Fractional Topogical Insulators: numerical evidences

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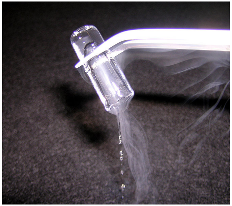
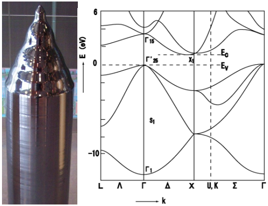
Motivations : Topological insulators

An insulator has a (large) gap separating a fully filled valence band and an empty conduction band

Atomic insulator : solid argon



Semiconductor : Si



How to define equivalent insulators ? Find a continuous transformation from one Bloch Hamiltonian $\mathcal{H}_0(\vec{k})$ to another $\mathcal{H}_1(\vec{k})$ without closing the gap

- Vacuum is the same kind of insulator than solid argon with a gap $2m_e c^2$
- Are all insulators equivalent to the vacuum ? No

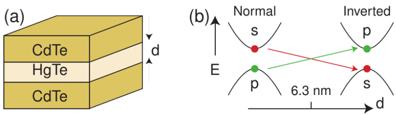
Motivations : Topological insulators

What is topological order ?

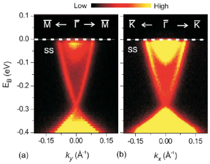
- cannot be described by symmetry breaking (cannot use Ginzburg-Landau theory)
- some physical quantities are given by a "topological invariant" (think about the surface genus)
- a bulk gapped system (i.e. insulator) system feeling the topology (degenerate ground state, cannot be lifted by local measurement).
- a famous example : Quantum Hall Effect (QHE)

TI theoretically predicted and experimentally observed in the past 5 years missed by decades of band theory

2D TI :



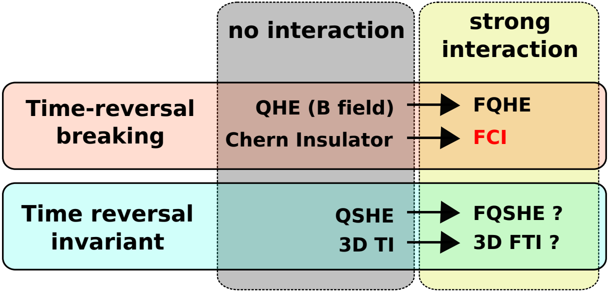
3D TI :



Motivations : FTI

A rich physics emerge when turning on strong interaction in QHE

What about Topological insulators ?

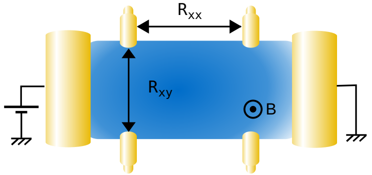


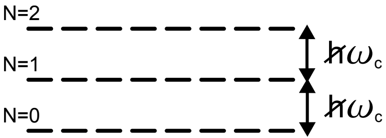
Outline

- Fractional Quantum Hall Effect
- Fractional Chern Insulators
- Entanglement spectroscopy
- FTI with time reversal symmetry

Fractional Quantum Hall Effect

Landau level



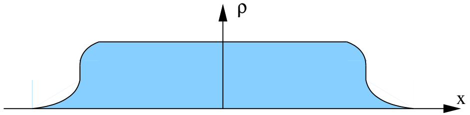


- Cyclotron frequency : $\omega_c = \frac{eB}{m}$
- Filling factor : $\nu = \frac{hn}{eB} = \frac{N}{N_\phi}$
- At $\nu = n$, n completely filled levels and a energy gap $\hbar\omega_c$
- Integer filling : a ($\mathbb{Z}$) topological insulator with a perfectly flat band / perfectly flat Berry curvature !
- Partial filling + interaction $\rightarrow$ FQHE
- Lowest Landau level ($\nu < 1$) : $z^m \exp\left(-|z|^2/4l^2\right)$
- N-body wavefunction : $\Psi = P(z_1, ..., z_N) \exp(-\sum |z_i|^2/4)$

The Laughlin wavefunction

A (very) good approximation of the ground state at $\nu = \frac{1}{3}$

$$\Psi_L(z_1, ...z_N) = \prod_{i<j}(z_i - z_j)^3 e^{-\sum_i \frac{|z_i|^2}{4l^2}}$$



- The Laughlin state is the unique (on genus zero surface) densest state that screens the short range (p-wave) repulsive interaction.
- **Topological state** : the degeneracy of the densest state depends on the surface genus (sphere, torus, ...)

