Entanglement spectrum in quantum many-body systems

V. Alba[1],M. Haque[1], A. Läuchli[2]

[1]Max Planck Institute for the Physics of Complex Systems, Dresden
[2]University of Innsbruck, Innsbruck

April 29, 2012

Outline

- Introduction
- Entanglement spectrum
- Entanglement spectra in gapped systems (1D) [arXiv:1107.1726]
- ES in the gapped $1D$ XXZ spin chain
  - The XXZ in the large Δ limit
  - Perturbative structure of the ES
  - Boundary-locality
  - Effective microscopic description (domain walls)
  - Comparison with the gapped $1D$ Bose-Hubbard
- ES of the $1D$ XXZ in the limit $\Delta \to -1^+$ (**In progress**)
- ES of $2D$ systems: the Bose Hubbard model (**In progress**)

Standard tools: entropies

- Consider a quantum system in d dimensions in the ground state $|\Psi\rangle$.

$$\rho \equiv |\Psi\rangle\langle\Psi|$$

- If the system is bipartite:

$$H = H_A \otimes H_B \rightarrow \rho_A = Tr_B \rho$$

- How to quantify the entanglement (quantum correlations) between A and B?
  - von Neumann entropy $S_A = Tr \rho_A \log \rho_A = -\sum_i \lambda_i \log \lambda_i$
  - Renyi's entropies $R_A^{(n)} = -\log Tr \rho_A^n = -\log \sum_i \lambda_i^n$
- Area law: away from criticality the entanglement is proportional to the surface area of subsystem A

$$\boxed{S_A \approx L^{d-1}} \qquad (1D \quad S \rightarrow const)$$

- For critical systems corrections to the area law [Calabrese, Cardy,2004]

$$1\text{D} \rightarrow \boxed{S_A = \frac{c}{3} \log L}$$

Entanglement spectrum

- Given the reduced density matrix ρ_A:

$$\{\lambda\} = \sigma(\rho_A) \quad \Rightarrow \quad \rho_A = e^{-\mathcal{H}_A} \quad \Rightarrow \quad \boxed{\sigma(\mathcal{H}_A) \equiv -\log(\{\lambda\})}$$

- Quantum Hall: the ES retains all the features of the critical edge modes (ES$\rightarrow$ edge spectra correspondence) [Li,Haldane, 2008]

[Läuchli et al,2010]

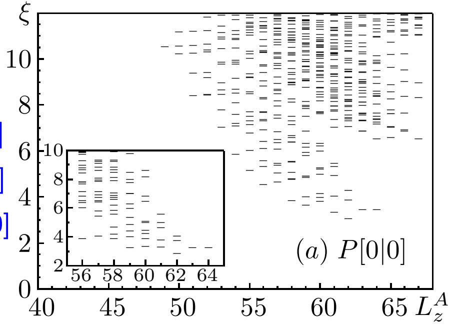
[Qi et al,2010]

[Thomale et al,2010]

[Regnault et al,2010]

[Pollmann et al,2010]

[Prodan et al,2010]

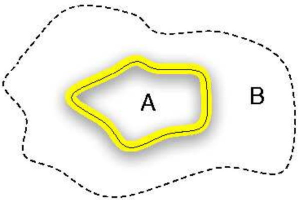


- More general: all the relevant information to describe the physics of a system is encoded in the entanglement spectrum.

ES in gapped systems

- In gapped sytems area law holds:

[Hastings,2007]



- Only the degrees of freedom near the boundary entangle the two subsystems.
- What is the consequence for the entanglement spectrum?
- the ES is a boundary local quantity.

