Boson Pairing and Unusual Criticality

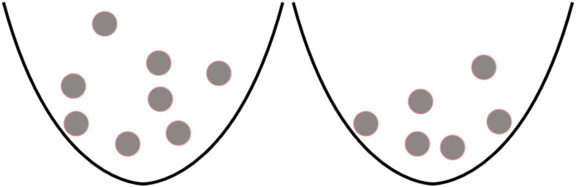
GGI, May 2012

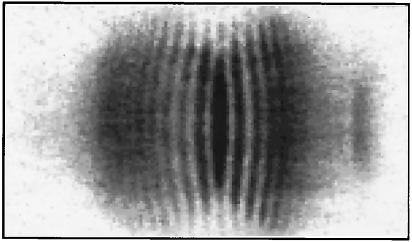
Austen Lamacraft

UNIVERSITY of VIRGINIA

Y. Shi, P. Fendley, AL, Phys. Rev. Lett. 107, 240601 (2011)
A. J. A. James, AL, Phys. Rev. Lett. 106, 140402 (2011)

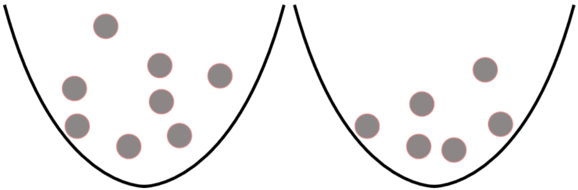
Interference of independent BECs





M. R. Andrews *et al.* (1997)

Interference of independent BECs

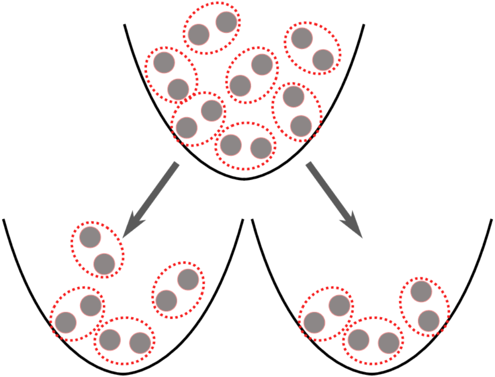


$$\text{Definite phase } \left|\Delta\theta\right\rangle = \sum_{n=0}^{N} e^{in\Delta\theta} \left|n\right\rangle_L \left|N-n\right\rangle_R$$

$$\text{Definite number } \left|n\right\rangle_L \left|N-n\right\rangle_R = \int_0^{2\pi} \frac{d\left(\Delta\theta\right)}{2\pi} \left|\Delta\theta\right\rangle e^{-in\Delta\theta}$$

Pair condensates: Ising variables in an XY system

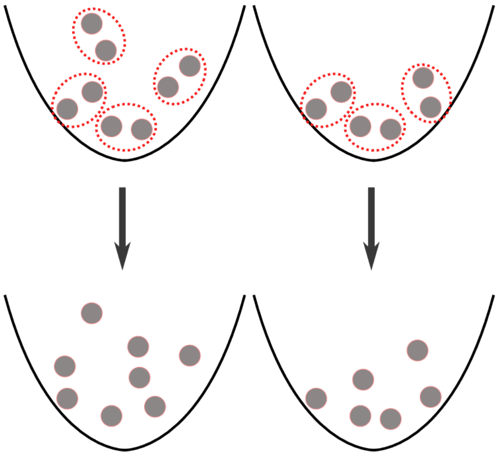
Take a condensate of molecules and *split it*



$$|\Psi\rangle = \sum_{n=0}^{N/2} |2n\rangle_L \, |N-2n\rangle_R$$

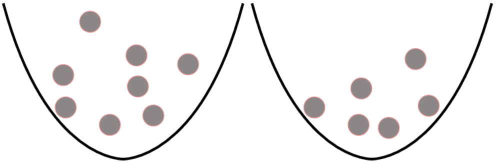
Pair condensates: Ising variables in an XY system

Dissociate pairs



Pair condensates: Ising variables in an XY system

What is the resulting state?