The "Laughlin wavefunction"

A (very) good approximation of the ground state at $\nu = \frac{1}{3}$

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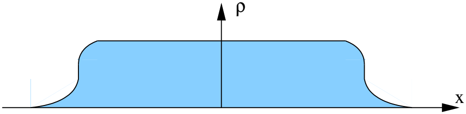

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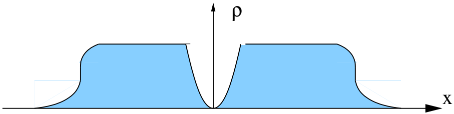
The “Laughlin wavefunction” : quasihole

Add one flux quantum at z_0 = one quasi-hole

$$\Psi_{qh}(z_1, ...z_N) = \prod_i (z_0 - z_i) \;\; \Psi_L(z_1, ...z_N)$$



- Locally, create one quasi-hole with fractional charge $\frac{\pm e}{3}$
- “Wilczek” approach : quasi-holes obey fractional statistics
- Adding quasiholes/flux quanta increases the size of the droplet
- For given number of particles and flux quanta, there is a specific number of qh states that one can write
- These numbers/degeneracies can be classified with respect some quantum number (angular momentum L_z) and are a fingerprint of the phase (related to the statistics of the excitations).

Fractional Chern Insulator

Interacting Chern insulators

- A Chern insulator is a zero magnetic field version of the QHE (Haldane, 88)
- Topological properties emerge from the band structure
- At least one band is a non-zero Chern number C, Hall conductance $\sigma_{xy} = \frac{e^2}{h}|C|$
- Basic building block of 2D $\mathbb{Z}_2$ topological insulator (half of it)
- Is there a zero magnetic field equivalent of the FQHE ? $\rightarrow$ Fractional Chern Insulator
- Here we will focus on the $C = \pm 1$.

From CI to FCI

To go from IQHE to FQHE, we need to :

- consider a single Landau level
- partially fill this level, $\nu = N/N_\Phi$
- turn on repulsive interactions

From CI to FCI

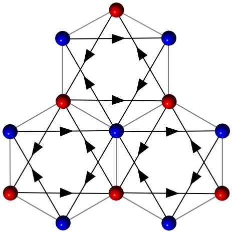
To go from IQHE CI to FQHE FCI, we need to :

- consider a single Landau level
  consider a single band
- partially fill this level, $\nu = N/N_{\Phi}$
  partially fill this band, $\nu = N/N_{\text{unit cells}}$
- turn on repulsive interactions
  turn on repulsive interactions

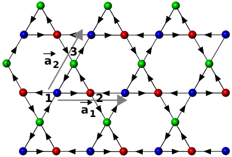
What QH features should we try to mimic to get a FCI ?

- Several proposals for a CI with nearly flat band that may lead to FCI
- But "nearly" flat band is not crucial for FCI like flat band is not crucial for FQHE (think about disorder)

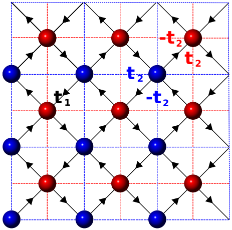
Four (almost) flat band models



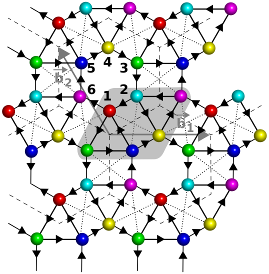
Haldane model,
Neupert et al. PRL (2011)



Kagome lattice,
E. Tang et al. PRL (2011)

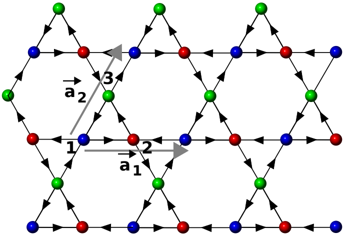


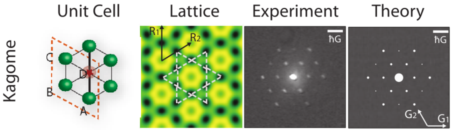
Checkerboard lattice,
K. Sun et al. PRL (2011).



