Dark Matter Clustering from the Renormalization Group and implications for cosmic acceleration

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(in collaboration with Sabino Matarrese)

- Motivations: BAO and all that
- Eulerian Perturbation Theory: Traditional and Compact forms. Results.
- RG approach: formulation and preliminary results

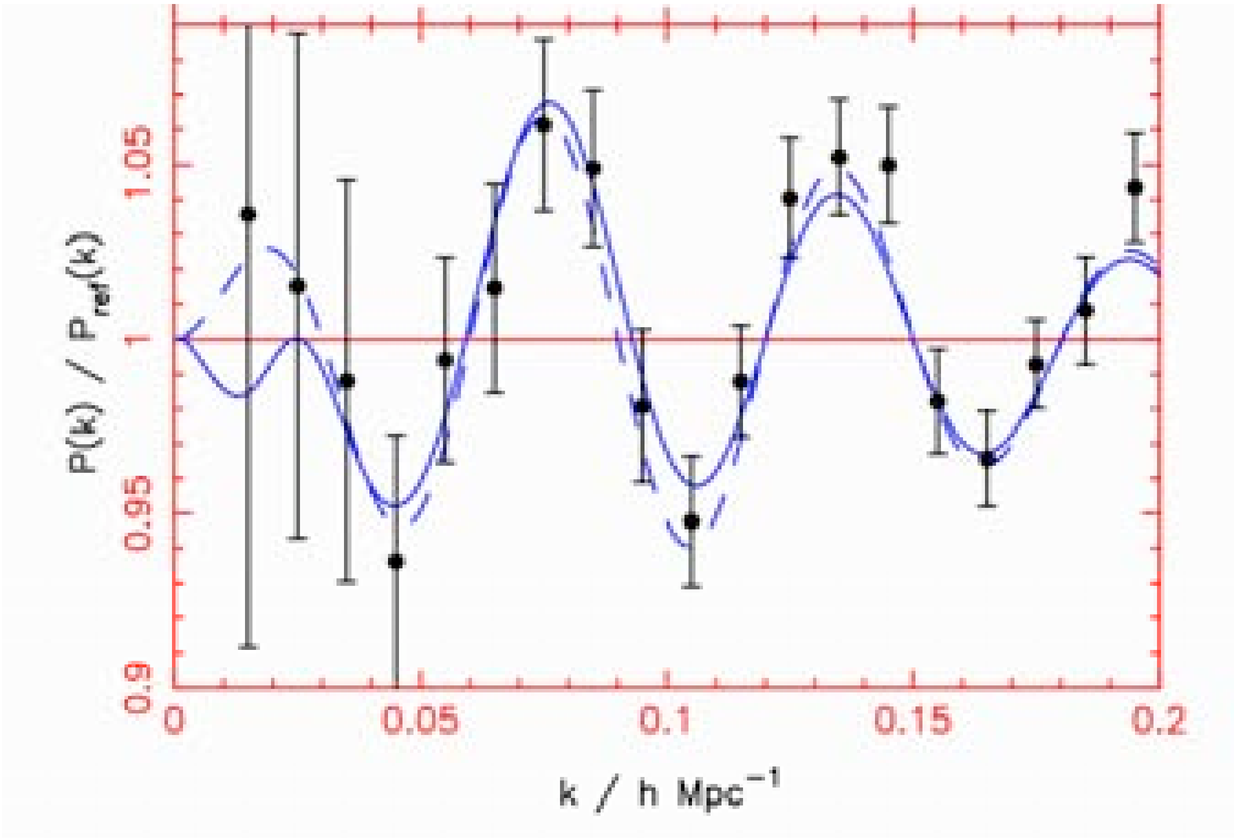
Motivations

Present and future probes of DE: BAO, Weak Lensing, Lyα, 21cm, ...

they all require improved computational techniques

Ex.: BAO from WFMOS

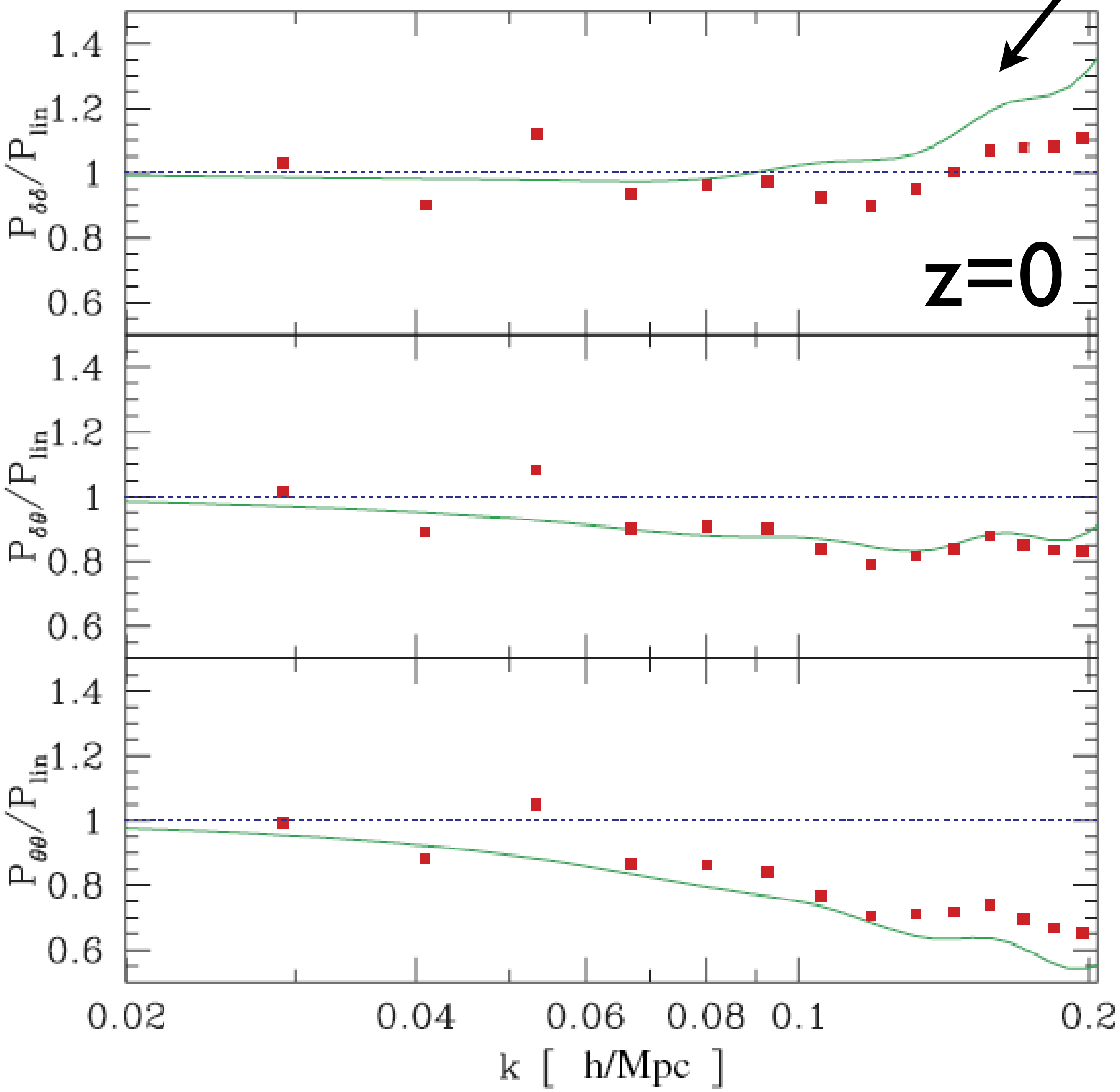
(2M galaxies at 0.5<z<1.3)



Goal: predict the LSS power spectrum to % accuracy.

