Understanding Entanglement in 2+1D Systems through Arrays of Coupled 1+1D Field Theories

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Outline

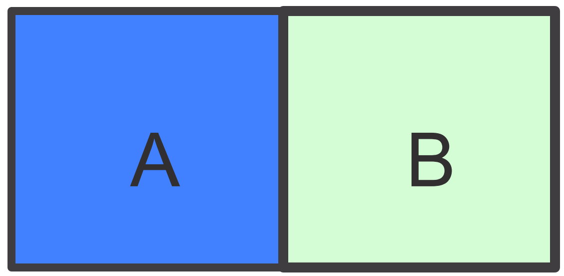
1) Overview of Entanglement in 1D and 2D

2) Overview of Methodology: DMRG on coupled 1D field theories

3) Results on the entanglement entropy and spectra of coupled quantum Ising chains

Entanglement Entropy

Take a system and divide it into two blocks A and B



Suppose we have a pure state, $|\psi\rangle$, of the system. Performing a Schmidt decomposition in terms of subsystem A and B:

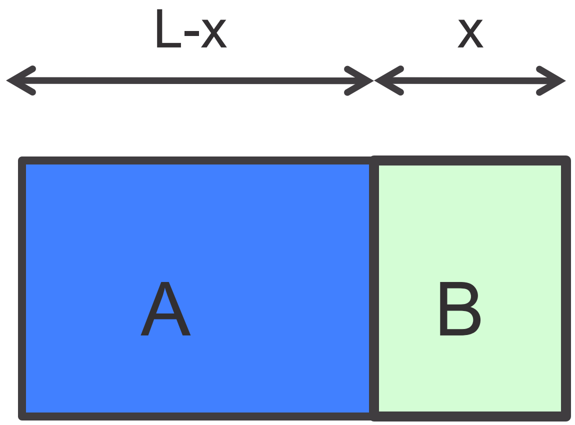
$$|\psi\rangle = \sum_{\alpha} c_{\alpha} |i_{\alpha}\rangle \otimes |j_{\alpha}\rangle \qquad i_{\alpha} \in \mathcal{H}_A, \;\; j_{\alpha} \in \mathcal{H}_B$$

We can form the reduced density matrix for subsystem A by tracing over $\mathcal{H}_B$:

$$\rho_A = \mathrm{Tr}_{j_{\alpha} \in \mathcal{H}_B} |\psi\rangle\langle\psi| = \sum_{\alpha} c_{\alpha}^2 |i_{\alpha}\rangle\langle i_{\alpha}|$$

The entanglement entropy is then: $S_A = -\sum_{\alpha} c_{\alpha}^2 \log(c_{\alpha}^2)$

Entanglement Entropy in One Dimension



P. Calabrese and J. Cardy
J.Stat.Mech.0406:P06002,2004

C. Holzhey, F. Larsen, and F. Wilczek,
Nucl. Phys. B 424 44 1994

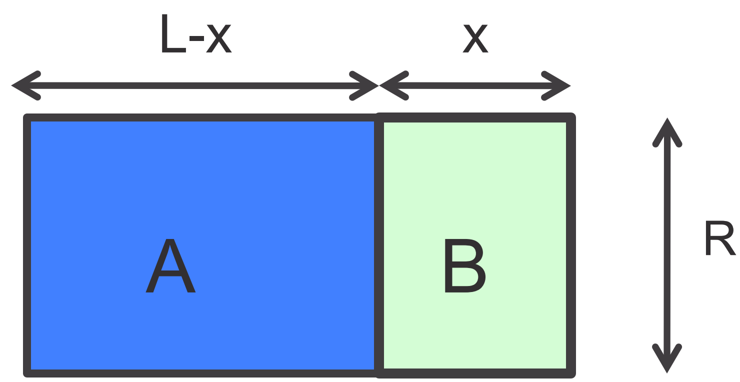
System is critical:

$$S_A = \frac{c}{6}\ln(L_{\text{eff}}), \qquad L_{\text{eff}} = \frac{L}{a\pi}\sin\frac{\pi x}{L}$$

System is gapped:

$$S_A = \frac{c}{6}\ln(L_{\text{eff}}), \qquad L_{\text{eff}} = \xi/a$$

Entanglement Entropy in Dimensions > 1



Generically systems obey an area law: $$S_A = c(R/a)^{d-1}$$

Eisert, Cramer, Plenio, Rev. Mod. Phys. 82, 277 (2010)

Sometimes systems also exhibit a subleading, universal correction

$$S_A = c(R/a)^{d-1} + \gamma(L, x, R)$$

Entanglement Entropy in Dimensions > 1

Examples of universal contributions:

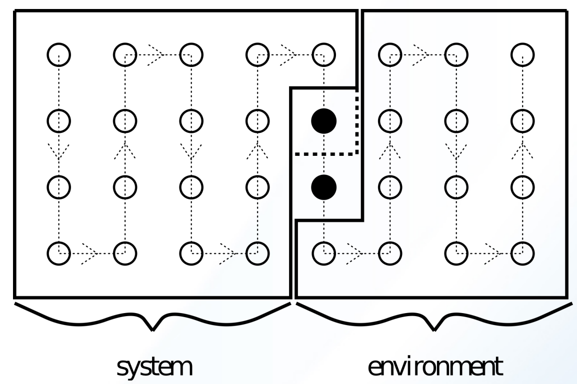
- Conformal quantum critical points: (Ardonne, Fendley, Fradkin Ann. Phys. (N.Y.) 310, 493 (2004))
  - These were shown to give a universal correction to the entanglement entropy (Fradkin and Moore, PRL 97, 050404 (2006); Zaletel, Bardarson, Moore, PRL 107, 020402 (2011).
- Certain quantum dimer models (G. Misguich, D. Serban, and V. Pasquier, PRL 89, 137202 (2002))/ topological field theories
  - These were shown to have a contribution to S_A coming from the quantum dimension. (Levin, Wen PRL 96, 110405 (2006); A. Kitaev, J. Preskill, PRL 96, 110404 (2006).
- Symmetry broken theories (Grover and Metlitski, arXiv:1112.5166;
  Kallin, Hastings, Melko, Singh, Phys. Rev. B 84, 165134 (2011)
  Ju, Kallin, Fendley, Hastings, Melko, PRB 85, 165212 (2012)
  - Logarithmic correction proportional to the number of Goldstone modes.

