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Exact results for quantum quenches in the Ising chain

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OUTLINE

- Introduction
- Correlation functions after a sudden quench in the TFIC
- Thermalization *vs.* GGE
- ... towards a stationary state

- Dynamics?

MF and P. Calabrese, Phys. Rev. A 78, 010306(R) (2008)

P. Calabrese, F. H. L. Essler, and MF, Phys. Rev. Lett. 106, 227203 (2011)

→ arXiv:1204.3911 (2012)

→ arXiv:1205.2211 (2012)

F. H. L. Essler, S. Evangelisti, and MF, in preparation

F. H. L. Essler and MF, in preparation

Quantum systems out of equilibrium

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle$$

main theoretical questions:

- Correlation functions
- Are there emergent phenomena?
  non-equilibrium steady states?
- How to evaluate averages?

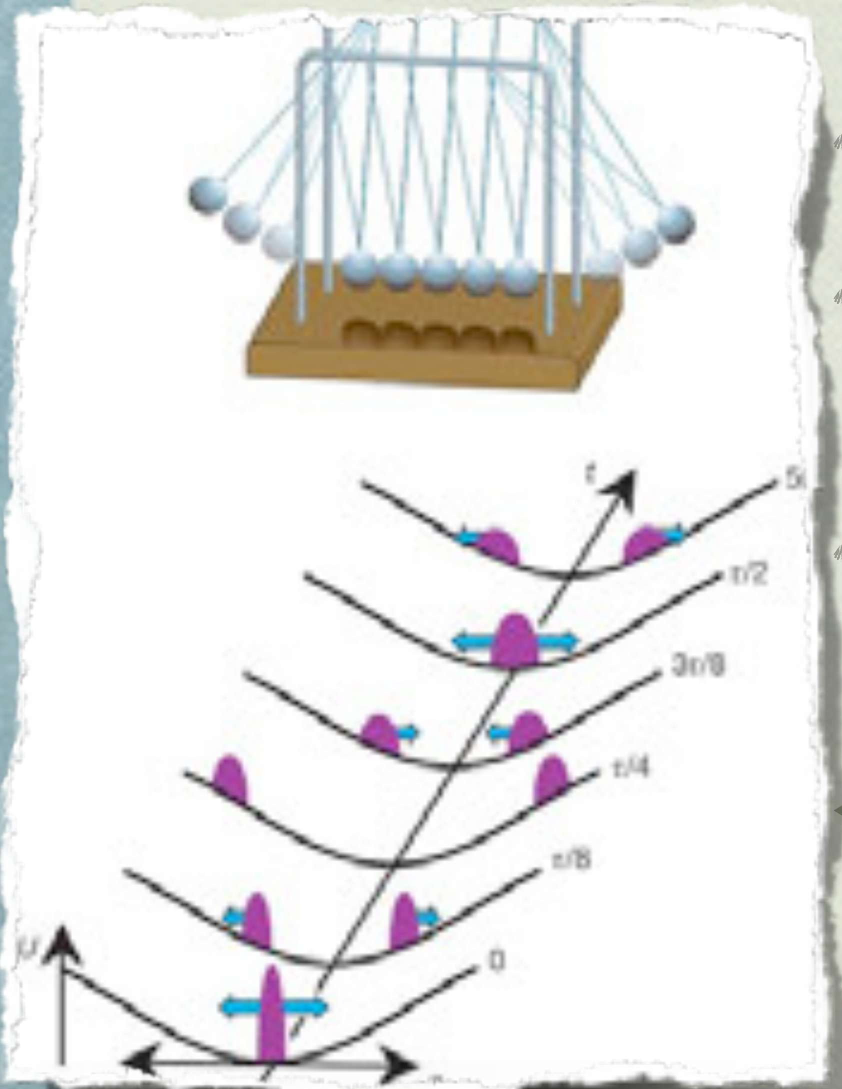
"macroscopic description"

- many-body physics with **ultracold atomic gases**
  interaction and external potentials can be changed **dynamically**
- **weakly** coupled to the environment → coherence for **long** times
  (also mesoscopic heterostructures, quantum dots, ...)

not only academic

Quantum Newton's Cradle

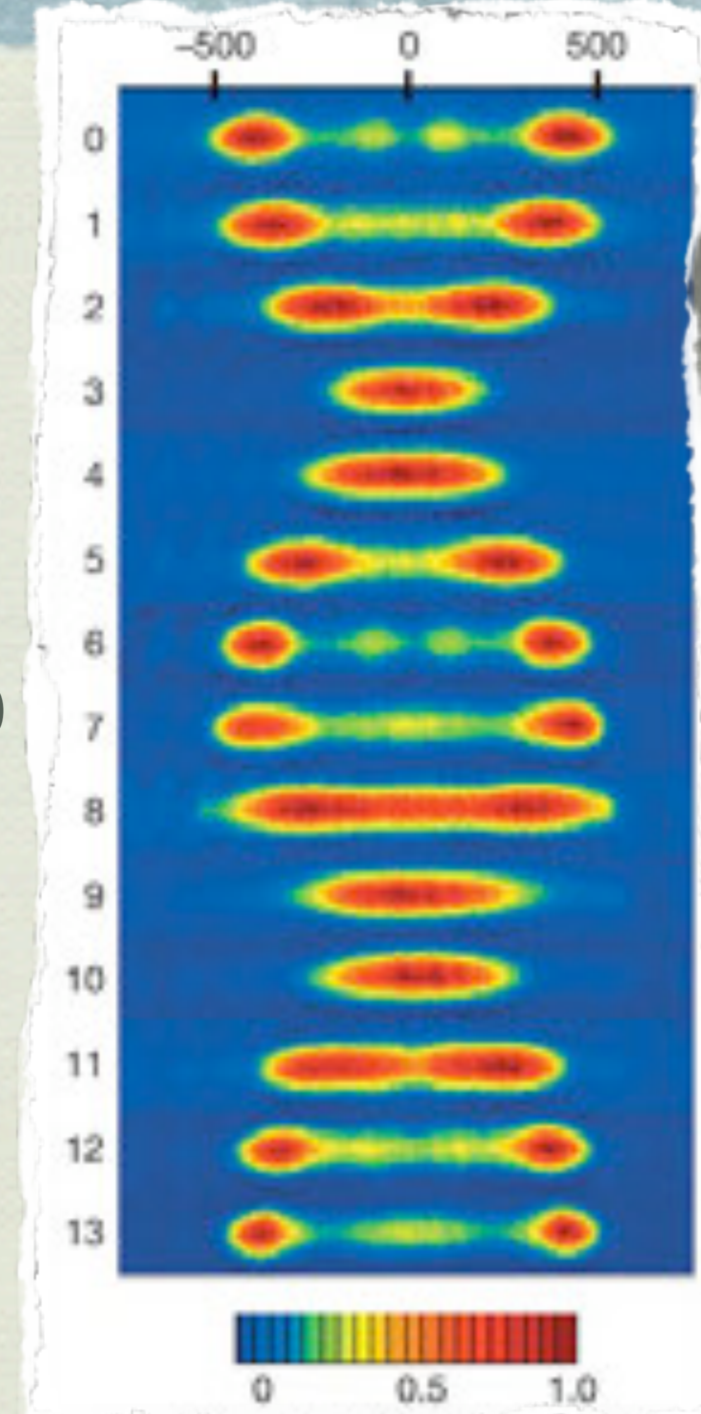
Kinoshita, Wenger, and Weiss (2006)