Superposition involves only *even* numbers of atoms

$$\sum_{n=0}^{N/2} |2n\rangle_L |N-2n\rangle_R = \frac{1}{2}\sum_{n=0}^{N} |n\rangle_L |N-n\rangle_R + \frac{1}{2}\sum_{n=0}^{N}(-1)^n |n\rangle_L |N-n\rangle_R$$
$$= \frac{1}{2}\left(|\Delta\theta = 0\rangle + |\Delta\theta = \pi\rangle\right)$$

Pair condensates: Ising variables in an XY system

What are the consequences for the phase diagram?

Outline

Outline

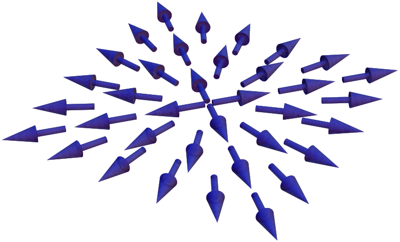
Pair condensates

Interplay of strings and vortices

Novel signatures

Vortices give a twist in 2D

Quantized vortices: phase increases by $2\pi \times q$ (Integer q)



$$\mathbf{v} = \frac{1}{m}\nabla\theta = \frac{1}{m}\frac{\hat{\mathbf{e}}_\theta}{r}$$

$$\text{Kinetic energy} = \frac{mn}{2}\int d\mathbf{r}\,\mathbf{v}^2 = \frac{\pi n}{m}\ln\left(\frac{L}{\xi}\right)$$

The Kosterlitz–Thouless transition

Consider free energy of a single integer vortex

$$\text{Entropy, } S = k_B \ln \left(\frac{L}{\xi}\right)^2$$

$$F = E - TS = \left(\frac{\pi n}{m} - 2k_B T\right) \ln \left(\frac{L}{\xi}\right)$$

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Vanishes at

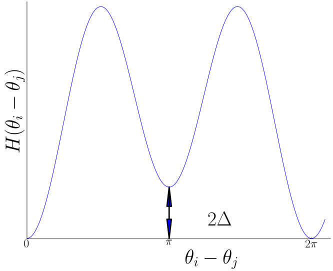
$$k_B T_c = \frac{\pi}{2} \frac{n}{m}$$

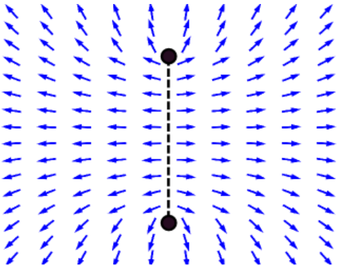
A simple model for pair condensates

$$H_{\mathrm{GXY}} = -\sum_{\langle ij\rangle}\Big[\overbrace{\Delta\cos(\theta_i-\theta_j)}^{\text{Hopping}} + \overbrace{(1-\Delta)\cos(2\theta_i-2\theta_j)}^{\text{Pair hopping}}\Big]$$

Korshunov (1985), Lee & Grinstein (1985)

- $\Delta = 1$ is usual XY; $\Delta = 0$ is π-periodic XY
- $\Delta > 0 \rightarrow$ metastable min. $\rightarrow$ line tension along π-phase jump





"Half" Kosterlitz–Thouless transition at $\Delta = 0$

$$E = \frac{mn}{2} \int d\mathbf{r}\, \mathbf{v}^2 = \frac{\pi n}{4m} \ln\left(\frac{L}{\xi}\right)$$

$$F = U - TS = \left(\frac{\pi n}{4m} - 2k_B T\right) \ln\left(\frac{L}{\xi}\right)$$

