Topological Order in Spin Systems with $\mathbb{Z}_n$ symmetry

Paul Fendley

Much ado about topological order

Systems with topological order in 2+1 dimensions typically have anyonic/fractionalized/spin-charge separated excitations.

These quasiparticles can even have non-abelian statistics, i.e. when ``braided'' around each other, the system can change state.

Local perturbations don't affect statistics, so this gives promise for topologically protected quantum computing.

What is topological order?

- Conceptually useful definition: the number of ground states depends on topological properties (e.g. genus) of space.

- Common characteristic: gapless edge modes

Free fermion cases are now well understood

For example, topological insulators and superconductors made from free fermions in arbitrary dimensions have been classified

via K theory: Kitaev; via edge theories: Ryu, Schnyder, Furusaki, Ludwig

A well-known example is the integer quantum Hall effect.

This, however, is not the simplest example...

The quantum Ising chain

Yes, the one Onsager solved in the '40s …

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Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition

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The partition function of a two-dimensional "ferromagnetic" with scalar "spins" (Ising model) is computed rigorously for the case of vanishing field. The eigenwert problem involved in the corresponding computation for a long strip crystal of finite width (n atoms), joined straight to itself around a cylinder, is solved by direct product decomposition; in the special case $n=\infty$ an integral replaces a sum. The choice of different interaction energies $(\pm J, \pm J')$ in the (0 1) and (1 0) directions does not complicate the problem. The two-way infinite crystal has an order-disorder transition at a temperature $T=T_c$ given by the condition

$$\sinh(2J/kT_c)\sinh(2J'/kT_c)=1.$$

The energy is a continuous function of T; but the specific heat becomes infinite as $-\log|T-T_c|$. For strips of finite width, the maximum of the specific heat increases linearly with $\log n$. The order-converting dual transformation invented by Kramers and Wannier effects a simple automorphism of the basis of the quaternion algebra which is natural to the problem in hand. In addition to the thermodynamic properties of the massive crystal, the free energy of a (0 1) boundary between areas of opposite order is computed; on this basis the mean ordered length of a strip crystal is

$$(\exp(2J/kT)\tanh(2J'/kT))^n.$$

57 years later, Kitaev made a trivial-but-profound observation:

Fermions exist in nature

The easiest way to solve the 2d Ising model is to follow Kaufmann and map it on to free fermions. In the 1d quantum chain limit, this amounts to a Jordan-Wigner transformation.

If your physical system is comprised of spins, then this is a mathematical trick.

But if your physical system is comprised of fermions…

The quantum Ising chain

The Majorana chain

Unpaired Majorana fermions in quantum wires

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Abstract

Certain one-dimensional Fermi systems have an energy gap in the bulk spectrum while boundary states are described by one Majorana operator per boundary point. A finite system of length L possesses two ground states with an energy difference proportional to $\exp(-L/l_0)$ and different fermionic parities. Such systems can be used as qubits since they are intrinsically immune to decoherence. The property of a system to have boundary Majorana fermions is expressed as a condition on the bulk electron spectrum. The condition is satisfied in the presence of an arbitrary small energy gap induced by proximity of a 3-dimensional p-wave superconductor, provided that the normal spectrum has an odd number of Fermi points in each half of the Brillouin zone (each spin component counts separately).

In terms of fermions, the quantum Ising chain includes a Cooper-pairing interaction $\psi^\dagger \psi^\dagger + \psi \psi$, so fermion number is only conserved mod 2.

As a consequence, the fermions are Majorana: they have a single fermi point. In old-folks language, there is no fermion doubling.

Open boundary conditions in the Ising ordered phase yield edge zero modes in the fermion picture.

Ising order corresponds to topological order!

Even more remarkably, these correspondences can be extended to 2d quantum models.

The Kitaev honeycomb model is a spin model with nearest-neighbor interactions. By exploiting a non-obvious gauge symmetry, it can be mapped on to free fermions with a background gauge field.

Breaking time-reversal symmetry results in topological order!

Majoranas are not the end of the story

Interacting systems such as the fractional quantum Hall effect exhibit still more interesting behavior: charge fractionalization, universal topological quantum computation…

Understanding the physics here is much more difficult.

A basic thing to do is to add local fermion interactions. In 1d, this results in a classification very similar to that of free-fermi systems.

Fidkowski and Kitaev

A different approach to topological order for interacting systems

- 1d ``spin'' systems with $\mathbb{Z}_n$ symmetry can be mapped onto parafermions. These in general are not perturbations of free fermions, and are strongly interacting.

- Nonetheless, in 1d they can have edge zero modes, just like Majorana/Ising.

