Boundary theories for spins in lattices

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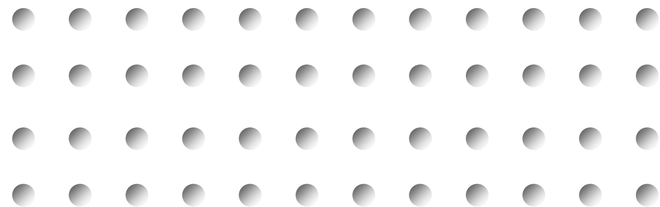
GGI, Florence, May 24, 2012

IC, Poilblanc, Schuch, and Verstraete, Phys. Rev. B 83, 245134 (2011)
Poilblanc, Schuch, Perez-Garcia, IC, arXiv:1202.0947
Schuch, Poilblanc, IC, Perez-Garcia, arXiv:1203.4816

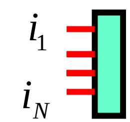
TENSOR NETWORKS

- N spins:



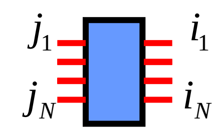
- States and observables can be written in terms of tensors

$$|\Psi\rangle = \sum_i c^{i_1,\dots,i_N} \, | i_1,\dots,i_N \rangle$$



$c^{i_1,\dots,i_N}$

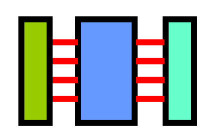
$$X = \sum_{i,j} X^{j_1,\dots,j_N}_{i_1,\dots,i_N} \, | j_1,\dots,j_N \rangle\langle i_1,\dots,i_N |$$



$X^{j_1,\dots,j_N}_{i_1,\dots,i_N}$

- Expectation values are tensor contractions:

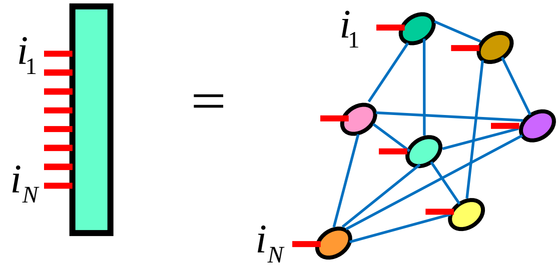
$$\langle\Psi | X | \Psi\rangle = \sum_{i,j} c^*_{j_1,\dots,j_N} \, X^{j_1,\dots,j_N}_{i_1,\dots,i_N} \, c^{i_1,\dots,i_N}$$



TENSOR NETWORKS

- Rewrite tensors in terms of smaller tensors:

STATES:



Tensor network states

or

Tensor product states

OBSERVABLES: similarly

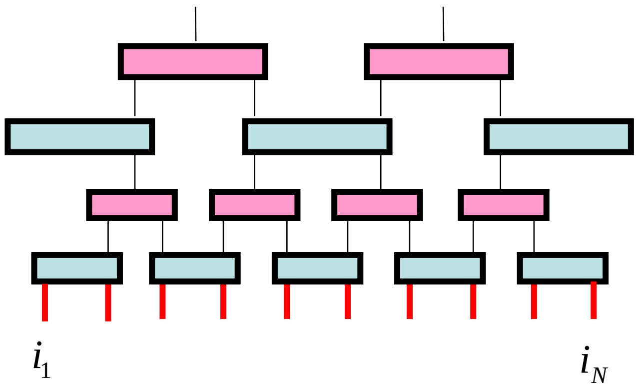
Why? Efficient description: $N^a d^b D^c$

- Guiding principle: entanglement

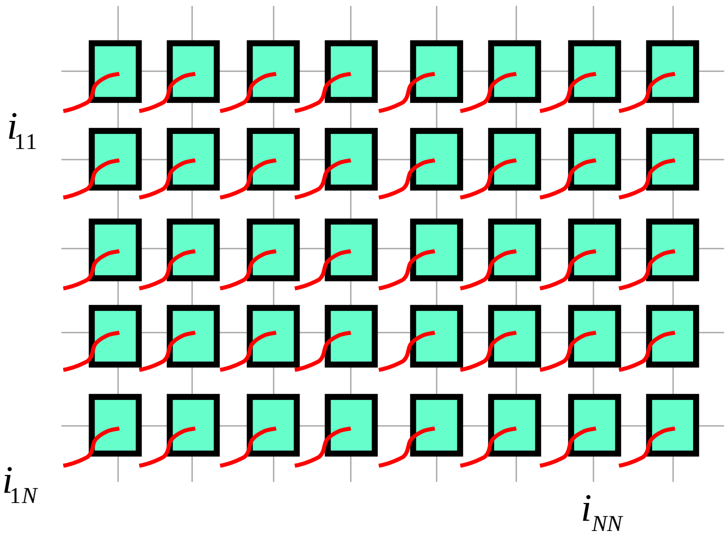
TENSOR NETWORKS
EXAMPLES

- Multi-scale ENTANGLEMENT renormalization ansatz
- Projected ENTANGLED-pair states

MERA: G.Vidal

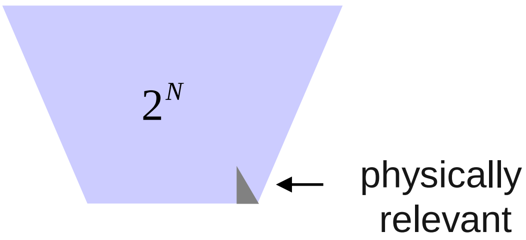


PEPS: F.Verstraete, I. Cirac



- Projected ENTANGLED-pair states
  - Thermal equilibrium
  - Local interactions
  - Arbitrary dimensions (Hastings)

➡ Numerical algorithms



- Projected ENTANGLED-pair states
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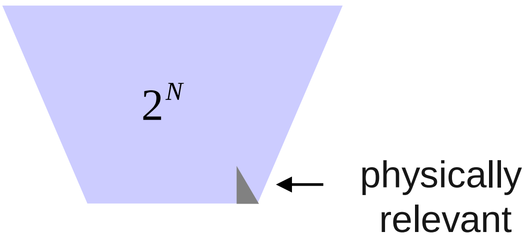
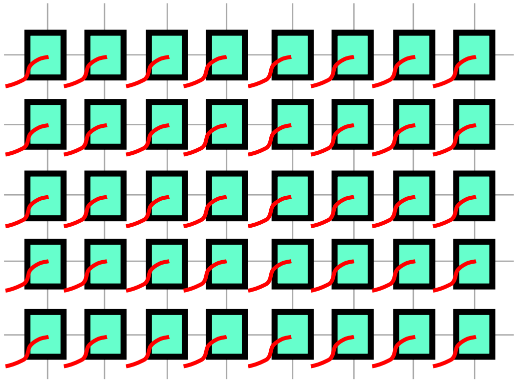
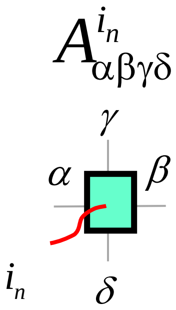