Ruby lattice, PRB (2011)

The Kagome lattice model





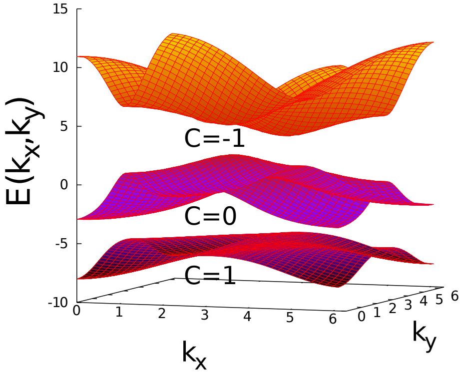
Jo et al. PRL (2012)

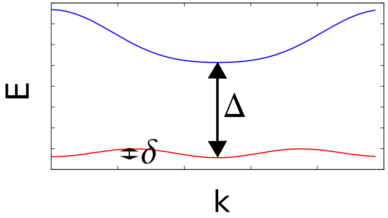
- three atoms per unit cell, spinless particles
- lattice can be realized in cold atoms
- only nearest neighbor hopping $e^{i\varphi}$
- three bands with Chern numbers $C = 1,\ C = 0$ and $C = -1$

$$\mathcal{H}(\mathbf{k}) = -t_1 \begin{bmatrix} 0 & e^{i\varphi}(1+e^{-ik_x}) & e^{-i\varphi}(1+e^{-ik_y}) \\ & 0 & e^{i\varphi}(1+e^{i(k_x-k_y)}) \\ \text{h.c.} & & 0 \end{bmatrix}$$

$k_x = \mathbf{k}.\mathbf{a_1}, k_y = \mathbf{k}.\mathbf{a_2}$

The flat band limit





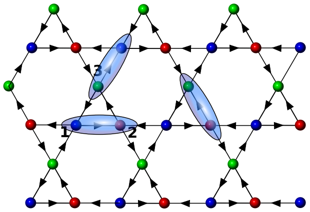
- $\delta \ll E_c \ll \Delta$ (E_c being the interaction energy scale)
- We can deform continuously the band structure to have a perfectly flat valence band
- and project the system onto the lowest band, similar to the projection onto the lowest Landau level

$$\mathcal{H}(\mathbf{k}) = \sum_{n=1}^{\text{nbr bands}} \mathcal{P}_n E_n(\mathbf{k})$$

$$\longrightarrow \quad \mathcal{H}^{FB}(\mathbf{k}) = \sum_{n=1}^{\text{nbr bands}} n\mathcal{P}_n$$

Two body interaction and the Kagome lattice

Our goal : stabilize a Laughlin-like state at $\nu = 1/3$.
A key property : the Laughlin state is the unique densest state that screens the short range repulsive interaction.



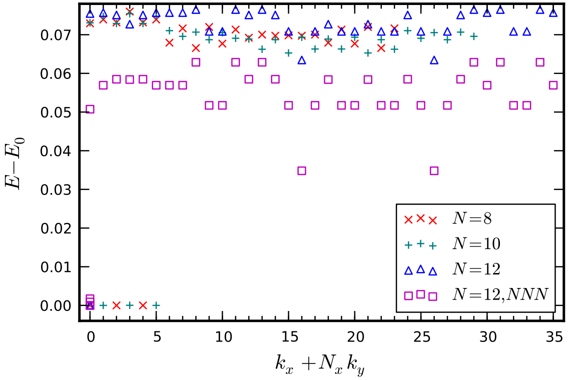
$$H^F_{\mathrm{int}} = U \sum_{<i,j>} : n_i n_j :$$

$$H^B_{\mathrm{int}} = U \sum_{i} : n_i n_i :$$

- A nearest neighbor repulsion should mimic the FQH interaction.
- We give the same energy penalty when two part are sitting on neighboring sites (for fermions) or on the same site (for bosons).
- On the checkerboard lattice : Neupert et al. PRL 106, 236804 (2011), Sheng et al. Nat. Comm. 2, 389 (2011), NR and BAB, PRX (2011)

The $\nu = 1/3$ filling factor

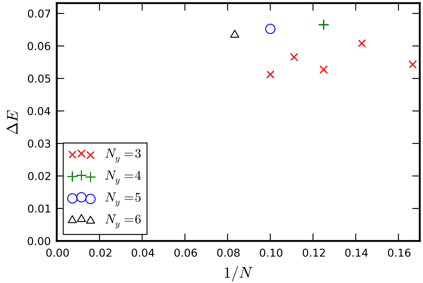
An **almost** threefold degenerate ground state as you expect for the Laughlin state on a torus (here lattice with periodic BC)



But 3fold degeneracy is not enough to prove that you have Laughlin-like physics there (a CDW would have the same counting).