- Moreover, there are 2d spin models a la Kitaev that map onto parafermions plus background $\mathbb{Z}_n$ gauge field.

Outline

- Edge/zero modes in the Ising/Majorana chain

- Edge/zero modes in the 3-state (chiral) Potts chain using parafermions
  an unusual form of integrability

- Coupling chains to make 2d $\mathbb{Z}_n$ gauge theories
  generalizing the Kitaev honeycomb model

How to fermionize the quantum Ising chain

$$H = \sum_j \left[f\sigma_j^x \; + J\sigma_j^z \sigma_{j+1}^z \right]$$

flip term interaction

Critical point is when $J = f$, ordered phase is $J > f$.

$\mathbb{Z}_2$ symmetry operator is flipping all spins:

$$\prod_j \sigma_j^x$$

Jordan-Wigner transformation in terms of Majorana fermions

$$\psi_j = \sigma_j^z \prod_{i<j} \sigma_i^x \,, \qquad \chi_j = \sigma_j^y \prod_{i<j} \sigma_i^x$$

string

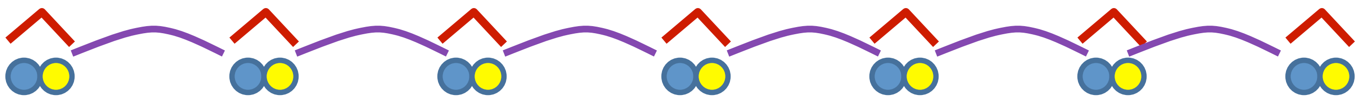
$$\{\psi_i \,, \psi_j\} = \{\chi_i \,, \chi_j\} = 2\delta_{ij}, \qquad \{\psi_i, \chi_j\} = 0$$

$\mathbb{Z}_2$ symmetry measures even or odd number of fermions:

$$(-1)^F = \prod_j \sigma_j^x = \prod_j (i\psi_j \chi_j)$$

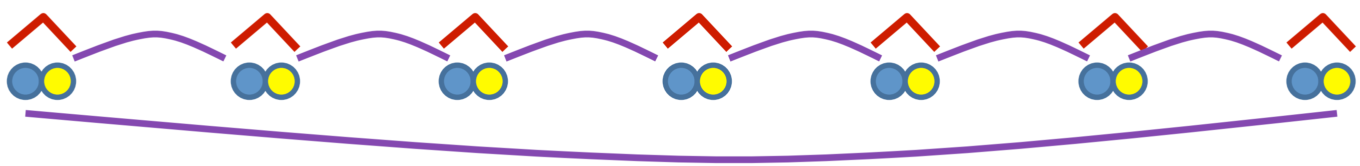
The Hamiltonian in terms of fermions

- with free boundary conditions:



$$H = if \sum_{j=1}^{N} \psi_j \chi_j + iJ \sum_{j=1}^{N-1} \chi_j \psi_{j+1}$$

- with periodic boundary conditions on the fermions:



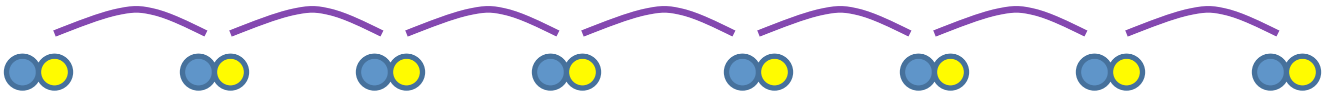
$$H = i \sum_{j=1}^{N} \left[f \psi_j \chi_j + J \chi_j \psi_{j+1} \right]$$

A catch: when written in terms of spins, this is twisted by $-(-1)^F$

Extreme limits:

- $J = 0$ (disordered in spin language):

- $f = 0$ (ordered in spin language):



The fermions on the edges, χ_1 and ψ_N, do not appear in H when $f = 0$. They commute with H!

Gapless edge modes → topological order

- When $f = 0$, the operators χ_1 and ψ_N map one ground state to the other – they form an exact zero mode.

- The gapless edge modes persist for all $f < J$: the series

$$\chi_1 + \frac{f}{J}\chi_3 + \left(\frac{f}{J}\right)^2 \chi_5 + \ldots$$

commutes with H.

- When $f < J$ this is is localized near the edge.

How does one characterize topological order with periodic boundary conditions?

Simple way for 1d: can show it depends on $(-1)^F$ of ground state.

Even fancier way: compute sign of Pfaffian.

For the experts: this is the 1d analog of the 2d Chern number.

On to the $\mathbb{Z}_n$ case:

- Fradkin and Kadanoff showed long ago that 2d spin models with $\mathbb{Z}_n$ symmetry can be written in terms of parafermions.