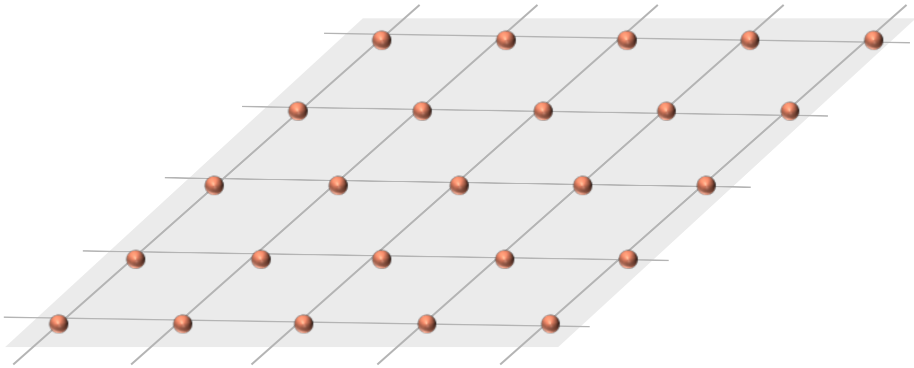
➡ Many-body physics





PEPS

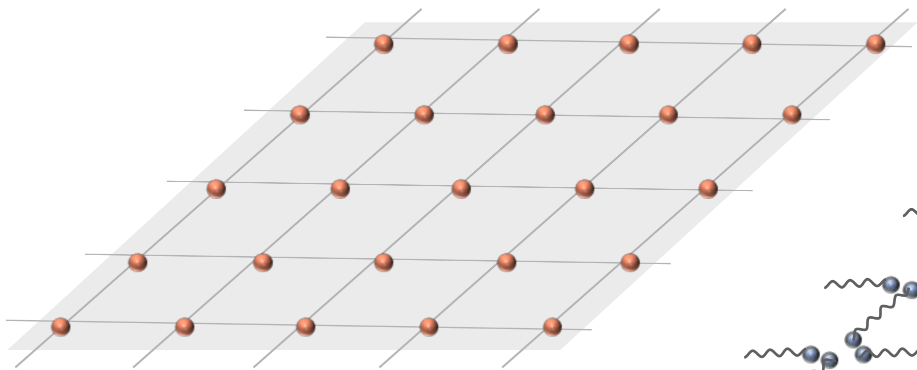
- Projected entangled-pairs:



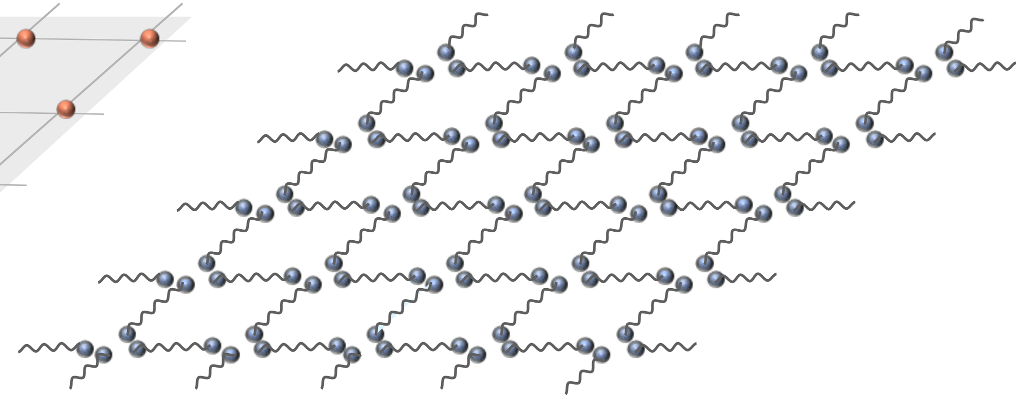
Physical spins: $|\Psi\rangle$

PEPS

- Projected entangled-pairs:



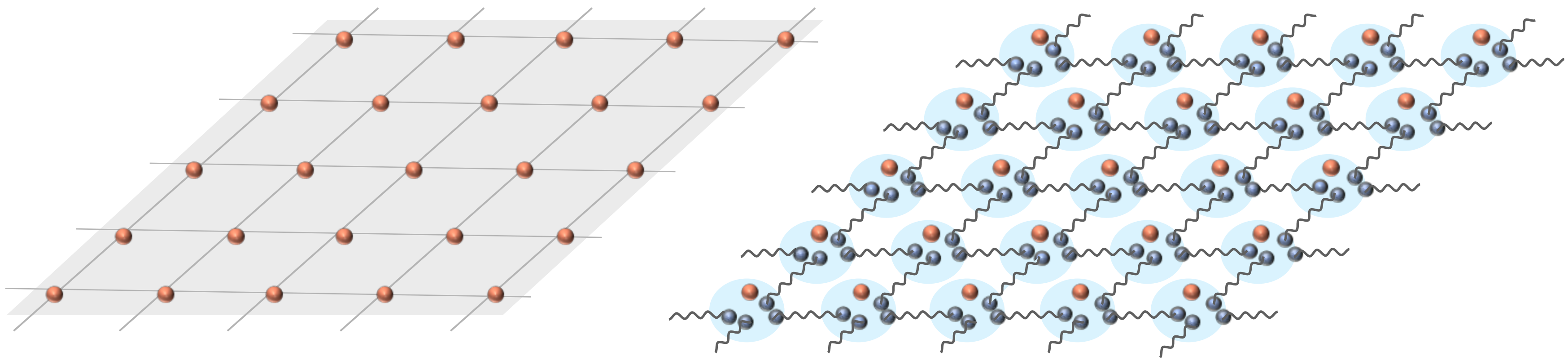
Physical spins: $|\Psi\rangle$



Auxiliary spins: $|\Phi\rangle^{\otimes N}$

PEPS

- Projected entangled-pairs:



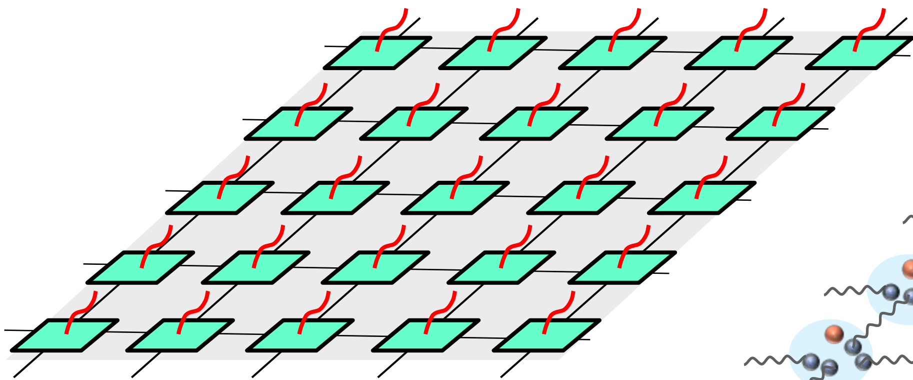
Physical spins: $|\Psi\rangle$ ⟵ Auxiliary spins: $|\Phi\rangle^{\otimes N}$

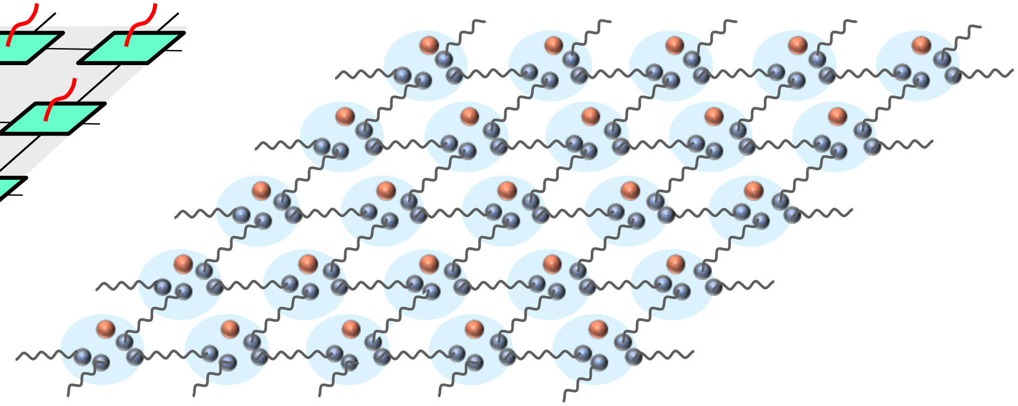
$$P = \sum A^{i}_{\alpha\beta\gamma\delta} \, |i\rangle\langle\alpha,\beta,\gamma,\delta|$$

- P's act locally
- Contain the information about the state
- Similar to AKLT construction

PEPS

- Projected entangled-pairs:

