Detection of symmetry protected topological phases in 1D

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FP. and A. M. Turner, arxiv:1204.0704

Florence, May. 29, 2012

Detection of symmetry protected topological phases in 1D

Overview

- Introduction: Symmetry protected topological phases
- Non-local order parameters
- Summary

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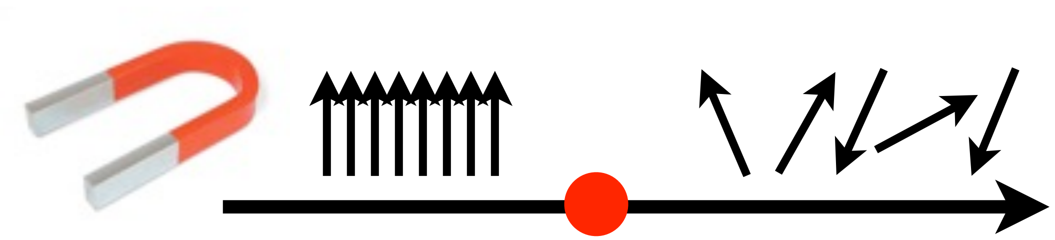
Symmetry protected topological phases

- **Quantum phases**: Two gapped quantum states belong to the same phase if they are adiabatically connected
- Phases in condensed matter are usually understood using **local order parameters** (“symmetry breaking”)
  - **Magnets**: spin rotation and TR symmetry broken
    ➡ Magnetization as order parameter

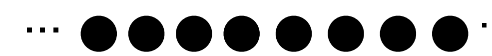


- **Topological phases not characterized by any symmetry breaking**
- We introduce **non-local order parameter for symmetry protected topological phases** in 1D

Symmetry protected topological phases

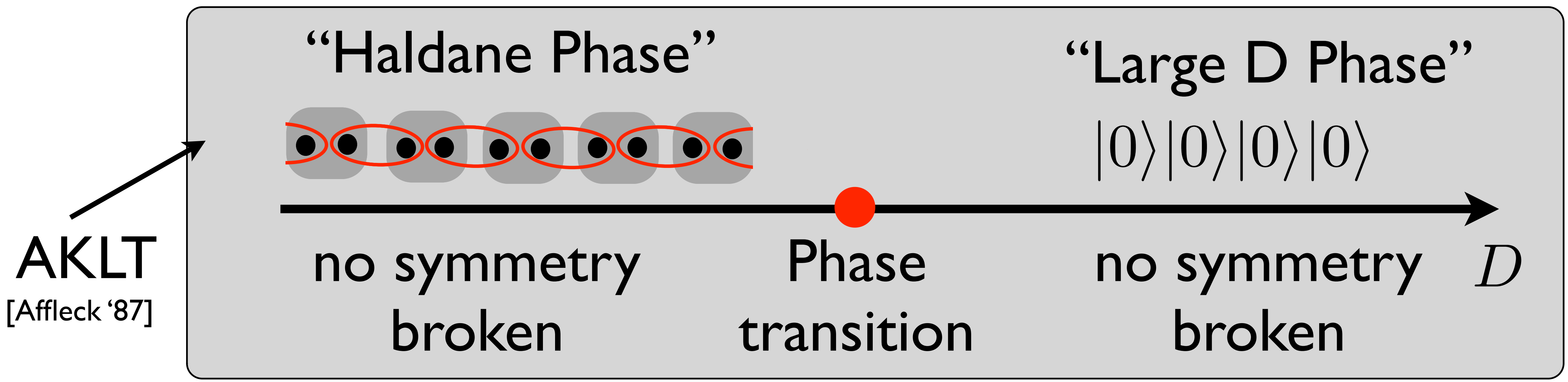
- Example: **Spin-1 chain** [Haldane '83]

$$H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D \sum_j (S_j^z)^2$$



$|S_z = \pm 1\rangle, |S_z = 0\rangle$

(time reversal, inversion, $Z_2 \times Z_2$ symmetry, ...)

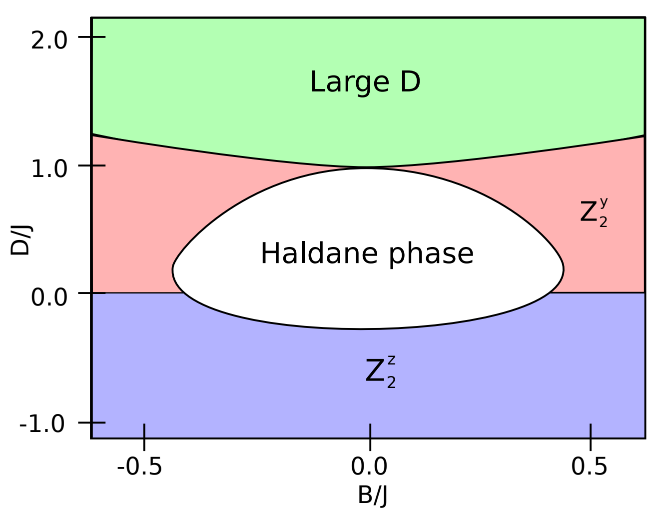


- *Hidden* $Z_2 \times Z_2$ symmetry breaking [Kennedy-Tasaki '92]
- String order parameter [den Nijs '89]

Symmetry protected topological phases

- **Spin-1 chain** with **less symmetries** [Gu et al. '09]

$$H = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D\sum_j (S_j^z)^2 + B_x \sum_j S_j^x$$



➡ **no** $Z_2 \times Z_2$ **symmetry**

➡ Haldane phase still well defined

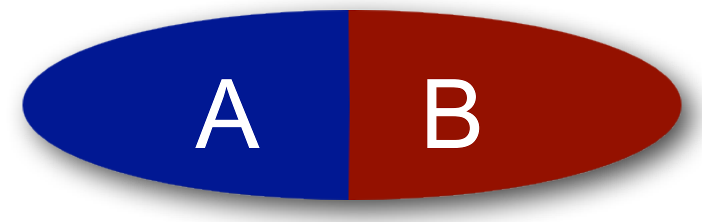
Which symmetries are required?
How to detect "topological" phases?

➡ **Idea: Use entanglement and matrix-product states** (capturing non-local properties)

Symmetry protected topological phases

Schmidt decomposition (SVD $C = UDV^\dagger$)

- Decompose a state $|\psi\rangle$ into a superposition of product states:



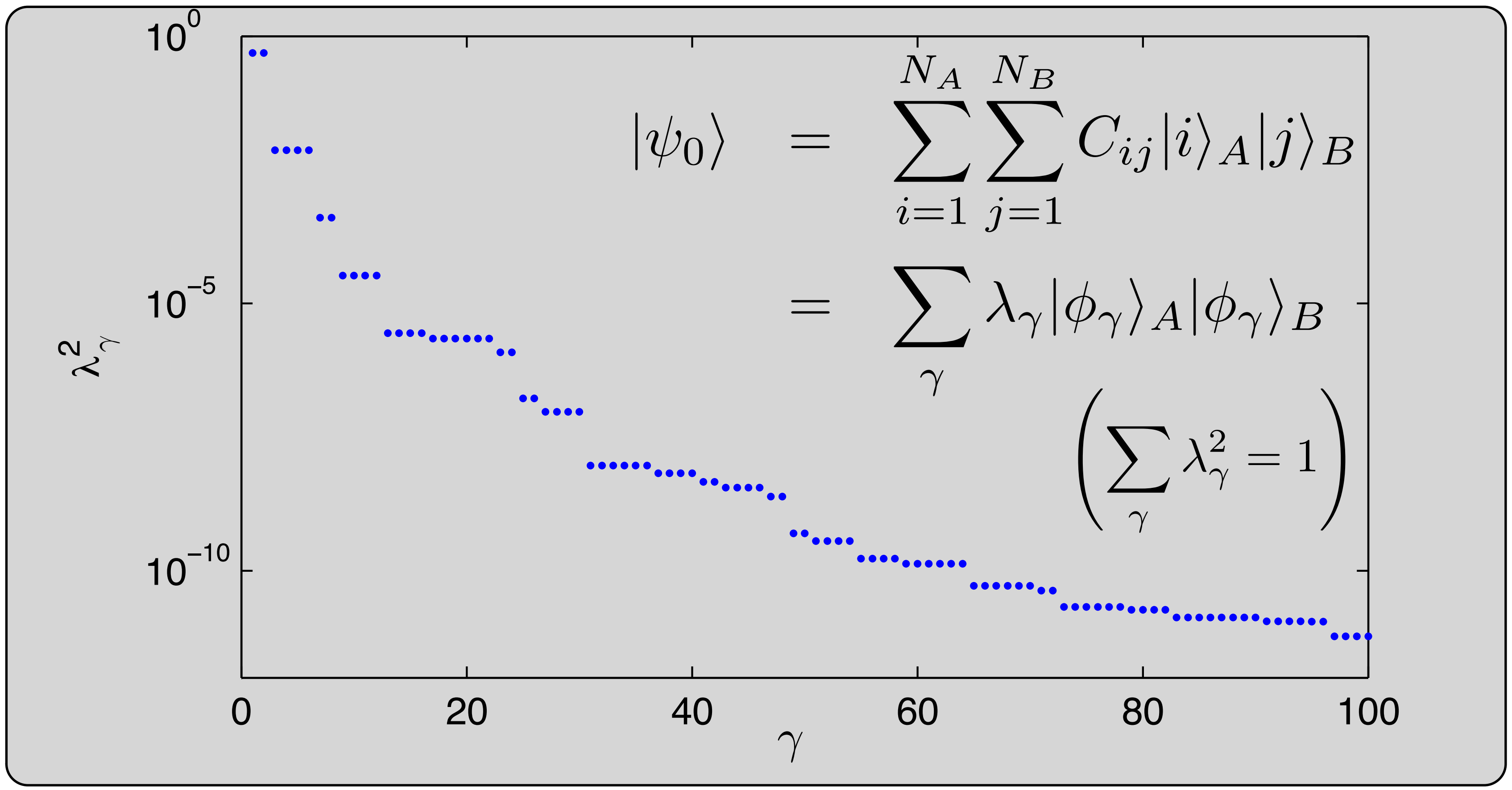
$$|\psi\rangle \quad = \quad \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} C_{ij} |i\rangle_A |j\rangle_B = \sum_{\gamma} \lambda_\gamma |\phi_\gamma\rangle_A |\phi_\gamma\rangle_B$$