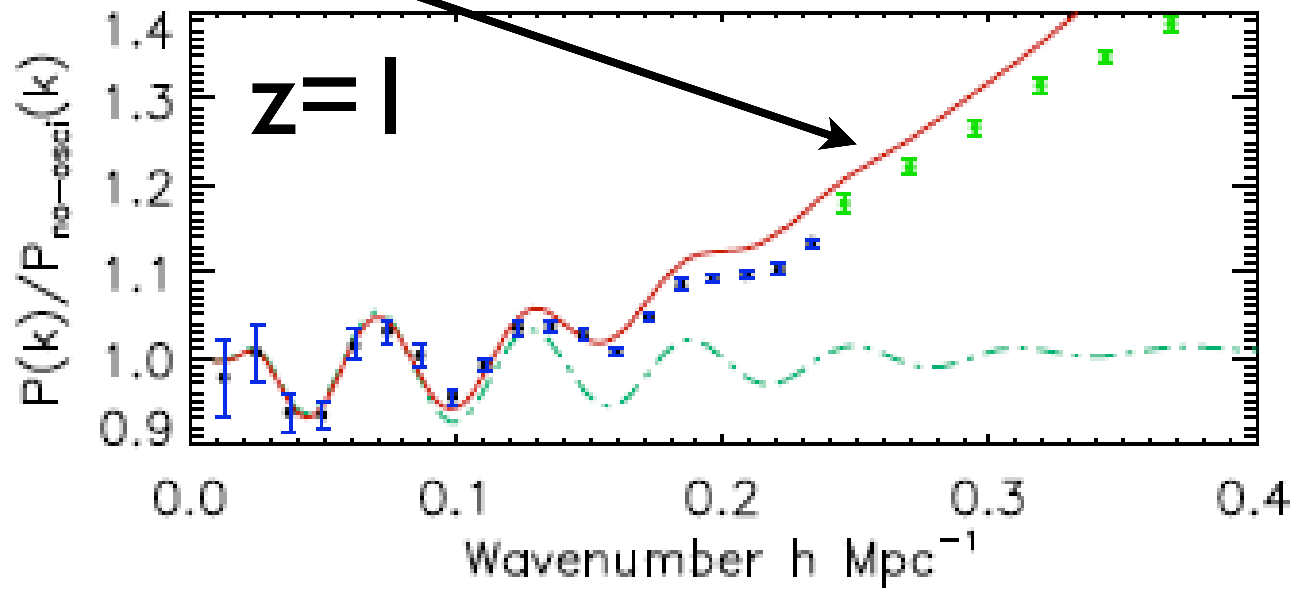
Present Status: Pert. Theory

1-loop PT



Scoccimarro, '04

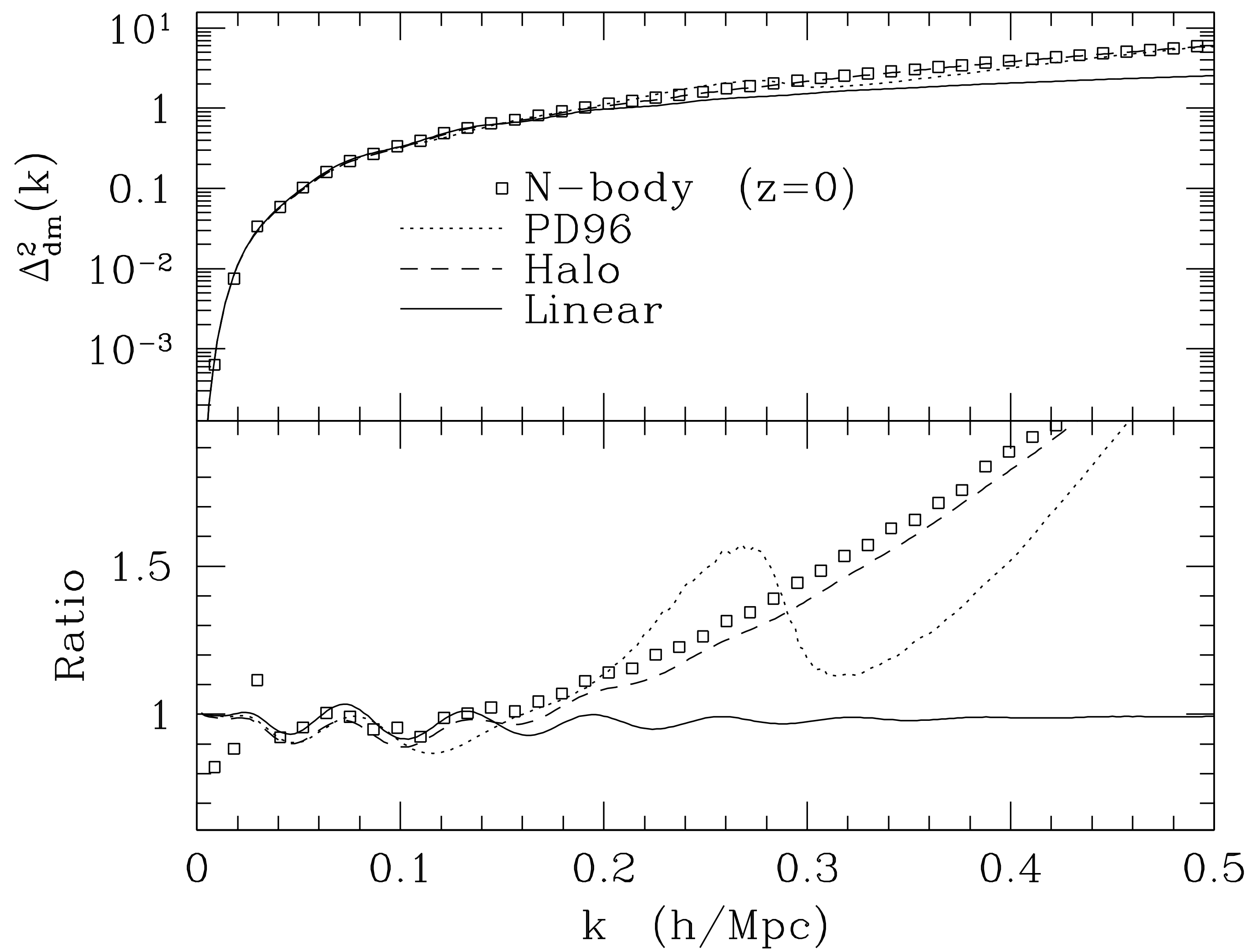
Jeong Komatsu, '06



Non-linearities becomes more and more

relevant in the DE-sensitive range 0<z<1

Present Status: N-body simulations+fitting functions



Huff et al, '06

~10% discrepancies between fitting functions and simulations

redshift-space distorsions quite hard

Goals

- Improve Pert. Theory towards lower z and higher k
- Study the effect of non-linearities on BAO
- Redshift-space distorsions

