Detection of symmetry protected topological phases in 1D

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Overview

- Introduction: Symmetry protected topological phases
- Non-local order parameters
- Summary
Symmetry protected topological phases

- **Quantum phases**: Two gapped quantum states belong to the same phase if they are adiabatically connected.

- Phases in condensed matter are usually understood using **local order parameters** ("symmetry breaking").
  - **Magnets**: spin rotation and TR symmetry broken. Magnetization as order parameter.

- **Topological phases** not characterized by any symmetry breaking.

- We introduce **non-local order parameter** for symmetry protected topological phases in 1D.
Symmetry protected topological phases

- Example: **Spin-1 chain** [Haldane '83]

\[
H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D \sum_j (S_j^z)^2
\]
(time reversal, inversion, \(Z_2 \times Z_2\) symmetry, ...)

- **Hidden** \(Z_2 \times Z_2\) symmetry breaking [Kennedy-Tasaki '92]

- **String order parameter** [den Nijs '89]
Symmetry protected topological phases

- Spin-1 chain with less symmetries [Gu et al. ‘09]
  \[ H = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D \sum_j (S_j^z)^2 + B_x \sum_j S_j^x \]

\[ \rightarrow \text{no } Z_2 \times Z_2 \text{ symmetry} \]

\[ \rightarrow \text{Haldane phase still well defined} \]

Which symmetries are required?
How to detect “topological” phases?

\[ \rightarrow \text{Idea: Use entanglement and matrix-product states} \text{ (capturing non-local properties)} \]
Symmetry protected topological phases

**Schmidt decomposition (SVD \( C = UDV^{\dagger} \))**

- Decompose a state \( |\psi\rangle \) into a superposition of product states:

\[
|\psi\rangle = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} C_{ij} |i\rangle_A |j\rangle_B = \sum_{\gamma} \lambda_\gamma |\phi_\gamma\rangle_A |\phi_\gamma\rangle_B
\]

- **Schmidt states:** \( |\phi_\gamma\rangle \), **Schmidt values:** \( \lambda_\gamma \)
- \( |\phi_\gamma\rangle \) are eigenstates of the reduced density matrix

\[
\rho_A = \text{Tr}_B |\psi\rangle \langle \psi| \quad \text{with} \quad \rho_A |\phi_\gamma\rangle_A = \lambda_\gamma^2 |\phi_\gamma\rangle_A
\]
Symmetry protected topological phases

- Example: Spin-1 Heisenberg chain \( H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} \)

\[
|\psi_0\rangle = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} C_{ij} |i\rangle_A |j\rangle_B
= \sum_{\gamma} \lambda_{\gamma} |\phi_{\gamma}\rangle_A |\phi_{\gamma}\rangle_B
\left( \sum_{\gamma} \lambda_{\gamma}^2 = 1 \right)
\]

- Schmidt values decay rapidly in ground states of gapped, local Hamiltonians (area law! [Hastings et al. ’07]): Matrix-Product representation
Symmetry protected topological phases

- **Matrix product state (MPS) representation**

\[
|\Psi\rangle = \sum_{j_1, \ldots, j_L} B^T A_{j_1} \cdots A_{j_L} B |j_1, \ldots, j_L\rangle
\]

- **Matrices not uniquely defined:** **Canonical Form**
  is directly related to the Schmidt decomposition: \( A_j = \Gamma_j \Lambda \)

[Vidal '02]
Symmetry protected topological phases

- Matrices are directly related to the Schmidt decomposition

\[
[\phi_\alpha]_{j_1,j_2\ldots} = \alpha \begin{array}{cccccccc}
\Gamma & \Lambda & \Gamma & \Lambda & \Gamma & \Lambda \\
\ast & & \ast & & \ast & \\
\end{array} \ldots \begin{array}{c}
A \hspace{1cm} B
\end{array}
\]

- Left/right **transfer matrices** $T$ have largest eigenvalue one with the identity as corresponding eigenstate

\[
T^\Sigma_{(\alpha\alpha')};(\beta\beta') = 1
\]
Symmetry protected topological phases

- Transformation of an MPS under symmetry operations [Perez-Garcia ’07]

\[
\tilde{\Gamma} = e^{i\theta} \cdot U^\dagger \Sigma \Gamma U\Sigma, \quad [U\Sigma, \Lambda] = 0
\]

...wave function only changes by a phase

- Time reversal \((\Gamma_j \rightarrow \Gamma^*_j)\) and inversion \((\Gamma_j \rightarrow \Gamma^{T}_j)\)

- Matrices \(U\Sigma\) are projective representations which tell us about topological phases [FP et al. ’10, Chen et al ’11]
Symmetry protected topological phases

Use projective representations to classify phases!