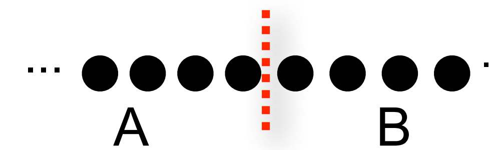
- **Schmidt states:** $|\phi_\gamma\rangle$, **Schmidt values:** λ_γ
- $|\phi_\gamma\rangle$ are eigenstates of the reduced density matrix

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$ with $$\rho_A |\phi_\gamma\rangle_A = \lambda_\gamma^2 |\phi_\gamma\rangle_A$$

Symmetry protected topological phases

- Example: Spin-1 Heisenberg chain $H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$





- **Schmidt values decay rapidly** in ground states of gapped, local Hamiltonians (**area law!** [Hastings et al. '07]): **Matrix-Product representation**

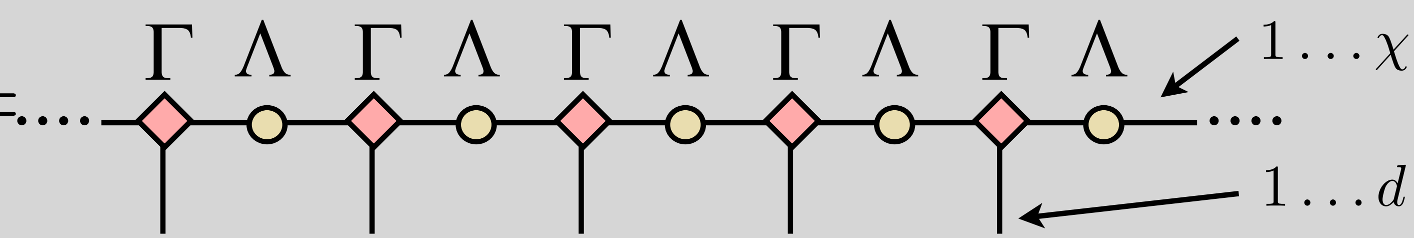
Symmetry protected topological phases

- **Matrix product state (MPS)** representation

$$|\Psi\rangle = \sum_{j_1,\ldots,j_L} \underbrace{B^T A_{j_1} \ldots A_{j_L} B}_{\psi_{j_1,\ldots,j_L}} |j_1,\ldots,j_L\rangle$$

- Matrices not uniquely defined: **Canonical Form** is directly related to the Schmidt decomposition: $A_j = \Gamma_j \Lambda$
[Vidal '02]

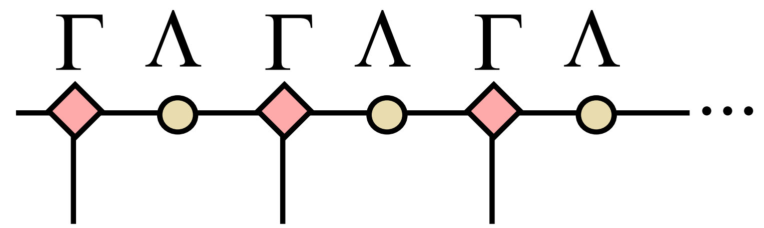
$$\psi_{\ldots,j_1,j_2,\ldots} = \ldots$$

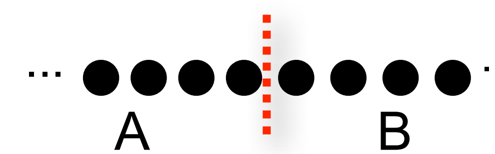


Symmetry protected topological phases

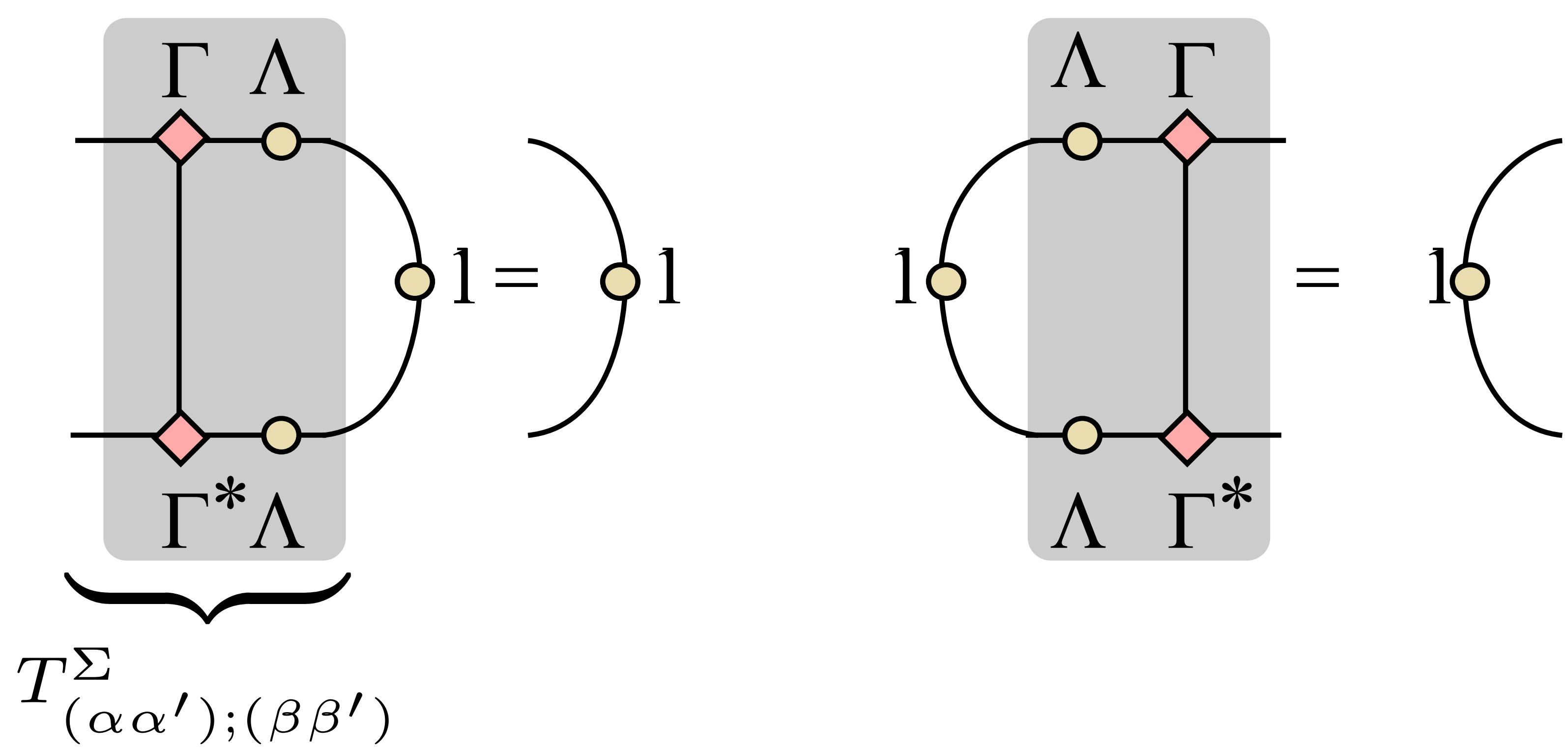
- Matrices are directly related to the Schmidt decomposition

$$[\phi_\alpha]_{j_1,j_2\ldots} = \alpha$$





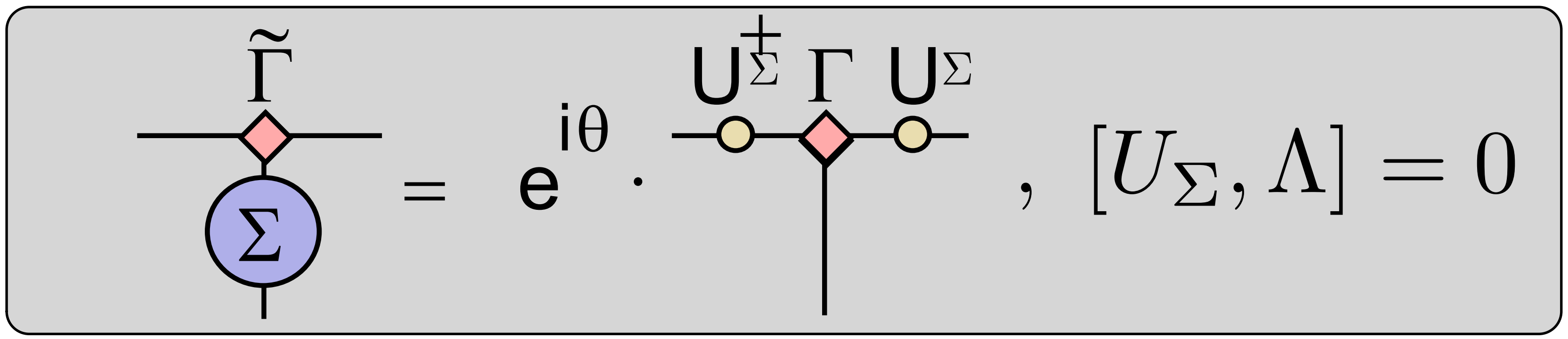
- Left/right **transfer matrices** T have largest eigenvalue one with the identity as corresponding eigenstate

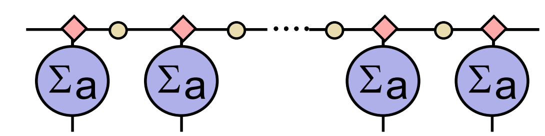


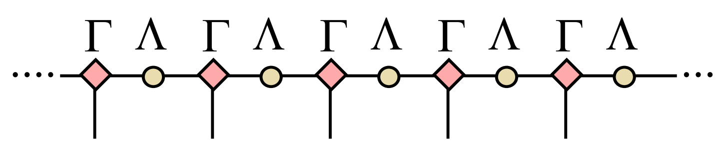
Symmetry protected topological phases

- Transformation of an MPS under **symmetry operations**

[Perez-Garcia '07]





...wave function 

only changes by a phase

- Time reversal ($\Gamma_j \to \Gamma_j^*$) and inversion ($\Gamma_j \to \Gamma_j^T$)

- Matrices U_Σ are **projective representations** which tell us about **topological phases** [FP et al. '10, Chen et al '11]

Symmetry protected topological phases

Use projective representations to classify phases!