Dark Matter Hydrodynamics

The DM particle distribution function, $f(\mathbf{x},\mathbf{p},\tau)$, obeys the Vlasov equation:

$$\frac{\partial f}{\partial \tau} + \frac{\mathbf{p}}{am}\cdot\nabla f - am\nabla\phi\cdot\nabla_{\mathbf{p}} f = 0$$

where $p = am\frac{d\mathbf{x}}{d\tau}$ and $$\nabla^2\phi = \frac{3}{2}\,\Omega_M\,\mathcal{H}^2\,\delta$$

Taking momentum moments, i.e.,

$$\int d^3\mathbf{p}\, f(\mathbf{x},\mathbf{p},\tau) \equiv \rho(\mathbf{x},\tau) \equiv \overline{\rho}(\tau)[1+\delta(\mathbf{x},\tau)]$$

$$\int d^3\mathbf{p}\,\frac{p_i}{am} f(\mathbf{x},\mathbf{p},\tau) \equiv \rho(\mathbf{x},\tau) v_i(\mathbf{x},\tau)$$

$$\int d^3\mathbf{p}\,\frac{p_i\,p_j}{a^2m^2} f(\mathbf{x},\mathbf{p},\tau) \equiv \rho(\mathbf{x},\tau) v_i(\mathbf{x},\tau) v_j(\mathbf{x},\tau) + \sigma_{ij}(\mathbf{x},\tau)$$

$$\cdots$$

and neglecting σ_{ij} and higher moments (single stream approximation), one gets...

Equations of motion for single-stream cosmology

$$\frac{\partial\,\delta}{\partial\,\tau} + \nabla\cdot[(1+\delta)\mathbf{v}] = 0\,, \qquad \frac{\partial\,\mathbf{v}}{\partial\,\tau} + \mathcal{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\phi$$

In Fourier space, (defining $\theta(\mathbf{x},\tau) \equiv \nabla\cdot\mathbf{v}(\mathbf{x},\tau)$),

$$\frac{\partial\,\delta(\mathbf{k},\tau)}{\partial\,\tau} + \theta(\mathbf{k},\tau) + \int d^3\mathbf{k_1} d^3\mathbf{k_2}\,\delta_D(\mathbf{k}-\mathbf{k_1}-\mathbf{k_2})\alpha(\mathbf{k_1},\mathbf{k_2})\theta(\mathbf{k_1},\tau)\delta(\mathbf{k_2},\tau) = 0$$

$$\frac{\partial\,\theta(\mathbf{k},\tau)}{\partial\,\tau} + \mathcal{H}\,\theta(\mathbf{k},\tau) + \frac{3}{2}\Omega_M\mathcal{H}^2\delta(\mathbf{k},\tau) + \int d^3\mathbf{k_1} d^3\mathbf{k_2}\,\delta_D(\mathbf{k}-\mathbf{k_1}-\mathbf{k_2})\beta(\mathbf{k_1},\mathbf{k_2})\theta(\mathbf{k_1},\tau)\theta(\mathbf{k_2},\tau) = 0$$

mode-mode coupling controlled by:

$$\alpha(\mathbf{k_1},\mathbf{k_2}) \equiv \frac{(\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{k_1}}{k_1^2}$$

$$\beta(\mathbf{k_1},\mathbf{k_2}) \equiv \frac{|\mathbf{k_1}+\mathbf{k_2}|^2(\mathbf{k_1}\cdot\mathbf{k_2})}{2k_1^2k_2^2}$$

Traditional Perturbation Theory

$\boxed{\text{Assume EdS, } \Omega_M = 1 \text{, then solutions have the form}}$

fastest growing mode only

$$\delta(\mathbf{k},\tau) = \sum_{n=1}^{\infty} a^n(\tau)\delta_n(\mathbf{k})$$

$$\theta(\mathbf{k},\tau) = -\mathcal{H}(\tau)\sum_{n=1}^{\infty} a^n(\tau)\theta_n(\mathbf{k})$$

fastest growing mode only

with

$$\delta_n(\mathbf{k}) = \int d^3\mathbf{q_1}\dots d^3\mathbf{q_n}\,\delta_D(\mathbf{k}-\mathbf{q_{1\dots n}})F_n(\mathbf{q_1},\dots,\mathbf{q_n})\delta_0(\mathbf{q_1})\dots\delta_0(\mathbf{q_n})$$

$$\theta_n(\mathbf{k}) = \int d^3\mathbf{q_1}\dots d^3\mathbf{q_n}\,\delta_D(\mathbf{k}-\mathbf{q_{1\dots n}})G_n(\mathbf{q_1},\dots,\mathbf{q_n})\delta_0(\mathbf{q_1})\dots\delta_0(\mathbf{q_n})$$

The Kernels F_n and G_n satisfy recursion relations, with $F_1 = G_1 = 1$, and $\delta_1 = \theta_1 = \delta_0$:

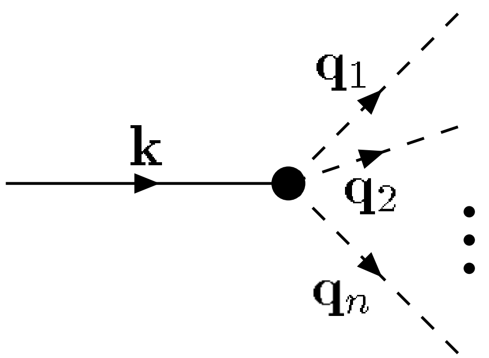
$$F_n(\mathbf{q_1},\dots,\mathbf{q_n}) = \sum_{m=1}^{n-1}\frac{G_m(\mathbf{q_1},\dots,\mathbf{q_m})}{(2n+3)(n-1)} \times\left[(2n+1)\alpha(\mathbf{k_1},\mathbf{k_2})F_{n-m}(\mathbf{q_{m+1}},\dots,\mathbf{q_n}) + 2\beta(\mathbf{k_1},\mathbf{k_2})G_{n-m}(\mathbf{q_{m+1}},\dots,\mathbf{q_n})\right]$$

$$G_n(\mathbf{q_1},\dots,\mathbf{q_n}) = \cdots$$

where $\mathbf{k_1} = \mathbf{q_1} + \ldots + \mathbf{q_m}$, $\mathbf{k_2} = \mathbf{q_{m+1}} + \ldots + \mathbf{q_n}$

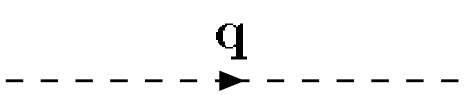
Traditional Diagrammar

Fry, '84
Goroff et al, '86
Wise, '88
Scoccimarro, Frieman, '96
....