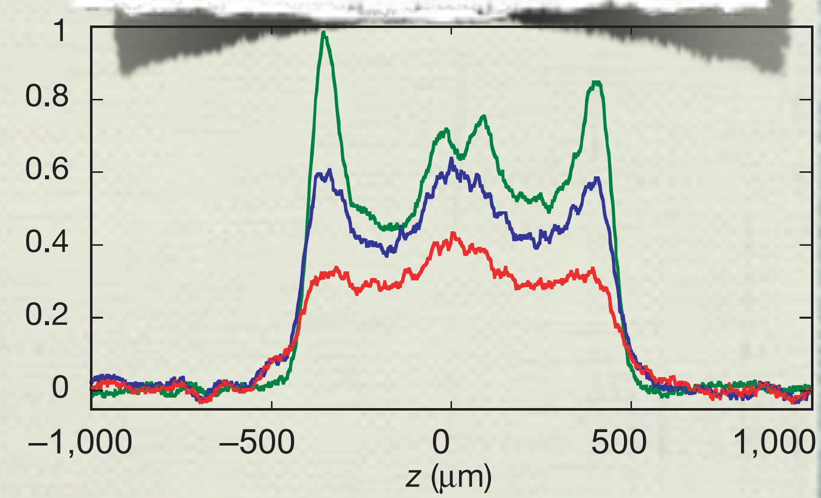


- 1D thousands of oscillations
- non-thermal momentum distribution (no thermalization?)
- 3D a few oscillations and then thermalization
- 1D close to integrability

(without trap)

$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{N \geq j > k \geq 1} \delta(x_j - x_k)$$





basic features (dimensionality, integrability, ...) play a central role in non-equilibrium physics

Sudden quench

consider a system in the ground state $|\varphi_0\rangle$ of a *local* Hamiltonian depending on certain parameters (magnetic field, interaction)

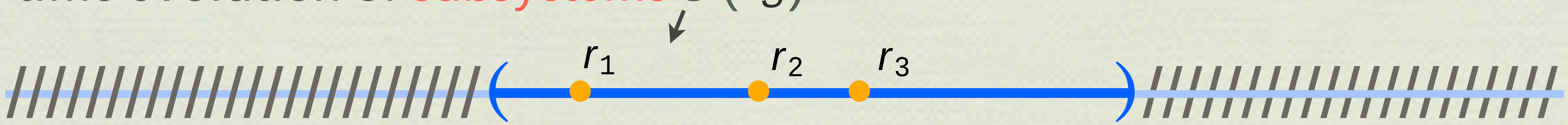
$$H(h_0,\dots)\,|\varphi_0\rangle = E_{G.S.}^{(h_0,\dots)}\,|\varphi_0\rangle$$

at a given time the parameters are changed

$$|\varphi_t\rangle = e^{-iH(h,\dots)t}\,|\varphi_0\rangle$$

extensive excess of energy
$[H(h_0,\dots), H(h,\dots)] \neq 0$ ⟶ global quench

- time evolution of (local) correlation functions $\langle\varphi_t|\hat{O}(r_1,\dots,r_i,\dots)|\varphi_t\rangle$
- time evolution of subsystems $S\ (\ell_S)$



$$\rho(t) \equiv e^{-iH(h,\dots)t}\,|\varphi_0\rangle\langle\varphi_0|\,e^{iH(h,\dots)t} \longrightarrow \rho_S(t) \equiv \mathrm{Tr}_{\bar{S}}\,\rho(t)$$

- late-time regime and emergence of stationary behavior

$v_{max}t \gg \ell_S, r$ $\quad\quad \exists \lim_{t\to\infty} \mathrm{Tr}[\rho_S(t)\hat{O}]$? $\quad\quad \exists \lim_{t\to\infty} \rho_S(t)$?

Transverse Field Ising chain

simplest paradigm of a quantum phase transition

$$H = -J \sum_{j=1}^{L} [\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z]$$

central charge c=1/2

0 $\langle \sigma_x \rangle \neq 0$ 1 $\langle \sigma_x \rangle = 0$ h

order parameter

spontaneously broken

Z_2 symmetry: rotation of π around z-axis

Mapping to free fermions

$$a_{2l} = \prod_{j<l} \sigma_j^z \sigma_l^z \qquad a_{2l-1} = \prod_{j<l} \sigma_j^z \sigma_l^y$$

$$\{a_l, a_n\} = 2\delta_{ln}$$

Jordan-Wigner transformation

two fermionic sectors $\left[\prod_l \sigma_l^z, H_{R(NS)}\right] = 0$

$$H = \frac{1 - \prod_l \sigma_l^z}{2} H_R + \frac{1 + \prod_l \sigma_l^z}{2} H_{NS}$$