The gapped XXZ spin chain

$$\mathcal{H} = \sum_i \Big(\underbrace{S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+}_{\mathcal{H}_p(i)} + \underbrace{\Delta S_i^z S_{i+1}^z}_{\mathcal{H}_0} \Big)$$

- In the limit ($\Delta \to \infty$) the ground state is a Neel state:

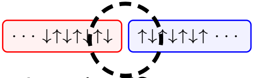
$$|N1\rangle \equiv |\cdots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \cdots\rangle \quad |N2\rangle \equiv |\cdots \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \cdots\rangle$$

- Large Δ, series expansion $|\Psi\rangle = \sum_{i\geq 0} \Delta^{-i} |\Psi_i\rangle$:
- How to calculate the entanglement spectrum:
  - (i) Let us bipartite the system

A $\cdots \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$ **B** $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \cdots$ $\Rightarrow$

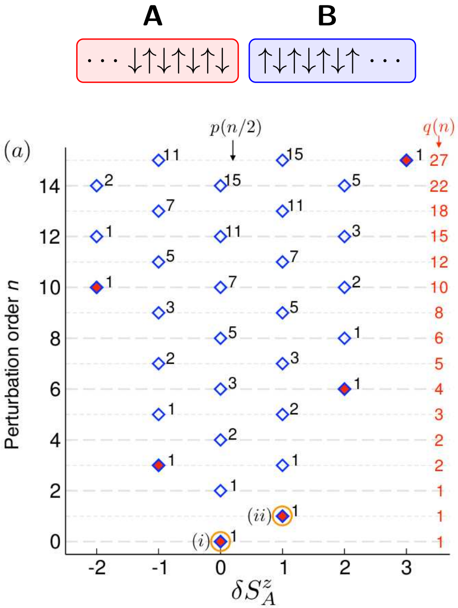
$$|\Psi\rangle = \sum_{i,j} \mathbf{M}_{i,j} |v_A^{(i)}\rangle \otimes |v_B^{(j)}\rangle$$

  - (ii) Perturbative expansion for the matrix $\mathbf{M}_{i,j}$.

Boundary locality $\Rightarrow$ 

  - (iii) SVD of $\mathbf{M}_{i,j}$ (Schmidt decomposition) gives the ES.

The perturbative ES: single boundary case



- Levels are organized in powers of Δ^{-1}: **higher levels in the ES $\Rightarrow$ higher orders.**
- The ES is not symmetric (we selected one Neel state.)
- In each δS^z_A sector at order n, non trivial **degeneracy** given by the **integer partitions** $p(n/2)$.
- The same 'degeneracy tower' for all the δS^z_A sectors.
- Total degeneracy at order n is given by the restricted integer partitions $q(n)$.

Boundary locality and domain walls

- Rephrase perturbation theory in terms of **domain walls**.

$$\mathcal{H}_p \to \cdots \downarrow\uparrow\downarrow \boxed{\uparrow\downarrow} \uparrow\downarrow\uparrow\downarrow \Rightarrow \cdots \downarrow\uparrow \boxed{\downarrow\downarrow}\ \boxed{\uparrow\uparrow} \uparrow\downarrow$$

- **Boundary locality**: levels higher in the ES are given by excitations farther from the boundary between **A** and **B** .

- Boundary locality gives the **rules** of the game:
  (i) Domain walls are created at the boundary.

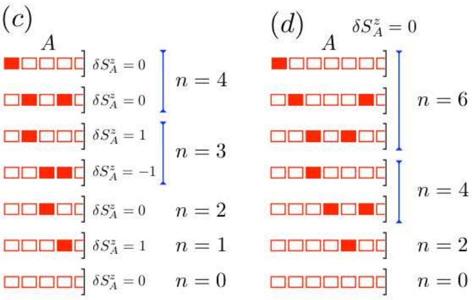
$$\cdots \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \boxed{\downarrow} \quad \boxed{\uparrow} \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \cdots \Rightarrow \cdots \downarrow\uparrow\downarrow\uparrow\downarrow \boxed{\uparrow\uparrow} \quad \boxed{\downarrow\downarrow} \uparrow\downarrow\uparrow\downarrow\uparrow \cdots \approx \Delta^{-1}$$