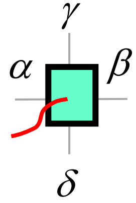




Physical spins: $|\Psi\rangle$ ← Auxiliary spins: $|\Phi\rangle^{\otimes N}$

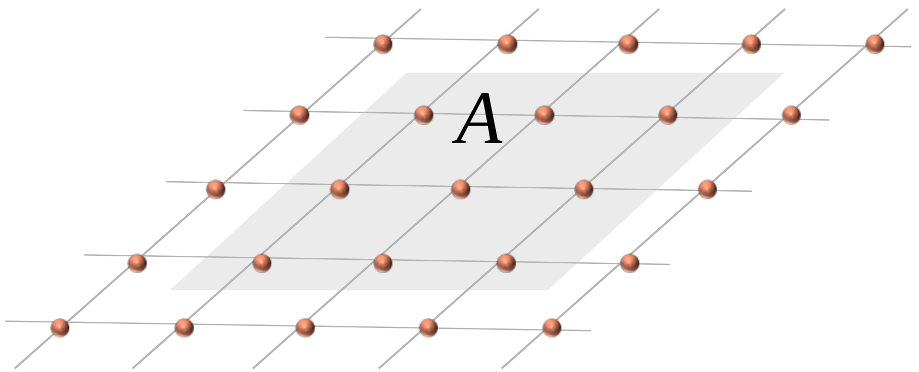
$$P = \sum A^{i}_{\alpha\beta\gamma\delta} \, |i\rangle\langle\alpha,\beta,\gamma,\delta|$$

$A^{i_n}_{\alpha\beta\gamma\delta}$



PEPS

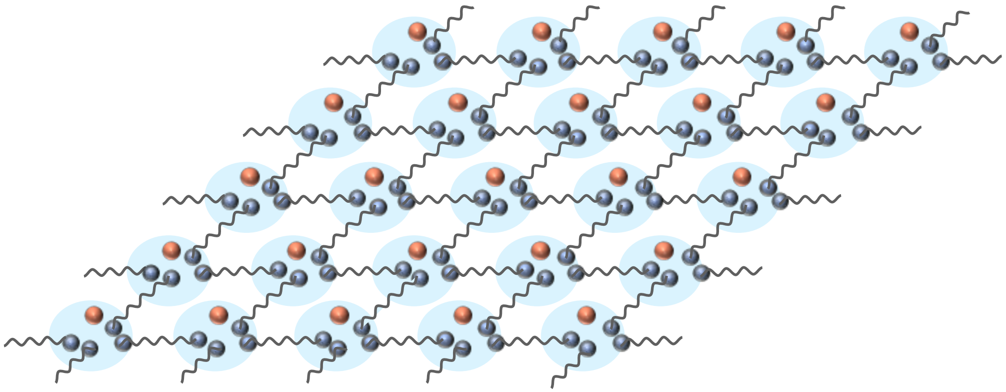
- Area law:



$$S(\rho_A) \sim N_{\partial A}$$

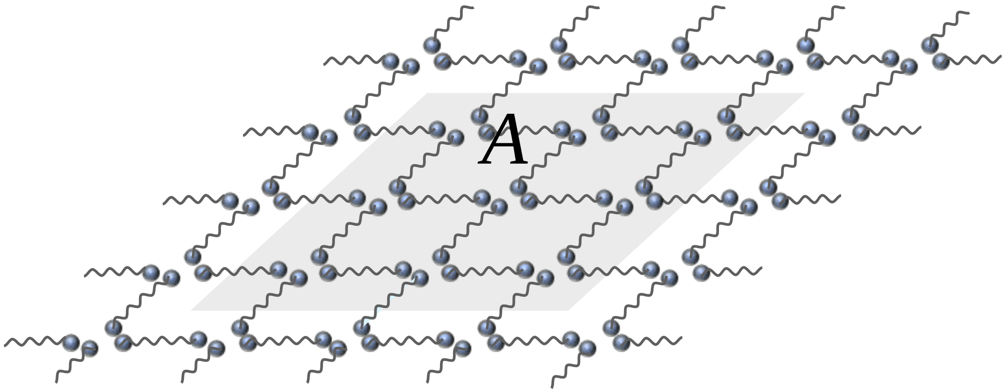
PEPS

- Area law:



PEPS

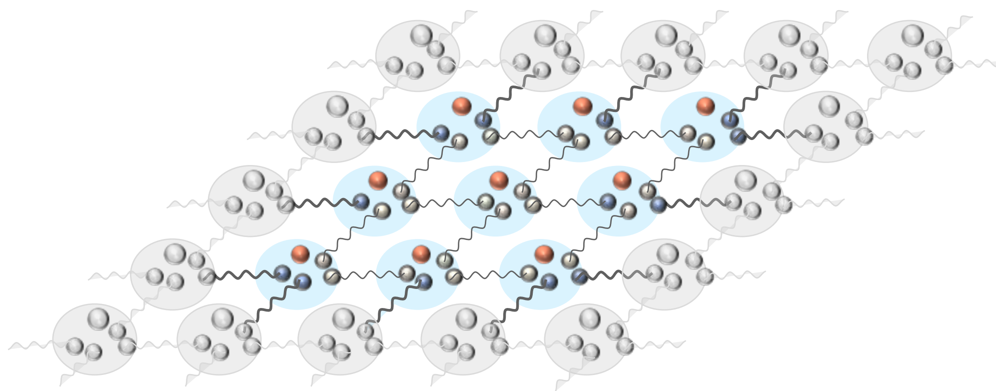
- Area law:



  - Only the auxiliary particles at the boundary contribute
  - Linear maps P cannot increase entanglement

PEPS

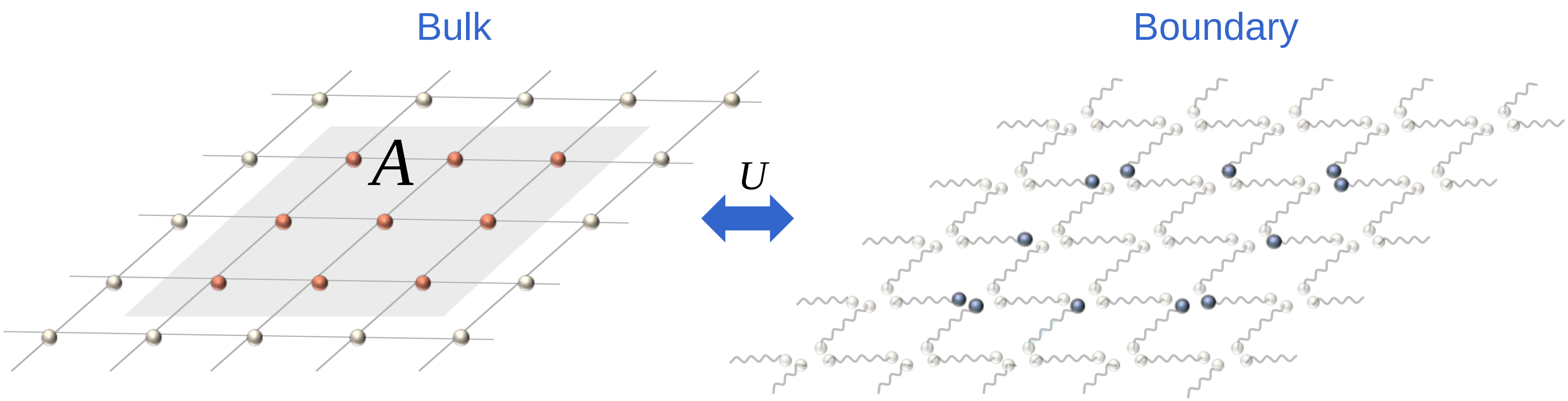
- Area law:



- Only the auxiliary particles at the boundary contribute
- Linear maps P cannot increase entanglement
- $S(\rho_A) \sim N_{\partial A}$
- # degrees of freedom in the bulk scale with the size of boundary

PEPS

- Bulk-boundary correspondence:



$$U : H^{\otimes A} \rightarrow h^{\otimes \partial A}$$ isommetry $$UU^{\dagger} = 1_h \qquad U^{\dagger}U = 1_H$$

$$\rho_A \qquad \sigma_{\partial A} = U \rho_A U^{\dagger}$$

$$X_A \qquad x_{\partial A} = U X_A U^{\dagger}$$

PEPS

- Bulk-boundary correspondence:

Bulk	Boundary
ρ_A	$\sigma_{\partial A} = U \rho_A U^\dagger$
X_A	$x_{\partial A} = U X_A U^\dagger$