- Ground state $|\psi_0\rangle$ is invariant under a symmetry group G with elements $g_1, g_2, \cdots, g_n$

- **Projective representation** U_{g_j} of the symmetry group

$$g_j g_k = g_l : \ U_{g_j} U_{g_k} = e^{i\phi_{jk}} U_{g_l}$$

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- **Phase ambiguities classify the phases** (Schur classes)

➡ **Complete classification of topological phases in 1D** [FP, A. Turner, E. Berg, M. Oshikawa '10, Chen et al '11]

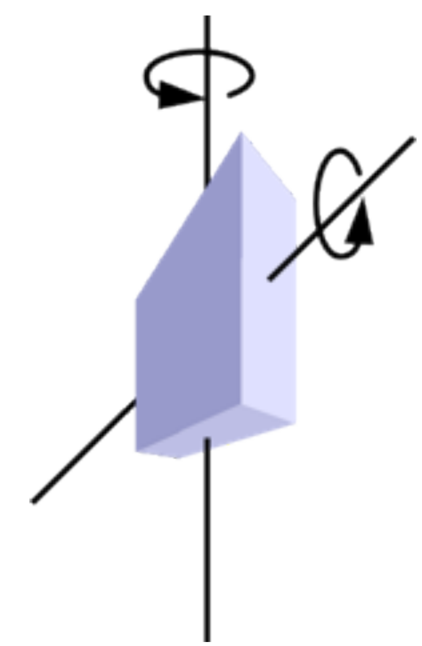
Symmetry protected topological phases

- **Which symmetries stabilize topological phases?**
- **Example** $\mathbb{Z}_n$: Rotation about single axis

$$R^n = \mathbb{1} \Rightarrow U_R^n = e^{i\phi}\mathbb{1}$$

➡ Redefining $\tilde{U}_R = e^{-i\phi/n} U_R$ removes the phase

- **Example** $\mathbb{Z}_2 \times \mathbb{Z}_2$: Phase for pairs

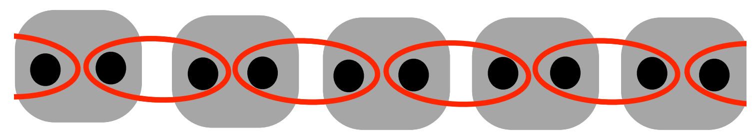


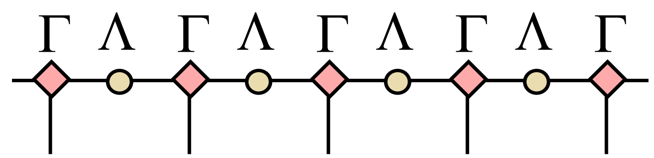
$$R_x R_y = R_y R_x \Rightarrow U_{R_x} U_{R_y} = e^{i\phi_{xy}} U_{R_y} U_{R_x}$$

➡ Phases $\phi = 0, \pi$ **cannot** be gauged away: **topological phases**

Symmetry protected topological phases

- **Example** S=1 AKLT state [Affleck '87]

➡ $|\psi\rangle =$ 



- Matrix-product state representation $\Gamma^i = \sigma_i, \;\; i = x, y, z$
- Rotations $\mathbb{Z}_2 \times \mathbb{Z}_2$ represented by Pauli matrices and thus $U_{R_x} U_{R_y} = -U_{R_y} U_{R_x}$
- Inversion symmetry with $U_{\mathcal{I}} = \sigma_y$: $\quad U_{\mathcal{I}} U_{\mathcal{I}}^* = -\mathbb{1}$
- Time reversal with $U_{\mathcal{TR}} = \sigma_y$: $\quad U_{\mathcal{TR}} U_{\mathcal{TR}}^* = -\mathbb{1}$

Symmetry protected topological phases

- Framework to classify topological phases in 1D by looking at the “entanglement states” / MPS

- **“Topological” phase** in a S=1 chain protected by
  - $\mathbb{Z}_2 \times \mathbb{Z}_2$
  - **Inversion symmetry**
  - **Time reversal symmetry**

  FP, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010)

- **Symmetry protected topological phases exist only in the presence of certain symmetries** (not topologically ordered!)

Symmetry protected topological phases

- **How can we detect which phase a given state belongs to?**

- **We discuss two ways to detect topological phases**:

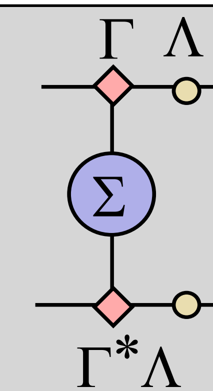
(1) Directly extract the projective representations from a matrix-product state representation (very useful for iTEBD [Vidal '07] / iDMRG [McCulloch '08])

(2) Non-local order parameters for inversion, and time reversal symmetry and a generalized string-order for internal symmetries

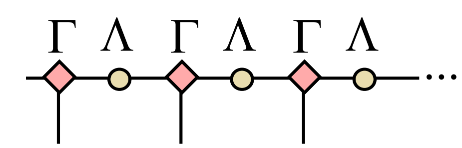
Non-local order parameter (I)

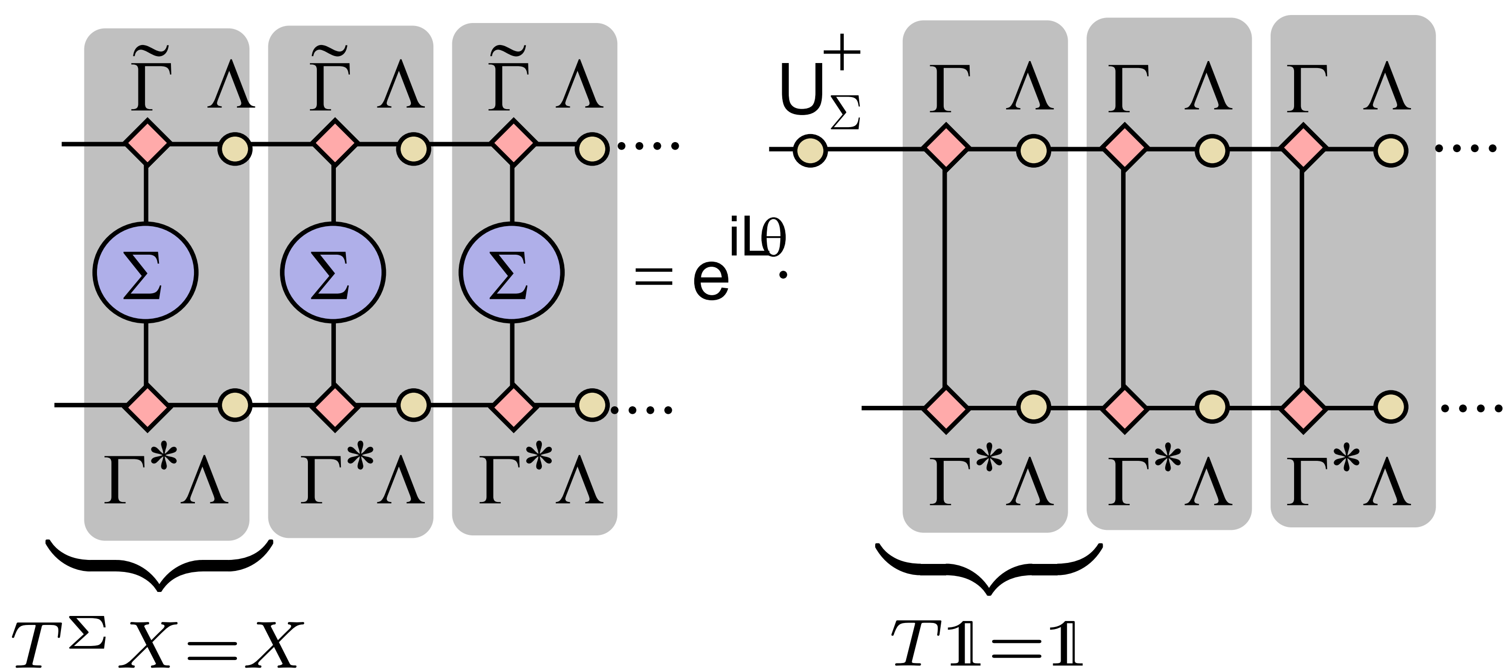
- Get U_Σ from the **dominant eigenvector** X of the **generalized transfermatrix** $(U_\Sigma = X^\dagger)$

$$T^{\Sigma}_{(\alpha\alpha');(\beta\beta')} = \sum_{j,j'} \Sigma_{jj'} \tilde{\Gamma}_{j',\alpha\beta} \Gamma^{*}_{j,\alpha'\beta'} \Lambda_{\beta} \Lambda_{\beta'}$$