Density Matrix Renormalization Group: Large Coupled Arrays of Continuum Chains

DMRG is ideally suited for single 1D chains:

2D systems are more difficult for DMRG:

What can be done is to convert a 2D system into a 1D one with long range interactions

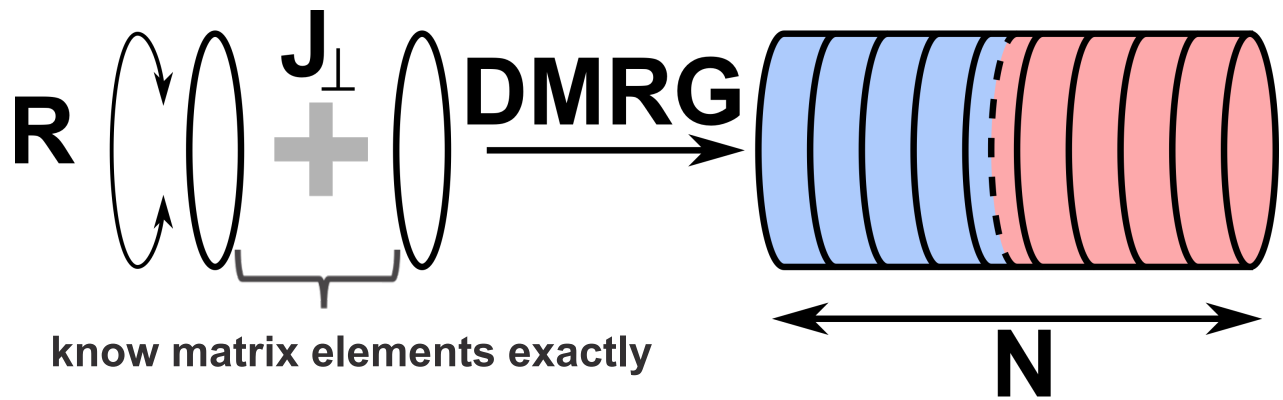


However the wider the system, the longer the range the interactions are, the more entangled the DMRG blocks become, with a consequent degradation of algorithm performance.

One route around these difficulties is to look to more clever generalizations of DMRG to higher dimensions, i.e. tensor network states (Cirac, Vidal, Verstraete, ….)

A Different 2D DMRG: Coupled Field Theoretic Chains

R, the length of the chain



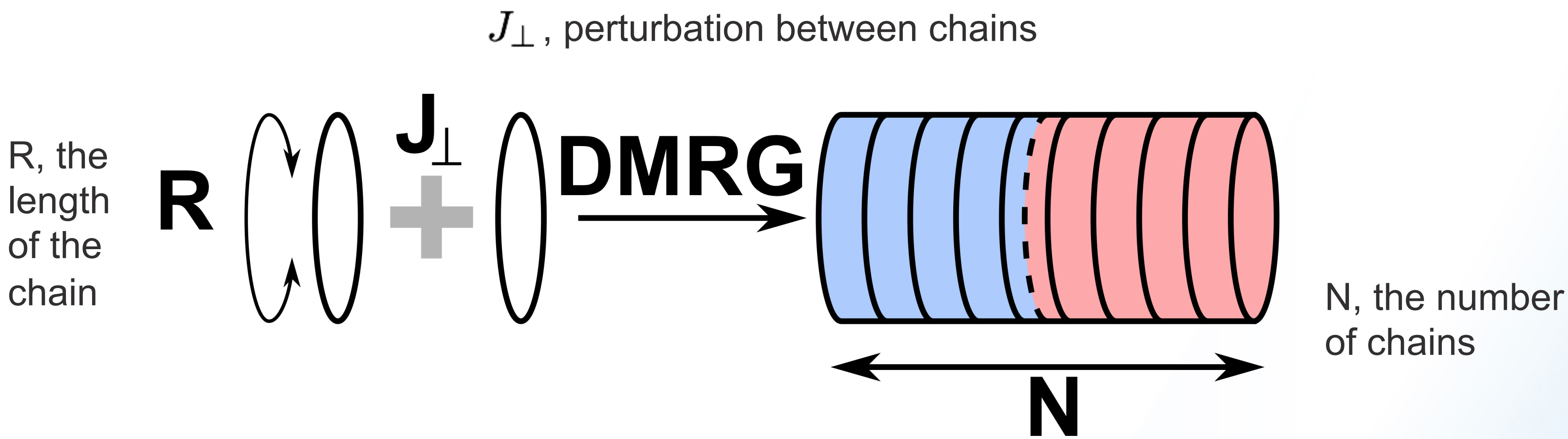
N, the number of chains

We instead run a 1D DMRG algorithm where the individual sites are 1D chains in a field theoretic representation.

Of course one could do the same with a 2D lattice. However for 1D chains with an appreciable number of sites the approach quickly breaks down because the size of the 1D Hilbert space explodes. (du Croo and van Leeuwen PRB 57, 8494 (1999), U Schollwöck, RMP 77, 259 (2005)).