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Vanishes at

$$k_B T_c = \frac{\pi}{8}\frac{n}{m}$$

"Half" Kosterlitz–Thouless transition at $\Delta = 0$

Possible Experiments on Two-dimensional Nematics

By P. G. de Gennes

Physique du Solide, Faculté des Sciences, 91 Orsay

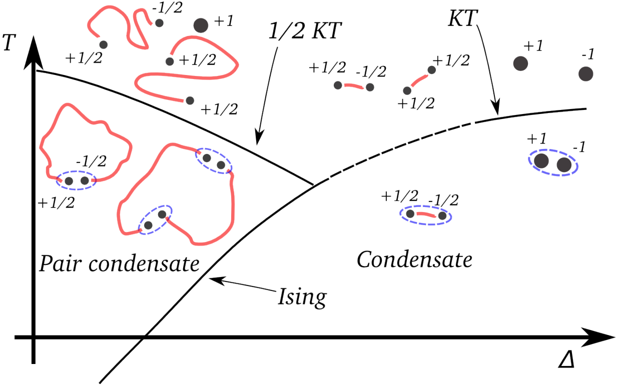
Received 9th July, 1971

Vanishes at

$$k_B T_c = \frac{\pi}{8} \frac{n}{m}$$

Schematic phase diagram

$$H_{\mathrm{GXY}} = -\sum_{\langle ij \rangle} \left[\Delta \cos(\theta_i - \theta_j) + (1-\Delta) \cos(2\theta_i - 2\theta_j)\right]$$



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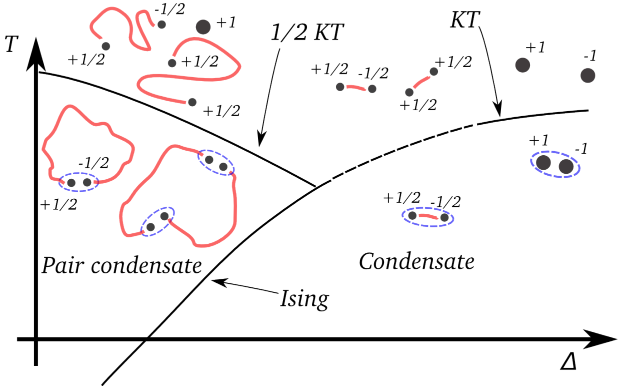
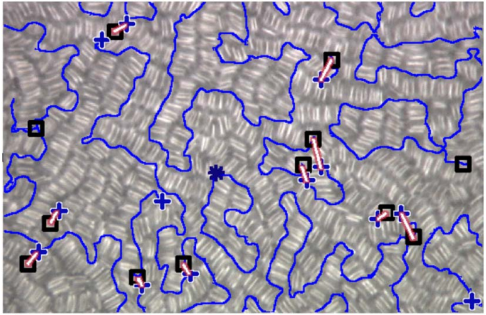
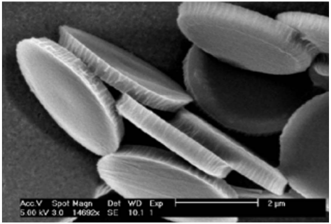


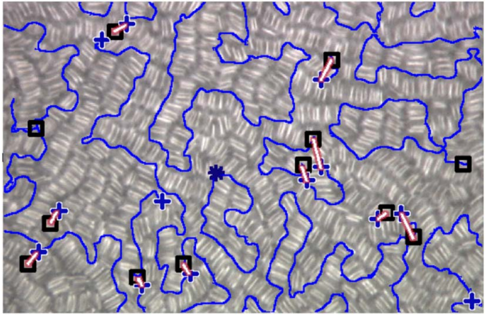
What is the nature of phase transition along dotted line?