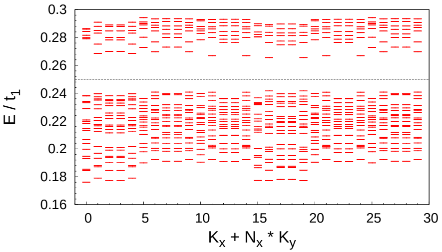
Gap

- Many-body gap can actually increase with the number of particles due to aspect ratio issues.
- Finite size scaling not and not monotonic reliable because of aspect ratio in the thermodynamic limit.
- The 3-fold degeneracy at filling 1/3 in the continuum exists for any potential and is not a hallmark of the FQH state. On the lattice, 3-fold degeneracy at filling 1/3 means more than in the continuum, but still not much



Quasihole excitations

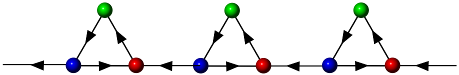
- The form of the groundstate of the Chern insulator at filling 1/3 is not exactly Laughlin-like. However, the universal properties SHOULD be.
- The hallmark of FQH effect is the existence of fractional statistics quasiholes.
- In the continuum FQH, Quasiholes are zero modes of a model Hamiltonians - they are really groundstates but at lower filling. In our case, for generic Hamiltonian, we have a gap from a low energy manifold (quasihole states) to higher generic states.

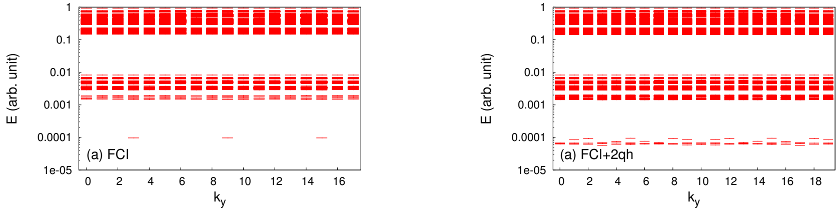


$N = 9,\ N_x = 5, N_y = 6$

The number of states below the gap matches the one of the FQHE !

The one dimensional limit : thin torus

- let's take $N_x = 1$, thin torus limit
- 
- the groundstate is just the electrostatic solution (1 electron every 3 unit cells)
- **a charge density wave** and not a Laughlin state

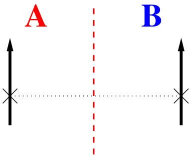


Can we differentiate between a Laughlin state and a CDW ?

Entanglement spectroscopy

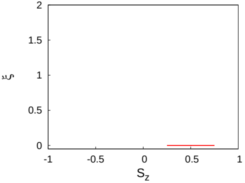
Entanglement spectrum - Li and Haldane, PRL (2008)

example : system made of two spins $1/2$

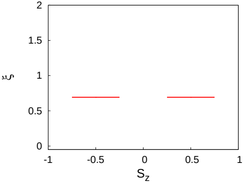


$\rho = |\Psi\rangle\langle\Psi|$, reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$
Entanglement spectrum : $\xi = -log(\lambda)$ (λ eigenvalues of ρ_A) as a function of S_z

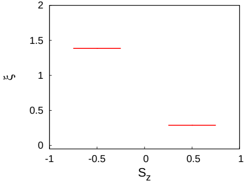
$|\uparrow\uparrow\rangle$



$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$



$\frac{1}{\sqrt{4}}|\uparrow\downarrow\rangle + \sqrt{\frac{3}{4}}|\downarrow\uparrow\rangle$



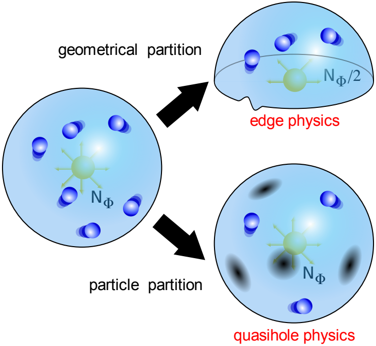
The counting (i.e the number of non zero eigenvalue) also provides informations about the entanglement

How to cut the system ?