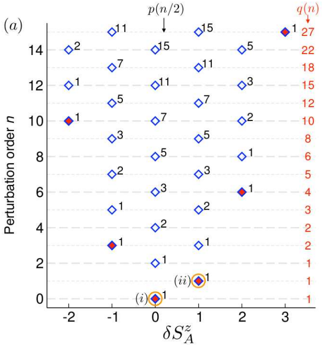
$$\cdots \downarrow\uparrow\downarrow\uparrow\downarrow \boxed{\uparrow\downarrow} \quad \boxed{\uparrow\downarrow} \uparrow\downarrow\uparrow\downarrow\uparrow \cdots \Rightarrow \cdots \downarrow\uparrow\downarrow\uparrow \boxed{\downarrow\downarrow} \uparrow \quad \downarrow \boxed{\uparrow\uparrow} \downarrow\uparrow\downarrow\uparrow \cdots \approx \Delta^{-2}$$

  (ii) Domain walls can be moved symmetrically in the bulk.

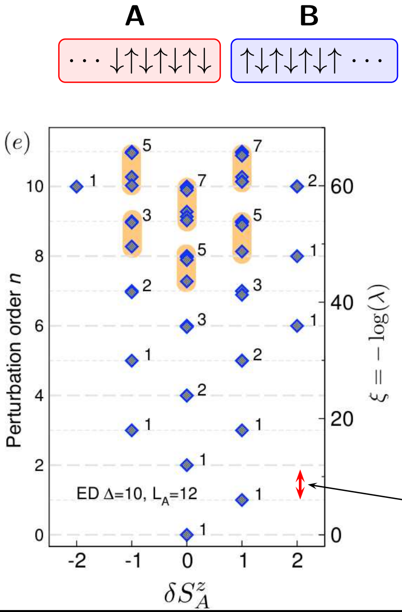
$$\cdots \downarrow\uparrow\downarrow\uparrow \boxed{\downarrow\downarrow} \uparrow \quad \downarrow \boxed{\uparrow\uparrow} \downarrow\uparrow\downarrow\uparrow \cdots \Rightarrow \cdots \downarrow\uparrow \boxed{\downarrow\downarrow} \uparrow\downarrow\uparrow \quad \downarrow\uparrow\downarrow \boxed{\uparrow\uparrow} \downarrow\uparrow \cdots \approx \Delta^{-2}$$

The domain walls picture at work





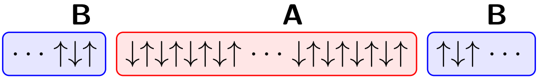
- Perturbative **hyerarchical structure** and degeneracy counting are correctly reproduced.
- More than an effective **microscopic picture**: the domain walls configurations give the leading contribution of the reduced density matrix eigenfunctions.



- Finite size entanglement spectrum with arbitrary precision data (10^{-90}).
- Good agreement with the perturbative picture:
  - (i) Same relation ES levels vs perturbative order.
  - (ii) Same degeneracy structure.
- **Finite size** corrections for levels very high in the spectrum (evident in the perturbative picture).
- The interlevels separation is constant and is given by $2\mathrm{arccosh}\Delta$.

Two boundaries: ES factorization

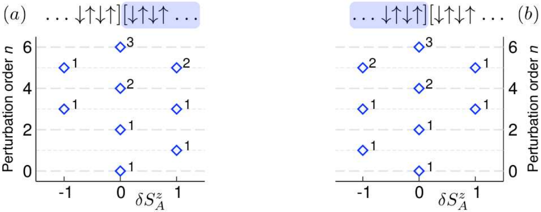
- Consider the case with two boundaries:



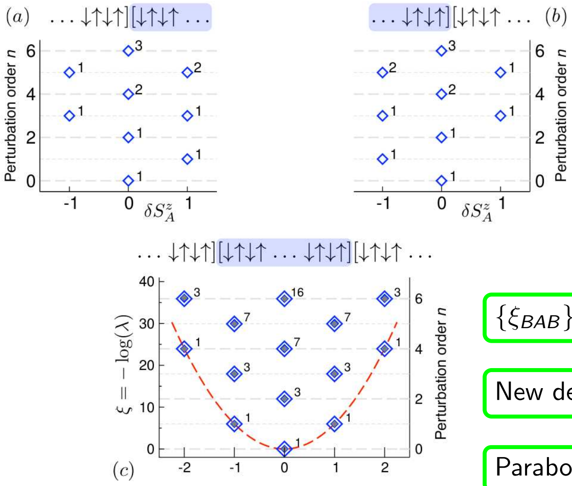
- Another consequence of the **boundary locality**: in the limit of large **A** the two boundaries are decoupled.