- Overlap with transformed Schmidt states





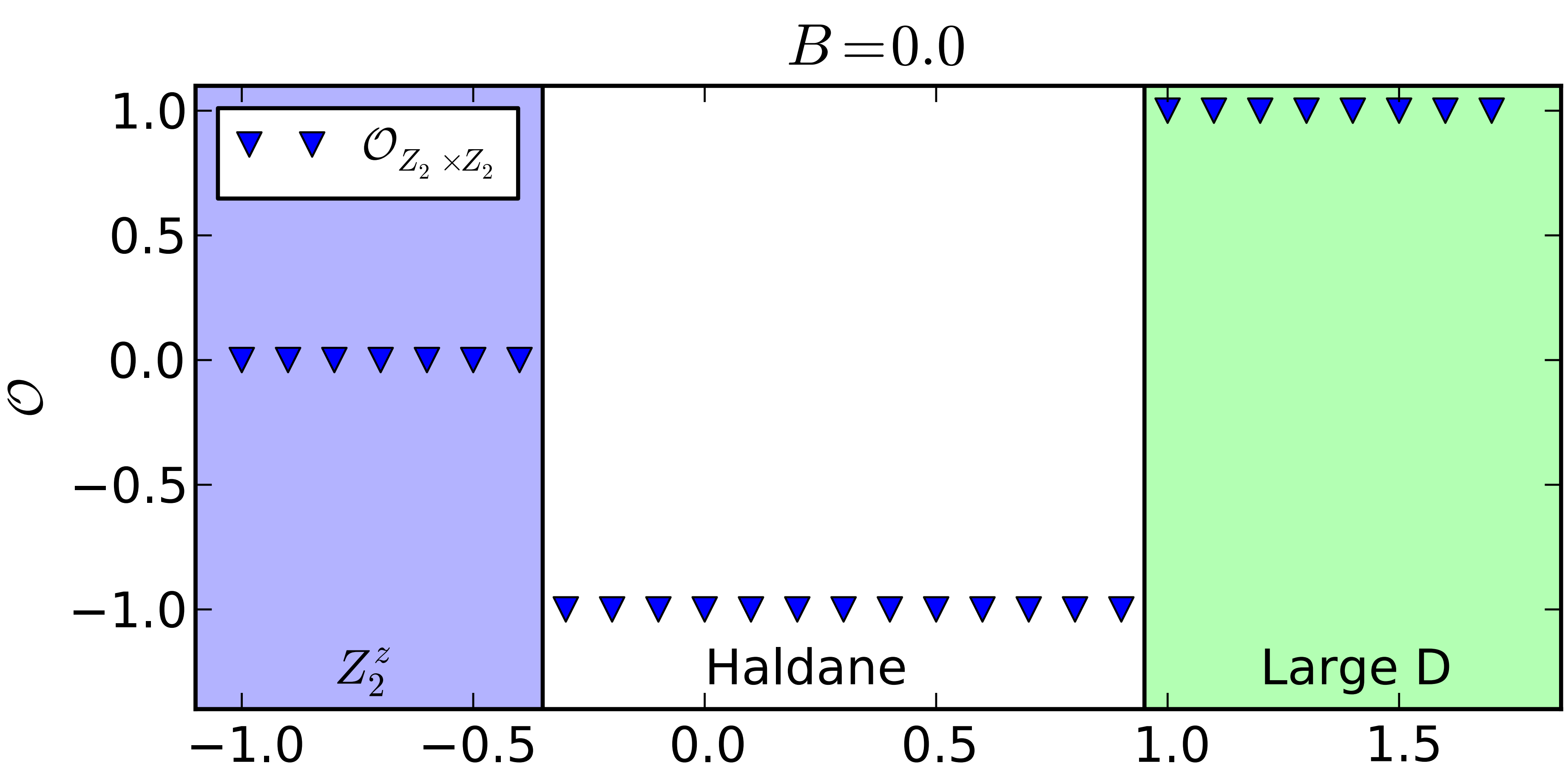
$\Leftrightarrow \quad U_\Sigma = X^\dagger$

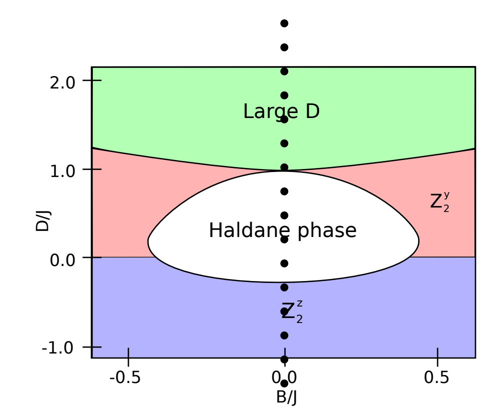
Non-local order parameter (I)

- S=1 chain $H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D \sum_j (S_j^z)^2 + B \sum_j S^x$

- $\mathbb{Z}_2 \times \mathbb{Z}_2$ stabilizes Haldane phase if $B = 0$

  and $\mathcal{O}_{Z_2 \times Z_2} = \begin{cases} 0 & \text{if symmetry broken} \\ \frac{1}{\chi} \mathrm{tr}\left(U_x U_z U_x^\dagger U_z^\dagger\right) & \text{if symmetry not broken} \end{cases}$.



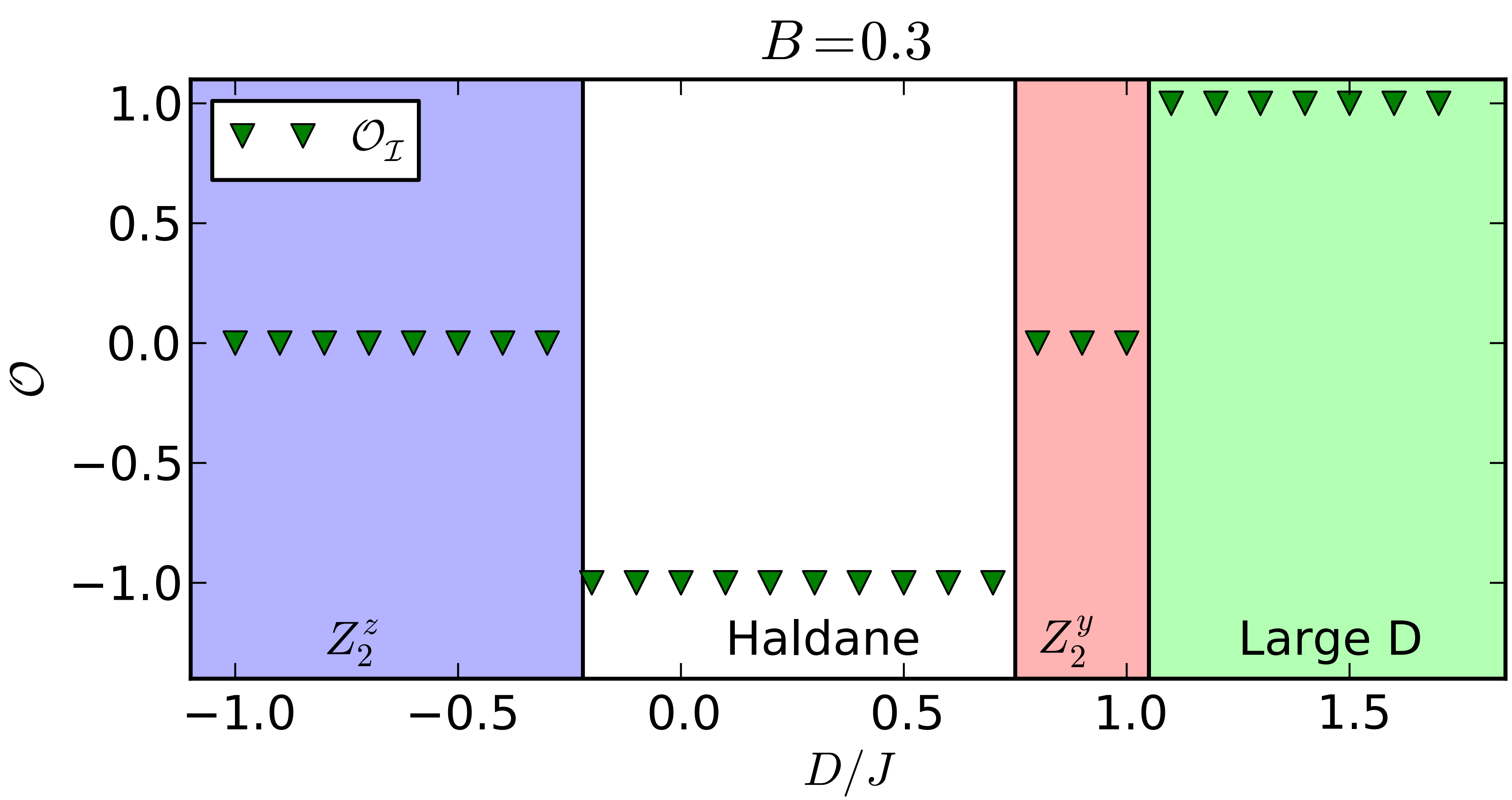


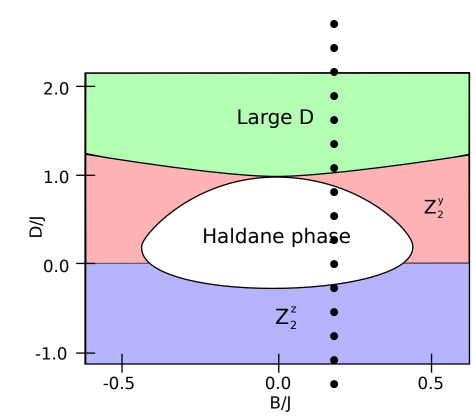
iMPS obtained using the iTEBD / iDRMG method

Non-local order parameter (I)

- S=1 chain $H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D \sum_j (S^z_j)^2 + B \sum_j S^x$

- **Inversion symmetry** stabilizes Haldane phase if $B \neq 0$ and $\mathcal{O}_{\mathcal{I}} = \begin{cases} 0 & \text{if symmetry broken} \\ \frac{1}{\chi}\mathrm{tr}\,(U_{\mathcal{I}} U^*_{\mathcal{I}}) & \text{if symmetry not broken}\ . \end{cases}$





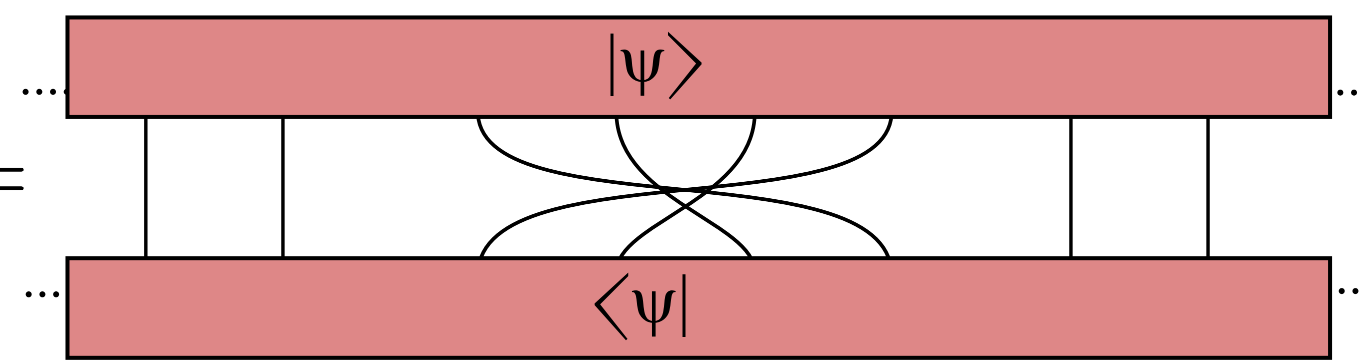
iMPS obtained using the iTEBD / iDRMG method

Non-local order parameter (2)

- What if we do not have access to the transfermatrix (i.e., Monte Carlo or experiments)?