The system can be cut in different ways :

- real space
- momentum space
- particle space

Each way may provide different information about the system (ex : trivial in momentum space but not in real space)



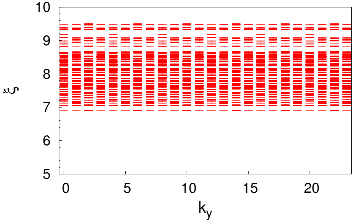
- **Orbital partitioning (OES)** : extracting the edge physics
- **Particle partitioning (PES)** : extracting the bulk physics

Particle entanglement spectrum

Particle cut : start with the ground state Ψ for N particles, remove $N - N_A$, keep N_A

$$\rho_A(x_1, ..., x_{N_A}; x'_1, ..., x'_{N_A})$$
$$= \int ... \int dx_{N_A+1}...dx_N \qquad \Psi^*(x_1, ..., x_{N_A}, x_{N_A+1}, ..., x_N)$$
$$\times \quad \Psi(x'_1, ..., x'_{N_A}, x_{N_A+1}, ..., x_N)$$

"Textbook expression" for the reduced density matrix.

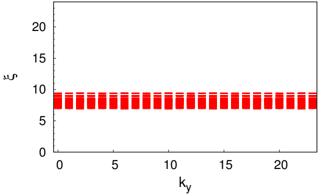


Laughlin $\nu = 1/3$ state $N = 8$, $N_A = 4$ on a torus

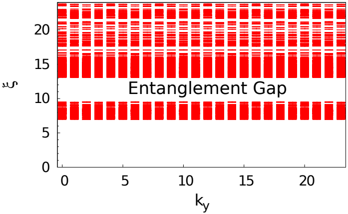
- Counting is the number of quasihole states for N_A particles on the same geometry
- the fingerprint of the phase.
- This information that comes from the bulk excitations is encoded within the groundstate !

Away from model states : Coulomb groundstate at $\nu = 1/3$

- Coulomb groundstate at $\nu = 1/3$ has the same universal properties than the Laughlin state
- The ES exhibits an entanglement gap.
- Depending on the geometry, this gap collapses after a few momenta away from the maximum one (the system "feels" the edge) or is along the full range of momenta (torus).
- The part below the gap has the same fingerprint than the Laughlin state : the entanglement gap protects the state statistical properties.



Laughlin on torus $\nu = 1/3$

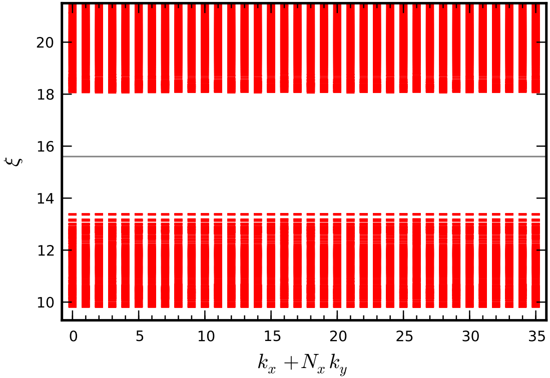


Coulomb on torus $\nu = 1/3$

Back to the FCI

Particle entanglement spectrum

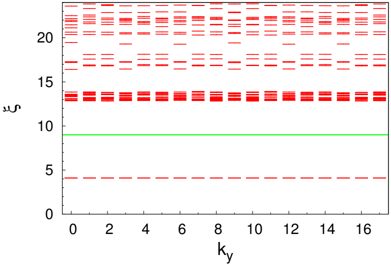
Back to the Fractional Chern Insulator



PES for $N = 12$, $N_A = 5$, 2530 states per momentum sector below the gap as expected for a Laughlin state

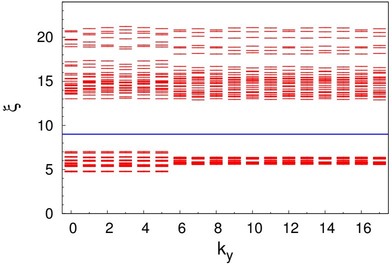
Particle entanglement spectrum : CDW

The PES for a CDW can be computed exactly and is not identical to the Laughlin PES