- Fateev and Zamolodchikov found integrable critical self-dual lattice spin models with $\mathbb{Z}_n$ symmetry. Later they found an elegant CFT description of the continuum limit.

- Read and Rezayi constructed fractional quantum Hall wavefunctions using the CFT parafermion correlators.

The 3-state (chiral) Potts model

The quantum chain version of the 3-state Potts model:

$$H = -\sum_j \left[f(\tau_j + \tau_j^\dagger) + J(\sigma_j^\dagger \sigma_{j+1} + \text{h.c.}) \right]$$

flip term potential

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}$$

$$\tau^3 = \sigma^3 = 1, \quad \tau^2 = \tau^\dagger, \quad \sigma^2 = \sigma^\dagger$$

$$\boxed{\tau\sigma = e^{2\pi i/3}\sigma\tau}$$

Define parafermions just like fermions:

In a 2d classical theory, they're the product of order and disorder operators. In the quantum chain,

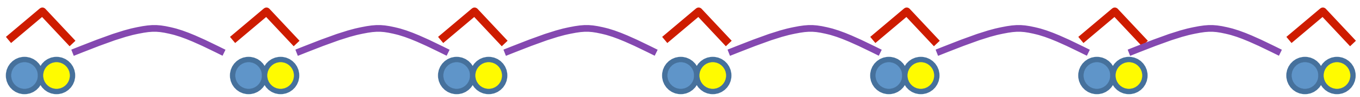
$$\psi_j = \sigma_j \prod_{i<j} \tau_i, \qquad \chi_j = \tau_j \sigma_j \prod_{i<j} \tau_i$$

$$\psi^3 = \chi^3 = 1, \quad \psi^2 = \psi^\dagger, \quad \chi^2 = \chi^\dagger$$

Instead of anticommutators, for $i < j$ and $\gamma = \chi$ or ψ :

$$\gamma_i \gamma_j = e^{2\pi i/3} \gamma_j \gamma_i$$

The Hamiltonian in terms of parafermions:



 $= f(\psi_j^\dagger \chi_j + \chi_j^\dagger \psi_j)$

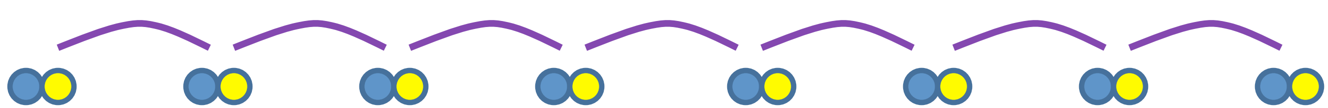
 $= J(\psi_{j+1}^\dagger \chi_j + \chi_j^\dagger \psi_{j+1})$

flip term

potential

These parafermions are not perturbations of free fermions – they cube to 1. The model isn't even integrable unless $J = f$.

However, when f = 0, there are edge zero modes!



Does the zero mode remain for $J > f > 0$?

We can't cheat like in the Ising chain and just solve the model.

First study an easier problem, and take periodic boundary conditions. Can we find an analogous ``zero'' mode?

A parafermion ``zero'' mode shifts the energy uniformly between $\mathbb{Z}_3$ sectors, i.e.

$$[H, \Psi] = (\Delta E)\, \Psi$$

There is a ``zero'' mode only if the couplings obey an interesting constraint!

Generalize to the chiral Potts model:

 $= f(e^{i\phi}\psi_j^\dagger \chi_j + e^{-i\phi}\chi_j^\dagger \psi_j)$

 $= J(e^{i\theta}\psi_{j+1}^\dagger \chi_j + e^{-i\theta}\chi_j^\dagger \psi_{j+1})$

Look for a ``zero'' mode linear in the parafermions:

$$\Psi = \sum_j [\alpha_j \psi_j + \beta_j \chi_j]$$

Then there is an exact ``zero" mode if the couplings obey:

$$f \cos(3\phi) = J \cos(3\theta)$$

$$f\cos(3\phi) = J\cos(3\theta)$$

This calculation is the world's easiest way of finding the couplings of the integrable chiral Potts chain.

Howes, Kadanoff and den Nijs; von Gehlen and Rittenberg; Albertini, McCoy, Perk and Tang; Baxter; Bazhanov and Stroganov

The integrable chiral Potts model is quite peculiar. The Boltzmann weights of the 2d classical analog are parameterized by higher genus Riemann surfaces instead of theta functions. They satisfy a generalized Yang-Baxter equation with no difference property. They are also 2d reductions of solvable 3d classical models.

The “superintegrable” line $\theta = \phi = \pi / 6$ is very special.