The approach works in our case because we represent the chains in the continuum limit as an integrable relativistic field theory.

Virtues of a Field Theory Representation for the Chain



Of course the Hilbert space of each chain is infinite.

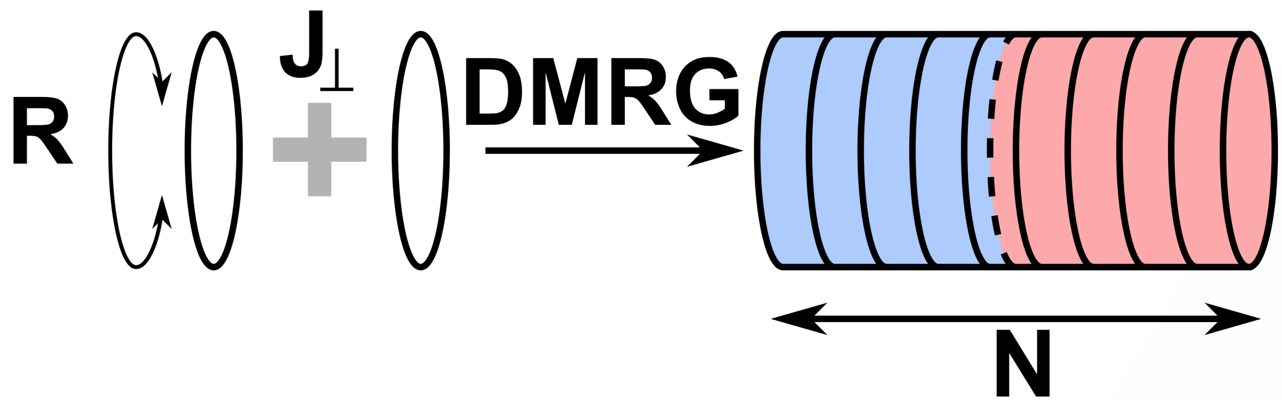
However we truncate this Hilbert space in energy. It has been shown for relativistic field theories, beginning with the pioneering work of Yurov and Zamolodchikov (Int. J. Mod. Phys. A 6, 4557 (1991)), that this truncation can be severe and still produce excellent results ***if*** the perturbation between chains is relevant.

We can truncate the Hilbert space of the continuum chains keeping only several hundred states per chain.

Virtues (continued) of a Field Theory Representation for the Chain

$J_\perp$, perturbation between chains

R, the length of the chain



N, the number of chains

Several hundred states per chain is still a lot.

However, at least for the example at hand, the number of states needed to be kept for an error of 10^{-6}, is no more than a 100 even close to a critical point.

We keep the entanglement entropy between blocks small by keeping R (the chain length small). Because finite size corrections in a gapped relativistic field theory are exponential suppressed, R can be kept small while remaining in the infinite volume limit.

2D Quantum Ising Model

matrix elements can be computed exactly

$$H^{2D} = \sum_i H_i^{1D} + J_\perp \sum_{\langle ij \rangle} \int_0^R dx \sigma_i(x)\sigma_j(x)$$

$$H^{1D} = \int \frac{dx}{8\pi} \left[iv(\bar{\psi}\partial_x\bar{\psi} - \psi\partial_x\psi) - 2i\Delta\bar{\psi}\psi\right]$$

each chain is a massive Majorana fermion

$$H^{1D}_{\text{lattice}} = -J\sum_m (\sigma^z_m \sigma^z_{m+1} + (1+g)\sigma^x_m)$$