$\nu = 1/3,\ N_x = 1,\ N = 6,\ N_A = 3$

59 states below the gap $\longrightarrow$ CDW

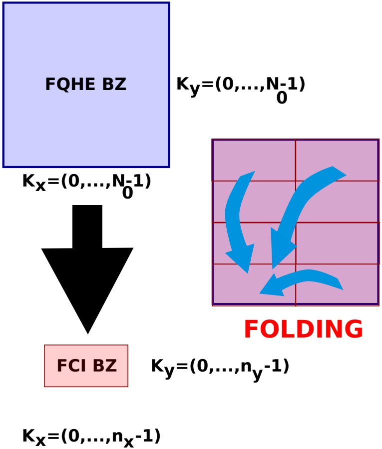


$\nu = 1/3,\ N_x = 6,\ N = 6,\ N_A = 3$

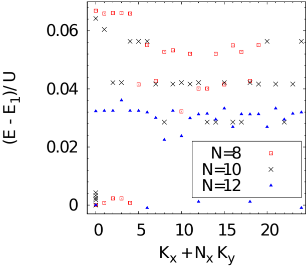
329 states below the gap $\longrightarrow$ Laughlin

Emergent Symmetries in the Chern Insulator

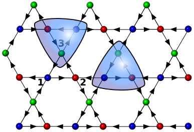
- in FQH, we have the magnetic translational algebra
- In FCIs, there is in principle no exact degeneracy (apart from the lattice symmetries).
- But both the low energy part of the energy and entanglement spectra exhibit an emergent translational symmetry.
- The momentum quantum numbers of the FCI can be deduced by folding the FQH Brillouin zone.
- FQH : $N_0 = GCD(N, N_\phi = N_x \times N_y)$
  FCI : $n_x = GCD(N, N_x)$,
  $n_y = GCD(N, N_y)$

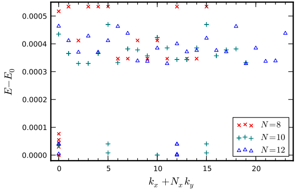


Beyond the Laughlin states



- A clear signature for composite fermion states at $\nu = 2/5$ and $\nu = 3/7$ (here Kagome at $\nu = 2/5$)
- Also observed for bosons at $\nu = 2/3$ and $\nu = 3/4$.

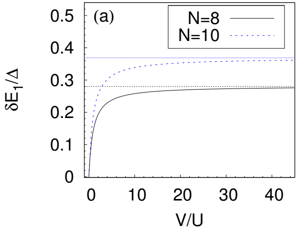




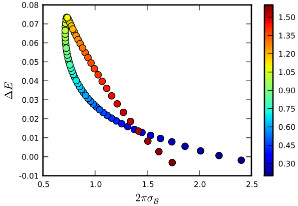
- Moore-Read state. Possible non-abelian candidate for $\nu = 5/2$ in the FQHE.
- MR state can be exactly produced using a three-body interaction.
- FCI require 3 body int.

FCI : a perfect world ?

- Not all models produce a Laughlin-like state
- Depends on the particle statistics : Haldane model fermions vs bosons
- Longer range interactions destabilize FCI
- Even more model dependent for the other states
- Does a flatter Berry curvature help ? Not really
- Situation is even worse for higher Chern numbers (Wang et al. arXiv :1204.1697)
- What are the key ingredients to get a robust FCI ?



Ruby+bosons $\nu = 2/3$

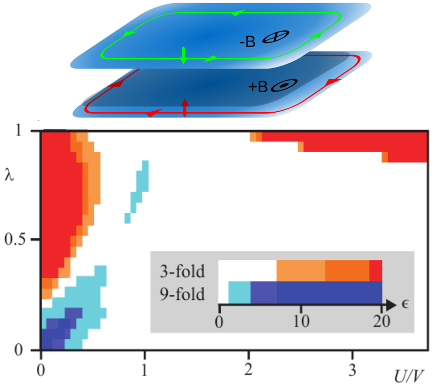


Kagome $N = 8$ and $N_x = 6, N_y = 4$

FTI with time reversal symmetry

From FCI to FQSH

- QSH can be built from two CI copies
- One can do the same for FQSH
- How stable if the FQSH wrt when coupling the two layers ? Coupling via interaction or the band structure
- Neupert et al., PRB 84, 165107 (2011) using the checkerboard lattice
- Not really conclusive for the FQSH : does it survive beyond the single layer gap ?