Here the ``zero’’ mode occurs for all values of f and J.

This is halfway between ferro and antiferromagnet, because

$$\sigma_i e^{i\pi/6} + \sigma_i^\dagger e^{-i\pi/6} = \begin{cases} e^{i\pi/6} + e^{-i\pi/6} = \sqrt{3} \\ e^{i\pi/6} e^{-i2\pi/3} + e^{-i\pi/6} e^{2i\pi 3} = 0 \\ e^{i\pi/6} e^{i2\pi/3} + e^{-i\pi/6} e^{-2i\pi 3} = -\sqrt{3} \end{cases}$$

and so the spectrum is invariant under $H \to -H$.

These are very good signs for the existence of an edge zero mode. An even better sign:

Label the sums of the flip and interaction terms as

$$A_0 = \sum_j (e^{i\pi/6}\sigma_j + e^{-i\pi/6}\sigma_j^\dagger)$$

$$A_1 = \sum_j (e^{i\pi/6}\tau_j\tau_{j+1}^\dagger + e^{-i\pi/6}\tau_j^\dagger\tau_{j+1})$$

Taking commutators of the these, one finds remarkable simplifications. They satisfy the Onsager algebra.

The identical algebra Onsager used to solve the Ising model originally also occurs in the chiral Potts models!

This allows the explicit construction of an infinite series of quantities commuting with the Hamiltonian

von Gehlen and Rittenberg

Onsager's original paper:

By (57) we have also

$$[A_m, A_{m+n}]=0.$$

If we exclude these trivial cases, the congruences

$$a\equiv b; \quad a+k\equiv b+m$$

will not be satisfied simultaneously when we compute

$$[A_k, A_m]=\sum_{a,b=1}^{n} [P_{a,a+k}, P_{b,b+m}]$$

$$=\sum_{c=1}^{n} [P_{xc}, P_{x,c+m-k}]+\sum_{a=1}^{n} [P_{ax}, P_{a-m+k,x}]$$

whence

$$[A_k, A_m]=4G_{k-m}, \qquad (60)$$

$$[G_m, A_k]=2A_{k+m}-2A_{k-m} \qquad (61)$$

which is also valid in the trivial case $m\equiv 0$. It is easily shown from (60) and (61) that the commutators $G_1, \cdots, G_{n-1}$ commute with each other. We have generally

$$[G_m, [A_k, A_0]]=[[G_m, A_k], A_0]+[A_k, [G_m, A_0]].$$

Using this together with (60) and (61) we find

$$\begin{aligned}[G_m, G_k]&=[(2A_{k+m}-2A_{k-m}), A_0]\\ &\quad+[A_k, (2A_m-2A_{-m})]\\ &=8G_{k+m}-8G_{k-m}\\ &\quad+8G_{k-m}-8G_{k+m}=0.\end{aligned} \qquad (61a)$$