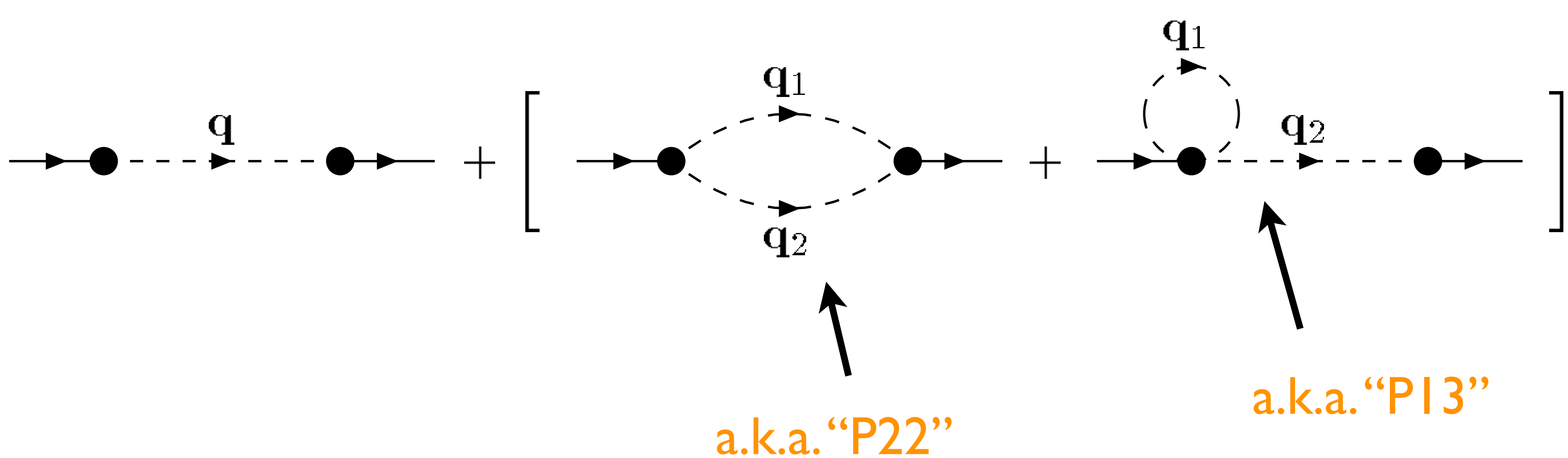
$\equiv \quad \delta_D(\mathbf{k} - \mathbf{q}_1 - \ldots - \mathbf{q}_n) F_n^{(s)}(\mathbf{q}_1, \ldots, \mathbf{q}_n)$

An infinite number of basic vertices!
<u>very redundant</u>!!

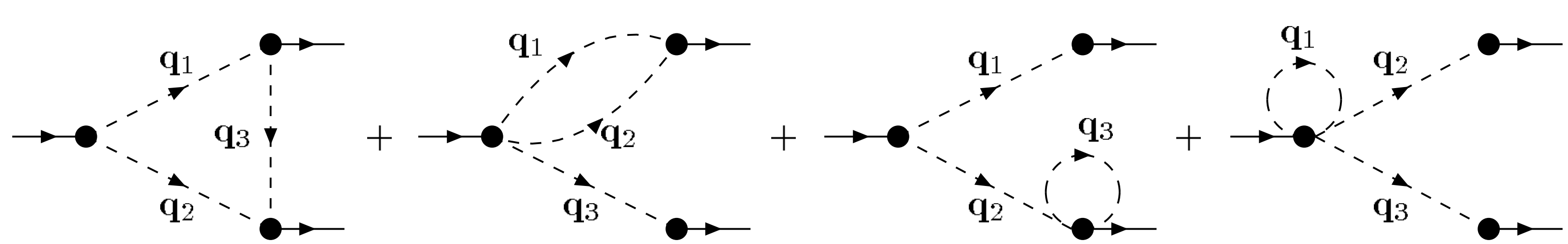


$\equiv \quad P_1(q, \tau)$

Example: 1-loop correction to the density power spectrum:



bispectrum:



Compact Perturbation Theory

Crocce, Scoccimarro '05

The hydrodynamical equations for density and velocity perturbations,

$$\frac{\partial\,\delta}{\partial\,\tau} + \nabla\cdot[(1+\delta)\mathbf{v}] = 0\,, \qquad \frac{\partial\,\mathbf{v}}{\partial\,\tau} + \mathcal{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\phi\,,$$

can be written in a compact form (we assume an EdS model):

$$(\delta_{ab}\partial_\eta + \Omega_{ab})\,\varphi_b(\eta,\mathbf{k}) = e^\eta\gamma_{abc}(\mathbf{k},\,-\mathbf{k_1},\,-\mathbf{k_2})\,\varphi_b(\eta,\mathbf{k_1})\,\varphi_c(\eta,\mathbf{k_2}) \qquad (1)$$

where $\begin{pmatrix} \varphi_1(\eta,\mathbf{k}) \\ \varphi_2(\eta,\mathbf{k}) \end{pmatrix} \equiv e^{-\eta}\begin{pmatrix} \delta(\eta\,,\mathbf{k}) \\ -\theta(\eta\,,\mathbf{k})/\mathcal{H} \end{pmatrix} \qquad \eta = \log\frac{a}{a_{in}} \qquad \mathbf{\Omega} = \begin{pmatrix} 1 & -1 \\ -3/2 & 3/2 \end{pmatrix}$

and the only non-zero components of the vertex are

$$\gamma_{121}(\mathbf{k_1},\,\mathbf{k_2},\,\mathbf{k_3}) = \gamma_{112}(\mathbf{k_1},\,\mathbf{k_3},\,\mathbf{k_2}) = \delta_D(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\,\frac{(\mathbf{k_2}+\mathbf{k_3})\cdot\mathbf{k_2}}{2k_2^2}$$

$$\gamma_{222}(\mathbf{k_1},\,\mathbf{k_2},\,\mathbf{k_3}) = \delta_D(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\,\frac{|\mathbf{k_2}+\mathbf{k_3}|^2\,\mathbf{k_2}\cdot\mathbf{k_3}}{2\,k_2^2\,k_3^2}$$

An action principle

Matarrese, M.P., '06

Eq. (1) can be derived by varying the action

$$S = \int d\eta_1 d\eta_2 \, \chi_a \, g^{-1}_{ab} \, \varphi_b - \int d\eta \, e^{\eta} \, \gamma_{abc} \, \chi_a \, \varphi_b \, \varphi_c$$

where the auxiliary field $\chi_a(\eta, \mathbf{k})$ has been introduced and $g_{ab}(\eta_1, \eta_2)$ is the retarded propagator:

$$(\delta_{ab}\partial_\eta + \Omega_{ab}) \, g_{bc}(\eta, \eta') = \delta_{ac} \, \delta_D(\eta - \eta')$$

so that $\varphi^0_a(\eta, \mathbf{k}) = g_{ab}(\eta, \eta')\varphi^0_b(\eta', \mathbf{k})$ is the solution of the ***linear*** equation

Explicitly, one finds: $\mathbf{g}(\eta_1, \eta_2) = \begin{cases} \mathbf{B} + \mathbf{A} \, e^{-5/2(\eta_1 - \eta_2)} & \eta_1 > \eta_2 \\ 0 & \eta_1 < \eta_2 \end{cases}$

$$\mathbf{B} = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{A} = \frac{1}{5}\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}$$

growing mode

decaying mode

Initial conditions: $\varphi^0_b(\eta', \mathbf{k}) \propto u_b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$

