Heat Flow in non-equilibrium CFT

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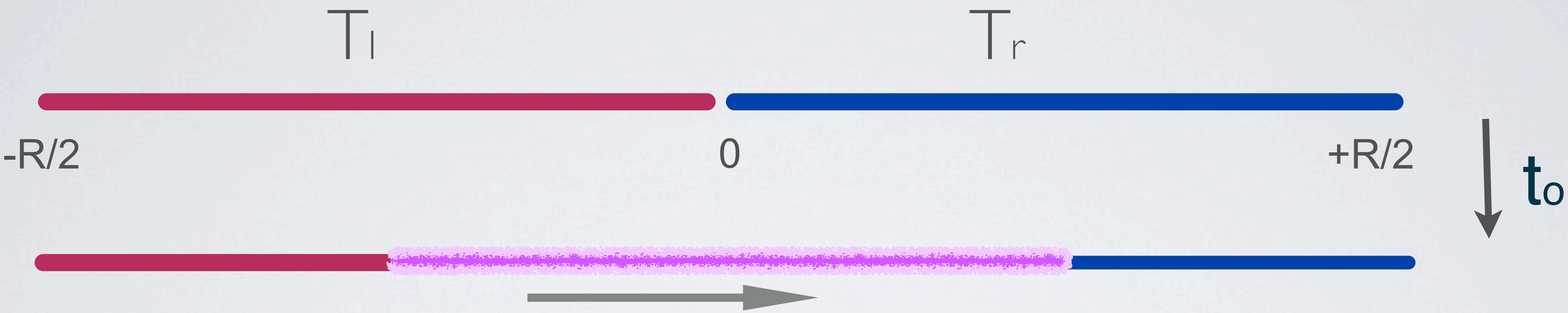
Firenze - May 2012

What are we describing?

Two isomorphic 1D gapless systems, prepared at different temperatures, put into contact.



Heat/Energy flow on domain of size = $v\ t_0$

Aim: properties of the (hopefully) steady flow across/near the contact.
Need: $R \gg v\ t_0 \gg$ observation domain.

Gapless systems = CFT with central charge c.

$\longrightarrow$ Non equilibrium CFT.

An example:

A (gap less) spin chain with an edge initial broken and then restored.....

$$H_l = \sum_{i \leq -1/2} \vec{S}_{i-1} \cdot \vec{S}_i \qquad\qquad H_r = \sum_{i \geq +1/2} \vec{S}_i \cdot \vec{S}_{i+1}$$

Initial thermal equilibirum: $\rho_0 \propto e^{-\beta_l H_l} \otimes e^{-\beta_r H_r}$

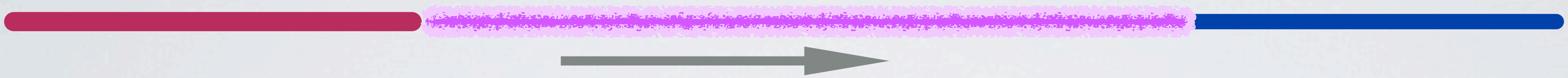
$$H = \sum_{i \leq -1/2} \vec{S}_i \cdot \vec{S}_{i+1} + \vec{S}_{-1/2} \cdot \vec{S}_{+1/2} + \sum_{i \geq +1/2} \vec{S}_i \cdot \vec{S}_{i+1} \quad = \sum_{\text{all } i} \vec{S}_i \cdot \vec{S}_{i+1}$$

After a long enough waiting time: $\rho_{\text{stat.}}$

To break and reconnect a finite number of links is equivalent in the continuum limit.

Heat flow/transfer.

$\Delta_t Q$ = heat transferred during time duration t in the stationary regime



1) Mean heat current: $J = -\partial_t(\Delta_t Q)$

$$\langle J \rangle = \frac{c\pi}{12\hbar}\, k_B^2 (T_l^2 - T_r^2)$$

(In the steady state)

Universal : independent of microscopic data

Let h and p be the energy and momentum densities: $\partial_t h + \partial_x p = 0$.
$Q = \int_{x<0} dx\, h(x)$ is the energy on the left and $\langle J \rangle = \langle p \rangle$.

2) Heat transfer large deviation function

Let $$F(\lambda) := \lim_{t\to\infty} t^{-1} \log\langle e^{i\lambda\Delta_t Q}\rangle$$

Definition of probability distribution of charge transfer done using quantum rules for measurements.

$$F(\lambda) = \frac{c\pi}{12\hbar}\left(\frac{i\lambda}{\beta_r(\beta_r - i\lambda)} - \frac{i\lambda}{\beta_l(\beta_l + i\lambda)}\right)$$

Universal!