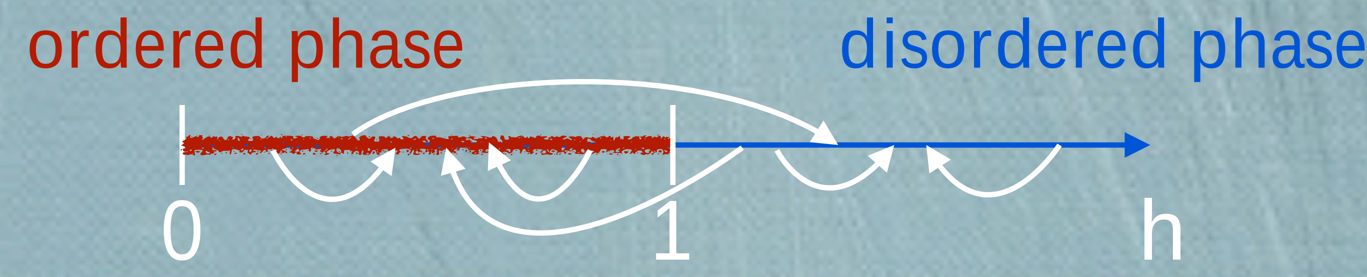
$$H_{R(NS)} = \sum_{l,n} \frac{a_l H_{ln}^{R(NS)} a_n}{4}$$

Wick theorem

imaginary antisymmetric

Quench dynamics in the TFIC

$H(h_0) \to H(h)$



Approach I: Block-Toeplitz determinants: *(in the thermodynamic limit)*

expectation values of even operators ➡ Pfaffians of structured matrices

$\langle \varphi_0 | \sigma_1^x(t) \sigma_{\ell+1}^x(t) | \varphi_0 \rangle \sim$ Block-Toeplitz matrix

multi-dimensional stationary phase approximation

Approach II: “Form-Factor” Sums:

1. large finite volume L
2. initial state
   - (ordered phase) $(|\bar{0}\rangle_R \pm |\bar{0}\rangle_{NS})/\sqrt{2}$
   - (disordered phase) $\bar{0}\rangle_{NS}$
3. express this in terms of the final Bogoliobov fermions

$$\bar{0}\rangle_{R(NS)} = \exp\Big(i \sum_{0<p\in R(NS)} K(p) \alpha_p^\dagger \alpha_{-p}^\dagger\Big) |0\rangle_{R(NS)} \qquad K(p) = \tan\Big(\frac{\theta_h(p) - \theta_{h_0}(p)}{2}\Big)$$

4. Lehmann representation in terms of the final Bogolioubov fermions

$${}_{NS}\langle \bar{0} | \sigma_1^x(t) | \bar{0} \rangle_R = \sum_{l,n\geq 0} \frac{1}{n!l!} \sum_{\substack{k_1,\dots k_n \\ p_1,\dots,p_l}} \prod_{j=1}^{n} K(k_j) e^{2it\varepsilon_{k_j}} \prod_{i=1}^{l} K(p_i) e^{-2it\varepsilon_{p_i}} \; {}_{NS}\langle \{-k,k\}_n | \sigma_1^x | \{p,-p\}_l \rangle_R$$

known exactly for the lattice model

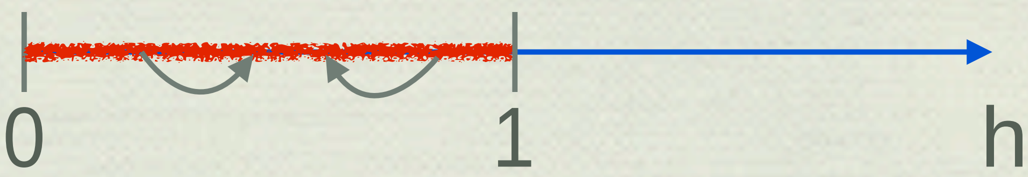
5. $K(p)$ as expansion parameter: *low density of excitations*

One-point function:

different from zero only for quenches from the ordered phase

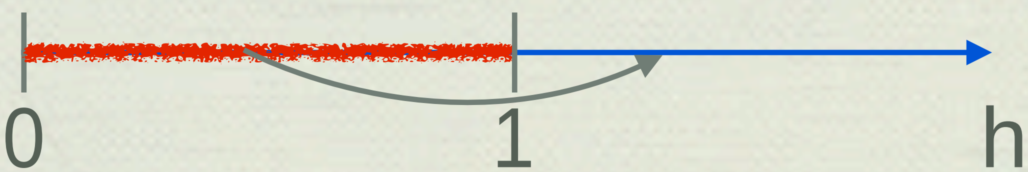
$$\Delta_k = \theta_h(k) - \theta_{h_0}(k)$$

... *to* the ordered phase



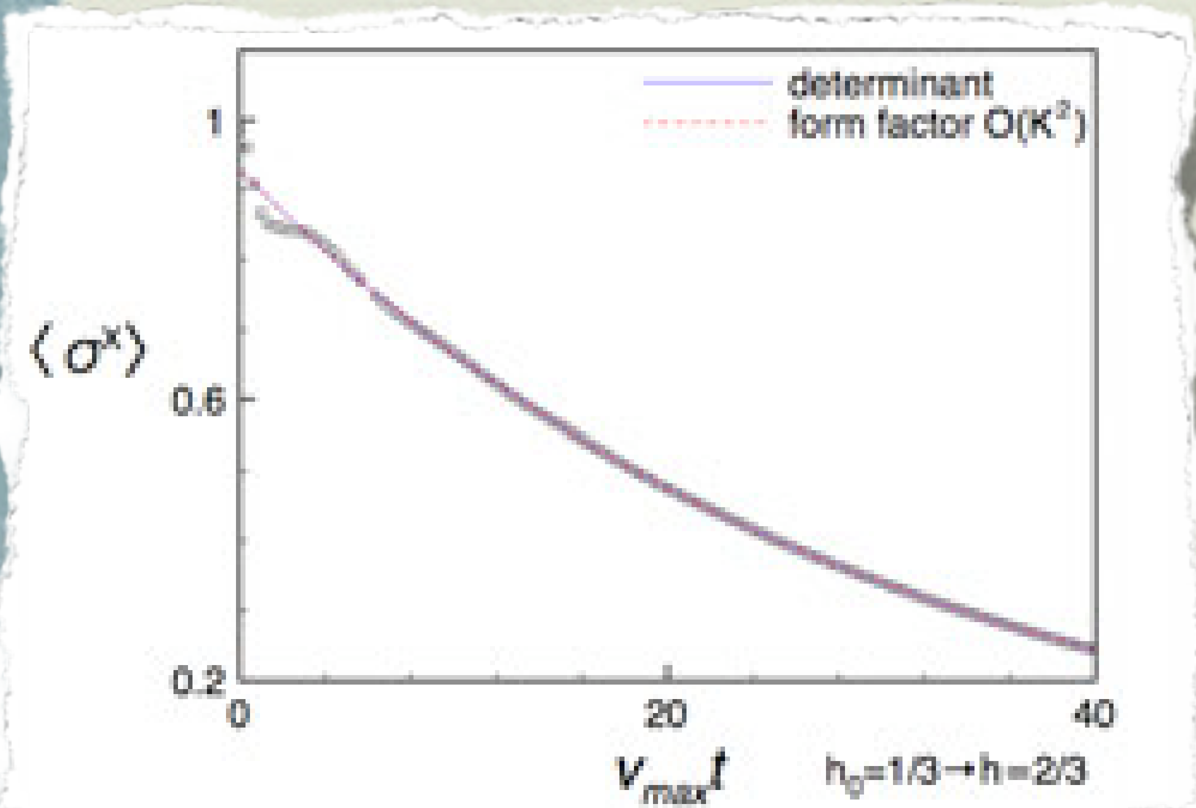
$$\langle \sigma_i^x(t) \rangle \simeq (\mathcal{C}_{\mathrm{FF}}^x)^{\frac{1}{2}} \exp\left[t \int_0^{\pi} \frac{dk}{\pi} \varepsilon_h'(k) \ln|\cos \Delta_k| \right]$$