- **Inversion** symmetry: Inverting part of the wave function

$$\langle \Psi | \mathcal{I}_{1,2n} | \Psi \rangle =$$



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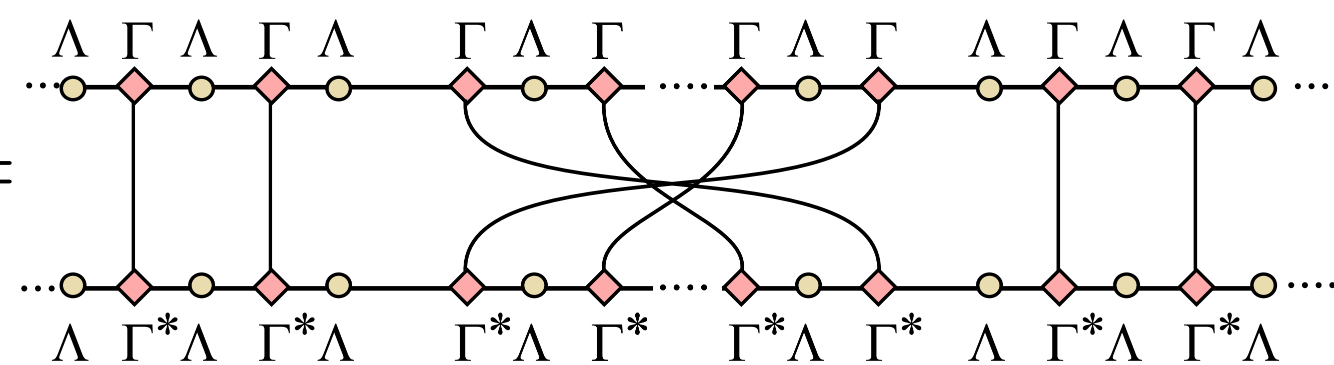

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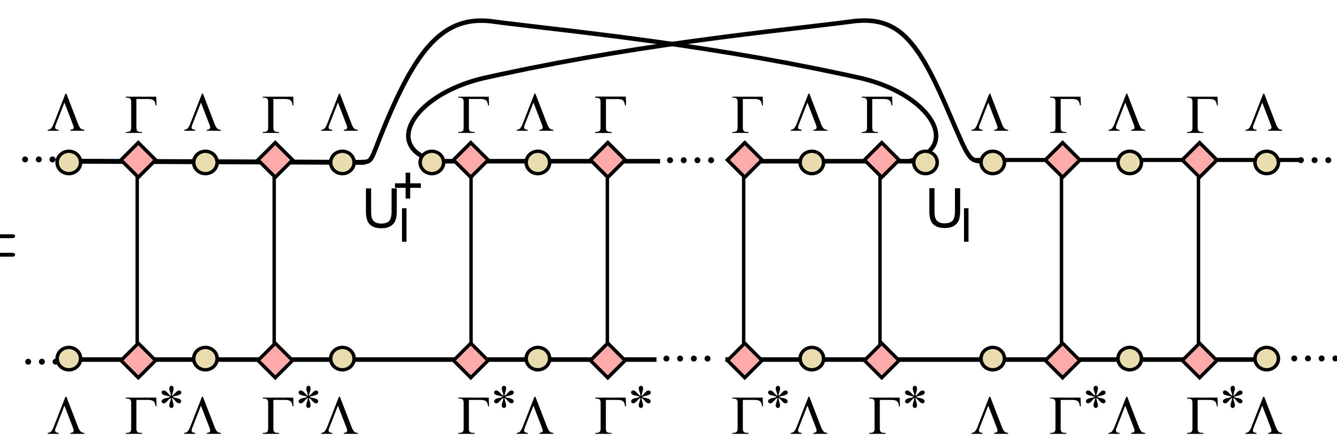

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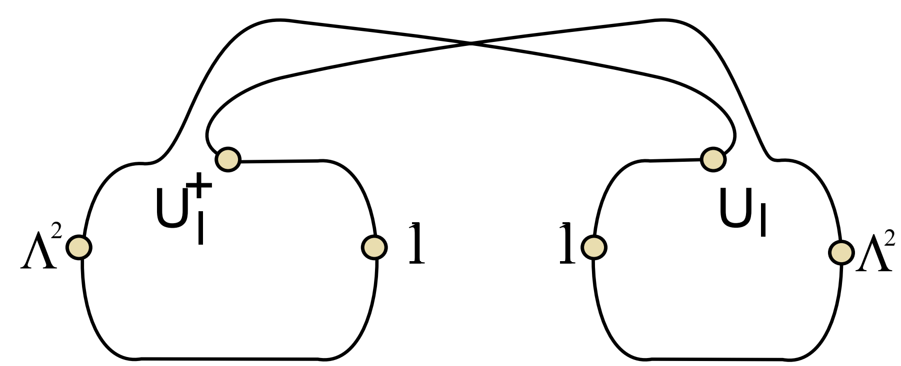

$$= \mathrm{Tr}(U_I^* U_I \Lambda^4)$$

Non-local order parameter (2)

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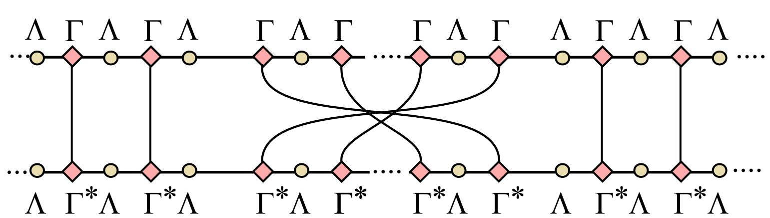
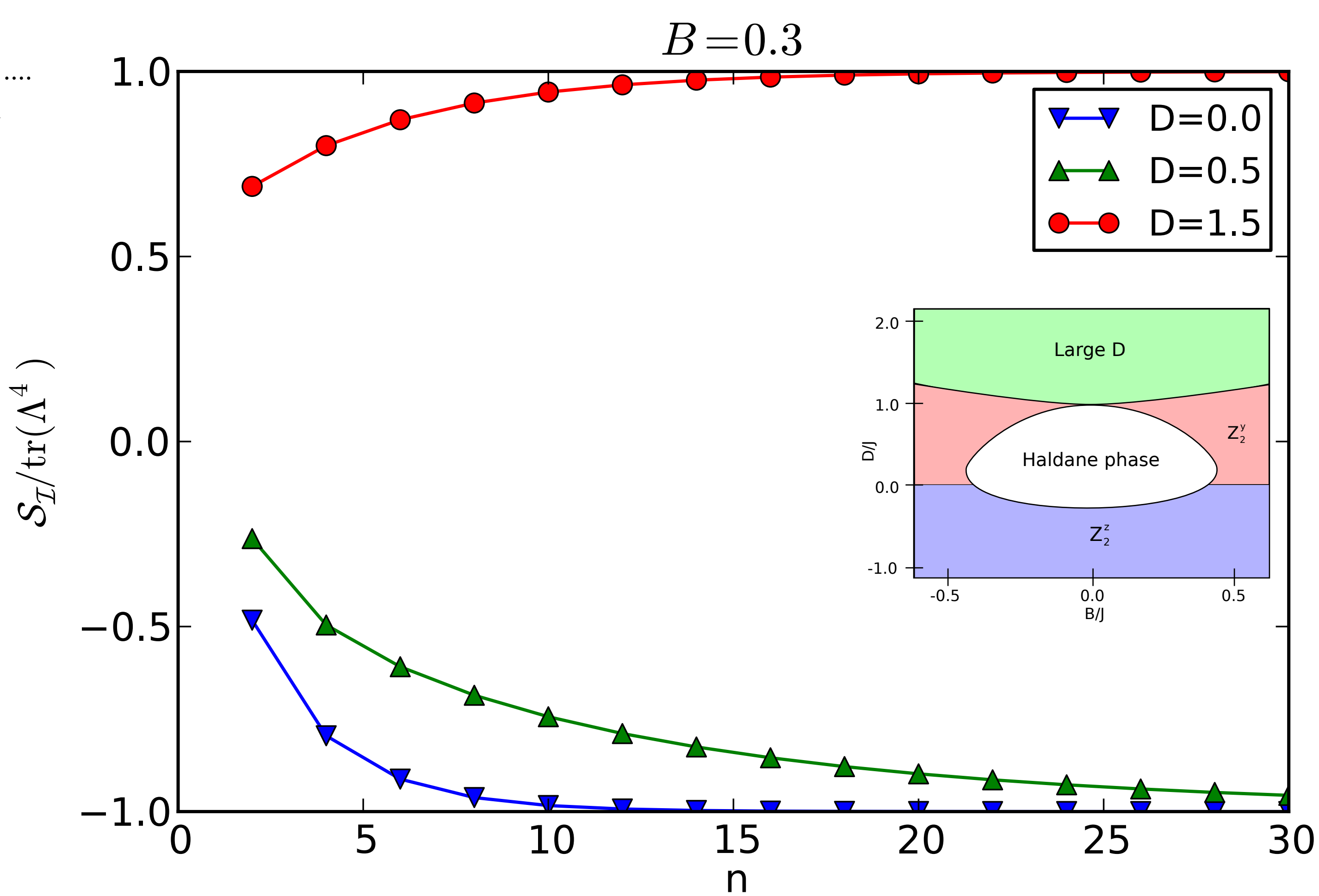
$\mathcal{S}_{\mathcal{I}} =$ 