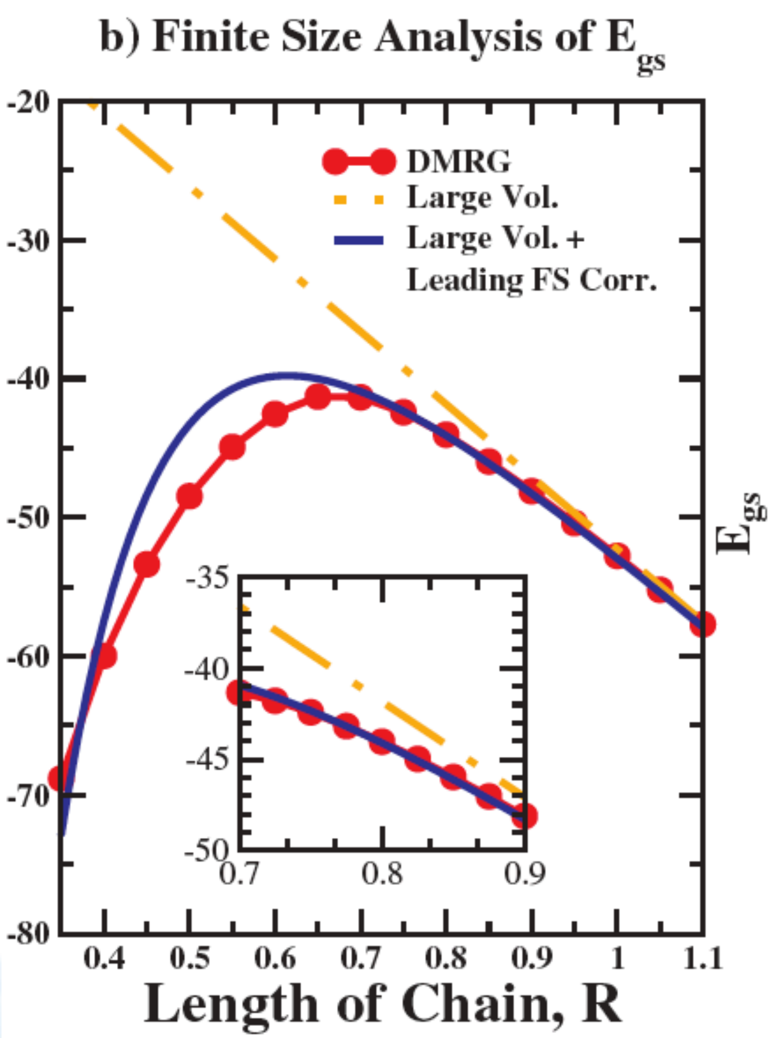
which is equivalent to

$$\Delta \sim gJ$$

$$v \sim Ja$$

Tests of DMRG Algorithm: 1) Exponential Finite Size Scaling

Ground State Energy, 100 Chains



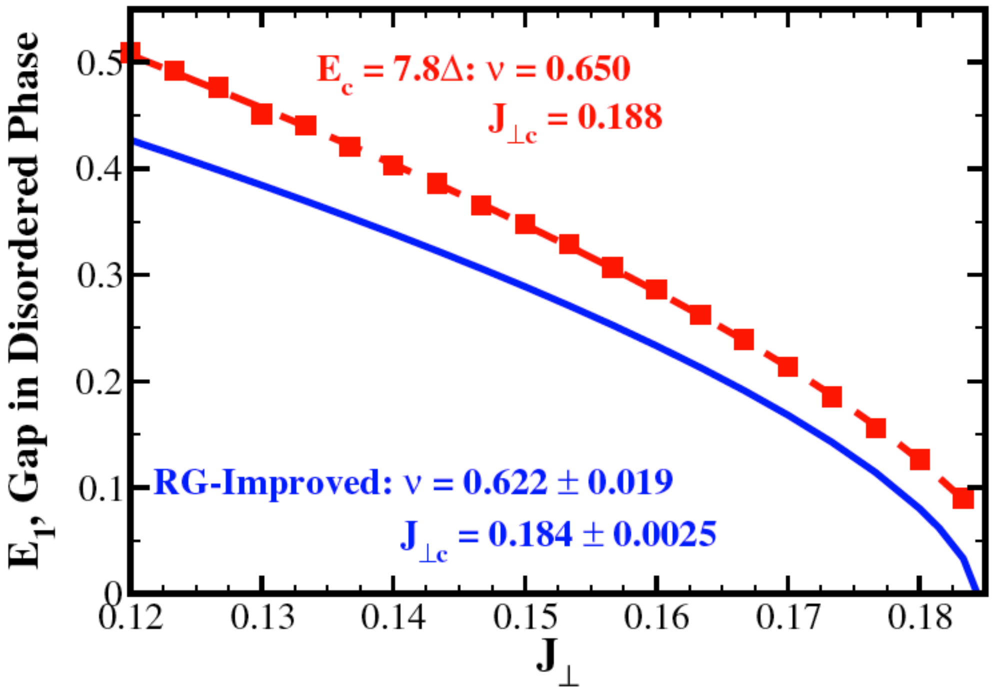
RMK and Y. Adamov, PRL 102, 097203 (2009)

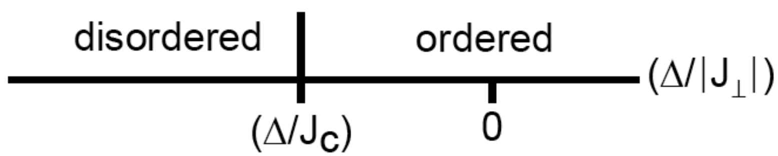
The ground state energy in finite volume is modified by the production of virtual excitations

This modification is controlled by the gap of the lower lying excitations and so can be analytically estimated

We thus have an independent check on our previous results

Tests of DMRG Algorithm: 2) Critical Behaviour





Phase diagram for the 2D quantum system
Critical point should be of the 3D *classical* Ising type

Gap should disappear as

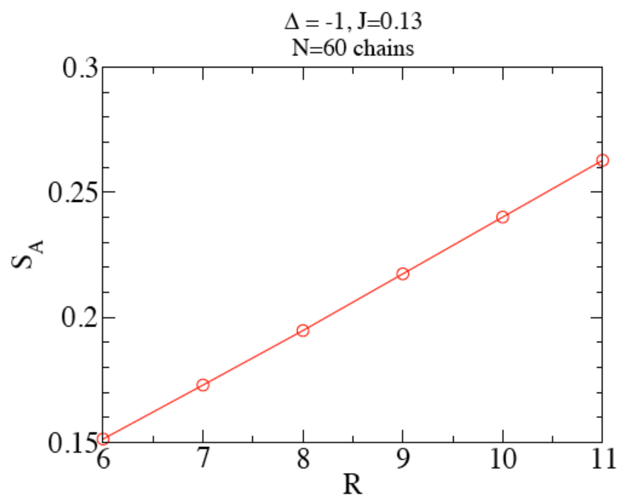
$$E_1 \sim |J_\perp - J_c|^\nu, \quad \nu = 0.630$$

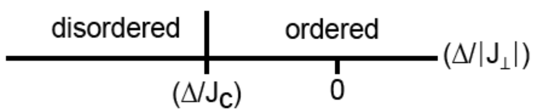
RMK and Y. Adamov, PRL 102, 097203 (2009)