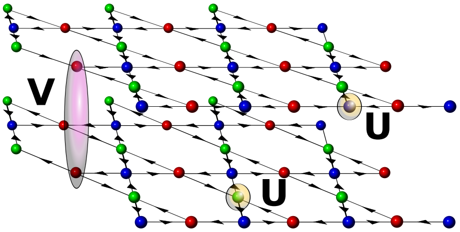


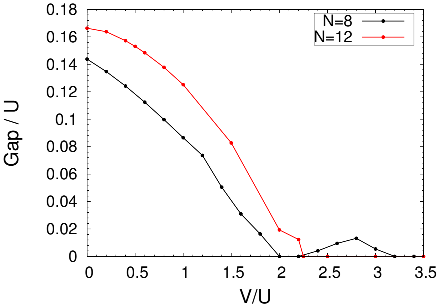
$N = 16$ at $\nu = 2/3$

V intralayer, U on-site interlayer, λ NN interlayer

From FCI to FQSH

- Two copies of the Kagome model with bosons.
- Hubbard model with two parameters for the interaction : U on-site same layer, V on-site interlayer





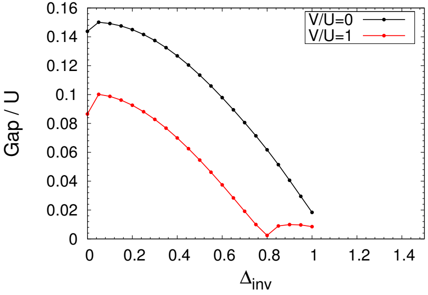
From FCI to FQSH

We can also couple the two layers through the band structure by adding an inversion symmetry breaking term.

$$H(\mathbf{k}) = \begin{bmatrix} h_{\mathrm{CI}}(\mathbf{k}) & \Delta_{inv} C \\ \Delta_{inv} C^{+} & h^{*}_{\mathrm{CI}}(-\mathbf{k}) \end{bmatrix}$$

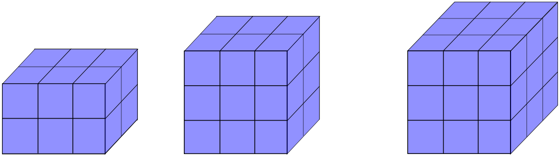
with $C = -C^t$, here

$$C_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$



3D FTI : a technical challenge

- an unknown territory : nature of the excitations (strings ?), effective theory for the surface modes (beyond Luttinger ?), algebraic structure (GMP algebra in 3D ?)
- is there a microscopic model ?
- example : Fu-Kane-Mele model with interaction for N electrons at filling $\nu = 1/3$



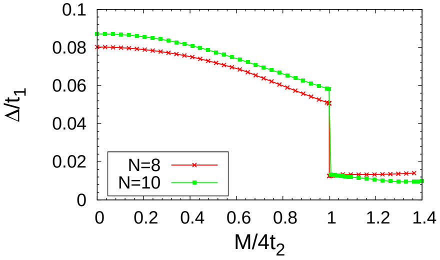
3x2x2	3x3x2	3x3x3
dim=61,413	dim=69,538,908	dim=3,589,864,780,047
960kb	1Gb	52Tb

Conclusion

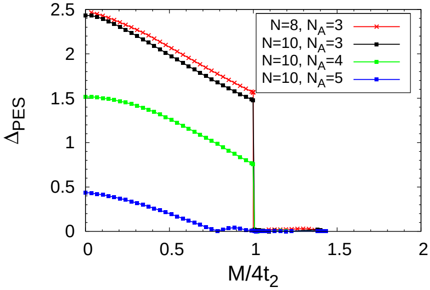
- Fractional topological insulator at zero magnetic field exists as a proof of principle.
- A clear signature for several states : Laughlin, CF and MR (using many body interactions)
- There is a counting principle that relates the low energy physics of the FQHE and the FCI
- What are the good ingredients for an FCI ? Does the knowledge of the one body problem is enough ?
- First time entanglement spectrum is used to find information about a new state of matter whose ground-state wavefunction is not known.
- Entanglement spectrum powerful tool to understand strongly interacting phases of matter.
- FQSH might be at hand...
- **Roadmap :** find one such insulator experimentally, 3D fractional topological insulators ?

From topological to trivial insulator

One can go to a trivial insulator, adding a $+M$ potential on A sites and $-M$ potential on B sites. A perfect (atomic) insulator if $M \to \infty$.



Energy gap



Entanglement gap

Sudden change in both gaps at the transition ($M = 4t_2$)