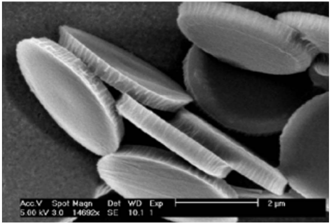
Almost tetratic phases of colloidal rectangles[1]

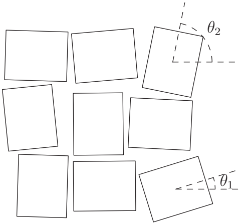
[1] Kun Zhao *et al.*, PRE (2007)

Almost tetratic phases of colloidal rectangles[1]





- $\langle \cos(2\theta) \rangle \neq 0$ *Nematic*
  disordered by π-disclinations
- $\langle \cos(4\theta) \rangle \neq 0$ *Tetratic*
  disordered by $\frac{\pi}{2}$-disclinations



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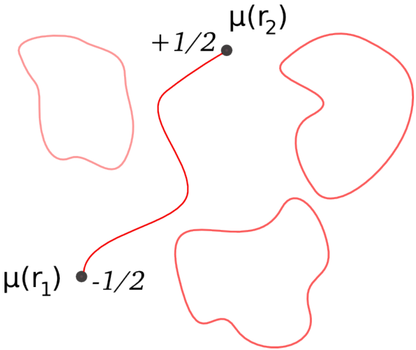
Outline

How things change on the Ising critical line

Redo KT argument accounting for string



Partition function with string connecting **x**, **y** defines $\langle \mu(\mathbf{x})\mu(\mathbf{y}) \rangle$

μ's are *disorder operators*

How things change on the Ising critical line

Disorder operators dual to $\sigma(\mathbf{x})$ of Ising model. At T_c

$$\langle \sigma(\mathbf{x})\sigma(\mathbf{y})\rangle = \langle \mu(\mathbf{x})\mu(\mathbf{y})\rangle = \frac{1}{|\mathbf{x}-\mathbf{y}|^{1/4}}$$

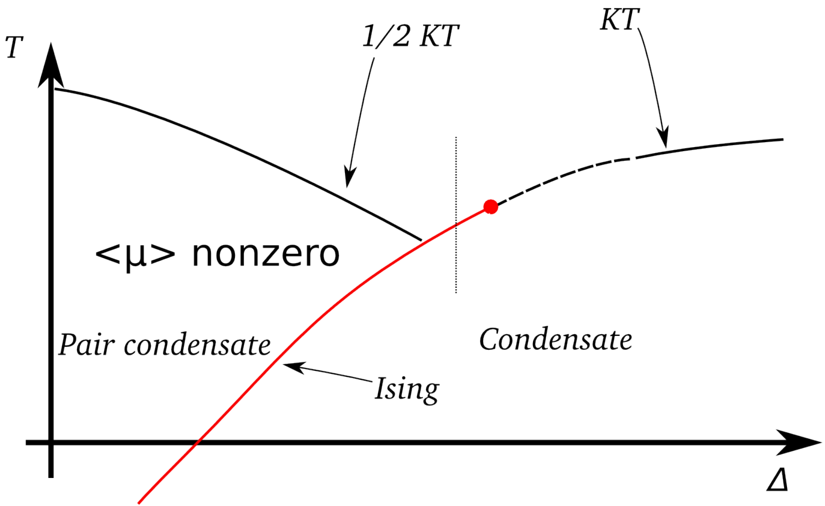
Take one end of string to ∞ i.e. edge of system: $\langle \mu(\mathbf{x})\rangle \to \frac{1}{L^{1/8}}$
Contributes $+\frac{k_B T}{8}\ln L$ to free energy $F = -k_B T \ln \mathcal{Z}$

$$F = U - TS = \left(\frac{\pi n}{4m} - \frac{15}{8}k_B T\right)\ln\left(\frac{L}{\xi}\right)$$

$$k_B T_c = \frac{2\pi}{15}\frac{n}{m}$$

Dissociation at higher temperatures than for ‘free’ half vortices

How things change on the Ising critical line



Direct Ising transition to low temperature phase!