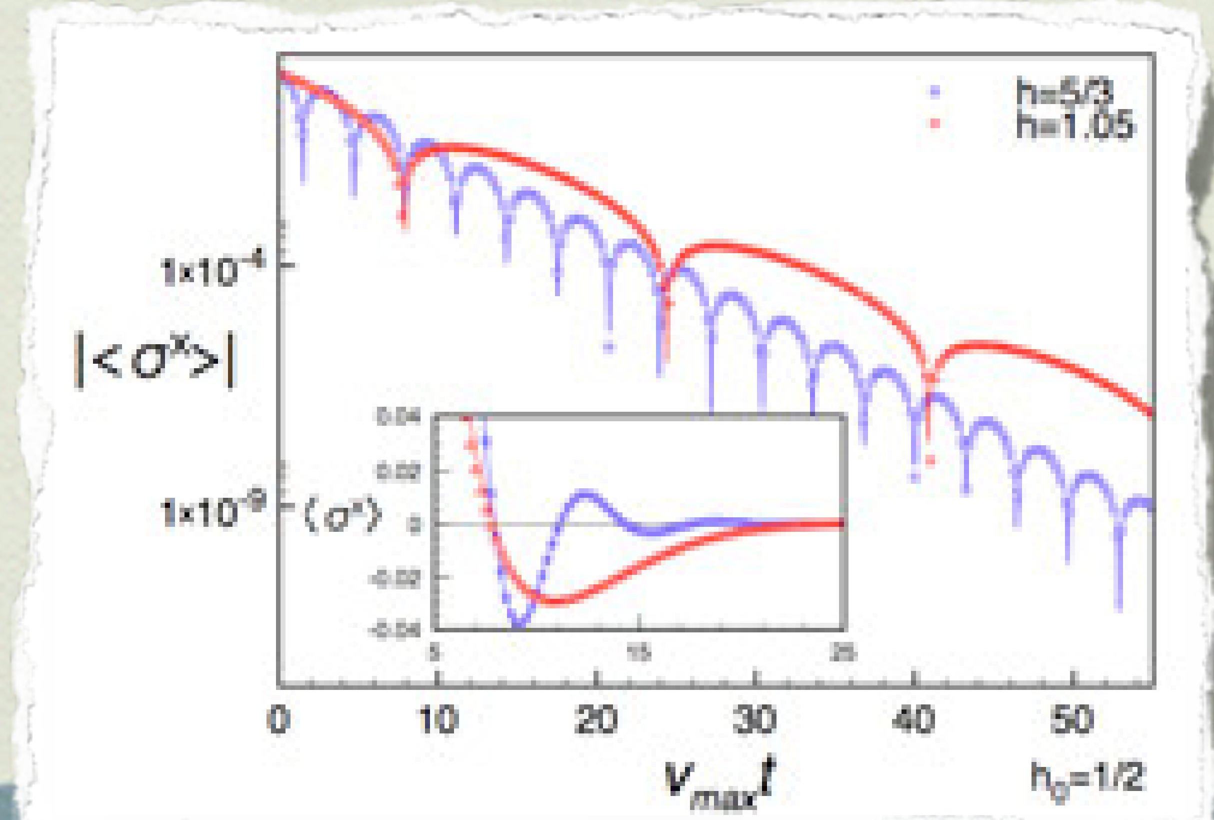
... *to* the disordered phase



$$\cos \Delta_{k_0} = 0$$

$$\langle \sigma_i^x(t) \rangle = (\mathcal{C}_{\mathrm{FP}}^x)^{\frac{1}{2}} [1 + \underline{\cos(2\varepsilon_h(k_0)t + \alpha)} + \dots]^{\frac{1}{2}} \exp\left[t \int_0^{\pi} \frac{dk}{\pi} \varepsilon_h'(k) \ln|\cos \Delta_k| \right]$$

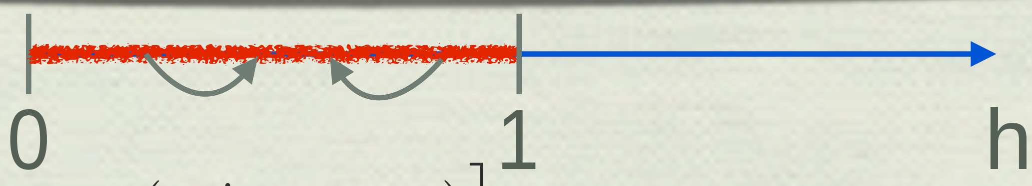




Two-point function:

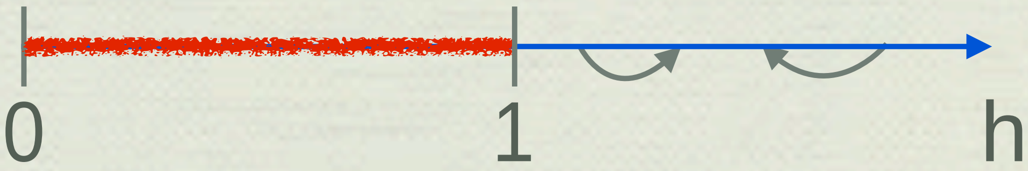
less clear behavior in quenches across the critical point

... *within* the ordered phase

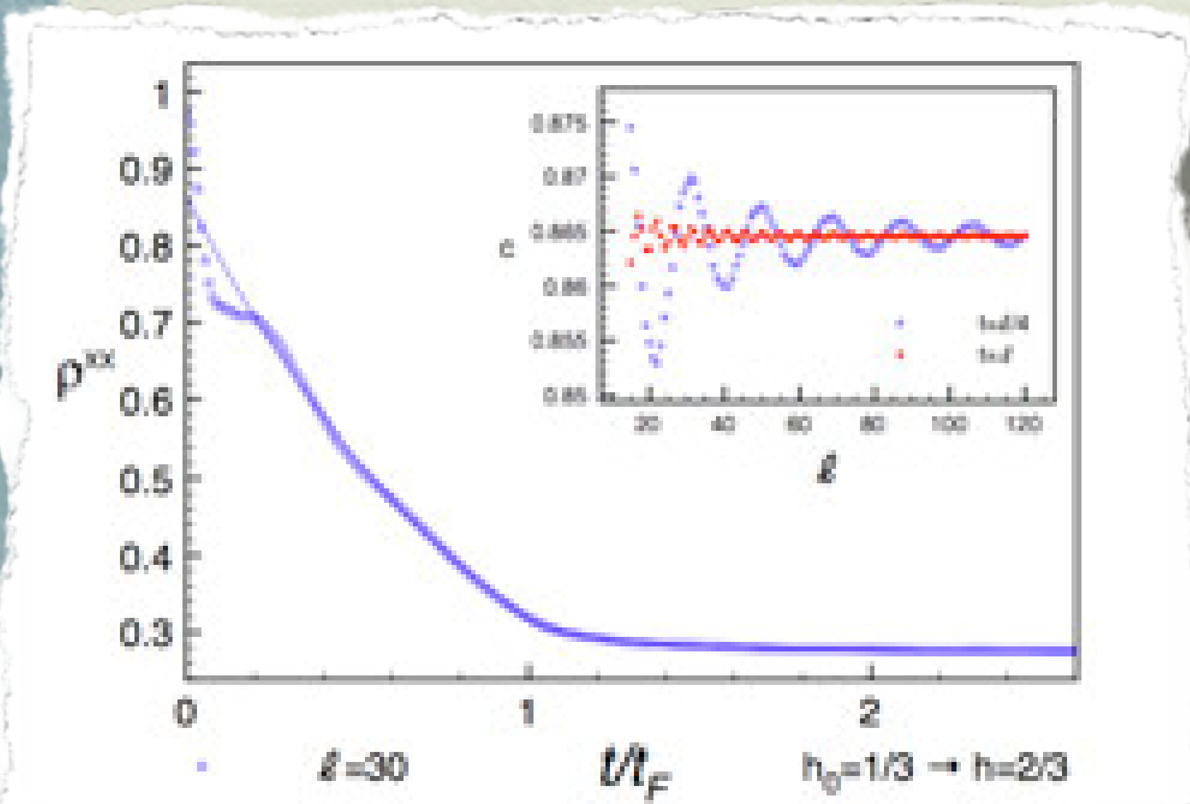


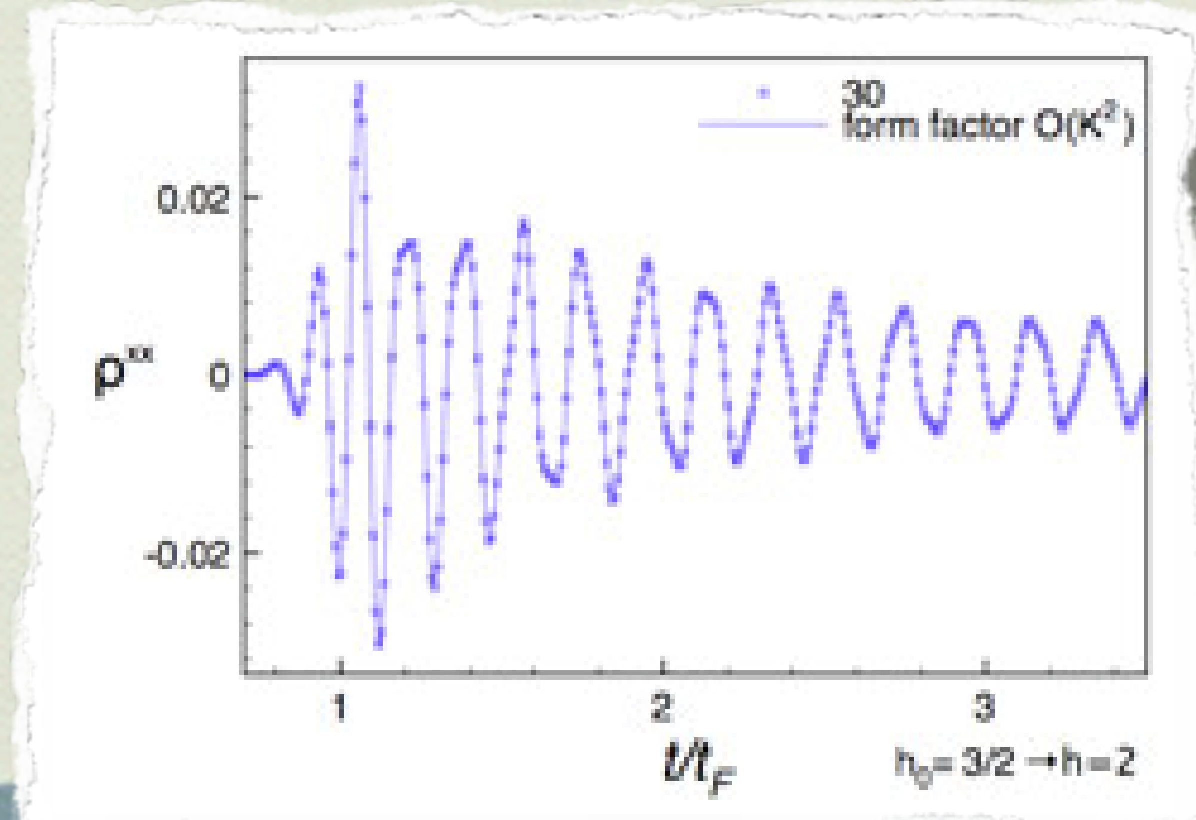
$$\rho^{xx}_{FF}(\ell,t) \simeq C^x_{FF} \exp\left[\ell \int_0^\pi \frac{dk}{\pi} \ln|\cos\Delta_k|\,\theta_H\left(2\varepsilon'_h(k)t-\ell\right)\right]$$
$$\times \exp\left[2t \int_0^\pi \frac{dk}{\pi} \varepsilon'_h(k) \ln|\cos\Delta_k|\,\theta_H\left(\ell-2\varepsilon'_h(k)t\right)\right]$$