Satisfies the (expected) fluctuation relation:

$$F(i(\beta_l - \beta_r) - \lambda) = F(\lambda)$$

The fluctuation relation relates the probabilities $P_t(\theta)$ and $P_t(-\theta)$ of opposite heat transfers $\Delta_t Q = \pm t\theta$ across the interface:

$$e^{-t\beta_l\theta} P_t(\theta)d\theta = e^{-t\beta_r\theta} P_t(-\theta)d\theta.$$

F is fully determined by the fluctuation relation plus a factorization property (see below) and the leading term (heat current).

Plan:

-- The steady state.
-- Heat transfer and its (quantum) statistics.
-- A (classical) Poissonian interpretation.

How to prepare the stationary state?

Two dynamics:

	$H_0 = H_l + H_r$	Before contact
	H	After contact

At time $-t_0$, Gibbs states: $\rho_0 \propto e^{-\beta_l H_l} \otimes e^{-\beta_r H_r}$

At time 0, $\rho = e^{-it_0 H}\, \rho_0\, e^{+it_0 H}$, evolution with the H dynamics

If the limit exists (see below) $\lim_{R \gg t_0 \to +\infty} e^{-it_0 H}\, \rho_0\, e^{+it_0 H}$ is stationary.

We shall argue that it factorizes on left/right movers.

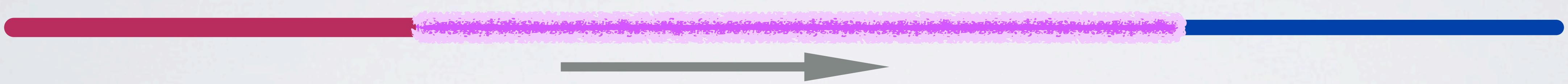
An heuristic description

Since the initial Gibbs state commutes with the H_o dymanics

$$e^{-it_0 H}\,\rho_0\, e^{+it_0 H} = e^{-it_0 H} e^{+it_o H_o}\,\rho_0\, e^{-it_o H_o} e^{+it_0 H}$$

Thus, after having (formally) taken the large time limit: cf. Ruelle.

$$\rho_{stat} = S\,\rho_0\,S^{-1}$$ with S the scattering matrix.



At least for integrable scattering preserving momentum,
Right movers are at temperature T_l.
Left movers are at temperature T_r.

$$\rho_{stat} = \rho_+(T_l) \otimes \rho_-(T_r)$$

The far left/right parts of the two sub-systems serve as effective reservoirs (for R large) which are different temperatures.

Field theory construction of the stationary state.

In CFT, the energy and momentum (chiral) densities are expressed in terms of their chiral component whose modes are the Virasoro generators:

$$h = h_+ + h_-, \quad p = h_+ - h_-, \quad \text{with} \quad h_\pm(x,t) = h_\pm(x \mp t)$$

The Ho and H dynamics differ by the boundary conditions:

Before: reflection

$$h_+(0^\pm) = h_-(0^\pm)$$

After: transmission

$$h_\pm(0^-) = h_\pm(0^+)$$

Alternative:
CFTs coupled with different defects.

Start from the definition of the (would be) stationary measure:

$$\lim_{R \gg t_o \to \infty} \langle \prod_j \phi_+^{(j)}(x_j, t_o) \phi_-^{(j)}(y_j, t_o) \rangle_{\rho_o}$$

For any given x_j, y_j there are $R \gg t_o$ large enough such that $x_j - t_o \in [-R/2, 0]$ and $y_j + t_o \in [0, R/2]$, so that the left/right movers have been moved into the two sub-systems.

$$\langle \prod_j \phi_+^{(j)}(x_j - t_o) \rangle_{\rho_o^l} \langle \prod_j \phi_-^{(j)}(y_j + t_o) \rangle_{\rho_o^r}. \qquad \text{for} \quad \phi_\pm = h_\pm \text{ or } Id$$

For large R, correlations of pure left/right movers are translation invariants. Thus, the stationary measure exists (at least when acting on h-densities). It is factorized on left/right movers.

Heat current.