  - Expectation values:

$$\langle X \rangle_A = \mathrm{tr}(X_A \rho_A) = \mathrm{tr}(U X_A U^\dagger U \rho_A U^\dagger) = \mathrm{tr}(x_{\partial A} \sigma_{\partial A}) = \langle x \rangle_{\partial A}$$

  - Boundary Hamiltonian:

$$\rho_A = e^{-H_A} \qquad \sigma_{\partial A} = e^{-H_{A\partial}}$$

$$H_{\partial A} = U H_A U^\dagger$$

  - Entanglement spectrum:: $\sigma(H_A) = \sigma(H_{\partial A})$

The standard ES is exactly the spectrum of the boundary Hamiltonian

The boundary Hamltonian has a physical meaning

PEPS

- Boundary theory:
  - The boundary operators can be determined using PEPS algorithms

$$\sigma_{\partial A} = e^{-H_{A\partial}}$$

  - The boundary Hamiltonian reflects properties of the original state $|\Psi\rangle$
    - Symmetries: $u_g |\Psi\rangle = e^{i\theta_g} |\Psi\rangle \quad \Rightarrow \quad U_g H_{\partial A} U_g^\dagger = H_{\partial A}$
    - Topology: Non-local projector
    - Criticality:

      If Ψ is the ground state of a GAPPED LOCAL Hamiltonian, then the boundary Hamiltonian is LOCAL

PEPS

- Examples:
  - AKLT model in 2D

    - Auxiliary particles s=1/2
    - Symmetry: $su(2)$
    - Finite correlation length

    ➡ $H_{\partial A}$ is the 1D Heisenberg Hamiltonian

    ⬇

    ES corresponds to c=1 CFT

  - Kitaev's toric code

    - Auxiliary particles s=1/2
    - Symmetry: Z_2
    - Finite correlation length
    - Topological

    ➡ $\sigma_{\partial A}$ is a non-local projector (Z_2)

    ⬇

    ES is flat

  - RVB on a Kagome lattice

    - Auxiliary particles s=1
    - Symmetry: $su(2)$
    - Finite correlation length
    - Topological

    ➡ $\sigma_{\partial A}$ contains a non-local projector (Z_2)

    $H_{\partial A}$ is a 1D t-J model

PEPS

- Z_2 spin liquids

Kitaev‘s toric code
(square lattice)

o-RVB (RK)
(Kagome lattice)

RVB
(Kagome lattice)

PEPS

- Z_2 spin liquids

Kitaev‘s toric code
(square lattice)

local
unitary

↔

o-RVB (RK)
(Kagome lattice)

local
invertible

RVB
(Kagome lattice)

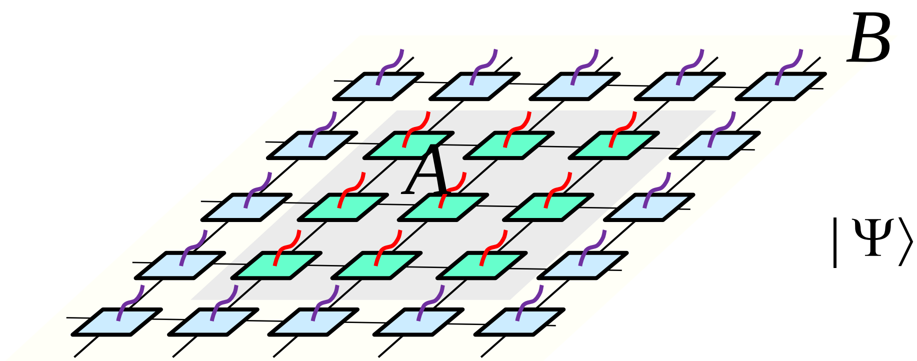
- They correspond to the same phase
- RVB is ground state of local (FF) Hamiltonian (4-fold degeneracy)

OUTLINE

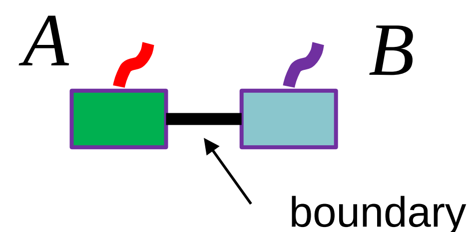
- How to determine the boundary theory for a PEPS
- Symmetries
- Finite correlation length
- Examples

PEPS
BOUNDARY THEORY

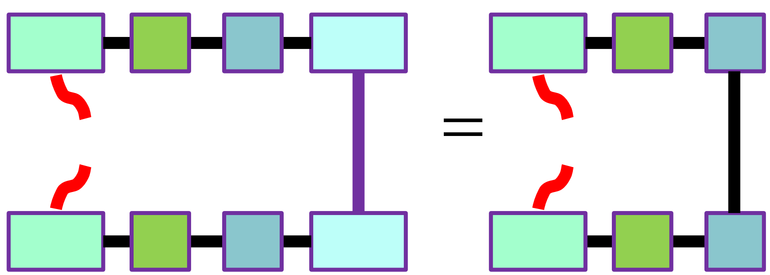
- Reduced state:



- Combine the tensors of regions A and B

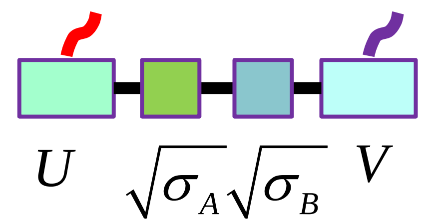


- Reduced state:



$$\rho_A = \mathrm{tr}_B(|\Psi\rangle\langle\Psi|) \qquad U\sqrt{\sigma_A}\sigma_B\sqrt{\sigma_A}U^\dagger$$

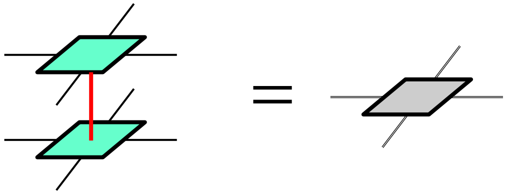
- Polar decomposition:



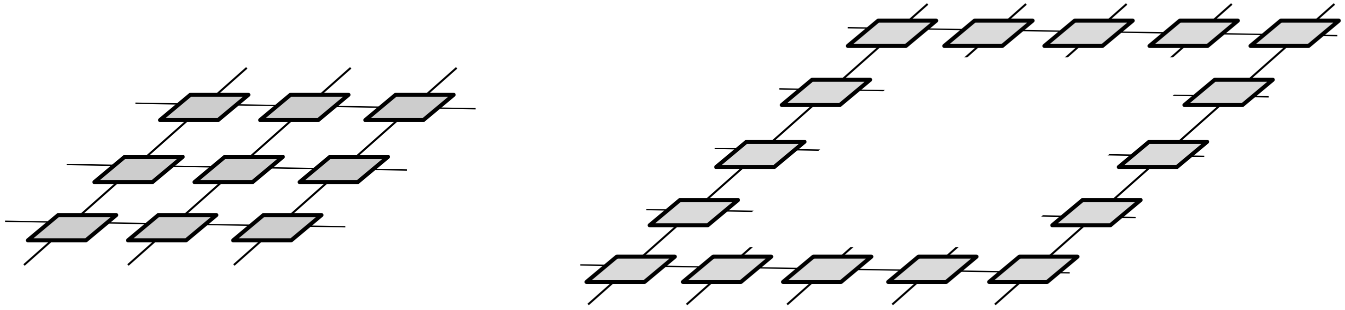
$$\Rightarrow \quad \sigma_{\partial A} = \sqrt{\sigma_A}\sigma_B\sqrt{\sigma_A}$$

- In practice:

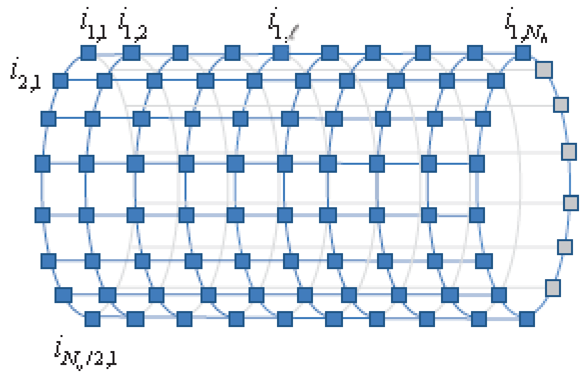
1.- Contract the tensor A with ist complex conjugate