A generating functional

The probability of the configuration $\varphi_a(\eta_f)$, given the initial condition $\varphi_a(\eta_i)$, is

$$P[\varphi_a(\eta_f);\varphi_a(\eta_i)] = \delta\left[\varphi_a(\eta_f) - \overline{\varphi}_a[\eta_f;\varphi_a(\eta_i)]\right]$$

solution of the e.o.m.

fixed extrema

$$\sim \int \mathcal{D}''\varphi_a \mathcal{D}\chi_b \exp\left\{ i\int_{\eta_i}^{\eta_f} d\eta\, \chi_a \left[(\delta_{ab}\partial_\eta + \Omega_{ab})\varphi_b - e^\eta \gamma_{abc}\varphi_b\varphi_c\right]\right\}$$

only tree-level (saddle point)

The generating functional ***at fixed initial conditions*** is

$$Z[J_a,\Lambda_b;\varphi_c(\eta_i)] = \int \mathcal{D}\varphi_a(\eta_f) \exp\left\{ i\int_{\eta_i}^{\eta_f} d\eta (J_a\varphi_a + \Lambda_b\chi_b)\right\} P[\varphi_a(\eta_f);\varphi_a(\eta_i)]$$

We are interested in ***statistical*** correlations, ***not in single solutions:***

$$Z[J_a, \Lambda_b; K's] = \int \mathcal{D}\varphi_c(\eta_i) W[\varphi_c(\eta_i); K's] Z[J_a, \Lambda_b; \varphi_c(\eta_i)]$$

where all the initial correlations are contained in

$$W[\varphi_c(\eta_i); K's] = \exp\left\{-\varphi_a(\eta_i; \mathbf{k}) K_a(\mathbf{k}) - \frac{1}{2}\varphi_a(\eta_i; \mathbf{k_a}) K_{ab}(\mathbf{k_a}, \mathbf{k_b}) \varphi_b(\eta_i; \mathbf{k_b}) + \cdots\right\}$$

In the case of Gaussian initial conditions: $\mathbf{(K(k))^{-1}_{ab} = P^0_{ab}(k) \equiv u_a u_b P^0(k)}$

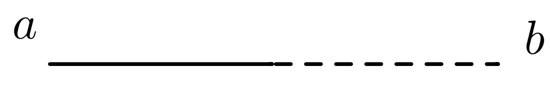
Putting all together...

$$\boxed{Z[\mathbf{J}, \mathbf{\Lambda}] = \int \mathcal{D}\varphi \, \mathcal{D}\chi \exp\left\{\int d\eta_1 d\eta_2 \left[-\frac{1}{2}\chi \, \mathbf{g}^{-1} \mathbf{P}^{\mathbf{L}} \mathbf{g}^{\mathbf{T}^{-1}} \chi + i \, \chi \, \mathbf{g}^{-1} \varphi\right] - i \int d\eta \left[\mathbf{e}^{\eta} \gamma \, \chi\varphi\varphi - \mathbf{J}\varphi - \mathbf{\Lambda}\chi\right]\right\}}$$

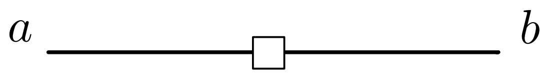
where the initial conditions are encoded in the linear power spectrum: $P^L_{ab}(\eta, \eta'; \mathbf{k}) \equiv \left(\mathbf{g}(\eta) \mathbf{P^0}(\mathbf{k}) \mathbf{g}^T(\eta')\right)_{ab}$

Derivatives of Z w.r.t. the sources J and Λ give all the N-point correlation functions (power spectrum, bispectrum, ...) and the full propagator (k-dependent growth factor)

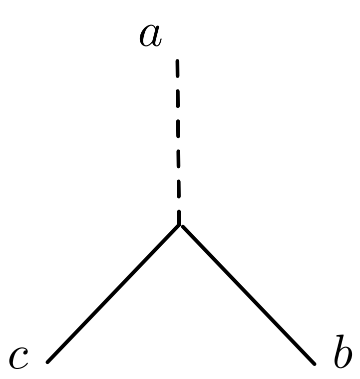
Compact Diagrammar



propagator (linear growth factor): $-i\, g_{ab}(\eta_a,\, \eta_b)$

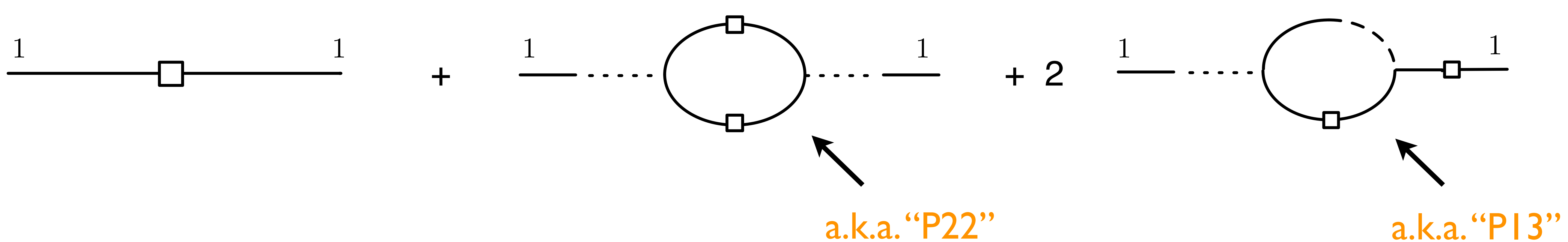


power spectrum: $P^L_{ab}(\eta_a,\, \eta_b;\, \mathbf{k})$



interaction vertex: $-i\, e^{\eta}\, \gamma_{abc}(\mathbf{k_a},\, \mathbf{k_b},\, \mathbf{k_c})$

Example: 1-loop correction to the density power spectrum:



All known results in cosmological perturbation theory are expressible in terms of diagrams in which <u>only a trilinear fundamental interaction</u> appears

1-loop PT: how good is it?

Makino et al.,'92

$$P(k,\tau) = D^2(\tau)P_{11}(k) + D^4(\tau)\left[P_{13}(k) + P_{22}(k)\right] + \ldots\,,$$

$$P_{13}(k) = \frac{k^3 P_{11}(k)}{252\,(2\pi)^2}\int_0^\infty dr\,P_{11}(kr)\left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3}\left(r^2-1\right)^3\left(7r^2+2\right)\ln\left|\frac{1+r}{1-r}\right|\right]$$

$$P_{22}(k) = \frac{k^3}{98\,(2\pi)}\int_0^\infty dr\,P_{11}(kr)\int_{-1}^{1} dx\,P_{11}\left[k\left(1+r^2-2rx\right)^{1/2}\right]\frac{\left(3r+7x-10rx^2\right)^2}{\left(1+r^2-2rx\right)^2}$$