... *within* the disordered phase

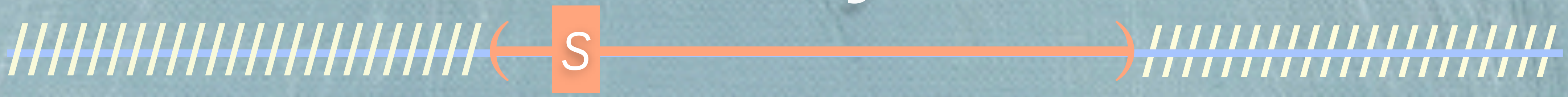


$$\rho^{xx}_{PP}(\ell,t) \simeq \rho^{xx}_{PP}(\ell,\infty) + (h^2-1)^{\frac{1}{4}}\sqrt{4J^2h}\,\boxed{\int_{-\pi}^{\pi}\frac{dk}{\pi}\frac{K(k)}{\varepsilon_k}\sin(2t\varepsilon_k-k\ell)}\times$$
$$\exp\left[-2\int_0^\pi \frac{dp}{\pi}K^2(p)\left(\ell+\theta_H(\ell-2t\varepsilon'_p)[2t\varepsilon'_p-\ell]\right)\right] + \dots$$



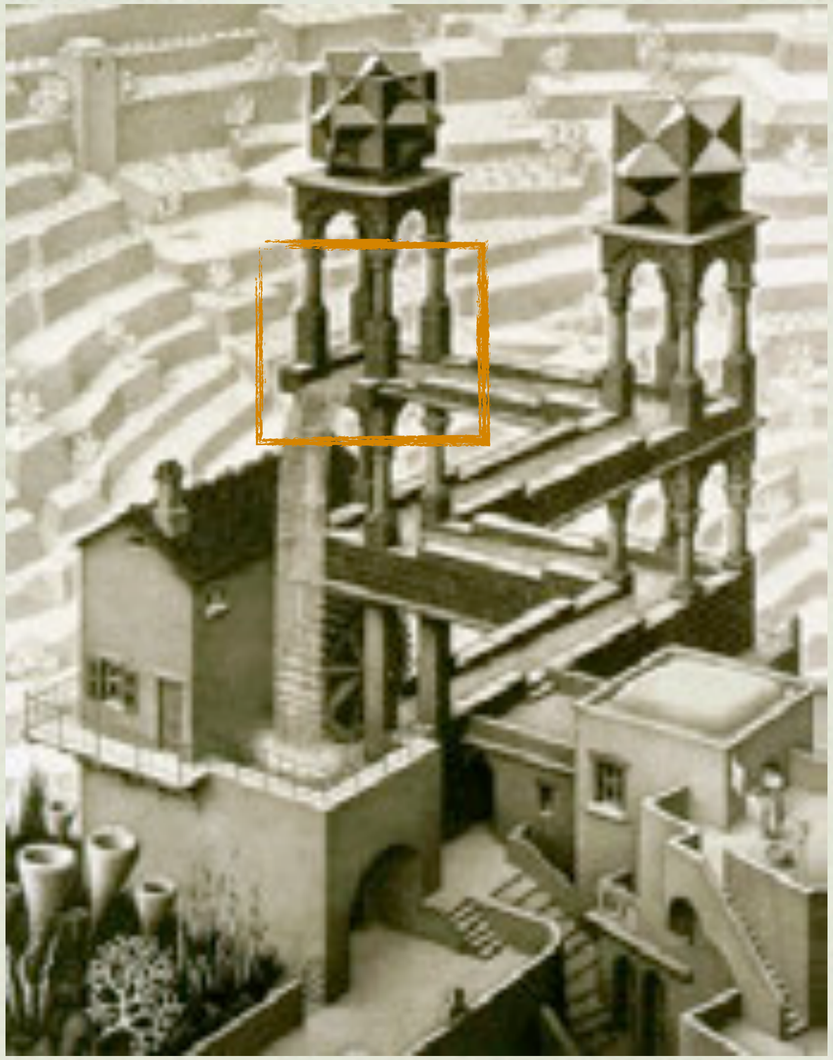


Stationary state



$$\rho(t) = e^{-iH(h,\dots)t} |\varphi_0\rangle \langle\varphi_0| e^{iH(h,\dots)t} \quad \longrightarrow \quad \rho_S(t) = \mathrm{Tr}_{\bar{S}}\rho(t)$$

- at large times after a quench correlations display stationary behavior
- entanglement entropies of subsystems become independent of time



M. C. Escher (1961)

effective density matrix?

$$\exists \lim_{t\to\infty} \mathrm{Tr}[\rho_S(t)\hat{O}] \quad ? \qquad \lim_{t\to\infty} \mathrm{Tr}_{\bar{S}}\rho(t) \to \bar{\rho}_S$$

Eigenstate Thermalization?

Deutsch (1991), Srednicki (1994)

intuitive argument: in an infinite system the rest of the system could act as a bath

integrable systems: conserved quantities! (more parameters)

$$\cancel{\bar{\rho} \overset{?}{\approx} \frac{e^{-H/T_{eff}}}{\mathrm{Tr}e^{-H/T_{eff}}}}$$

non-integrable systems?