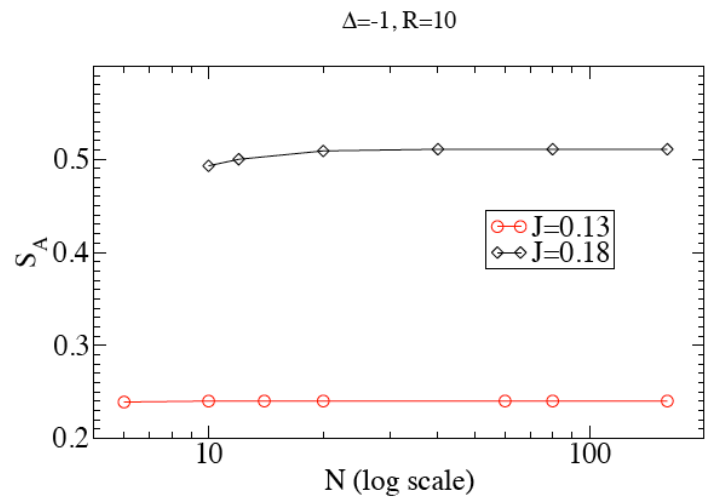
Entanglement Entropy in Disordered Phase

Area law is obeyed:

Area law $S_A \sim R$





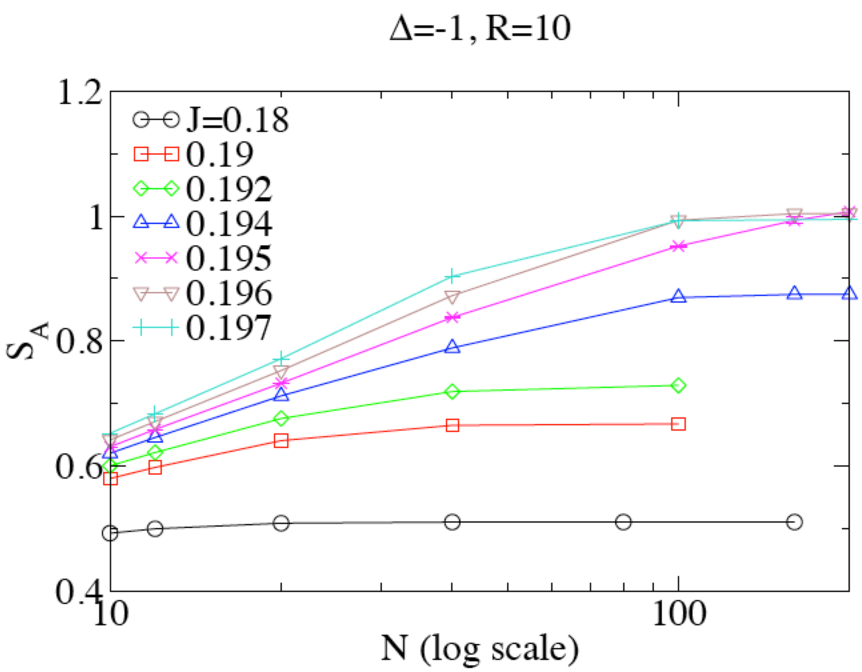


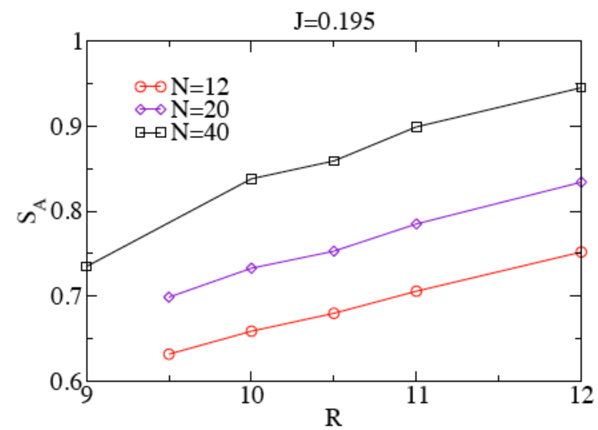
Massive phase

S_A saturates for large $N \sim L \gg \xi$

Entanglement Entropy Close to Criticality

Area law is obeyed but we see the development of a logarithmic term as a function of N, (the number of chains in the system with a block size of N/2):