All follows from the Jacobi identity and the two commutators

$$[A_0,[A_0,[A_0,A_1]]]=[A_0,A_1]$$

$$[A_1,[A_1,[A_1,A_0]]]=[A_1,A_0]$$

Dolan and Grady; Davies

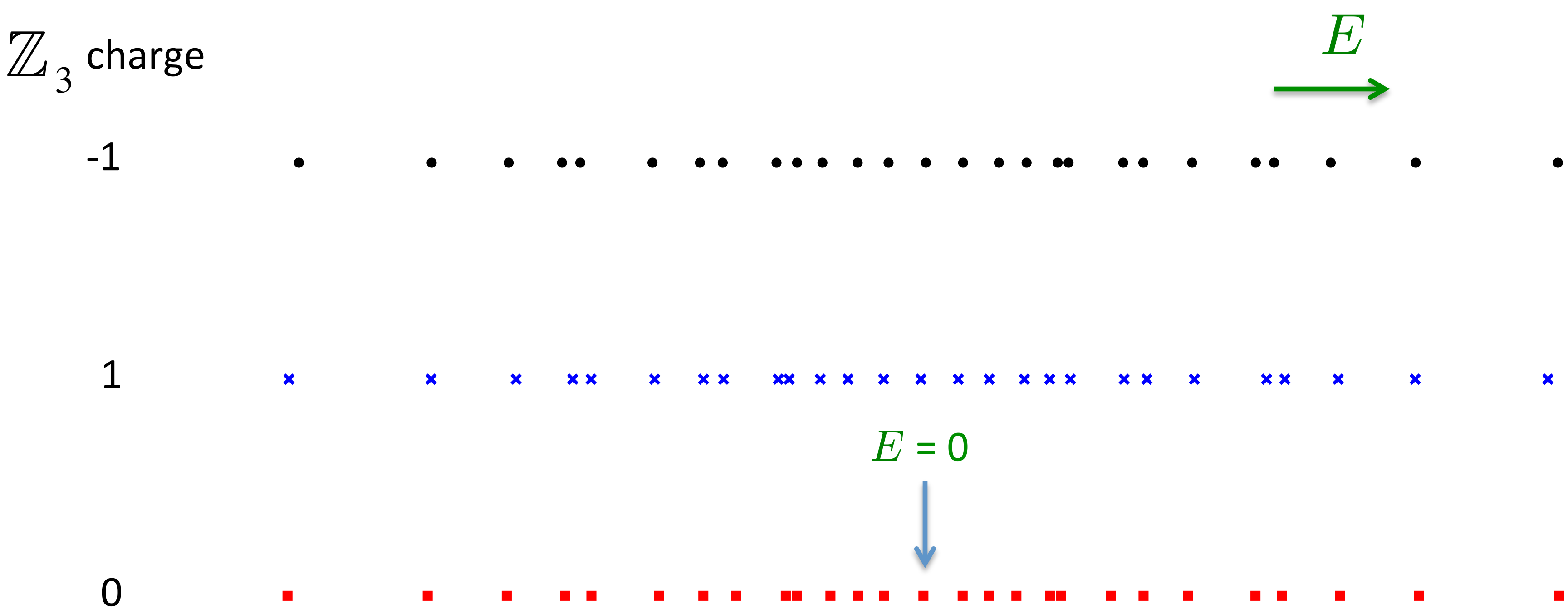
He didn't know this, but in the Ising case, the Onsager algebra is simply that of (zero-momentum) fermion bilinears!

Thus a miracle of the superintegrable chiral Potts model is that despite its not being a free-fermion theory, the algebra generated by the two parts of the Hamiltonian is identical to that of fermion bilinears.

Maybe there is a Pfaffian-ish formula to detect topological order ?!? The Read-Rezayian?!?

The open superintegrable chain

The energy levels for f = J/2 for 4 sites:



The levels in three sectors get exponentially close as the number of sites is increased. There is an edge zero mode!

Unfortunately, finding the edge zero mode explicitly does not seem to be as easy as for Ising. Perturbatively:

$$\Psi = \chi_1 + \frac{f}{2J}(\psi_1 - \omega\,\chi_1^\dagger\psi_1^\dagger + \chi_2 + \omega\psi_1^\dagger\chi_2^\dagger - \omega\chi_1^\dagger\chi_2^\dagger - \bar{\omega}\chi_1^\dagger\psi_1\chi_2) + \ldots$$

where $\omega = e^{2\pi i/3}$.

I have computed up to third order in f/J, and also found the ``leading'' piece at all orders.

Lots of interesting formal things going on: can look for the zero mode by deriving a (parity-invariant) ``Hamiltonian'' acting on the space of parafermions, and looking for its zero-energy state.

But I still need a boundary Onsager algebra…

Topological order in 2d

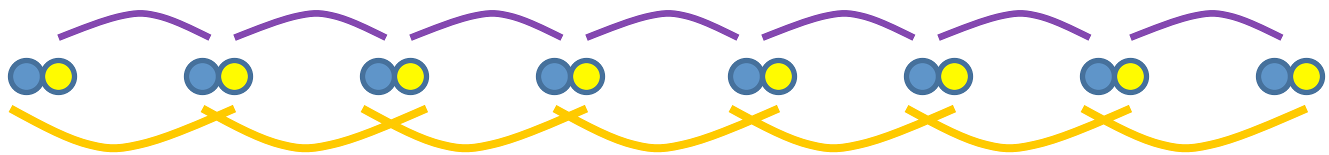
The Ising/Majorana chain has an elegant generalization to 2d via the Kitaev honeycomb model.

This is a spin model that can be mapped to free fermions coupled to a background gauge field.

I'll describe the analog for parafermions.