Quenches to integrable systems

operators commuting with the final Hamiltonian are stationary:
expectation values are conserved $[H, I] = 0 \Rightarrow \langle \varphi_t | I | \varphi_t \rangle = \langle \varphi_0 | I | \varphi_0 \rangle$

charge with local density → conservation law also for the reduced system

M. C. Escher (1956)

at late times

persistent oscillations

stationary behavior:
each local charge gives a constraint
$\rho_\infty = \rho_\infty(I_1, I_2, \ldots)$

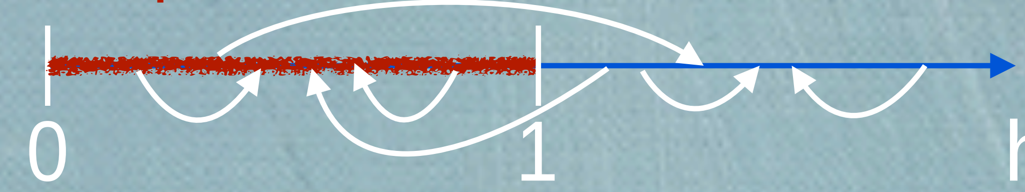
conjecture (GGE): $\bar{\rho}_S \xrightarrow{|S|\to\infty} \frac{e^{-\sum_m \beta_m I_m}}{\mathrm{Tr}[e^{-\sum_m \beta_m I_m}]}$

Rigol, Dunjko, Yurovsky, and Olshanii (2007)

Infinite time after a quench in the TFIC

$$H(h_0) \to H(h)$$

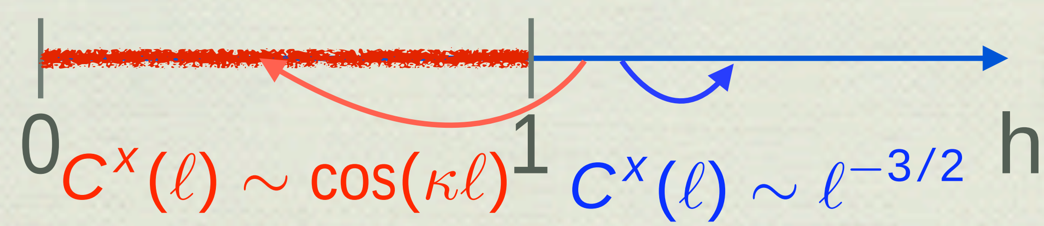
(in the thermodynamic limit)



- equal-time correlations are stationary (almost all excitations have nonzero velocity (no localized excitations); *counterexample: quench to* $H = \sum_l \sigma_l^z$)

- exponential decay with distance (possibly "dressed" with power-law and oscillatory factors)

$$\rho^{xx}(\ell \gg 1, t = \infty) = C^x(\ell) e^{-\ell/\xi}$$



$$\rho_c^{zz}(\ell \gg 1, t = \infty) \simeq C^z \ell^{-\alpha^z} e^{-\ell/\xi_z} \qquad \alpha^z = \frac{1 + \theta_H(|\log h| - |\log h_0|)}{2}$$

- local properties described exactly by a GGE (equal-time fermionic two-point functions have thermal structure corresponding to an Hamiltonian that commutes with the final one)

“Pair Ensemble” vs GGE: the role of locality

$$\overline{0}\rangle_{R(NS)} = \exp\Big(i \sum_{0<p\in R(NS)} K(p)\alpha^\dagger_p\alpha^\dagger_{-p}\Big)\, |0\rangle_{R(NS)}$$

$$n_k = \alpha^\dagger_k\alpha_k$$

$$\langle n_k n_{-k}\, O\rangle_t = \langle n_k\, O\rangle_t$$

$\downarrow$

factorization in pair of quasiparticles

off-diagonal elements do not contribute $\longrightarrow$ Pair Ensemble

different from GGE !

(GGE in the Ising model is Gaussian)

Pair Ensemble and GGE are locally indistinguishable

my feeling (when GGE works): we CAN construct the effective density matrix considering only *local charges*

... towards the stationary state:

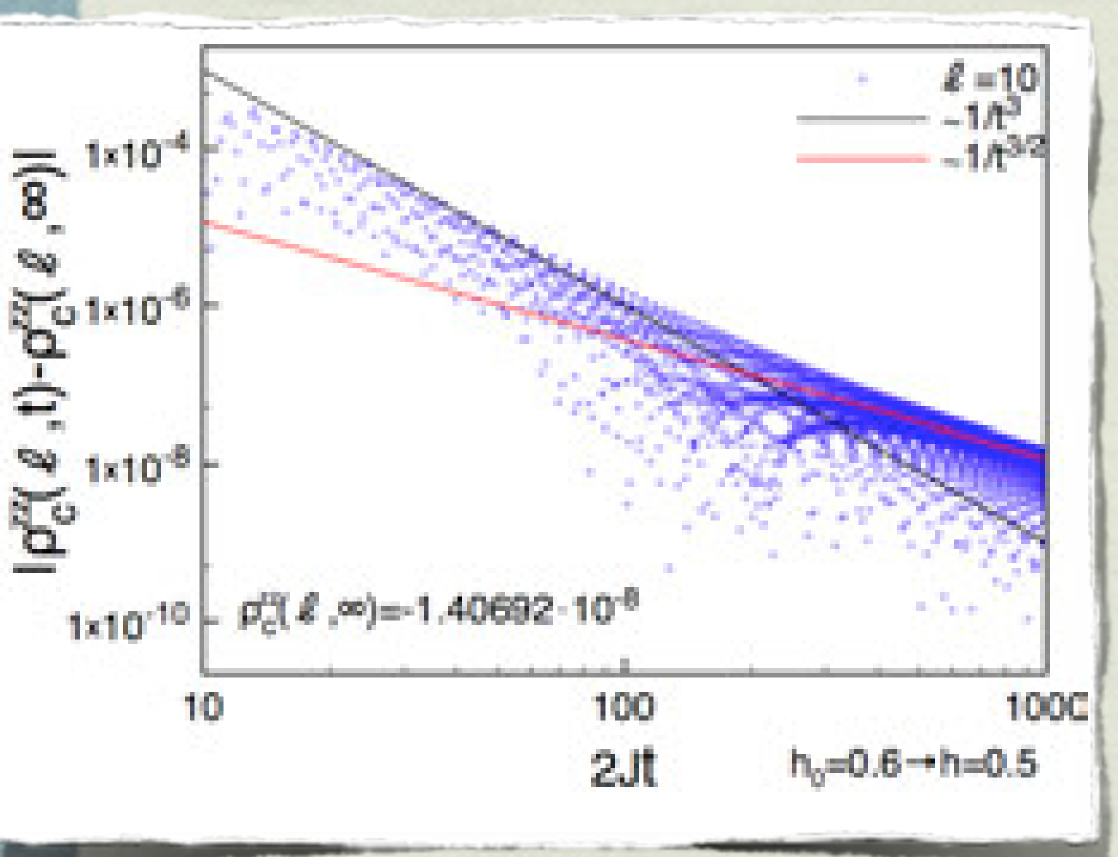
"practical meaning" of the infinite time limit