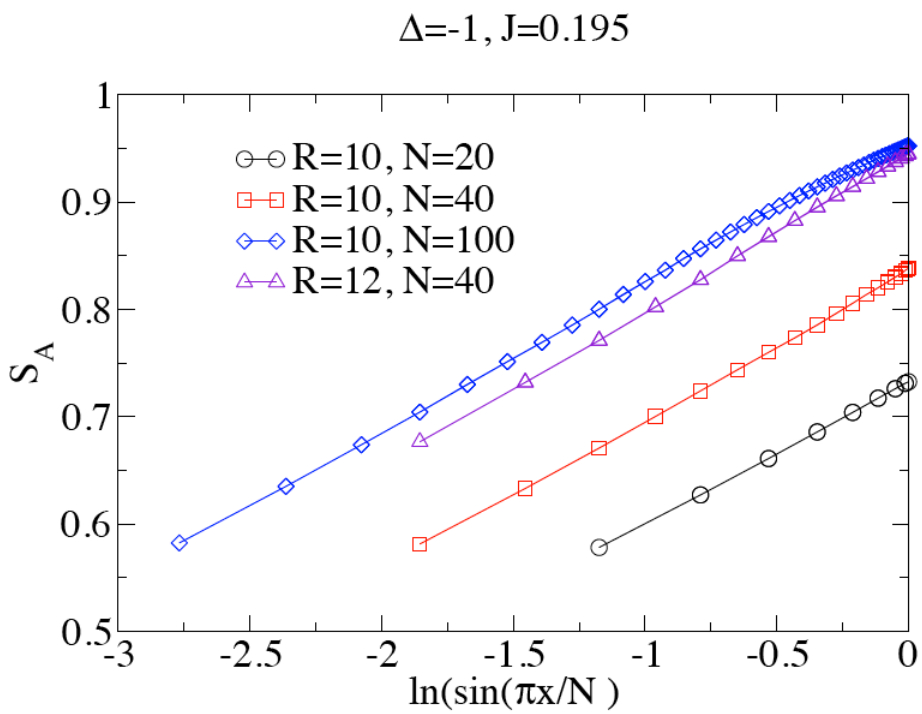


Still have $S_A \sim R$

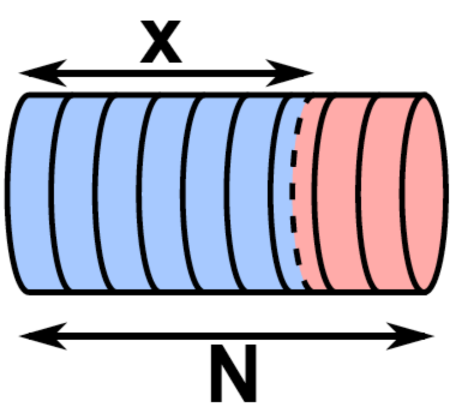
though some departure for small R

Chord Scaling Close to Criticality

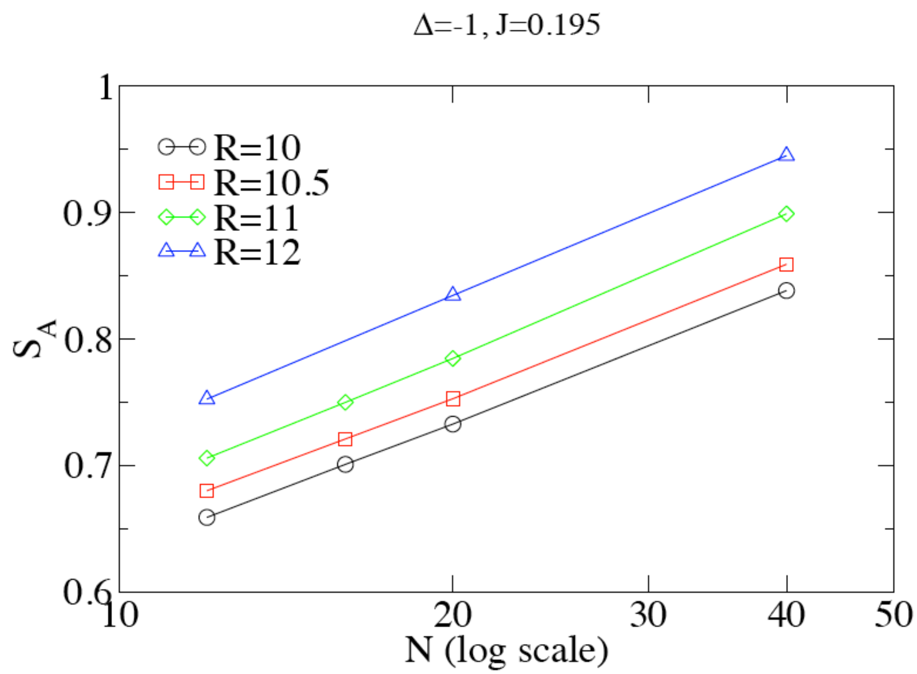
For different block lengths, we see 1D like chord scaling. We find c_{eff} ~ 1 for different system sizes. Similar chord scaling is seen in Ju, Fendley, Hastings, and Melko for 2D Heisenberg (PRB 85, 165121 (2012)).



$$S_A \sim \frac{c_{eff}}{6} \ln(\sin(\pi x/N))$$



Dependence of Entanglement Entropy on R and N



$$S_A \sim BR + \frac{c_{eff}}{6}\log(N) + \ldots, \text{ with } \frac{c_{eff}}{6} \approx 0.16 \Rightarrow c \sim 1$$

Together with chord scaling, the logarithmic dependence suggests 1D-like features to entanglement entropy.

Entanglement Spectra

The entanglement spectra arises from writing the reduced density matrix as

$$\rho_{reduced} = e^{-H_{entanglement}}$$

and finding the spectrum of $H_{entanglement}$.

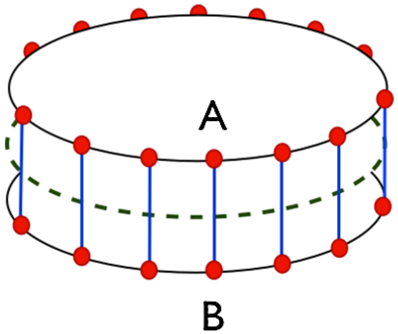
The interest in entanglement spectra begins with work by Haldane and Li (PRL 101, 010504 (2008)) where it was shown this spectra encodes information about the topological nature of quantum Hall states.