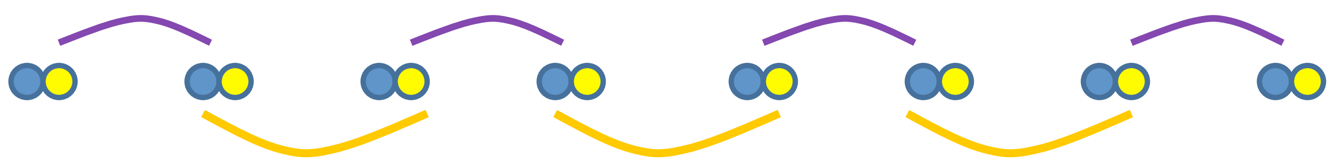
View the 2d model as coupled 1d chains

The quantum YZ chain



 $= J_z \sigma_i^z \sigma_{i+1}^z$  $\equiv J_y \sigma_i^y \sigma_{i+1}^y$

is comprised of two commuting Hamiltonians:



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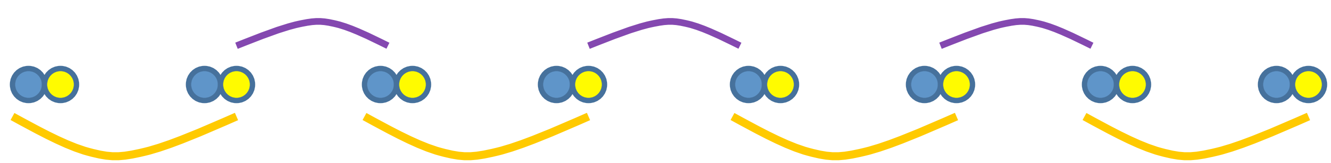
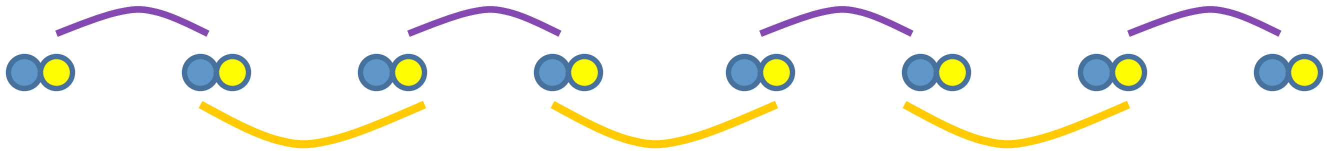


Consider one of these Hamiltonians:

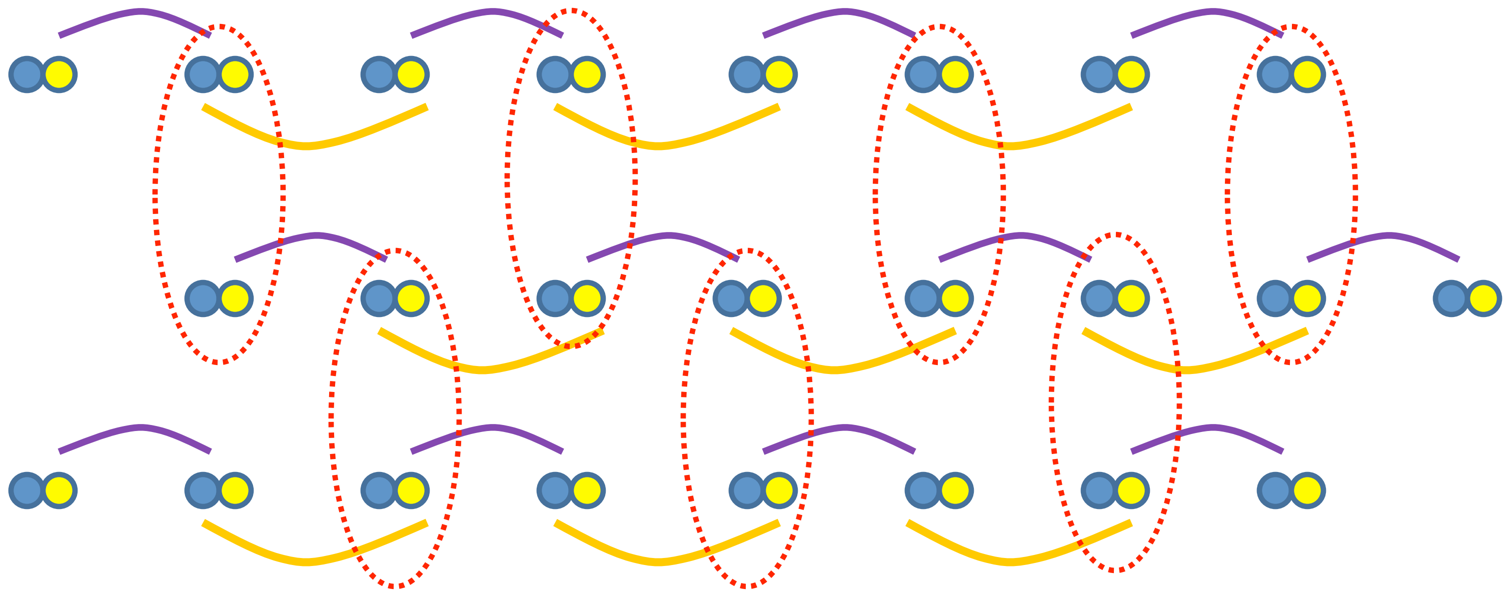


$$H^{(1)} = \sum_j \left[\sigma^z_{2j-1}\sigma^z_{2j} + \sigma^y_{2j}\sigma^y_{2j+1}\right] = i\sum_j \left[\chi_{2j-1}\psi_{2j} + \psi_{2j}\chi_{2j+1}\right]$$

Just like the edge modes, none of the fermions χ_{2j-1} or ψ_{2j} appear!