fundamental question: how long we need to wait to "see" the observables converging to their stationary values

finite systems: GGE → *time* must be

- smaller (enough) than the system's size (in units of the maximal velocity)
- much larger *(how much?)* than the typical length (distance in 2-point functions, subsystem's length, ...)

strong limits to the lengths of subsystems that relax!



$$\rho_c^{zz}(\ell,t) \sim \rho_c^{zz}(\ell,\infty) + \frac{E^z(t)\ell e^{-\ell/\tilde{\xi}_z}}{t^{3/2}} + \frac{D^z(t)\ell^2}{v_{\max}^2 t^3} + \dots$$

generally $Jt \gg (\ell/a)^\alpha e^{\kappa\ell/a}$

at fixed distance,

exponentially large times

the behavior depends on the observable

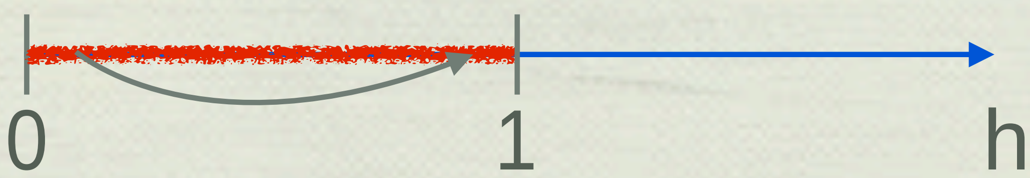
Dynamical correlations
at infinite times after the quench

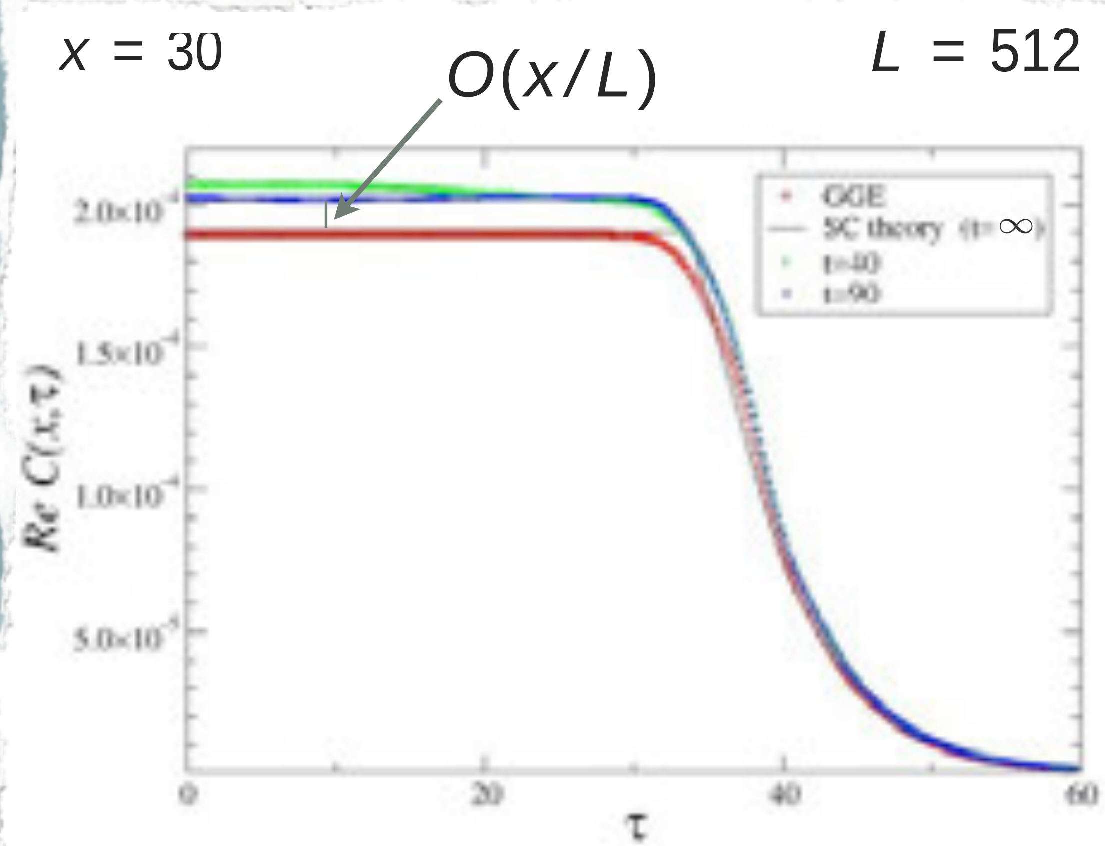
- numerical analysis + general physical arguments

$$\lim_{t\to\infty} \langle O_1(t+\tau) O_2(t) \rangle = \mathrm{Tr}[\rho_{GGE}\, O_1(\tau) O_2]$$

FDT

with less symmetries
w.r.t. the thermal case





$$C(x,\tau) = \langle \sigma^x_{(L-x)/2}(t+\tau)\, \sigma^x_{(L+x)/2}(t) \rangle$$

in practice $\ell, \tau \ll t \lesssim L$

exact results?

Conclusions

- Quantum quenches display a rich phenomenology
- Many open problems:
  - ... thermalization (non-integrable systems)
  - ... GGE (integrable systems)
- Importance to have analytic results
- More general initial states?
- ... and when subsystems do not relax?

Thank you

for your attention