$$\rho(\mathbf{B}\ \mathbf{A}\ \mathbf{B}) \approx \rho(\mathbf{B}\ \mathbf{A}) \otimes \rho(\mathbf{A}\ \mathbf{B})$$

$$\rho(\mathbf{B}\ \mathbf{A}) = \rho(\mathbf{A}\ \mathbf{B})|_{S^z_A \to -S^z_A}$$



The two ES combination



$\{\xi_{BAB}\} = \{\xi_{BA}\} + \{\xi_{AB}\}$

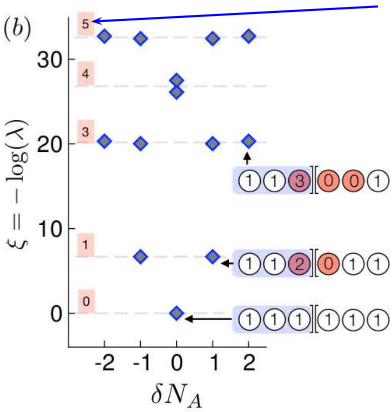
New degeneracies (1,3,7,16,...)

Parabolic envelope

The $1D$ Bose Hubbard

$$\mathcal{H} = J\sum_i \underbrace{(b_i^\dagger b_{i+1} + h.c.)}_{\mathcal{H}_p} + \frac{U}{2}\sum_i n_i(n_i - 1)$$

- In the Mott insulating phase (large U).
- Expansion in J/U starting from unit filling $|\Psi_0\rangle = |\cdots 111111 \cdots\rangle$

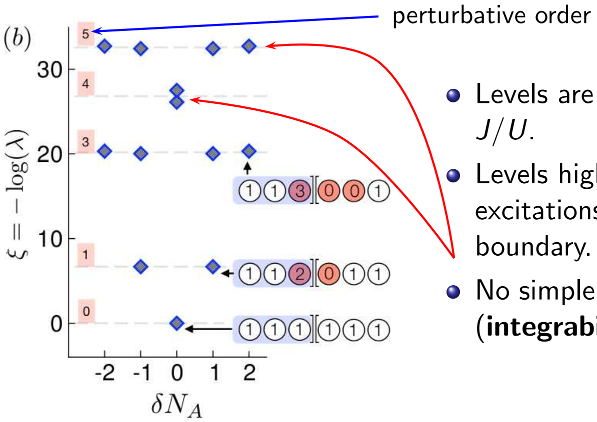


- Levels are organized in powers of J/U.
- Levels higher in the spectrum $\Rightarrow$ excitations farther from the boundary.

The 1D Bose Hubbard

$$\mathcal{H} = J\sum_i \underbrace{(b_i^\dagger b_{i+1} + h.c.)}_{\mathcal{H}_p} + \frac{U}{2}\sum_i n_i(n_i - 1)$$

- In the Mott insulating phase (large U).
- Expansion in J/U starting from unit filling $|\Psi_0\rangle = |\cdots 111111\cdots\rangle$



- Levels are organized in powers of J/U.
- Levels higher in the spectrum $\Rightarrow$ excitations farther from the boundary.
- No simple rules or exact degeneracy (**integrability breaking**).

Summary

- The entanglement spectrum for gapped systems is dominated by the physics at the boundaries (**boundary-locality**).
- Boundary-locality provides a **perturbative scheme** for calculating entanglement spectra in gapped systems.
- The study of entanglement spectrum in the XXZ allowed to unveil a beautiful combinatorial structure (**integrability**).
- The entanglement spectrum is a useful tool to analyse the structure (correlations, symmetries) of the wave functions (**"state tomography"**).