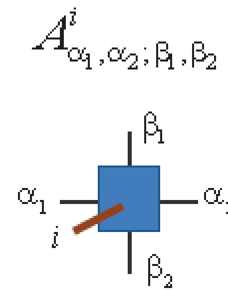


2.- Determine σ_A, σ_B

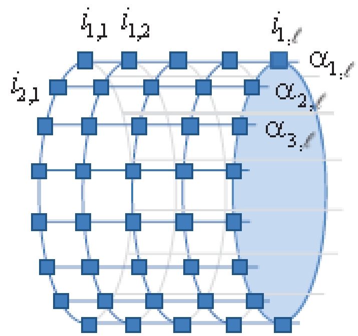


- Cylinder:





A



- Exact calculations $N \times \infty$
- Reflection symmetry: $\sigma_A = \sigma_B$

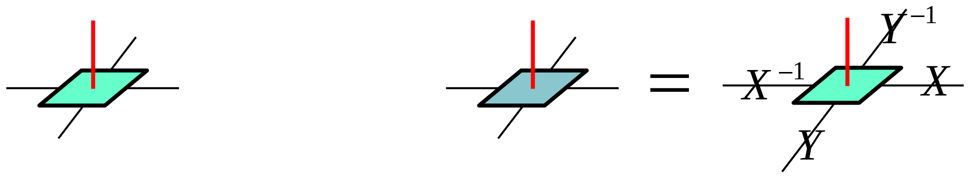
$$\sigma_{\partial A} = \sigma_A^2$$

with MPS algorithms $\infty \times \infty$

PEPS

SYMMETRIES

- Gauge :



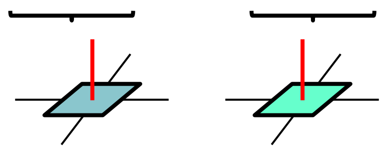
$A^{i_n}_{\alpha\beta\gamma\delta}$ $B^{i_n}_{\alpha\beta\gamma\delta}$

- Different tensors give rise to the same state:
- Under general conditions, the above is the only possibility

  (Perez-Garcia, Sanz, Gonzalez, Wolf, IC, 2009)

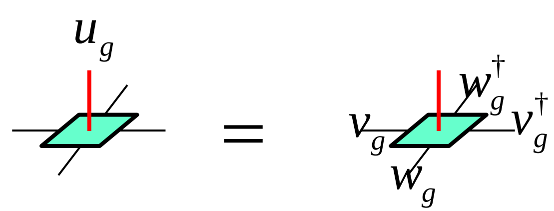
PEPS
SYMMETRIES

- Symmetry : $U_g \,|\Psi\rangle = e^{i\theta_g}\,|\Psi\rangle$



must be related by a local Gauge trafo

  - Global symmetry: $U_g = u_g^{\otimes N}$

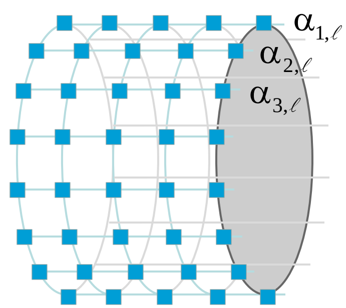


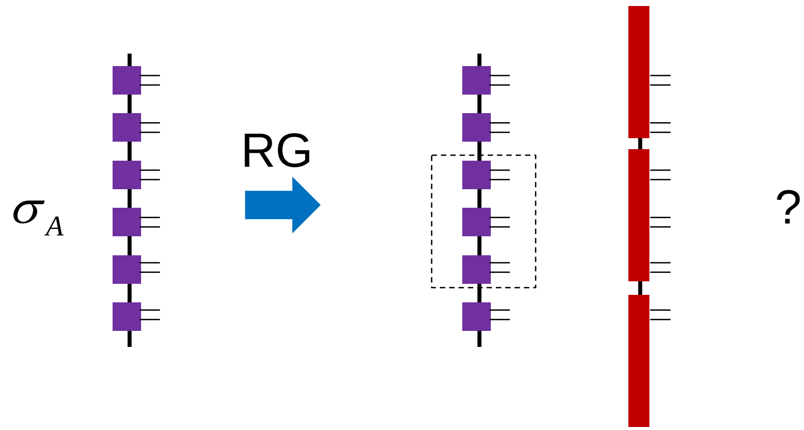
where the v's and w's are (projective) representations of the same group

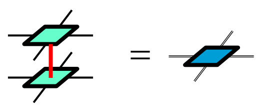
  - Boundary operator has the same symmetry: $\sigma_{\partial A} = v_g^{\otimes \partial A}\sigma_{\partial A} v_g^{\dagger \otimes \partial A}$

PEPS
FINITE CORRELATION LENGTH

- Cylinder :





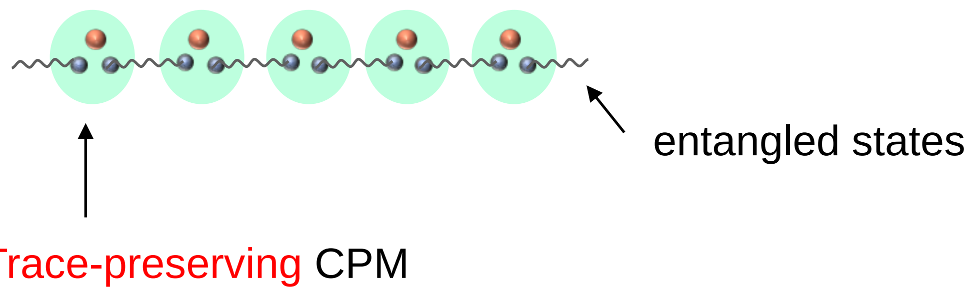


- For pure states (MPS): full classification

  (Verstraete, IC, Latorre, Rico, Wolf, 2005)

PEPS
FINITE CORRELATION LENGTH

- RG for mixed states MPDO :



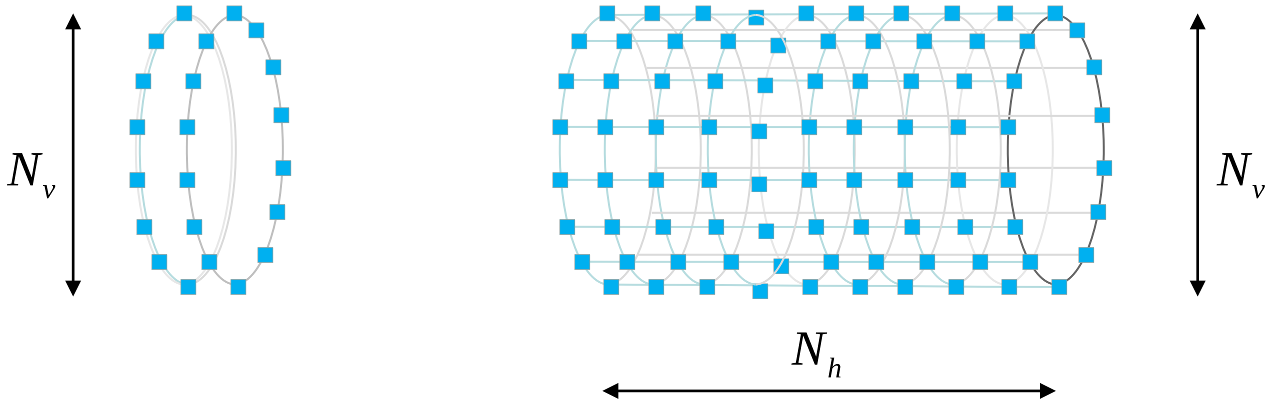
- Boundary theory: $\sigma_{\partial A} = \oplus e^{-\sum h_{n,n+1}}$

Local Hamiltonian

Degeneracy and topology

PEPS
EXAMPLES

- 2D AKLT in a 2-leg ladder/square lattice:



- spin $S=2$
- Deformed AKLT Hamiltonian $H = \sum_{<n,m>} Q_n(\Delta)Q_m(\Delta)P_{n,m}Q_m(\Delta)Q_n(\Delta)$
- Symmetry: $su(2)/u(1)$

nematic deformation

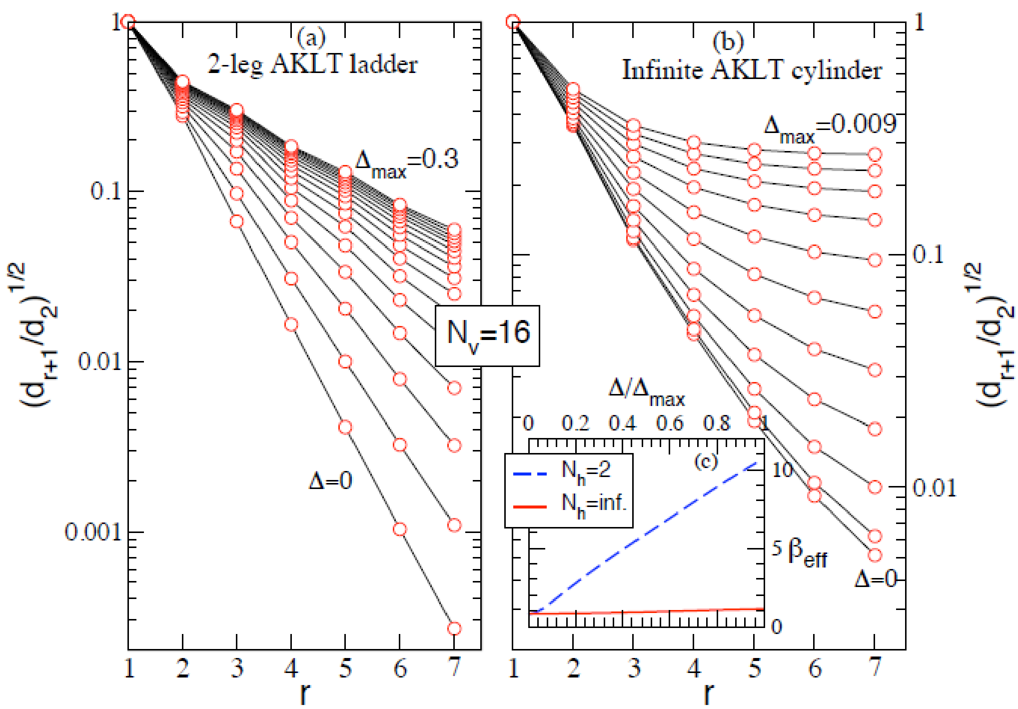
$Q(\Delta) = e^{-8\Delta S_z^2}$

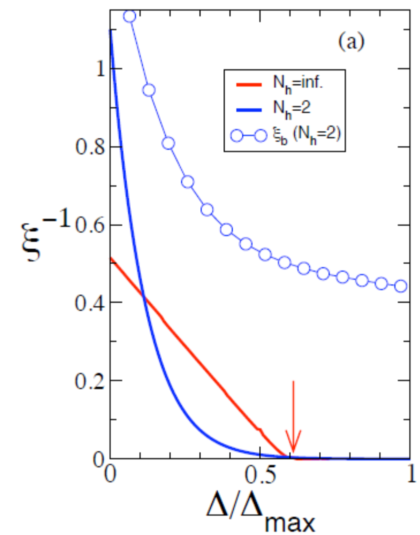
projector onto S=4 subspace

- Ground state: PEPS with D=2
- Boundary Hamiltonian: $H_{\partial A} = \sum_r d_r \sum$ all possible terms with range-r interacctions

PEPS
EXAMPLES

- 2D AKLT in a 2-leg ladder/square lattice:

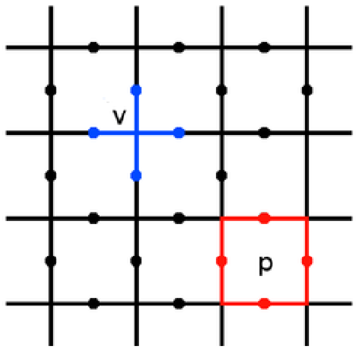




- For AKLT the boundary Hamiltonian is s=1/2 Heisenberg
- Similar results with other models

PEPS
EXAMPLES

- Kitaev toric code:



- Degenerate ground state. Gapped.
- Symmetry: Z_2
- Ground state: PEPS with D=2

- Boundary state

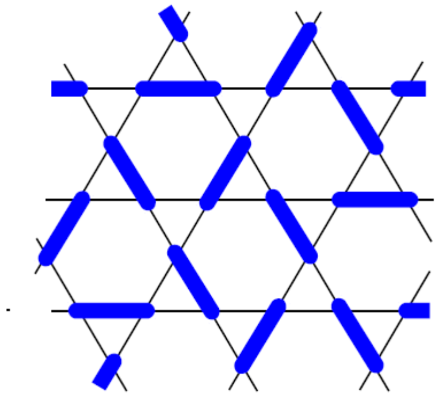
$$\sigma_{\partial A} = P_{even} \oplus P_{odd}$$

Non-local operator

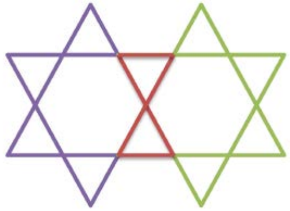
- Boundary Hamiltonian: trivial (up to the projector)

PEPS
EXAMPLES

- 2D RVB on a Kagome lattice:



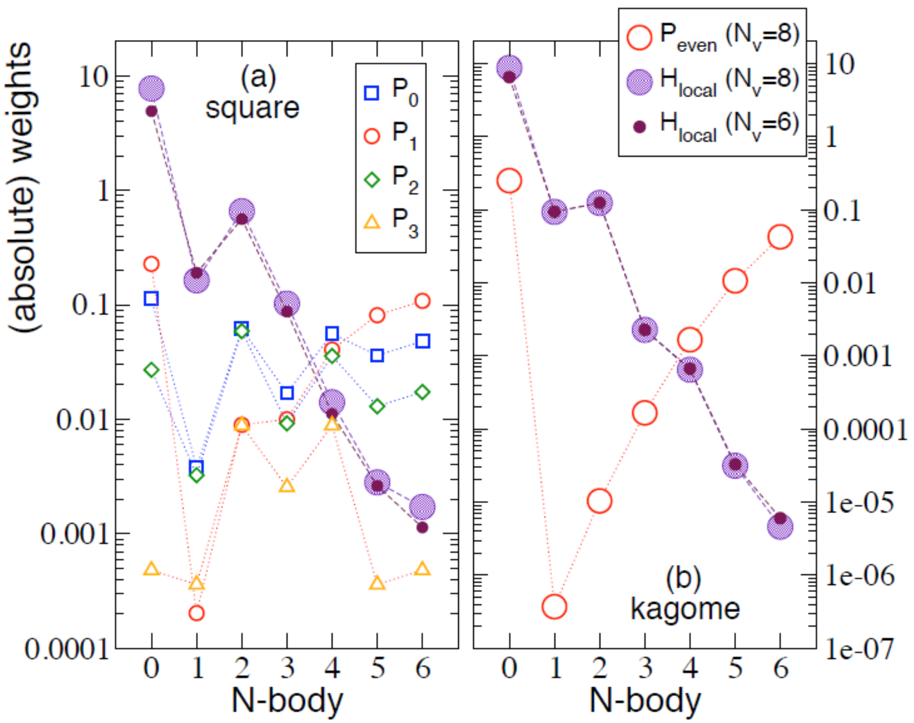
1. single spin ½ at each edge
   - Parent Hamiltonian acting on two stars



   - PEPS with D=3
   - $su(2)$ $\frac{1}{2}\oplus 0$ representation
   - Boundary Hamiltonian: t-J model