• Ground state $|\psi_0\rangle$ is invariant under a symmetry group $G$ with elements $g_1, g_2, \ldots, g_n$

• **Projective representation** $U_{g_j}$ of the symmetry group $g_j g_k = g_l : U_{g_j} U_{g_k} = e^{i\phi_{jk}} U_{g_l}$
Symmetry protected topological phases

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- Phase ambiguities classify the phases (Schur classes)

  ➡ Complete classification of topological phases in 1D [FP, A. Turner, E. Berg, M. Oshikawa ’10, Chen et al ’11]
Symmetry protected topological phases

- **Which symmetries stabilize topological phases?**

- **Example** $\mathbb{Z}_n$ : Rotation about single axis
  \[ R^n = 1 \Rightarrow U_R^n = e^{i\phi} 1 \]
  ➞ Redefining $\tilde{U}_R = e^{-i\phi/n} U_R$ removes the phase

- **Example** $\mathbb{Z}_2 \times \mathbb{Z}_2$ : Phase for pairs
  \[ R_x R_y = R_y R_x \Rightarrow U_{R_x} U_{R_y} = e^{i\phi_{xy}} U_{R_y} U_{R_x} \]
  ➞ Phases $\phi = 0, \pi$ cannot be gauged away: topological phases
Symmetry protected topological phases

- **Example** $S=1$ AKLT state  \[\text{[Affleck '87]}\]

\[|\psi\rangle = \begin{array}{c}
\Gamma \Lambda \Gamma \Lambda \Gamma \Lambda \Gamma \Lambda \Gamma \\
\end{array}\]

- Matrix-product state representation

\[\Gamma^i = \sigma_i, \ i = x, y, z\]

- Rotations $\mathbb{Z}_2 \times \mathbb{Z}_2$ represented by Pauli matrices and thus $U_{Rx} U_{Ry} = -U_{Ry} U_{Rx}$

- Inversion symmetry with $U_\mathcal{I} = \sigma_y : \ U_\mathcal{I} U_\mathcal{I}^* = -1$

- Time reversal with $U_{\mathcal{T\mathcal{R}}} = \sigma_y : \ U_{\mathcal{T\mathcal{R}}} U_{\mathcal{T\mathcal{R}}}^* = -1$
Symmetry protected topological phases

- Framework to classify topological phases in 1D by looking at the “entanglement states” / MPS

- “Topological” phase in a $S=1$ chain protected by
  - $\mathbb{Z}_2 \times \mathbb{Z}_2$
  - Inversion symmetry
  - Time reversal symmetry


- Symmetry protected topological phases exist only in the presence of certain symmetries (not topologically ordered!)
Symmetry protected topological phases

• How can we detect which phase a given state belongs to?

• We discuss two ways to detect topological phases:
  (1) Directly extract the projective representations from a matrix-product state representation (very useful for iTEBD [Vidal ’07] / iDMRG [McCulloch ’08])
  (2) Non-local order parameters for inversion, and time reversal symmetry and a generalized string-order for internal symmetries
Non-local order parameter (1)

- Get $U_\Sigma$ from the **dominant eigenvector** $X$ of the **generalized transfermatrix** ($U_\Sigma = X^\dagger$)

$$T^\Sigma_{(\alpha\alpha';(\beta\beta')} = \sum_{j,j'} \frac{\alpha_\beta}{j,j'} \tilde{\Gamma}_{j,j',\alpha\beta} \Gamma^*_{j,\alpha'\beta'} \Lambda_{\beta} \Lambda_{\beta'}$$