RG flow

Keep track of

1. Stiffness J (or $J_* = J^{-1}$)
2. Deviation from Ising critical line $\kappa = K - K_c$
3. Half vortex fugacity z_1

$$\frac{dz_1}{dl} = \left(\frac{15}{8} - \frac{\pi}{4J_*}\right) z_1 - \frac{\kappa z_1}{2}$$

$$\frac{dJ_*}{dl} = \frac{\pi^2 z_1^2}{4}$$

$$\frac{d\kappa}{dl} = \kappa - \frac{z_1^2}{4}$$

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Keep track of

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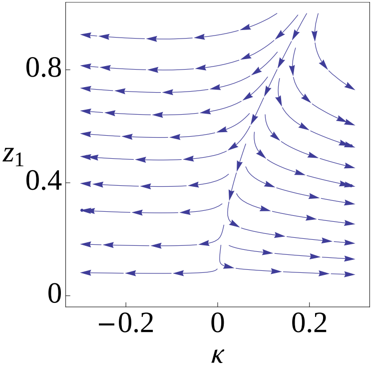
$J > \frac{15}{2\pi}$



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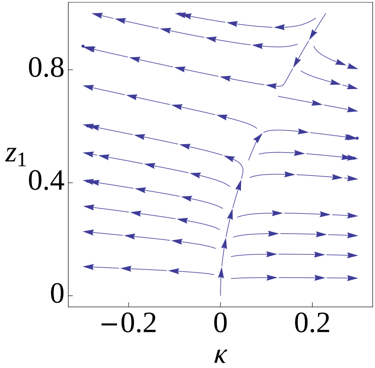
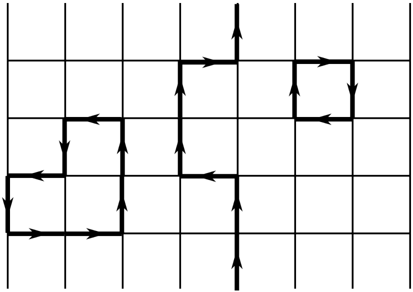


Numerical simulation using worm algorithm

$$\mathcal{Z} = \sum_{\substack{\{n_{ij}\} \\ \nabla\cdot n=0}} \exp\left(-\frac{J_*}{2} \sum_{\langle ij \rangle} n_{ij}^2 + \frac{K_*}{2} \sum_{\langle ij \rangle} (-1)^{n_{ij}} \right)$$



Simulate worldlines of bosons, not spin variables

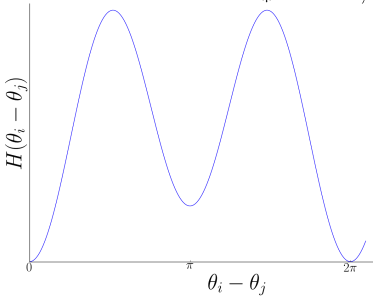
Numerical simulation using worm algorithm

$$\mathcal{Z} = \prod_c \int_{-\pi}^{\pi} \frac{d\theta_c}{2\pi} \prod_{\langle ab \rangle} w(\theta_a - \theta_b)$$

$w(\theta)$ written in Villain form: $w(\theta) \equiv w_V(\theta) + e^{-K} w_V(\theta - \pi)$

$$w_V(\theta) \equiv \sum_{p=-\infty}^{\infty} e^{-\frac{J}{2}(\theta + 2\pi p)^2} \propto \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-\frac{J_*}{2} n^2}$$

$J_* = J^{-1}$, $\sinh K_* \sinh K = 1$



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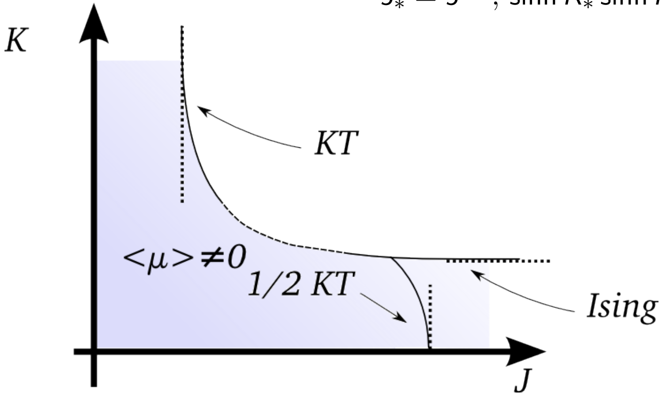