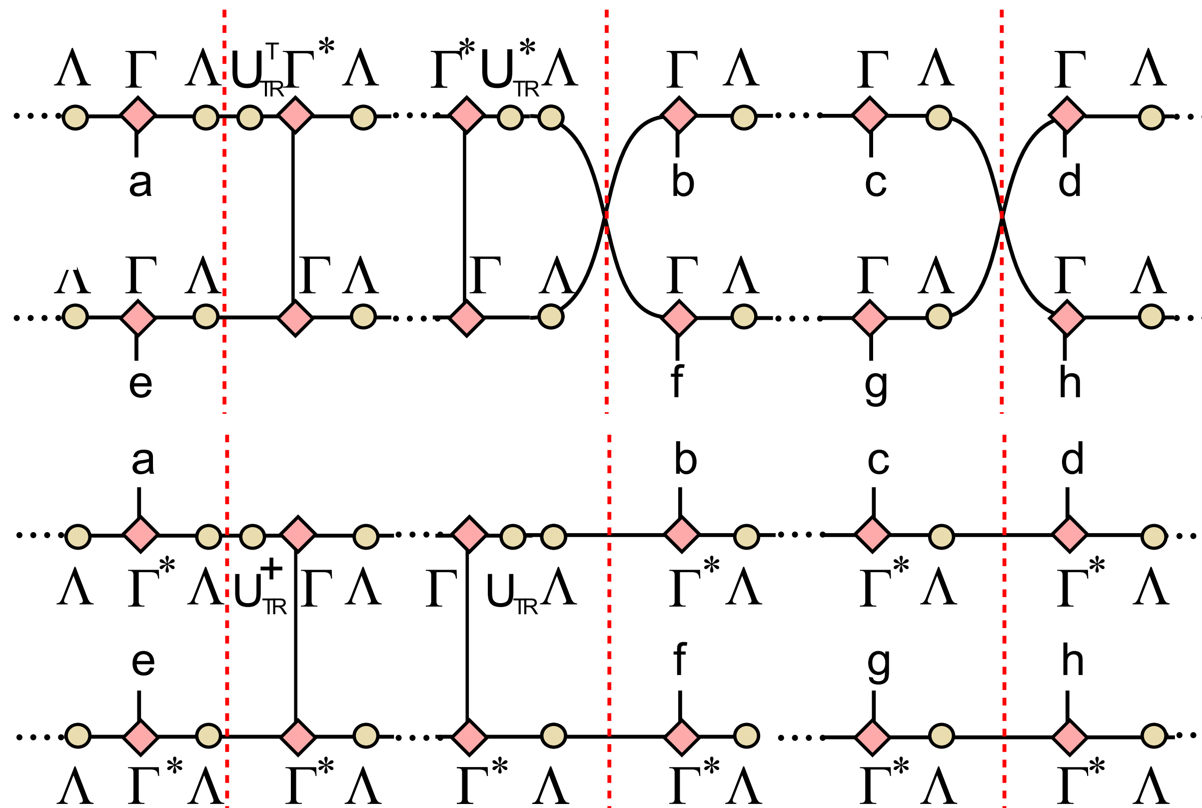
- Distinguishes the Haldane phase from the trivial phase in presence of inversion symmetry



Non-local order parameter (2)

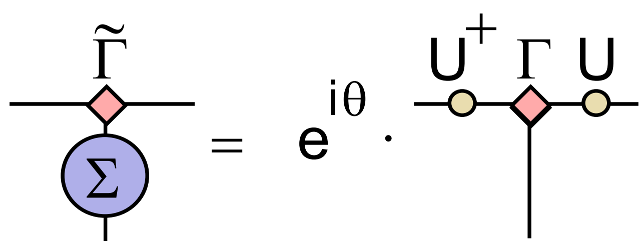
- **Time reversal** symmetry: two copies of the wave function with swapping operators [Isakov et al '11] and $\Sigma = e^{-i\pi S^y}$

$$S_{\mathcal{TR}} =$$



$\langle R_{1n}|\mathrm{Swap}_{n+1,2n}|\psi_2\rangle$

$\langle \psi_2|R_{1n}\rangle$



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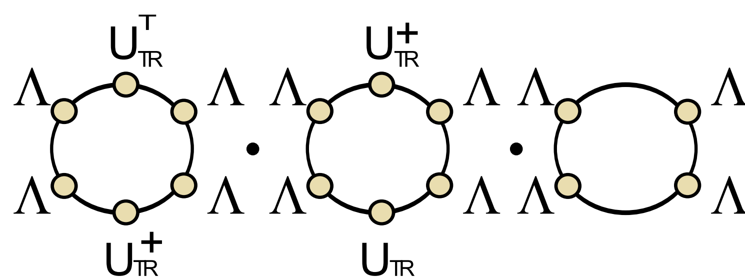
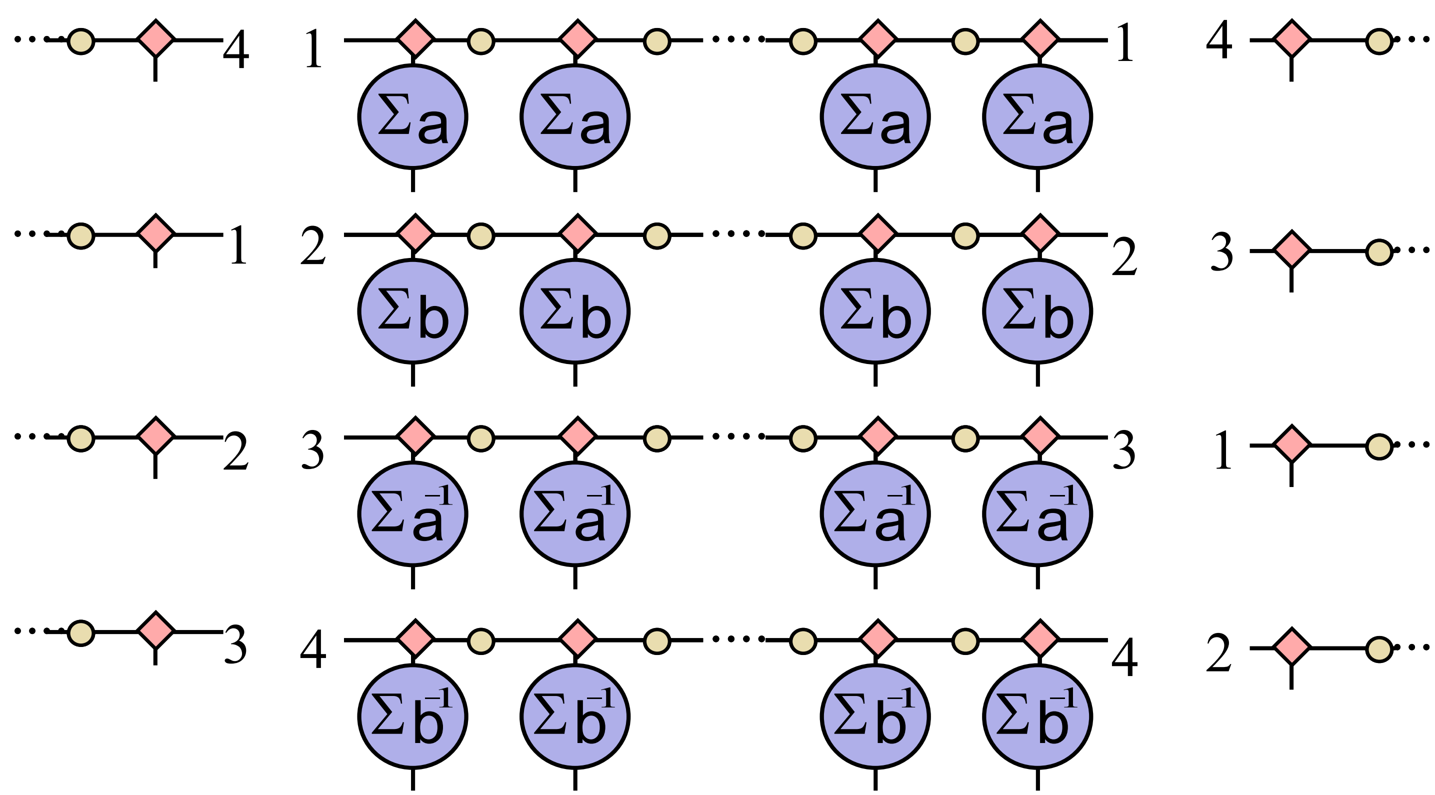


$$= \mathrm{Tr}(U_{TR} U_{TR}^{*} \Lambda^4)\,\mathrm{Tr}(\Lambda^4)^2$$

Non-local order parameter (2)

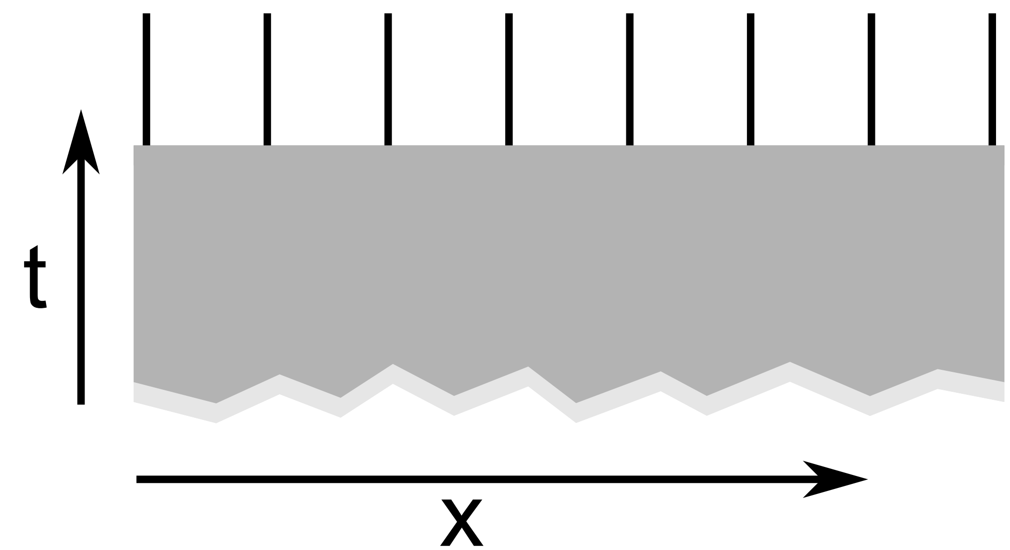
- **General internal symmetries** symmetry: multiple copies of the wave function
- Example: $S_{\mathcal{G}} = \mathrm{Tr}(U_a U_b U_a^{-1} U_b^{-1})$