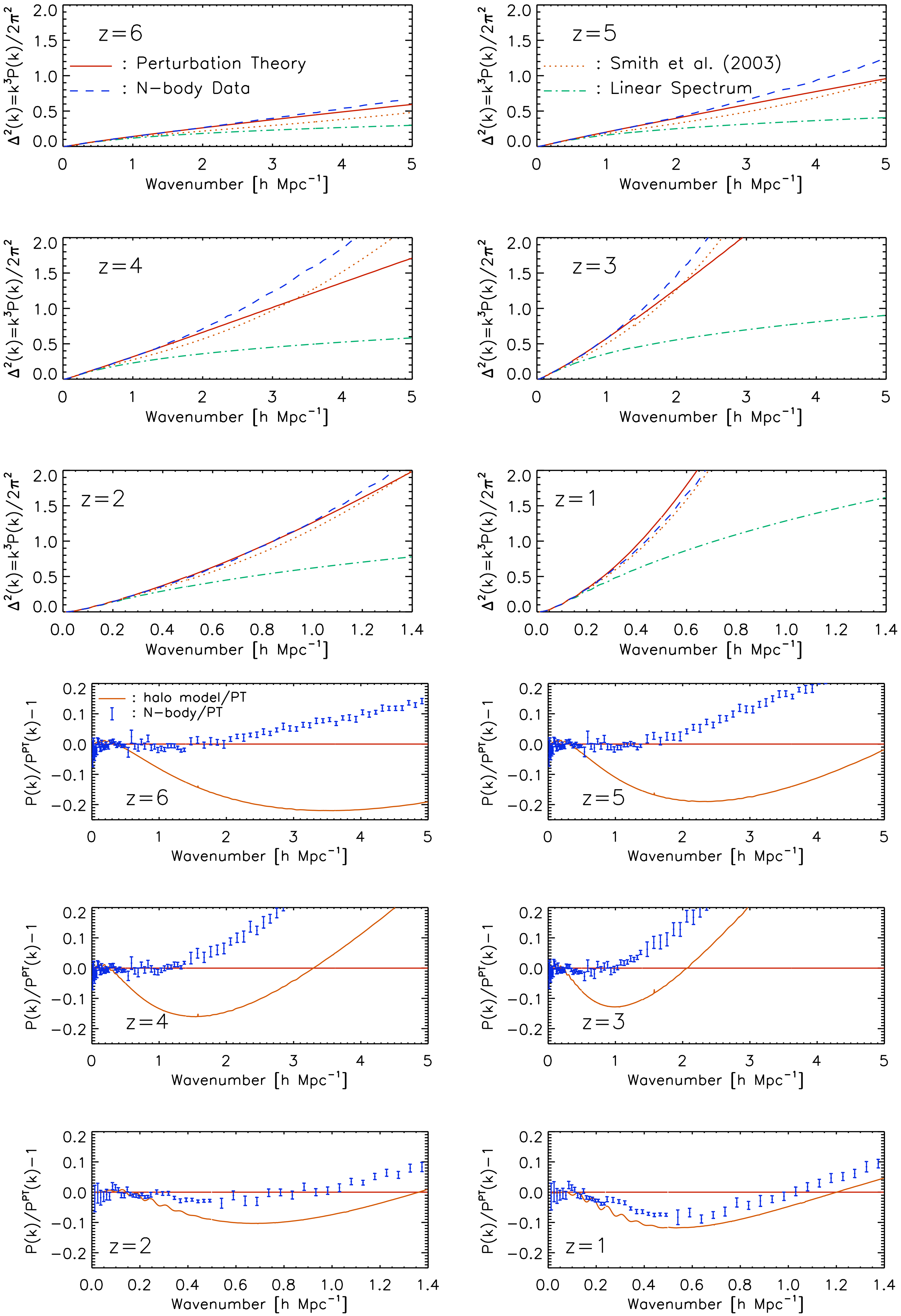
$$\boxed{D(\tau) = \delta_1(\tau)/\delta_{\mathrm{initial}}}$$

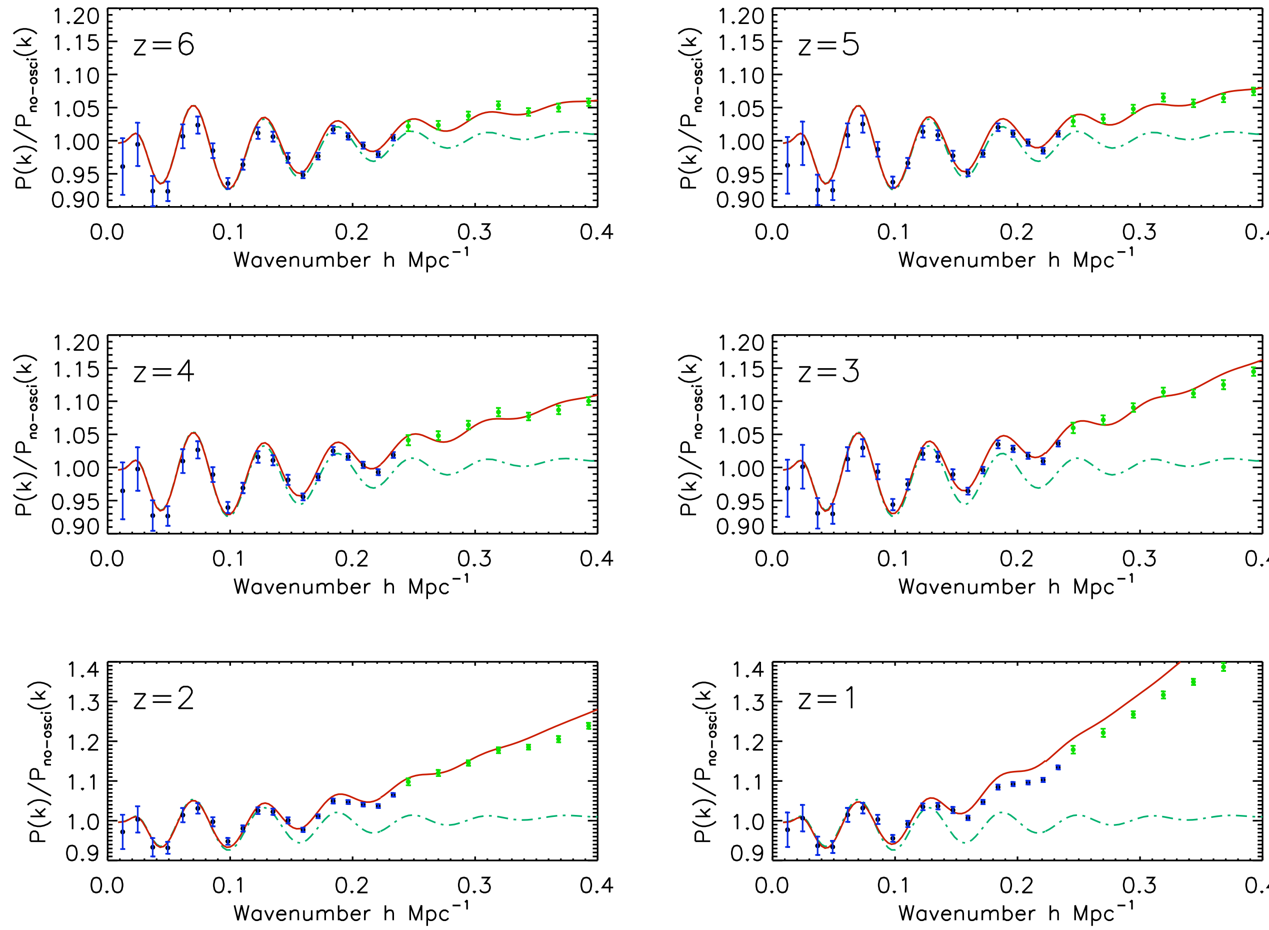
Linear growth factor: encodes different cosmologies at best than % level

Ex: $P_{22}(\Lambda\mathrm{CDM})/P_{22}(\mathrm{EdS}) \sim 1.006 \;\; (z=0)$ (Jeong Komatsu, '06)

Notice: the 1-loop corrections at any time depend on the initial power spectrum ($P_{11}(k) = P^0(k)$)!