2. Three spins ½ at each edge: dimers are orthogonal (related to KR model)

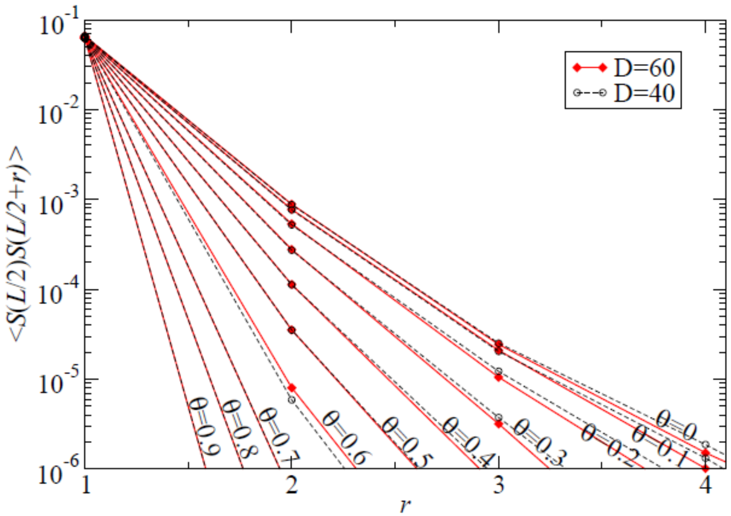
- 2D RVB on a Kagome lattice:
    1. single spin ½ at each edge



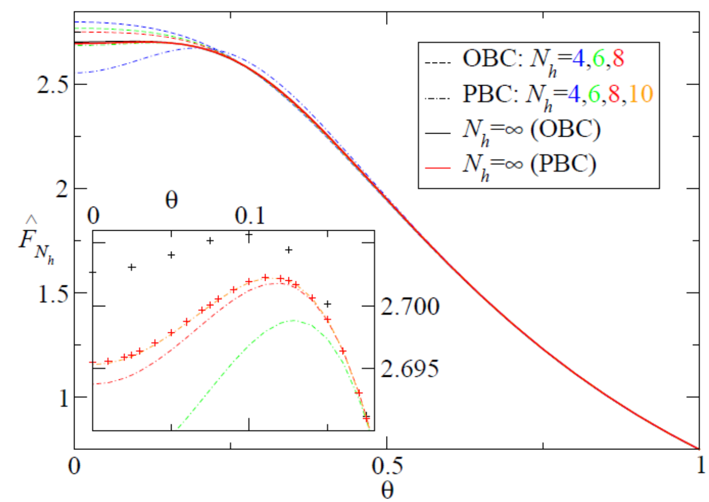
- 2D: interpolation RVB-oRVB: $|\Psi(\theta)\rangle$

$$|\Psi(0)\rangle = |RVB\rangle \qquad |\Psi(1)\rangle = |oRVB\rangle$$

correlation function



fidelity



RVB and toric code seem to be in the same phase

EXAMPLES

- ‚Uncle‘ Hamiltonians

  Fernandez, Schuch, Wolf, IC, Perez-Garcia, arXiv:1111.5817

- An order parameter for gapped phases in 1D

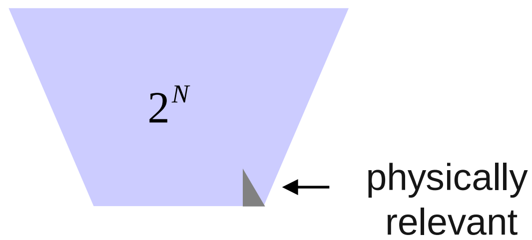
  Haegeman, Perez-Garcia, IC, Schuch, arXiv:1201.4174

$$o = \langle \Psi \,|\, (u_g^{\otimes N_1} \otimes u_g^{\otimes N_2} \otimes 1^{\otimes N_3})\mathbf{F}_{13}(1^{\otimes N_1} \otimes u_h^{\otimes N_2} \otimes u_h^{\otimes N_1}) \,|\, \Psi \rangle$$

- Parent Hamiltonian for Laughlin spin state in a lattice

  Anne Nielse, IC, German Sierra, arXiv:1201.3096 ➡ poster

SUMMARY and CONCLUSIONS



CONCLUSION:

- Thermal equilibrium and local interaction spins can be efficiently described by PEPS
  - Numerical algorithms
  - New perspective

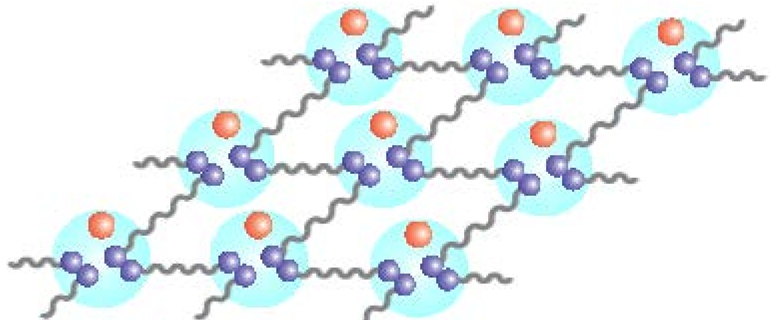
HERE:

- Area law: bulk-boundary correspondence
- Boundary reflects properties of the bulk: criticality, topology, etc
- Finite correlation length implies locality of boundary Hamiltonian
- Locality + symmetries dictate entanglement spectrum

Applicaton: contraction of PEPS is efficient

TENSOR NETWORKS
PROJECTED ENTANGLED-PAIR STATES

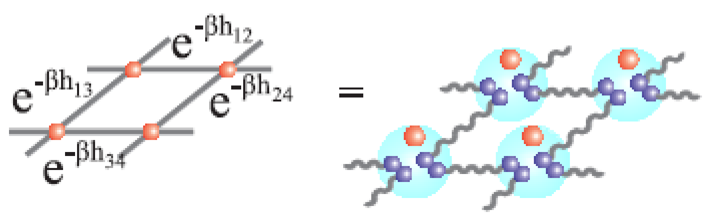
- Physical interpretation:



$$P = \sum A^{i}_{\alpha\beta\gamma\delta} \,|\, i\rangle\langle \alpha,\beta,\gamma,\delta \,|$$

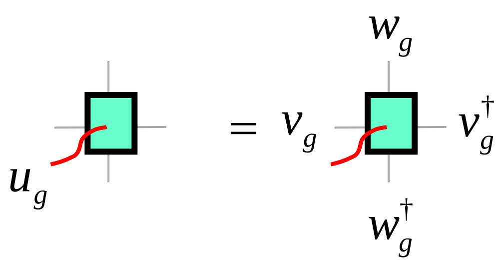
- Why do they provide efficient descriptions? $H = \sum_{<n,m>} h_{n,m}$

$$|\Psi_0\rangle \prec \lim_{\beta\to\infty} e^{-\beta H}\,|\varphi\rangle^{\otimes N} = \left(\prod e^{-\frac{\beta}{M} h_{n,m}} \right)^{M} |\varphi\rangle^{\otimes N}$$