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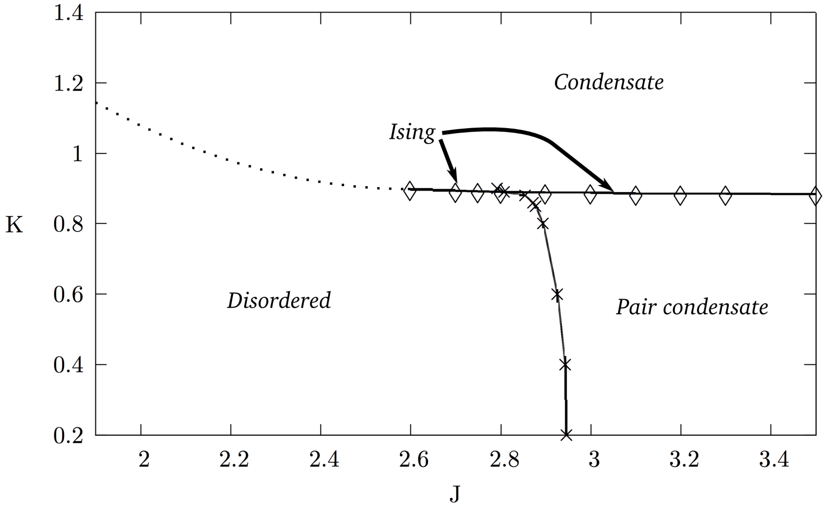
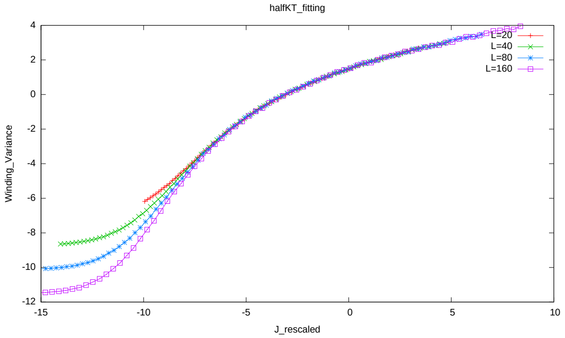


Use of sectors

Calculation of superfluid stiffness $\Upsilon = \frac{\partial^2 F}{\partial \theta^2}$ 2

$$\Upsilon = \frac{T}{2}\langle \mathbf{W}^2 \rangle$$

$\mathbf{W} = (W_x, W_y)$, vector of windings

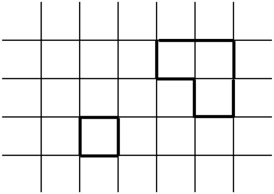


2 Pollock & Ceperley (1987)

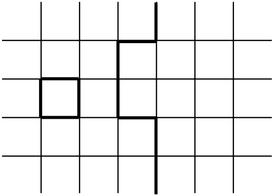
Use of sectors

Keeping track of *parity* of winding gives sectors of Ising model

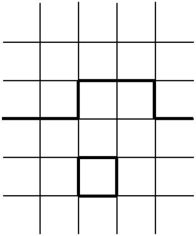
$\mathcal{Z}_{\mathcal{P}\mathcal{P}}$



$\mathcal{Z}_{\mathcal{P}\mathcal{A}}$



$\mathcal{Z}_{\mathcal{A}\mathcal{P}}$



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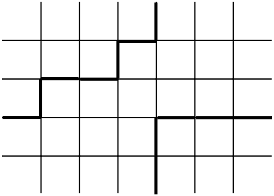


Use of sectors

Critical ratios known from CFT[2]

