily consecutive!).

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Asymptotics

$$S_m = \frac{1}{2}\log\left(\frac{\pi m}{2}\right) + \frac{1}{2} + O\left(m^{-1}\right)$$

$$S_m^{(n)} = \frac{1}{2}\log\left(\frac{\pi m}{2}\right) + \frac{\log(n)}{2(n-1)} + O\left(m^{-1}\right)$$

- Both entropies scale logarithmically, which reminds us of the scaling behaviour of the entropy at critical points.

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- Both entropies scale logarithmically, which reminds us of the scaling behaviour of the entropy at critical points.
- However the coefficient of the log-terms is not what is expected for critical points.

The spin-$\frac{1}{2}$ XXX ferromagnetic chain II

- We have also studied the asymptotic behaviour of these entropies, and this is what we found:

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- Both entropies scale logarithmically, which reminds us of the scaling behaviour of the entropy at critical points.
- However the coefficient of the log-terms is not what is expected for critical points.
- What does this asymptotic behaviour actually mean?

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- All the states that they (and we) considered are permutation symmetric (more later!).
- These results are however just special cases of a more general behaviour which we have (for the first time) interpreted geometrically.

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- The meaning of this "dimension" is relatively simple to explain for the spin-$\frac{1}{2}$ XXX chain.
- An infinite subset of these ground states are factorizable (zero-entropy states) corresponding to choosing all spins to point in a particular direction $\vec{v}$.

Ground states of the spin-$\frac{1}{2}$ XXX spin chain

- Every factorizable ground state $|\Psi\rangle_{\vec{v}}$ may be labeled by a unit vector $\vec{v}$, which takes any possible direction inside the two-dimensional sphere S^2 and such that $\vec{\sigma} \cdot \vec{v}|\Psi\rangle_{\vec{v}} = |\Psi\rangle_{\vec{v}}$.

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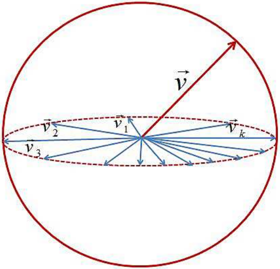
- The particular ground state whose entropies we have computed corresponds to an infinite linear combination where all coefficients c_α are equal to each other and where the vectors $\vec{v}_\alpha$ are such as to generate a great circle on the unit sphere. Graphically...

Different geometries



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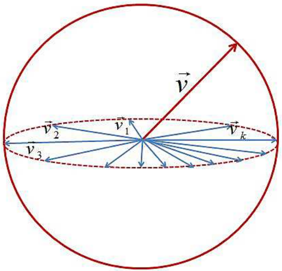

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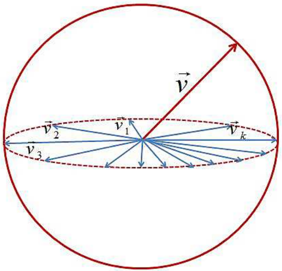

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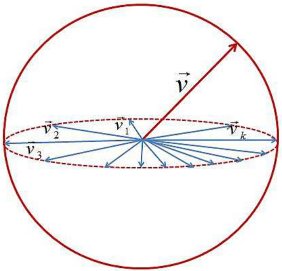

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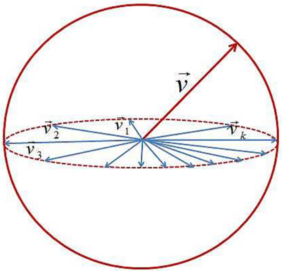

- The asymptotic behaviour of the entropy $\approx \frac{d}{2}\log(m)$ for m large has $d = 1$, the geometric dimension of the circle.
- More complex linear combinations of basic states give ground states with entropy asymptotics characterized by $0 < d < 2$ and $d \in \mathbb{R}$.
- The maximum value of d in this case is 2 (the geometric dimension of the sphere and four times the spin).
- This is consistent with [Ercolessi et al. '10].

Higher spins

- As mentioned earlier [Popkov, Salerno & Shütz '04] have considered the case of permutation symmetric states in higher spin systems. For a particular choice of the initial (pure) state they found that the entropy scaled as $s \log m$.

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- The maximum value of $4s$ can also be seen as the number of free parameters which characterize the state. For permutation symmetric states, the spin projection is the same at each site. For spin s at each site we have a $(2s+1)$-dimensional vector with complex entries and norm 1. This gives a total of $4s+1$ parameters minus a phase, that is, at most $4s$ degrees of freedom.

Conclusions

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- The approach is based on the use of replica cyclic permutation operators and their correlation functions.
- Employing this approach we have identify a particular type of universal behaviour of the entanglement entropy which may be found in theories with infinitely degenerate ground states spanned by a basis of zero-entropy permutation symmetric states.
- This universal behaviour is characterized by the geometric structure of the support of these states, which may be a fractal one.

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- More generally, the role of permutation symmetric states in the context of quantum information is quite prominent. Their symmetry makes then easy to treat analytically and they have even been realized in the lab (see e.g. Prevedel et al. '09).
- It seems that this kind of states is regarded as one which maximizes entanglement. Our work and that Popkov et al. show that there is a limit to the amount of entanglement that can be stored in such states.

Outlook II

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- The simplest example of this is the Random bond Ising model with $H = \sum J_i \vec{\sigma}_i \vec{\sigma}_{i+1}$ where J_i are couplings which are chose randomly. Once the choice is made it is known what the ground state is characterized by pairs of spins forming singlet states:

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- The entropy of such a state scales as $\frac{\ln 2}{3} \log m$ which is quite similar to the kind of behaviour we have found for systems with degenerate ground states. It would be interesting to see if the coefficient of the log-term does admit also a geometric interpretation in this case.