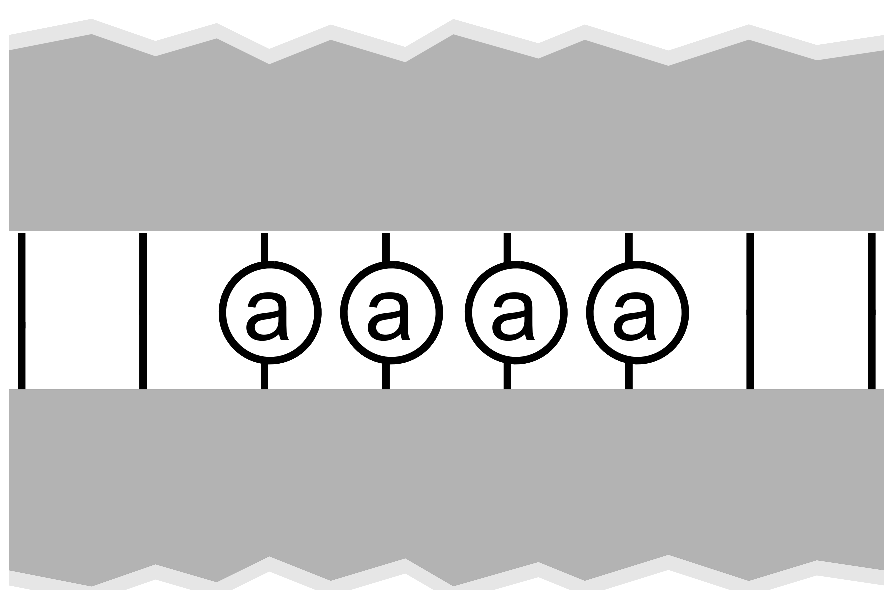
[See also: Haegemann et al.: arXiv:1201.4174]

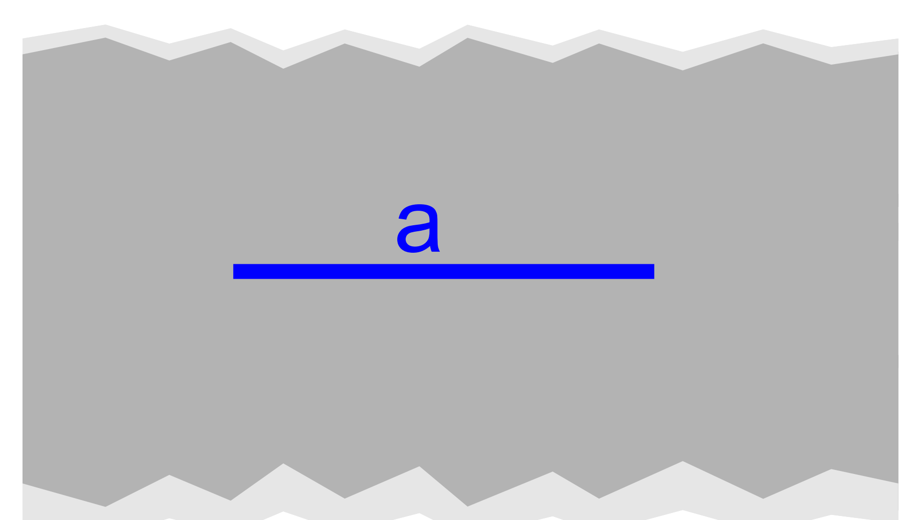
Non-local order parameter (2)

- **If phases are topological, where is the torus??**
- Express wave function as partition function on the half plane:



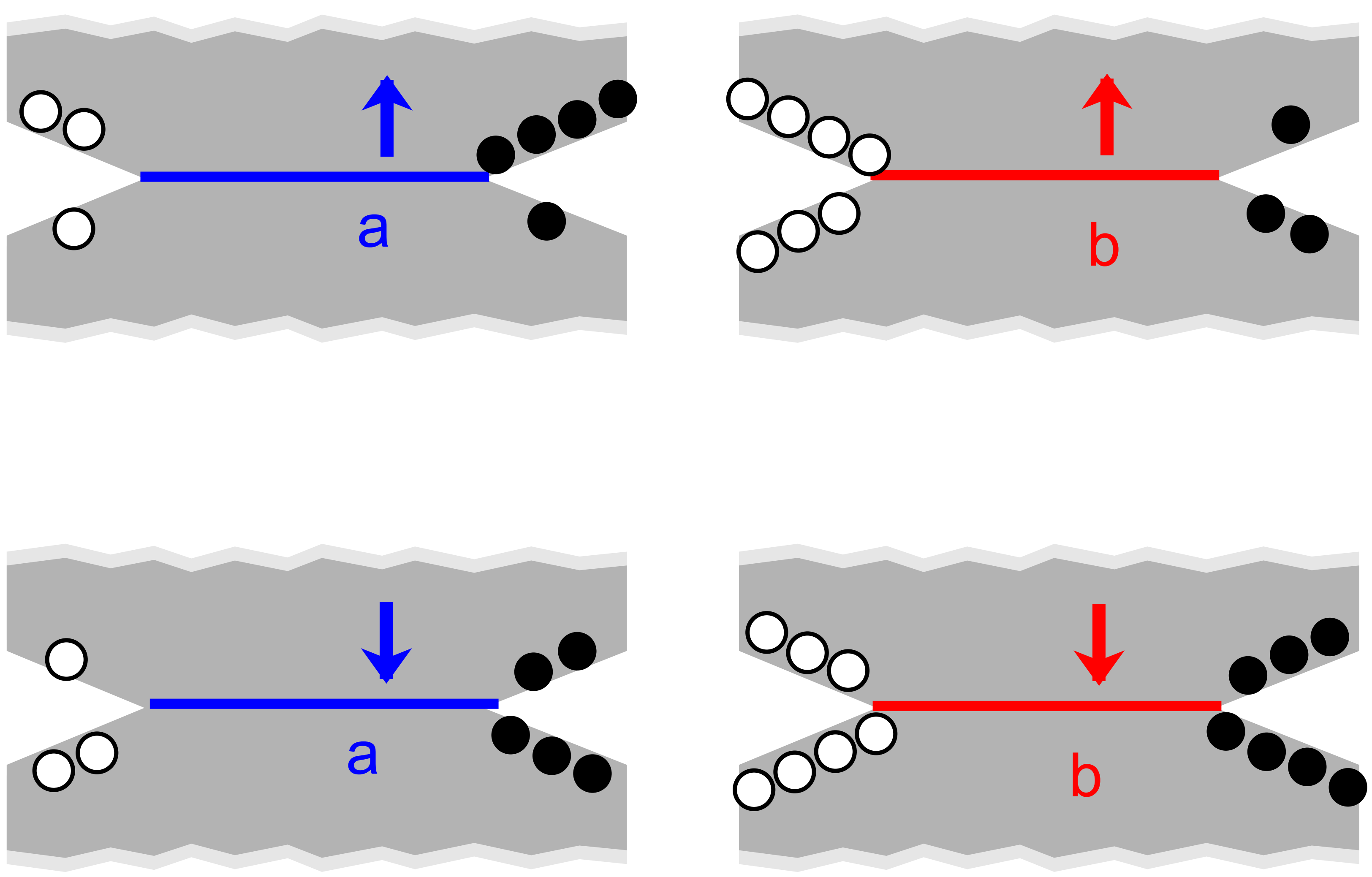
- Sandwich of a string operator:





Non-local order parameter (2)

- **If phases are topological, where is the torus??**
- Sandwich the symmetry string operators **a** / **b,** then deform them and finally glue them together



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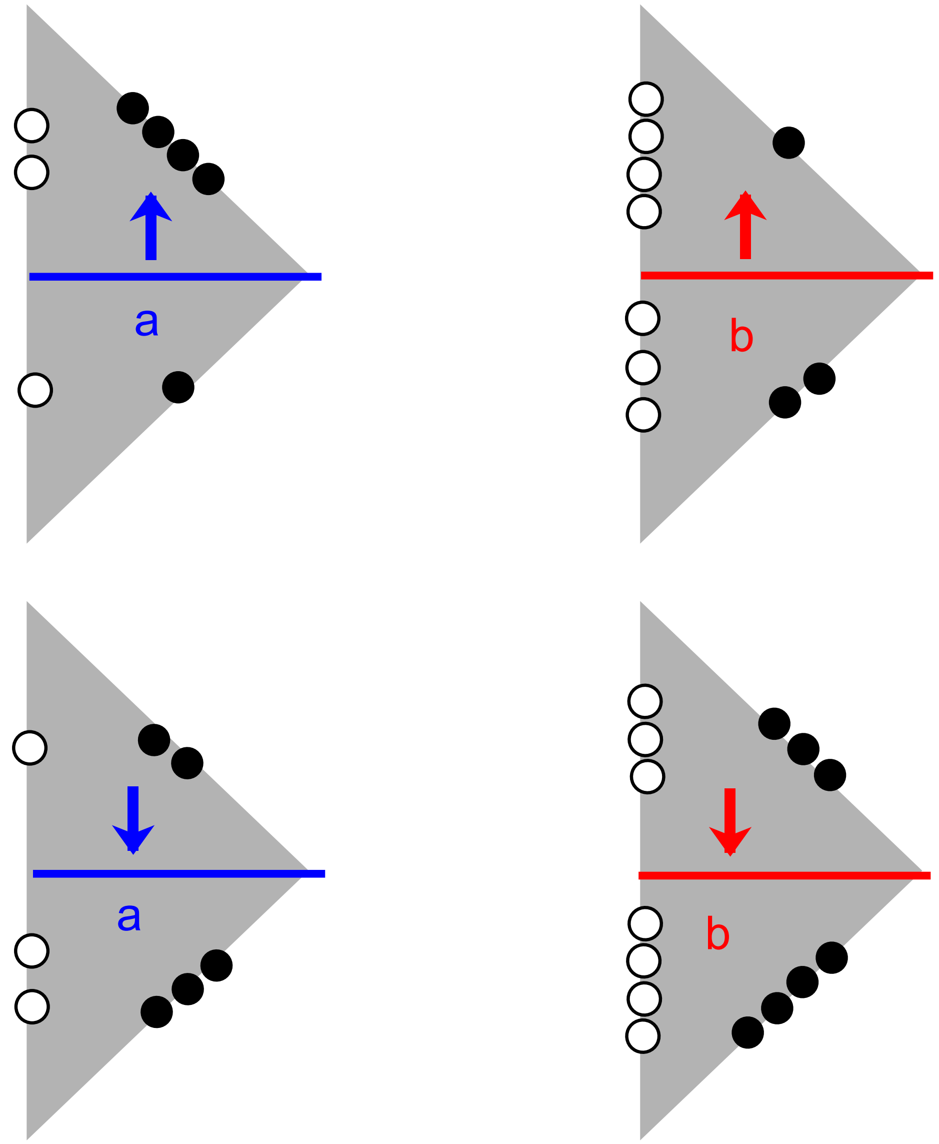