However it (now) extends to systems which are not obviously topological:

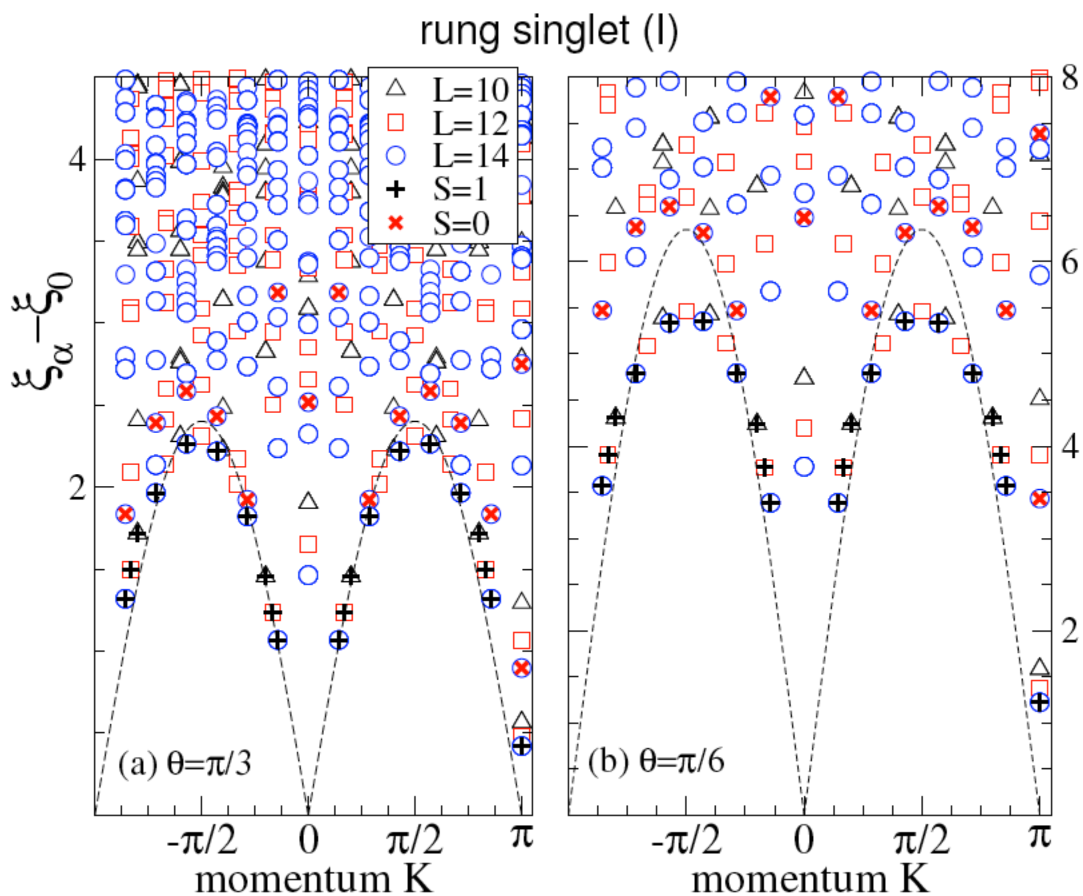
Spin ladders: D. Poilblanc, PRL 105, 077202 (2010)
A. M. Läuchli and J. Schliemann, PRB 85, 054403 (2012)

Key finding: entanglement spectra can resemble that of individual chains

Entanglement spectra of a spin-1/2 ladder



D. Poilblanc, PRL 105, 077202 (2010)



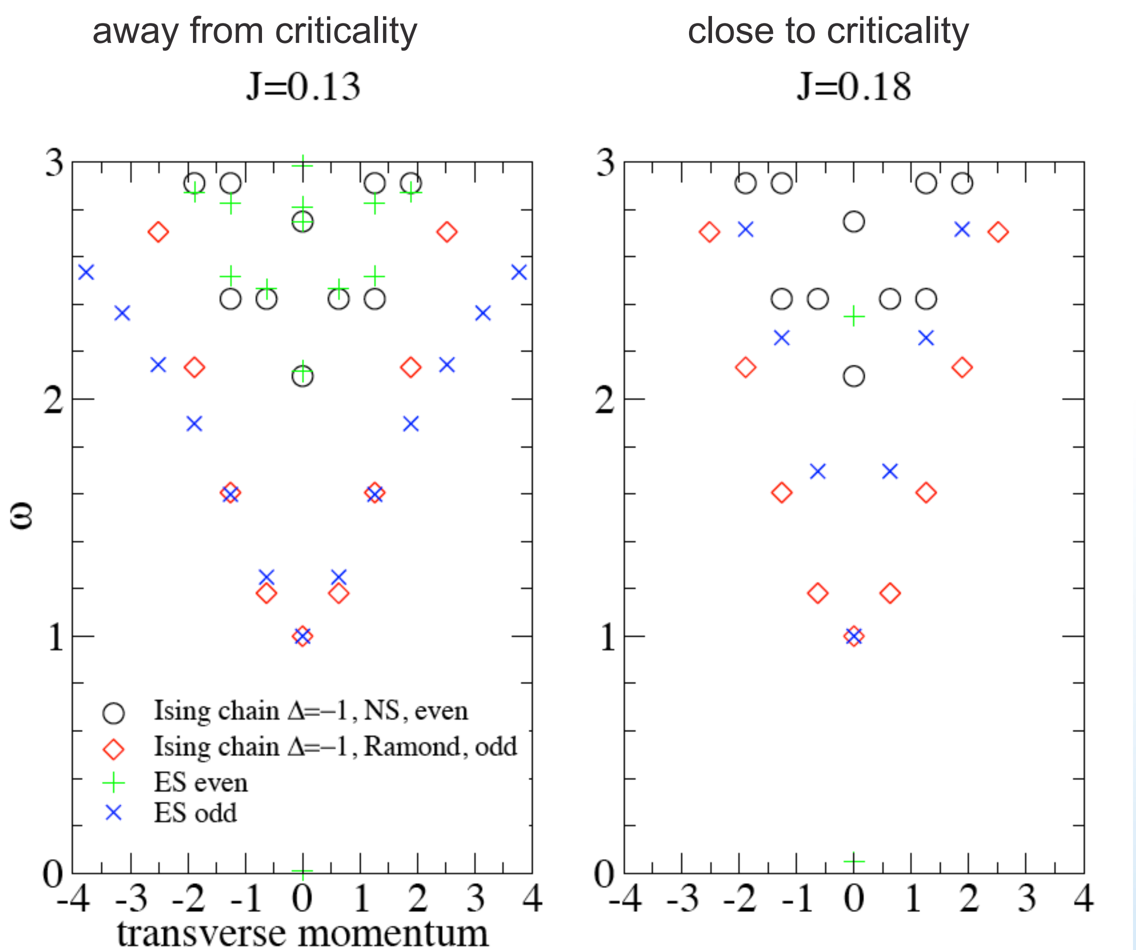
Spectrum arises from dividing the ladder system parallel to the ladder's legs.