This will change in the RG...





1-loop PT performs quite well for z >1 (better than halo approach)

Baryonic peaks modeled at few %

Things get ***much*** worse at z<1...

Jeong Komatsu,'06

Beyond perturbation theory: the renormalization group

Inspired by applications of Wilsonian RG to field theory,
here the RG parameter is the log of redshift : $\eta = \log \frac{a}{a_{in}}$

Recipe: define a cut-off propagator as $\mathbf{g}_{\bar\eta}(\eta,\,\eta') = \mathbf{g}(\eta,\,\eta')\,\Theta(\bar\eta - \eta)$ (step function)

then, plug it into the generating functional: $Z[\mathbf{J},\,\mathbf{\Lambda}] \longrightarrow Z_{\bar\eta}[\mathbf{J},\,\mathbf{\Lambda}]$

$$Z_{\bar\eta}[\mathbf{J},\,\mathbf{\Lambda}] = \int \mathcal{D}\varphi\,\mathcal{D}\chi\,\exp\left\{\int d\eta_1 d\eta_2\left[-\frac{1}{2}\chi\,\mathbf{g}_{\bar\eta}^{-1}\mathbf{P}^{\mathbf{L}}_{\bar\eta}\mathbf{g}_{\bar\eta}^{\mathbf{T}^{-1}}\chi + i\,\chi\,\mathbf{g}_{\bar\eta}^{-1}\,\varphi\right] - i\int d\eta\,[\mathbf{e}^{\eta}\gamma\,\chi\varphi\varphi - \mathbf{J}\varphi - \mathbf{\Lambda}\chi]\right\}$$

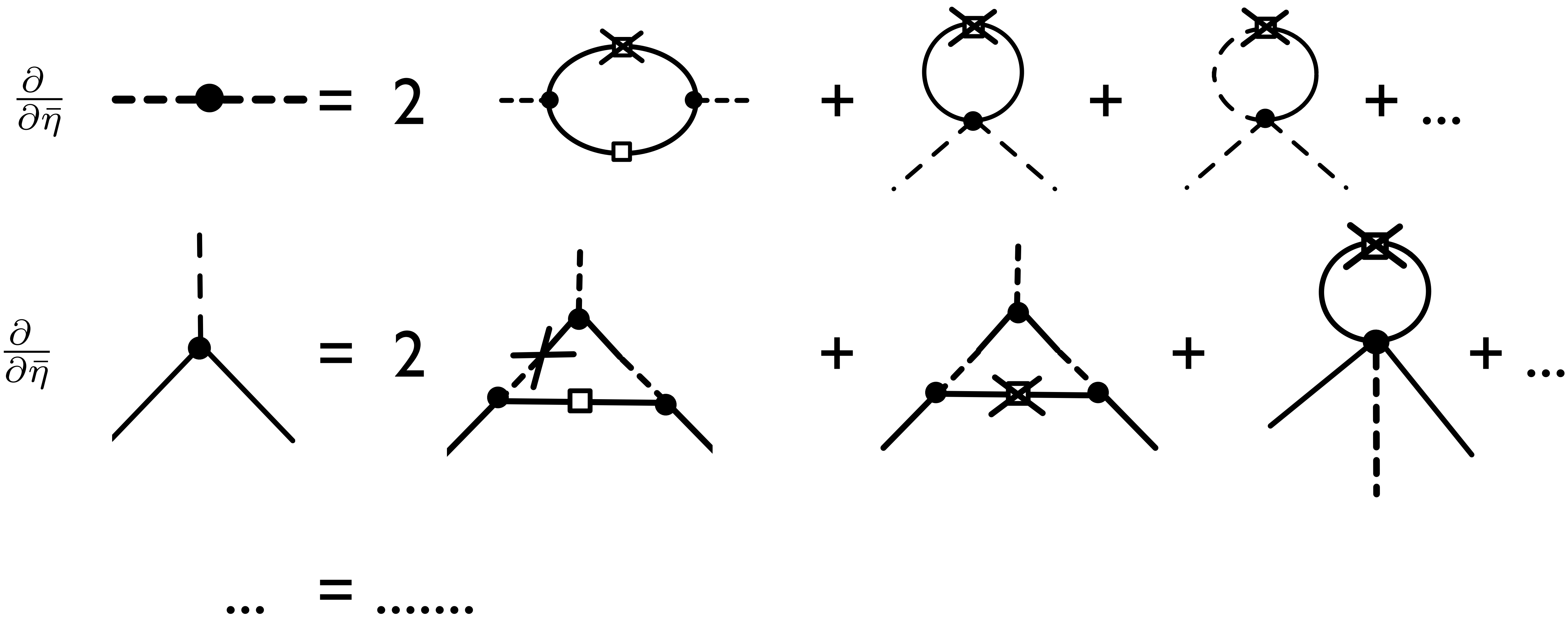
this object generates all the N-point functions for the Universe in which
the growth of perturbation has been frozen at $\bar\eta$

The evolution from $\bar\eta = 0$ to $\bar\eta = \eta_0$ can be described non-perturbatively by RG equations:

$$\frac{\partial}{\partial\bar\eta}Z_{\bar\eta} = \int d\eta\,d\eta'\left[\frac{1}{2}\frac{\partial}{\partial\bar\eta}\left(g_{\bar\eta}^{-1}P^L_{\bar\eta}\,g_{\bar\eta}^{-1\,T}\right)_{ab}\frac{\delta^2 Z_{\bar\eta}}{\delta\Lambda_b\delta\Lambda_a} - i\frac{\partial}{\partial\bar\eta}g^{-1}_{ab,\,\bar\eta}\frac{\delta^2 Z_{\bar\eta}}{\delta J_b\delta\Lambda_a}\right]$$

the RG eq. for the power spectrum is obtained by deriving twice wrt. the source J, the
bispectrum by deriving three times, and so on...

In pictures...



Thick lines and bold circles represent full (i.e. non-perturbative) propagators, power-spectrums, and vertices. Crosses represent the RG kernel.

Notice that an infinite number of vertices (3-linear, 4-linear,...) are generated. The infinite hierarchy of equations has to be truncated.

The equations can also be solved perturbatively. PT is fully reproduced.