Consider Q = half the energy difference in the two sub-systems.

$$Q(t) := \frac{1}{2}(H^l(t) - H^r(t)) \quad \text{with} \quad H^r(t) = \int_0^{+R/2} dx\, h(x,t) \qquad H^l(t) = \int_{-R/2}^{0} dx\, h(x,t)$$

The time evolution is that of the coupled system (H dynamics).
(the energy passes through the origin but is reflected at the extreme boundaries):

$$Q(t) = Q + \int_0^t dx\,(h_-(x) - h_+(-x))$$

The mean heat current is: $J = \langle h_+(-t) - h_-(t)\rangle_{\text{stat}}$

The mean is computed as for finite size effect:

$$\langle J\rangle = \frac{c\pi}{12\hbar} k_B^2 (T_l^2 - T_r^2)$$

Recall, that on left or right movers: $\rho_{stat} \propto e^{-\beta_l \frac{2\pi}{R} L_0}$ or $e^{-\beta_r \frac{2\pi}{R}\overline{L}_0}$

and $h_{+}^{l,r}(x) = \frac{2\pi}{R^2} T_R^{l,r}(x)$ with $T_R^{l,r}(x) := -\frac{c}{24} + \sum_{n\in Z} L_n^{l,r} e^{-2\pi i n x/R}$

Heat transfer and its large deviation function.

Defined by a two-step measurements:
(i) measure charge/energy Q at time 0, find q_0 with probability;
-- let the system evolves during time t;
(ii) measure again the charge/energy Q find q with probability:

$$P_t(q,q_0) = \mathrm{Tr}(P_q e^{-itH} P_{q_0} \rho_{\mathrm{stat}} P_{q_0} e^{itH} P_q)$$

The energy/heat transferred is $q-q_0$. Its generating function is:

$$\langle e^{i\lambda \Delta_t Q} \rangle := \sum_{q,q_0} e^{i\lambda(q-q_0)}\, P_t(q,q_0)$$

This admits an integral representation, $\langle e^{i\lambda \Delta_t Q} \rangle = \int \frac{d\mu}{2\pi}\, \mathcal{Z}_t(\lambda,\mu)$ with

$$\mathcal{Z}_t(\lambda,\mu) := \langle e^{-i(\frac{\lambda}{2}-\mu)Q} e^{i\lambda Q(t)} e^{-i(\frac{\lambda}{2}+\mu)Q} \rangle_{\mathrm{stat}}$$

Since the stationary measure factorizes, this factorizes.
To compute the large time behavior of each factor is reduced to a computation in CFT (via Virasoro algebra) ==> the announced formula.

And, factorization and the expected fluctuation relation, determines the large deviation function (without computation).

Classical Poissonian interpretation.

Recall: $F(\lambda) := \lim_{t\to\infty} t^{-1} \log\langle e^{i\lambda \Delta_t Q}\rangle$

$$F(\lambda) = \frac{c\pi}{12\hbar}\Big(\frac{i\lambda}{\beta_r(\beta_r - i\lambda)} - \frac{i\lambda}{\beta_l(\beta_l + i\lambda)}\Big)$$

This large deviation function admits a decomposition as sum of Poisson processes (as the Levy-Kintchin decomposition for infinitely divisible processes)

$F(\lambda) = F^r(\lambda) - F^l(-\lambda)$ with $F^{l,r}(\lambda) = \int d\nu^{l,r}(\varepsilon)\,(e^{i\lambda\varepsilon} - 1)$

with measure/intensity $(\varepsilon > 0)$ $d\nu^{l,r}(\varepsilon) = \frac{c\pi}{12\hbar}\, e^{-\beta_{l,r}\varepsilon}\, d\varepsilon$

The (quantum) large deviation coincides with that of (classical) Poisson processes:

$\mathbb{E}[e^{i\lambda\mathcal{E}_t}] = \exp[tF(\lambda)]$ with $d\mathcal{E}_t = \int \varepsilon[dN_t^r(\varepsilon) - dN_t^l(\varepsilon)]$

The jumps of the process are in correspondence with the transfers of energy quanta (particles) across the junction.
The intensity (= probability of transfer during dt) are proportional to the Boltzmann weight: $e^{-\beta_{l,r}\varepsilon}d\varepsilon dt$

Last comments.

The complete probability distribution of heat transfer is universal
(e.g. independent of v and other microscopic data).

Quite surprising/interesting: the (quantum) large deviation function admits a (classical) Poisson representation with a nice (universal?) interpretation.

Generalization?
-- with non-trivial/non-topological defects (reflection/transmission)
-- meaning (classical or not?) of the probability distribution of charge transfer if two non-commuting charges are measured?

Thank you.