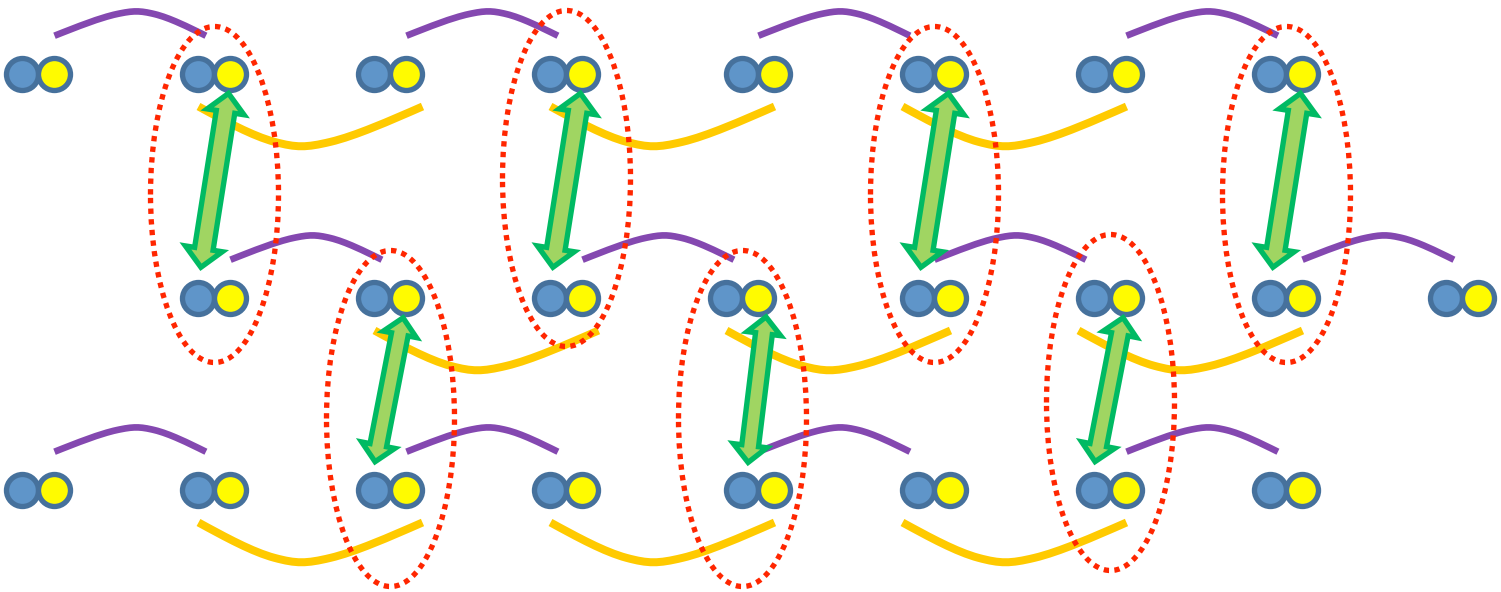
They commute with each individual term in $H^{(1)}$.

Now couple chains together into a honeycomb lattice:



$$H = \sum \psi\psi\chi\chi + \sum \psi\chi + \sum \chi\psi$$

$$H = \sum \sigma^x\sigma^x + \sum \sigma^y\sigma^y + \sum \sigma^z\sigma^z$$



Each fermion bilinear  commutes with each term in H.

The $\mathbb{Z}_2$ gauge flux is the product   $= \sigma^z \sigma^x \sigma^y \sigma^z \sigma^y \sigma^x$
around a hexagon.

The flux through each plaquette can be chosen individually, and is not dynamical.

Thus the Kitaev honeycomb model is simply free fermions coupled to a background $\mathbb{Z}_2$ gauge field.