One finds something akin to the spinon spectrum of a spin-1/2 chain.

Entanglement Spectra

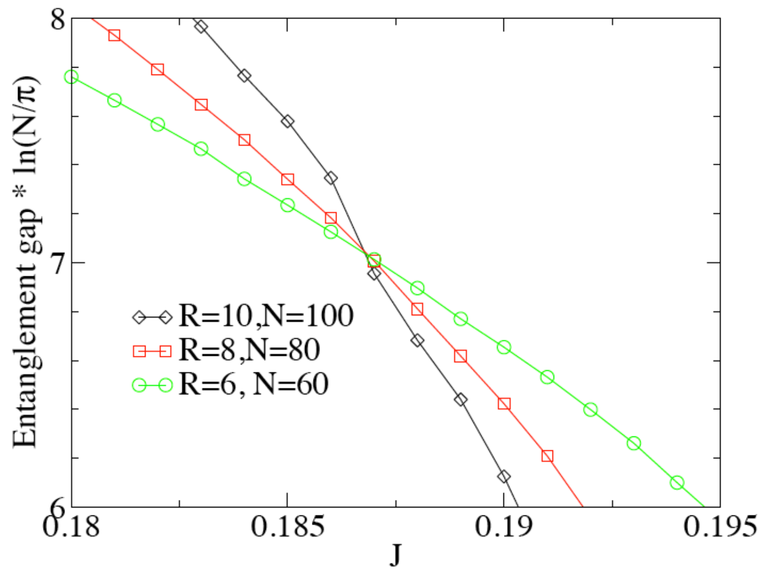
Away from criticality low lying entanglement spectrum appears similar to that of a single chain.

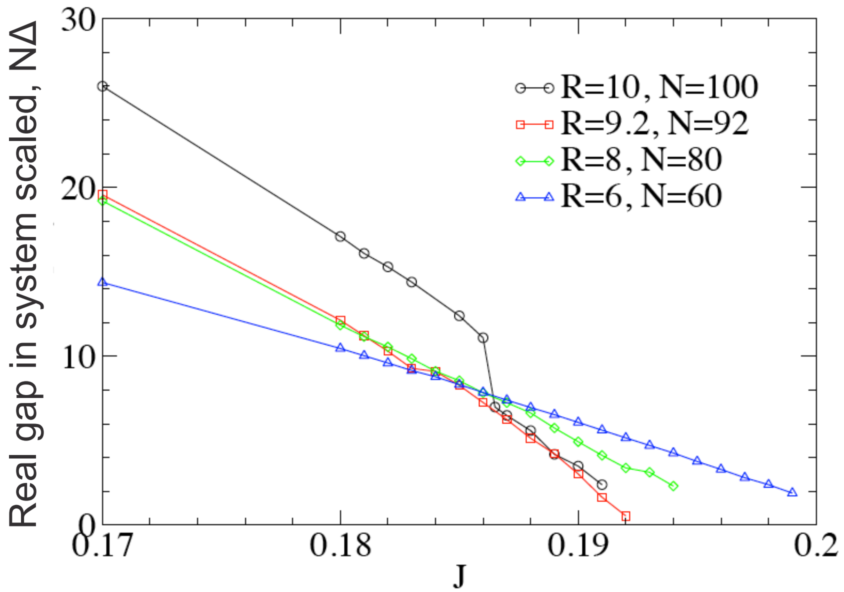
Perturbation theory shows the lowest branch behaves as $E_{ent}(k) = 2\log(\Delta^2+k^2)$ + const.



Does a vanishing of entanglement gap signal a critical point?

Finite size scaling of real and entanglement gap in vicinity of critical point:





Scaling of entanglement gap determined by: L_{eff} = ln(N/π)
P. Calabrese and A. Lefevre, PRA 78, 032329 (2008)
F. Pollmann and J. E. Moore, New J. Phys. 12, 025006.

Larger corrections to the finite size scaling function for real gap than for entanglement gap.

Again we see the entanglement properties of our system behaving like a 1D system.

What is going on?

A suggestion can be found in looking at the entanglement entropy of free fermions in 2D:

$$S = L\ln(L)$$

D. Gioev and I. Klich
PRL 96, 100503 (2006)

If we imagine this free fermionic system as a system of coupled chains, this result can be immediately arrived at by saying **all** the chains are critical and using the results for S in a 1D critical system.

In this light our finding can be interpreted as a finite number of the quantum Ising chains becoming critical. However simply because c=1 does not necessarily mean effectively that two Ising chains are critical.

Conclusions

- DMRG can be used successfully in 2D by applying it to coupled one dimensional field theories.

- For 2+1D quantum Ising, the entanglement entropy appears to have a universal subleading term:

$$S = b(R/a) + \frac{c_{\rm eff}}{6}\ln(N) + \cdots\ , \qquad c_{\rm eff} \approx 1$$

- In this system, the vanishing of the entanglement gap seems to coincide with the critical point and at least for 2+1D quantum Ising is a more robust indicator of the critical point than the vanishing of the real gap.