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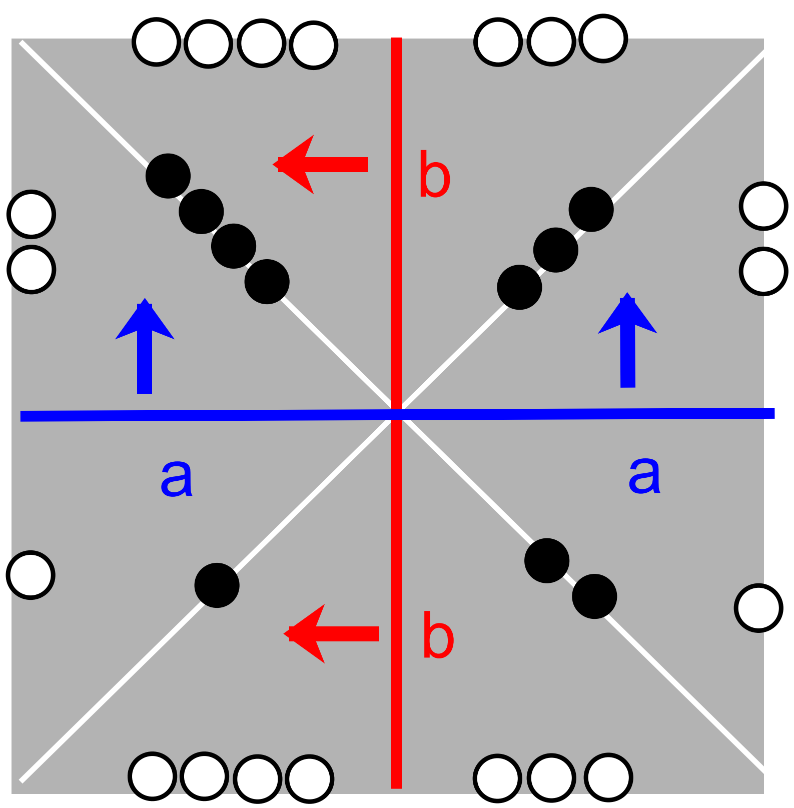
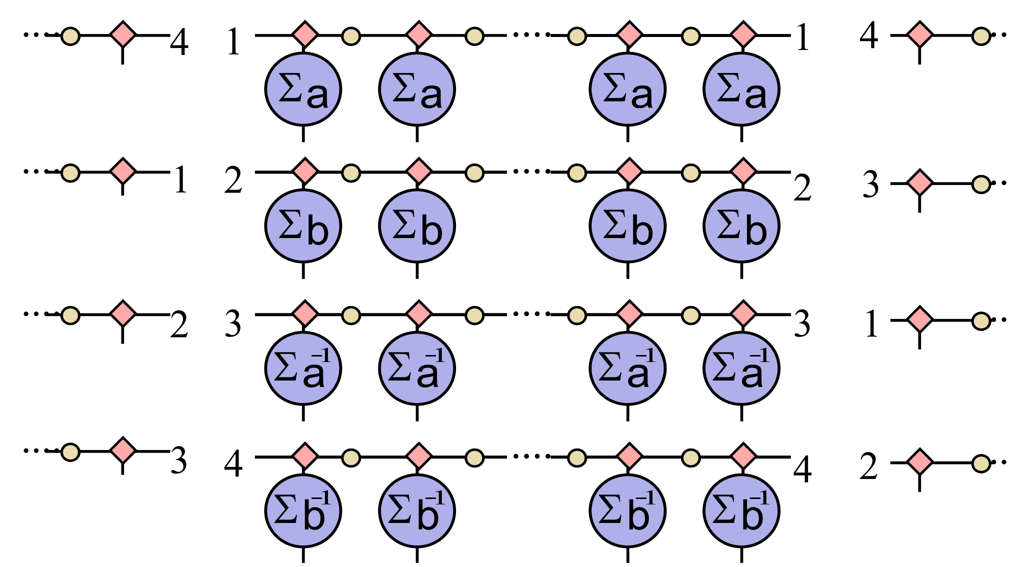
$$S_{\mathcal{G}} = \mathrm{Tr}(U_a U_b U_a^{-1} U_b^{-1})$$





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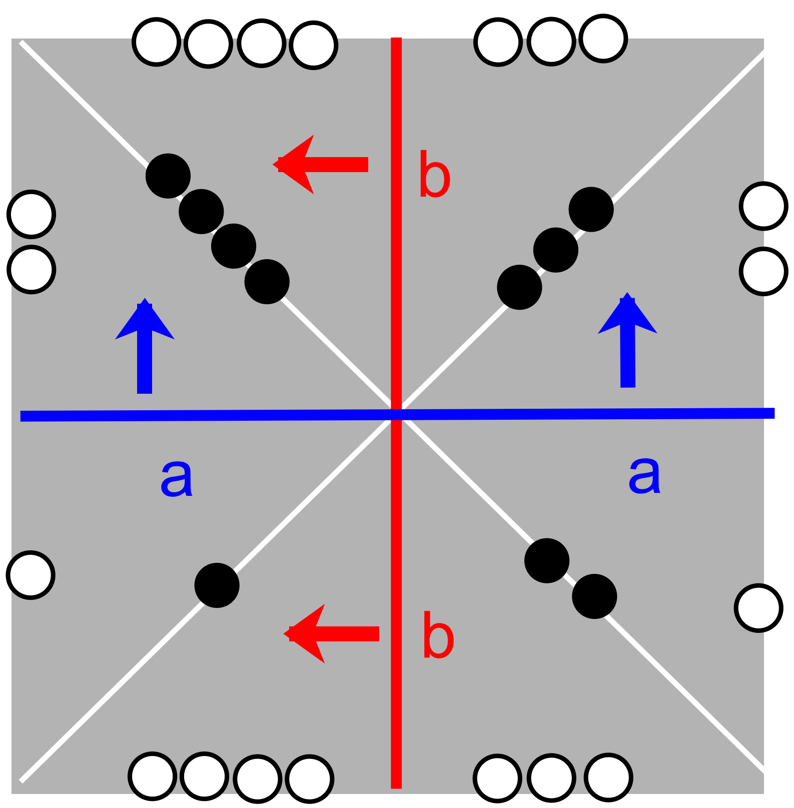

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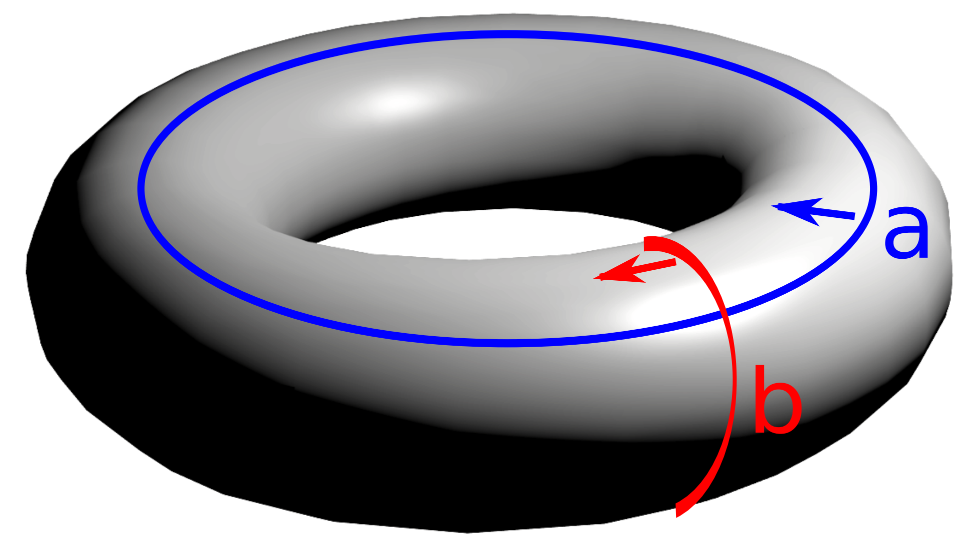
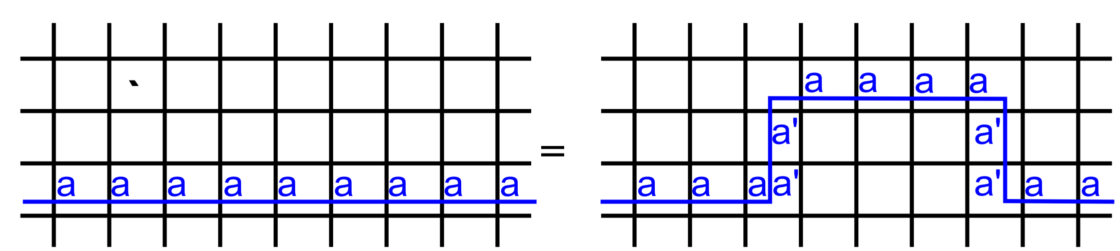
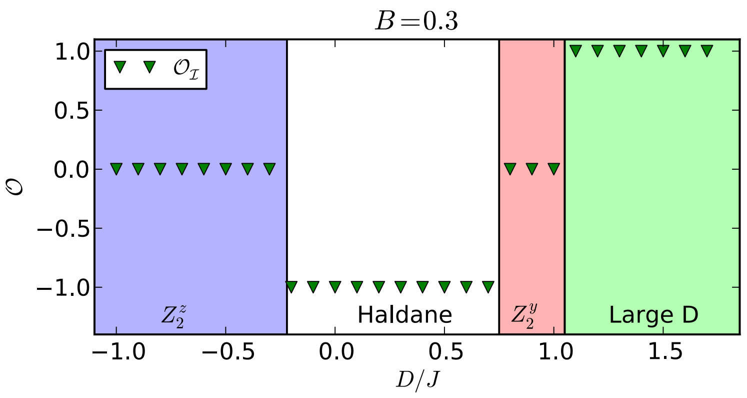




Summary

- Derivation of **non-local order parameters** which can be used to **detect/distinguish all symmetry protected topological phases in 1D**
  - Can be obtained directly from a generalized transfermatrix
  - Expressions which can be evaluated using any numerical methods, e.g., Quantum Monte Carlo



- Measuring string order experimentally: High-resolution imaging of low-dimensional quantum gases

[M. Endres et al '11]

FP. and A. M. Turner, arxiv:1204.0704