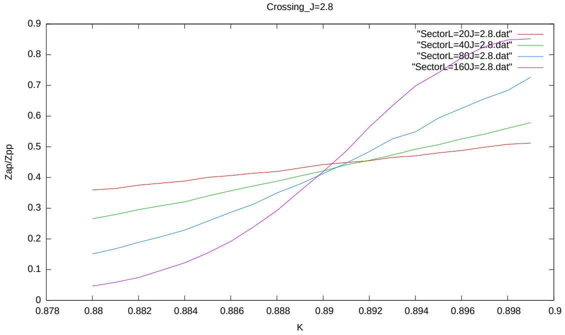
$$\mathcal{Z}_{PP} = 1.8963\ldots$$
$$\mathcal{Z}_{AP} = \mathcal{Z}_{PA} = \frac{1}{\sqrt{2}}$$
$$\mathcal{Z}_{AA} = 0.4821\ldots$$

Use to locate phase transition from finite size scaling

[2] P. Ginsparg, Les Houches lectues (1989)

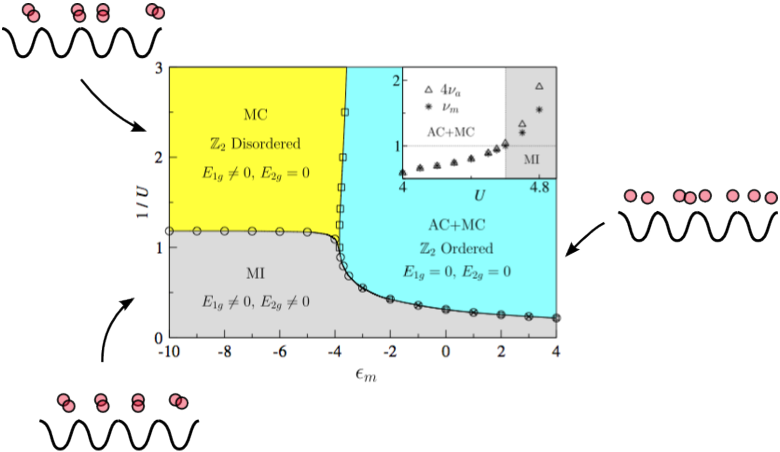
Use of sectors

$$\frac{\mathcal{Z}_{AP} + \mathcal{Z}_{PA}}{2\mathcal{Z}_{PP}} = 0.372\ldots$$



Quantum Transition

2D classical picture also applies to (1+1)D quantum transition



Ejima *et al.* (2011)

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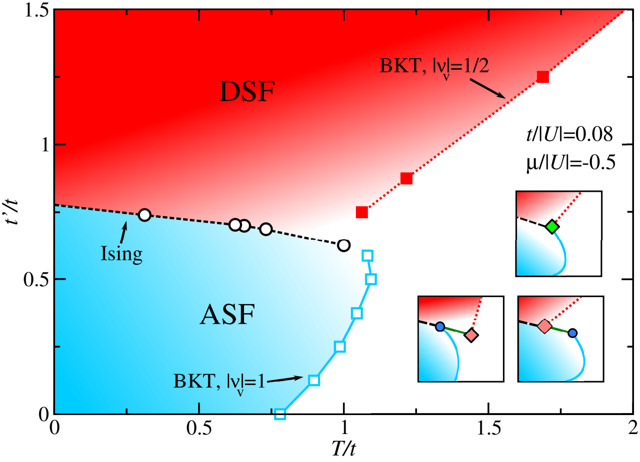

Bonnes & Wessel (2012)