- Overlap with transformed Schmidt states

$$T^\Sigma X = X \quad U^\dagger = e^{i\mu} \quad \Leftrightarrow \quad U_\Sigma = X^\dagger$$
Non-local order parameter (1)

- S=1 chain

\[ H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D \sum_j (S^z_j)^2 + B \sum_j S^x_j \]

- $\mathbb{Z}_2 \times \mathbb{Z}_2$ stabilizes Haldane phase if $B = 0$

and $O_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \begin{cases} 0 & \text{if symmetry broken} \\ \frac{1}{\chi} \text{tr} \left( U_x U_z U_x^\dagger U_z^\dagger \right) & \text{if symmetry not broken} \end{cases}$.

iMPS obtained using the iTEBD / iDRMGG method
Non-local order parameter (1)

- S=1 chain

\[ H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D \sum_j (S^z_j)^2 + B \sum_j S^x \]

- **Inversion symmetry** stabilizes Haldane phase if \( B \neq 0 \)

and \( O_I = \begin{cases} 0 & \text{if symmetry broken} \\ \frac{1}{\chi} \text{tr} (U_I U_I^*) & \text{if symmetry not broken} \end{cases} \).

**Non-local order parameter (1)**

\[ iMPS \text{ obtained using the iTEBD / iDRMG method} \]
Non-local order parameter (2)

• What if we do not have access to the transfer matrix (i.e., Monte Carlo or experiments)?

• **Inversion** symmetry: Inverting part of the wave function

\[
\langle \psi | \mathcal{I}_{1,2n} | \psi \rangle =
\]

...
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\[
\langle \Psi | \mathcal{I}_{1,2n} | \Psi \rangle = \Lambda \Lambda^2 = \text{Tr}(U_i^* U_i \Lambda^4)
\]
Non-local order parameter (2)

• What if we do not have access to the transfermatrix (i.e., Monte Carlo or experiments)?

• **Inversion** symmetry: Inverting part of the wave function

\[ S_L = \]

\[ S_L/\text{tr}(A^4) \]

\[ B=0.3 \]

• Distinguishes the Haldane phase from the trivial phase in presence of inversion symmetry
Non-local order parameter (2)

- **Time reversal** symmetry: two copies of the wave function with swapping operators [Isakov et al.'11] and \( \Sigma = e^{-i\pi S^y} \)

\[
S_{TR} = \langle R_{1n}|\text{Swap}_{n+1,2n}|\psi_2\rangle \quad \langle \psi_2|R_{1n}\rangle
\]

\[
\Sigma = e^{i\theta} \cdot U^+ \cdot \Gamma \cdot U
\]
Non-local order parameter (2)

- **Time reversal** symmetry: two copies of the wave function with swapping operators [Isakov et al '11] and \( \Sigma = e^{-i\pi S^y} \)

\[
S_{TR} = \sum = \text{Tr}(U_{ir}U_{ir}^{\dagger})\text{Tr}(\Lambda^4)^2
\]
Non-local order parameter (2)

- **General internal symmetries** symmetry: multiple copies of the wave function

- Example: $S_G = \Tr(U_a U_b U_a^{-1} U_b^{-1})$

[See also: Haegemann et al.: arXiv:1201.4174]
Non-local order parameter (2)

- If phases are topological, where is the torus??

- Express wave function as partition function on the half plane:

- Sandwich of a string operator:
Non-local order parameter (2)

- If phases are topological, where is the torus??
- Sandwich the symmetry string operators $a / b$, then deform them and finally glue them together
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- If phases are topological, where is the torus??
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$$S_G = \text{Tr} \left( U_a U_b U_a^{-1} U_b^{-1} \right)$$
Summary

- Derivation of **non-local order parameters** which can be used to **detect/distinguish all symmetry protected topological phases in 1D**
  - Can be obtained directly from a generalized transfer matrix
  - Expressions which can be evaluated using any numerical methods, e.g., Quantum Monte Carlo

- Measuring string order experimentally: High-resolution imaging of low-dimensional quantum gases
  
  [M. Endres et al ’11]

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