XXZ: the ES in the limit $\Delta \to -1^+$

XXZ: the ES in the limit $\Delta \to -1^+$

XXZ: the ES in the limit $\Delta \to -1^+$

The XXZ in the limit $\Delta \to -1^+$

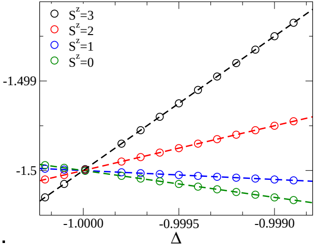
$$\mathcal{H} = \sum_i^L \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \boxed{-S_i^z S_{i+1}^z}\right)$$

- $\Delta = -1$ critical point (**non conformal**)
- Unitarily equivalent to the Heisenberg ferromagnet.

$$\mathcal{H} = -\sum_i \left(\tilde{S}_i^+ \tilde{S}_{i+1}^- + \tilde{S}_i^- \tilde{S}_{i+1}^+ + \tilde{S}_i^z \tilde{S}_{i+1}^z\right)$$

- The ground state is in the **symmetric** sector:

$$|\Psi_0\rangle = |\cdots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \cdots\rangle \Rightarrow \begin{cases} S_{tot}^- & |\cdots \boxed{\downarrow} \cdots\rangle \\ (S_{tot}^-)^2 & |\cdots \boxed{\downarrow} \cdots \boxed{\downarrow} \cdots\rangle \\ \vdots & \end{cases}$$



- The ground state is highly degenerate $(L+1)$.
- "**mean field**" like structure: no notion of distance the down spins are delocalized.

The entanglement spectrum at $\Delta = -1$

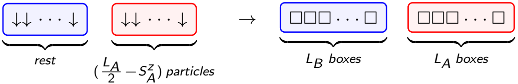
- The entanglement spectrum at $\Delta = -1$ has a simple combinatorial structure:

[Popkov,2005]

[Doyon,2011]

$$\{\xi_i\} = -2\log\sqrt{\frac{\binom{L_A}{\frac{L_A}{2}-S^z_A}\binom{L-L_A}{\frac{L-L_A}{2}-S^z+S^z_A}}{\binom{L}{\frac{L}{2}-S^z}}}$$

- One entanglement level for each sector S^z_A.
- Interpretation:

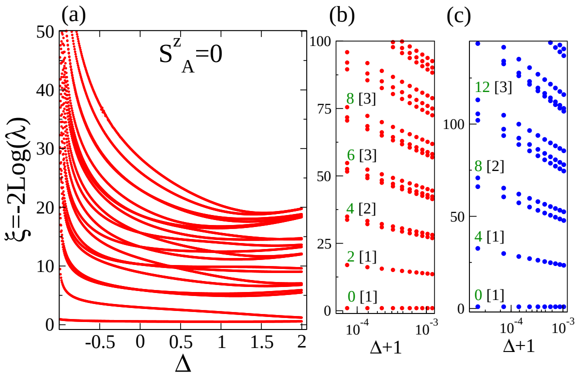


- $S^z = 0$ (i.e. the ground state at $\Delta > -1$), unusual scaling of von Neumann entropy:

$$S_A = \boxed{\frac{1}{2}}\log\left(\frac{\pi L_A}{2}\right) + \frac{1}{2} + \mathcal{O}(1/L_A)$$

- Log scaling without central charge.

The entanglement spectrum in the limit $\Delta \to -1^+$



$\xi \approx \alpha \log(\Delta + 1)$

(b) periodic b.c. $\alpha = 0, 2, 4, \ldots$

(c) open b.c. $\alpha = 0, 4, 8, \ldots$

- All the ES levels diverge (only one finite level for each S_A^z sector).
- The information about the state is encoded in the ES (“**state tomography**”)
- Simple multiplicity structure emerging.
- Similar behavior for all the states in the symmetric sector (same α).