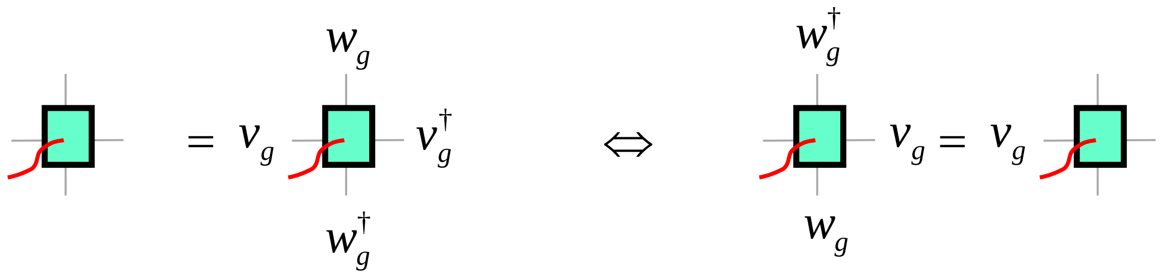


TOPOLOGICAL PHASES

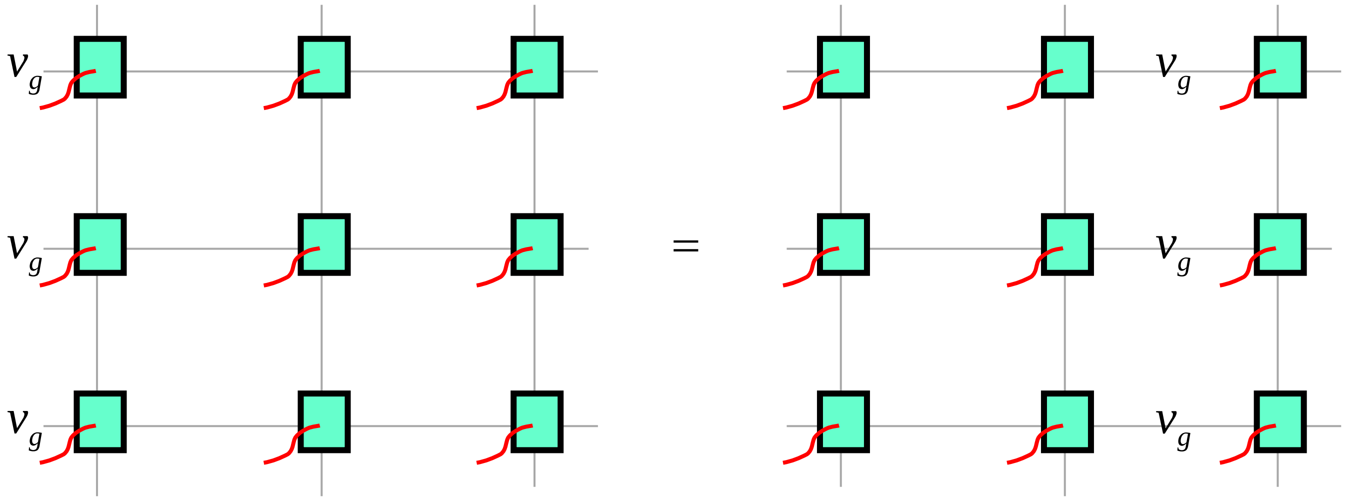
- Symmetries:

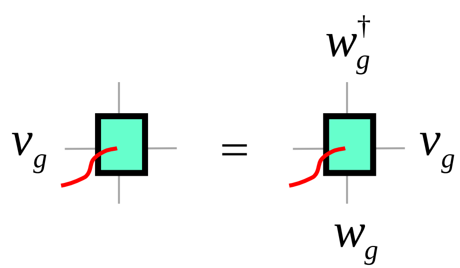


- Gauge symmetries:



- Wilson strings:

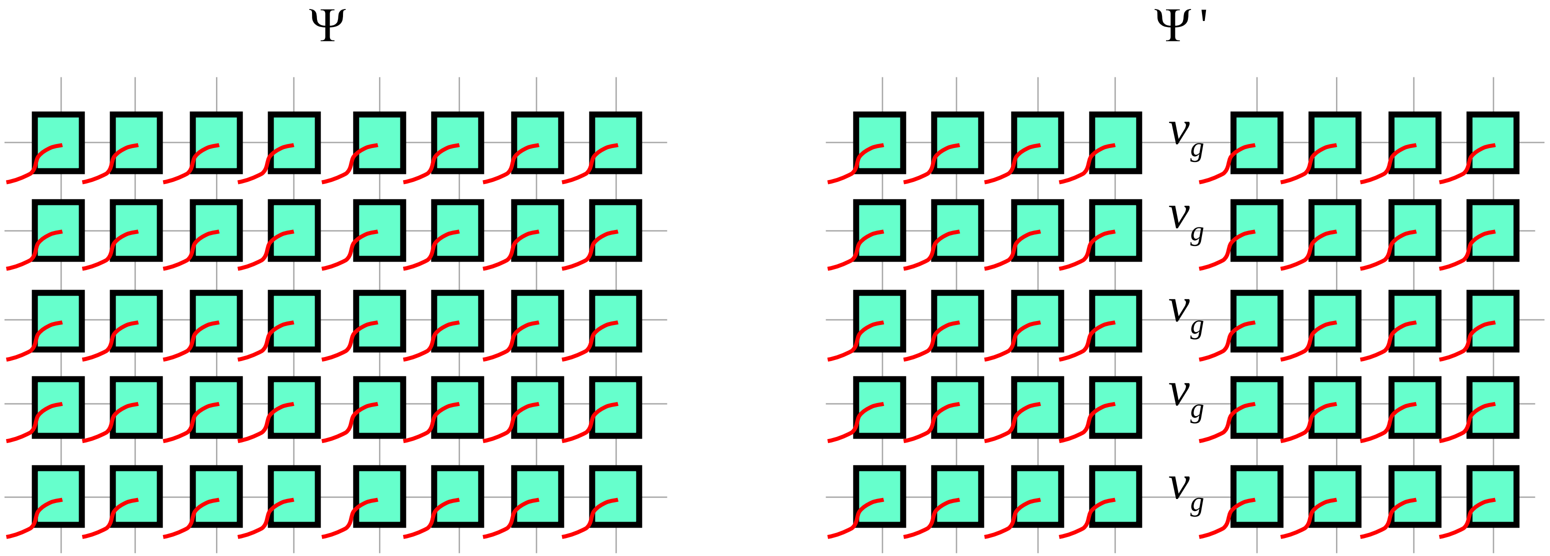




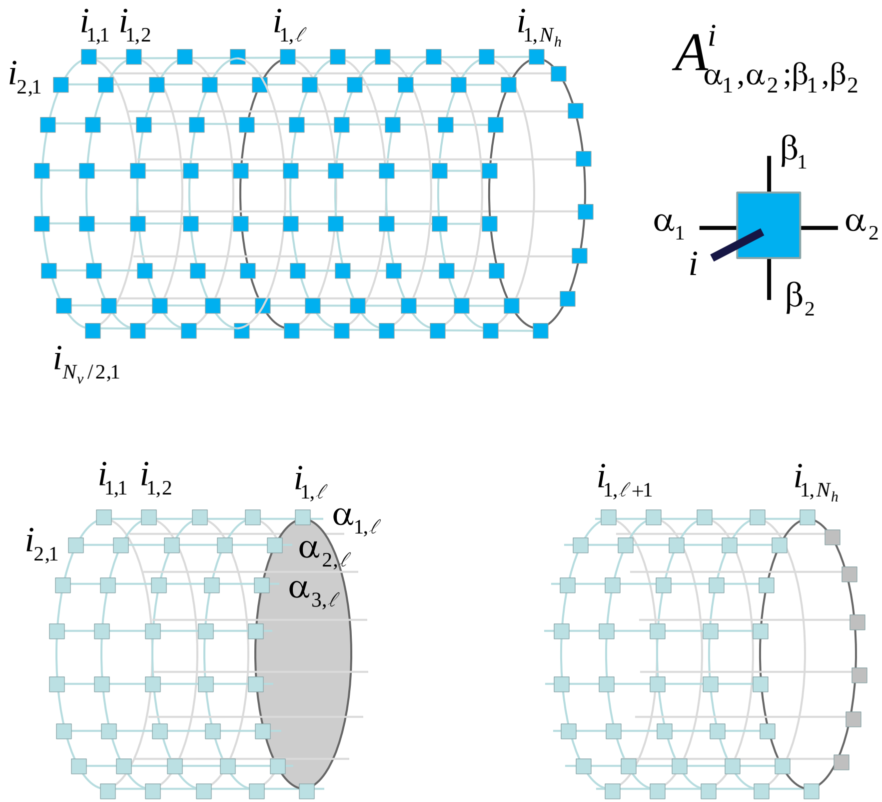
they can be moved to any column

TOPOLOGICAL PHASES

- Ground state degeneracy:



  - Are locally indistinguishable.
  - Any Hamiltonian for which one is the ground state is degenerate.



$i_{1,1}$ $i_{1,2}$ $i_{1,\ell}$ i_{1,N_h}
$i_{2,1}$
$i_{N_v/2,1}$
$A^{i}_{\alpha_1,\alpha_2;\beta_1,\beta_2}$
β_1
α_1
α_2
i
β_2
$i_{1,1}$ $i_{1,2}$ $i_{1,\ell}$
$i_{2,1}$
$\alpha_{1,\ell}$
$\alpha_{2,\ell}$
$\alpha_{3,\ell}$
$i_{1,\ell+1}$ i_{1,N_h}