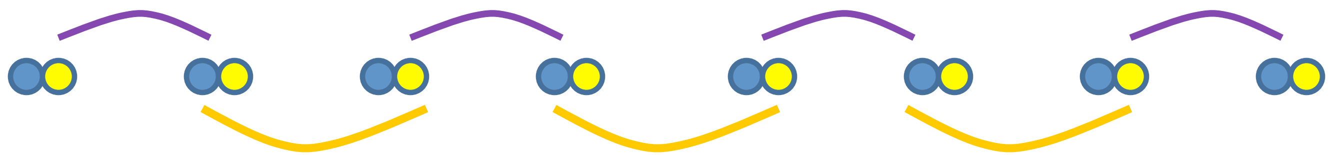
A magnetic field destroys the solvability, but causes non-abelian topological order.

On the Fisher lattice, non-abelian topological order occurs without the magnetic field; time-reversal symmetry is spontaneously broken.

Yao and Kivelson

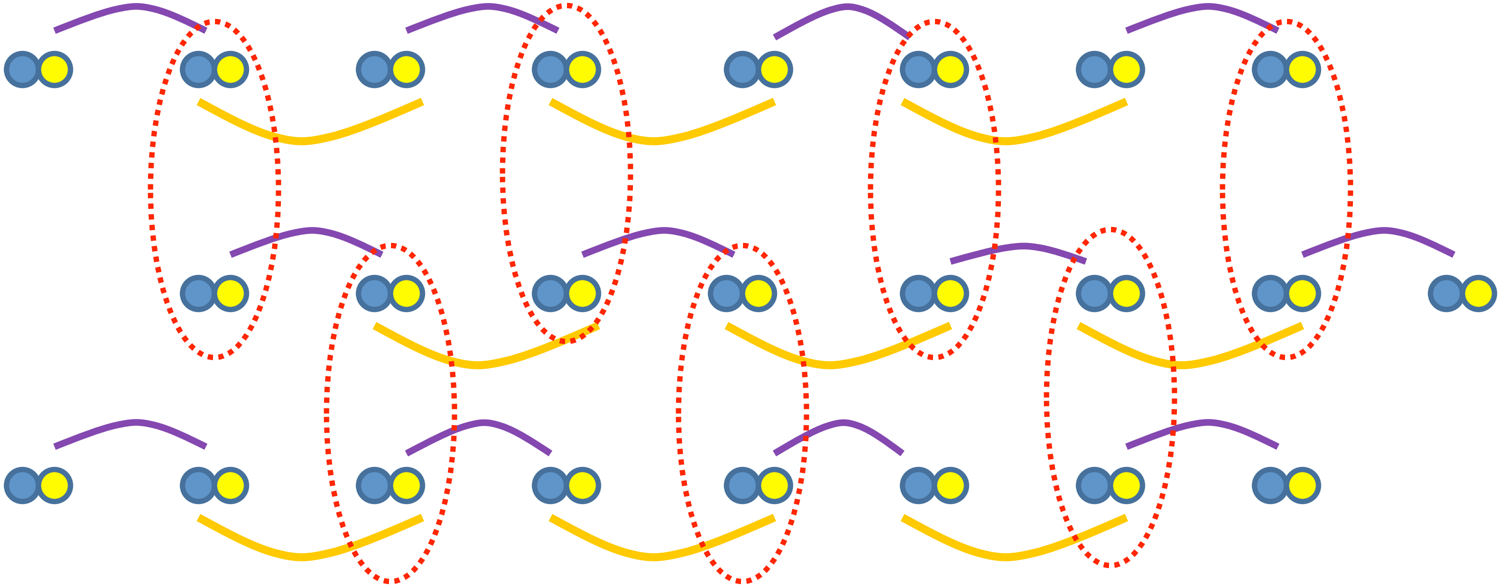
So what about parafermions?

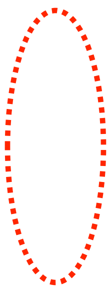
The same trick yields a “YZ” Hamiltonian that doesn’t involve half the parafermions:



 $= (\tau_i)^\dagger \tau_{i+1} + \text{h.c.}$

 $= (\tau_i \sigma_i)^\dagger \tau_{i+1} \sigma_{i+1} + \text{h.c.}$





$$= \sigma_k^\dagger \sigma_{k+1} + \text{h.c.}$$

The $\mathbb{Z}_3$ gauge flux is = $\begin{matrix} \tau^\dagger & \sigma & (\tau\sigma) \\ (\tau^\dagger\sigma^\dagger) & \sigma^\dagger & \tau \end{matrix}$ around a hexagon.

Questions

- The handwaving arguments for topological order work for parafermions. Presumably non-abelian?
- Is there a formula for the parafermions generalizing the Pfaffian/Chern number for fermions?
- Is there a connection to 2+1d integrable models?
- Should work for all $\mathbb{Z}_n$, what about $U(1)$?
- What's with the Onsager algebra?