Application: the power spectrum

The full propagator has the structure: $G_{\bar{\eta},ab}(\eta,\,\eta',\,\mathbf{k}) = (g_{\bar{\eta}}^{-1} - \Sigma_{\bar{\eta}})^{-1}_{ab}(\eta,\,\eta',\,\mathbf{k})$

and the full power spectrum: $P_{\bar{\eta},\,ab}(\eta,\,\eta',\,\mathbf{k}) = (\mathbf{G}_{\bar{\eta}}\mathbf{g}_{\bar{\eta}}^{-1}\mathbf{P}^{\mathbf{L}}_{\bar{\eta}}\mathbf{g}_{\bar{\eta}}^{\mathbf{T}^{-1}}\mathbf{G}^{\mathbf{T}}_{\bar{\eta}})_{ab} + (\mathbf{G}_{\bar{\eta}}\mathbf{\Phi}_{\bar{\eta}}\mathbf{G}^{\mathbf{T}}_{\bar{\eta}})_{ab}$

<u>Simple truncation scheme</u>: take $\Sigma_{\bar{\eta},\,ab} = 0\,,\qquad \Phi_{\bar{\eta},\,ab}(\eta,\,\eta';\,\mathbf{k}) = \Phi_{\bar{\eta}}(\mathbf{k})\,u_a u_b\,\delta(\eta)\,\delta(\eta')$

(u_a is proportional to the initial conditions.
for the growing mode $u_1 = u_2$)

then $P_{\bar{\eta},\,ab}(\eta,\,\eta',\,\mathbf{k}) = g_{\bar{\eta},\,ac}(\eta,\,0)\,u_c(P^0 + \Phi_{\bar{\eta}})(\mathbf{k})\,u_d\,g_{\bar{\eta},\,bd}(\eta',\,0)$

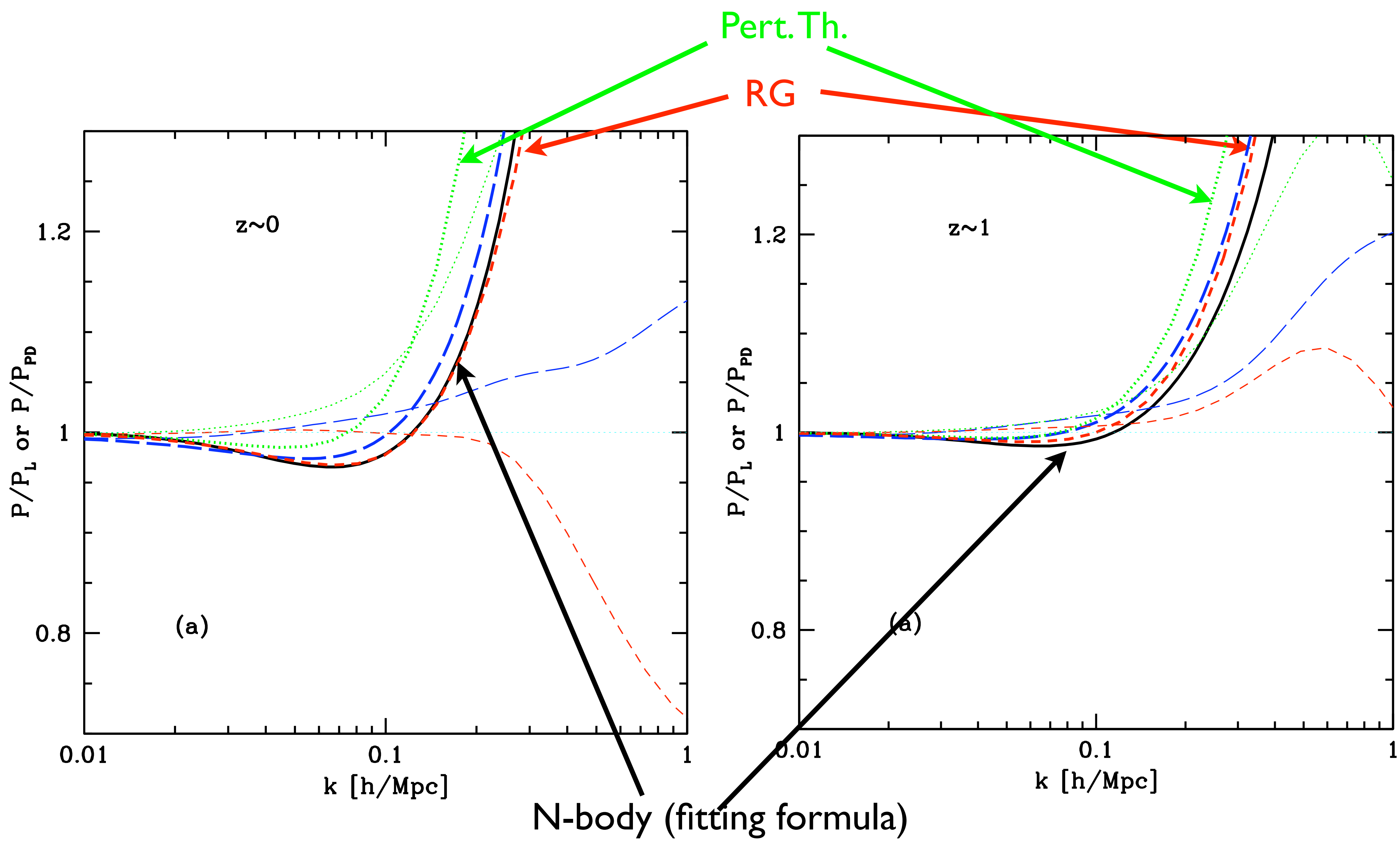
renormalized power spectrum

$\Phi_{\bar{\eta}}(\mathbf{k})$ evolves according to the following RG equation:

$$\frac{\partial}{\partial\bar{\eta}}\Phi_{\bar{\eta}}(\mathbf{k}) = e^{2\bar{\eta}}\frac{k^3}{(2\pi)^2}\int_0^\infty dr(P^0+\Phi_{\bar{\eta}})(kr)\left\{\int_{-1}^{1}dx(P^0+\Phi_{\bar{\eta}})(k(1+r^2-2rx)^{1/2})\frac{x^2(1-rx)^2}{(1+r^2-2rx)^2}\right.$$
$$\left.+\frac{1}{84}(P^0+\Phi_{\bar{\eta}})(k)\left[\frac{18}{r^2}-142+30r^2-18r^4+\frac{9(r^2-1)^3}{r^2}(1+r^2)\log\left|\frac{1+r}{1-r}\right|\right]\right\}$$

with the initial condition: $\Phi_{\bar{\eta}}(\mathbf{k}) = 0 \quad for \quad \bar{\eta} = 0$

At this level of approximation, the exact RG equation reduces (almost) exactly to that considered by McDonald [2], which already shows a remarkable improvement on 1-loop perturbation theory:



from McDonald, astro-ph/0606028

Conclusions

0) It is very important to quantify departures from linear theory in order to compare cosmological models with future galaxy surveys. The $0<z<1$ range is the most delicate for DE studies;

1) The compact perturbation theory formulated by Crocce and Scoccimarro is a very convenient starting point for applying RG techniques to cosmology;

2) Exact RG equations can be derived for any kind of correlation function and for the scale-dependent growth factor;

3) Systematic approximation schemes, based on truncations of the full hierarchy of equations, can be applied, borrowing the experience from field theory;

4) A simple approximation scheme already improves on 1-loop perturbation theory at $z=0$;

5) Immediate lines of development include: computation of the bispectrum and of the scale-dependent growth factor, improved truncations for the power spectrum, redshift-space distorsions, non-gaussian initial conditions.