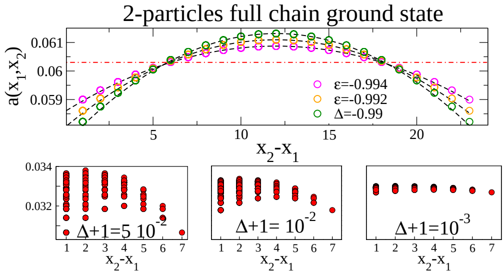
The physical interpretation

- Consider the state with only two particles (down spins).

$$|\cdots\boxed{\downarrow}\cdots\boxed{\downarrow}\cdots\rangle$$

- Expand the full chain wavefunction in the vicinity of $\Delta = -1$ ($\epsilon \equiv \Delta + 1$).

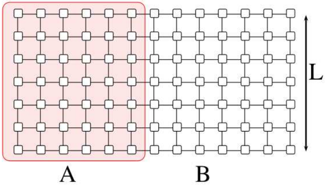
$$|\Psi\rangle = \sqrt{\frac{2}{L(L-1)}}\sum_{x_1<x_2}[1 + A_2\epsilon - \frac{\epsilon}{L-1}(x_1 - x_2 + L/2)^2]$$



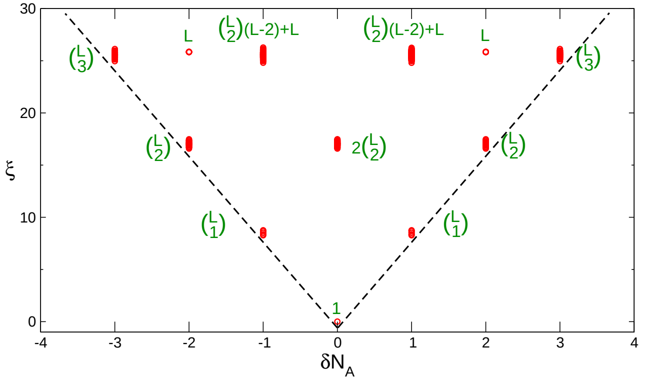
- At $\epsilon \neq 0$ the particles are not deconfined anymore.
- Easy to generalize to the case with more particles.
- The ES contains the information about the geometry.

2D Bose-Hubbard on a cylinder at filling 1

$$\mathcal{H} = -\sum_{\langle ij \rangle}(b_i^\dagger b_j + h.c.) + \frac{U}{2}\sum_i n_i(n_i - 1)$$



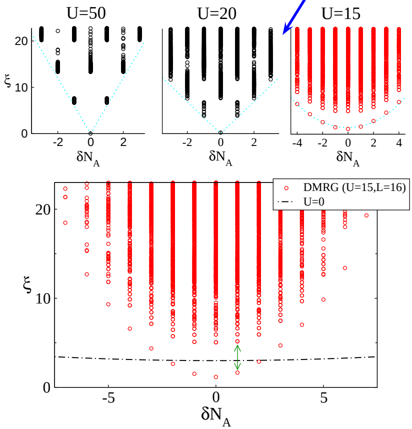
- Deep in the Mott insulating phase ($U = 100$).



- "Cone like" (**linear**) envelope.
- The ES levels show non trivial **dispersions**.
- Intriguing **multiplicity** structure emerging.

The superfluid phase

Mott-superfluid transition



- Drastic change at the Mott-superfluid transition ($U_c \approx 16$).
- In the superfluid **parabolic envelope**.
- **Entanglement gap** between the envelope and the rest of the ES.
- In the $U \to 0$ limit only the envelope survives (all the bosons are in the **condensate**).

Summary

- The ES show dramatic signatures of the phase transition at $\Delta = -1$.
- The symmetry of the wave function at $\Delta = -1$ is encoded in the simple structure of the ES.
- The ES is a useful tool to understand how the wave function evolves in the gapless phase ($\Delta = -1$).

- In the $2D$ Bose-Hubbard the ES show different behavior in the superfluid and Mott insulating phase.
- In the superfluid phase the ES shows signature of the condensate wavefunction.
- The formation of a gap in the ES provides a way to highlight the superfluid- Mott insulator transition.