Quantum Transition

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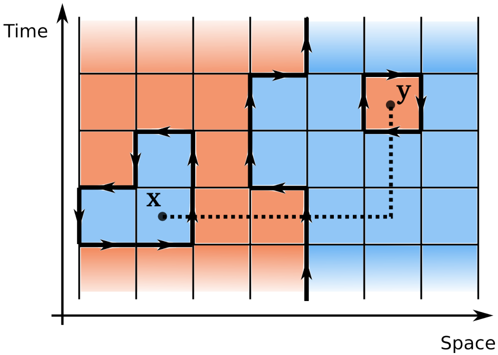
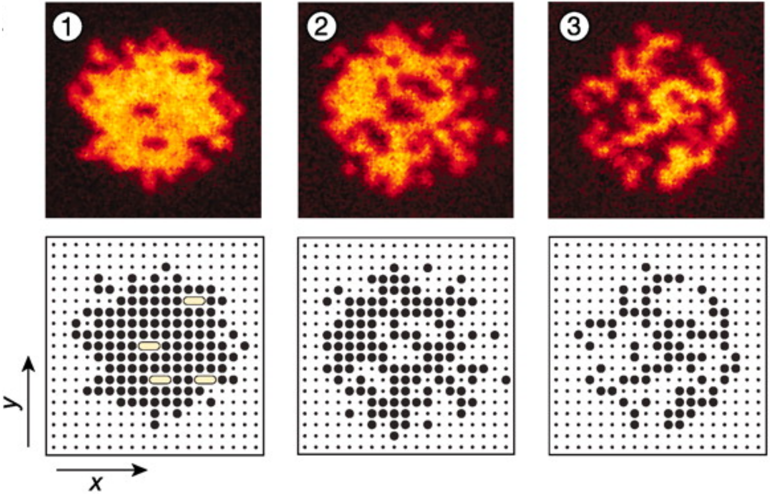


Disorder variables correspond to domains bounded by worldlines

$$\langle \mu(\mathbf{x})\mu(\mathbf{y})\rangle = \langle (-1)^{\#\text{ of lines crossed from }\mathbf{x}\text{ to }\mathbf{y}}\rangle$$

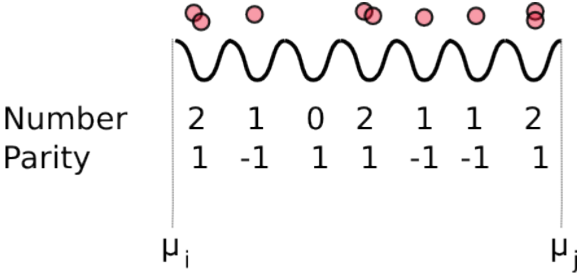
Outline

Site parity measurements from Bloch group



M. Endres *et al.* (2011)

Site parity measurements from Bloch group

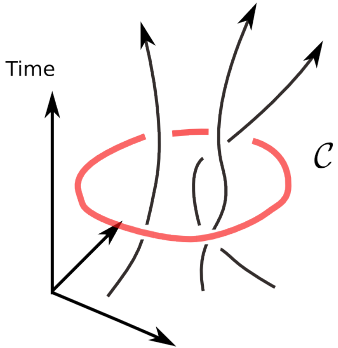


$$\langle \mu_i \mu_j \rangle = \langle \prod_{i \le k < j} (-1)^{n_k} \rangle$$

Develops long range order in the paired phases (Ising disordered)

$(2+1)D$

What is the analogous construction in $2+1D$?



$$\langle \mathcal{W}(\mathcal{C}) \rangle \equiv \langle (-1)^{\text{Lines piercing surface bounded by } \mathcal{C}} \rangle$$

$\mathcal{W}(\mathcal{C})$ is *Wilson loop* of Ising gauge theory

$(2+1)D$

$$\langle \mathcal{W}(\mathcal{C}) \rangle \equiv \langle (-1)^{\text{Lines piercing surface bounded by } \mathcal{C}} \rangle$$

$$\langle \mathcal{W}(\mathcal{C}) \rangle \propto \begin{cases} \exp\left(-\text{const. Area}\right) & \text{in unpaired phase} \\ \exp\left(-\text{const. Perimeter}\right) & \text{in paired phases} \end{cases}$$

- In (1+1)D half vortices are points & carry a disorder operator
- In (2+1)D half vortices are loops & carry Ising gauge charge