Equilibration times in closed long-range quantum spin models

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"New quantum states of matter in and out of equilibrium"

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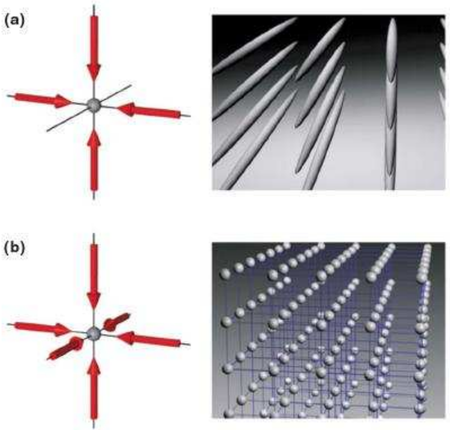
Cold atoms or ions arranged in lattices

- Cold atoms aligned in optical lattice; or trapped ions arranged in a Coulomb crystal.
- Tune interactions via Feshbach resonances, microwave radiation, ...
- Controlled engineering of condensed-matter Hamiltonians.



(from: I. Bloch *et al.*, Rev. Mod. Phys. **80** (2008) 885–964)

Cold atoms in optical lattices

Statistical description

After the cooling is switched off:

- conservation of energy and
- conservation of particle number

For pure s-wave scattering: (low temperature, no permanent dipole moment)

- conservation of magnetization.

$\Longrightarrow$ Closed-system dynamics
$\Longrightarrow$ Equilibration in closed quantum systems?
$\Longrightarrow$ Statistical description in the microcanonical ensemble

It will depend on the type of system studied whether there are significant differences to the standard open-system, canonical situation.

$\Longrightarrow$ Long-range makes a big difference!

Long-range Ising model

Chain of N interacting spin-$1/2$ particles in a magnetic field,

$$H_N = \mathcal{N}_N \sum_{i=1}^{N} \sum_{j=1}^{N/2} \frac{\sigma_i^z \sigma_{i+j}^z}{j^{\alpha}} - h \sum_{i=1}^{N} \sigma_i^z .$$

$$\alpha > 1 \implies \sum_{j=1}^{\infty} j^{-\alpha} < \infty: \quad \text{short-range}$$

$$0 < \alpha < 1 \implies \sum_{j=1}^{\infty} j^{-\alpha} = \infty: \quad \text{long-range}$$

Normalization $\mathcal{N}_N = \left(2 \sum_{j=1}^{N/2} j^{-\alpha} \right)^{-1}$ to make energy extensive.

Time evolution of the long-range Ising model

$$H_N = \mathcal{N}_N \sum_{i=1}^{N} \sum_{j=1}^{N/2} \frac{\sigma_i^z \sigma_{i+j}^z}{j^\alpha} - h \sum_{i=1}^{N} \sigma_i^z$$

Goal: Study time evolution of expectation value $\langle A \rangle(t)$, where

$$A(a_1, \dots, a_N) = \sum_{i=1}^{N} a_i \, \sigma_i^x, \qquad a_i \in \mathrm{R}.$$

with respect to initial state operators ϱ_0 which are diagonal in the σ_i^x-eigenbasis.

Inspired by G. G. Emch, J. Math. Phys. **7**, 1198 (1966), C. Radin, J. Math. Phys. **11**, 2945 (1970).
Experimental motivation by magnetic resonance experiments.

$$\begin{aligned} \langle A \rangle(t) &= \mathrm{Tr}\left[\mathrm{e}^{-\mathrm{i}H_N t} A \, \mathrm{e}^{\mathrm{i}H_N t} \rho(0)\right] = \cdots = \\ &= \langle A \rangle(0) \cos(2ht) \prod_{j=1}^{N/2} \cos^2\left(\frac{2\mathcal{N}_N t}{j^\alpha}\right) \end{aligned}$$

Calculation very similar to G. G. Emch, J. Math. Phys. **7**, 1198 (1966).

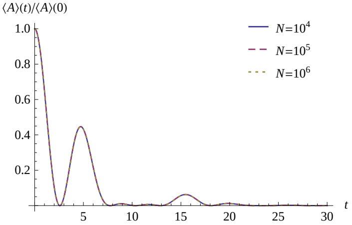
For simplicity: $h = 0 \Longrightarrow$ no Larmor precession $\cos(2ht)$.

Approach to equilibrium in the long-range Ising model?

$$\langle A\rangle(t) = \langle A\rangle(0) \prod_{j=1}^{N/2} \cos^2\left(\frac{2\mathcal{N}_N t}{j^\alpha}\right)$$

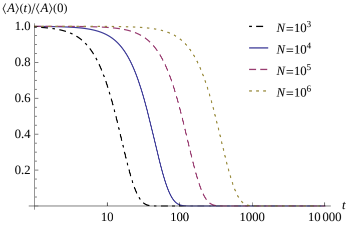
N finite: $\langle A\rangle(t)$ is quasiperiodic $\Longrightarrow$ Poincaré recurrences

N infinite: Get inspiration from plots…



short-range

upper bound?



long-range

lower bound?

Upper bound on $\langle A\rangle(t)$ in the thermodynamic limit

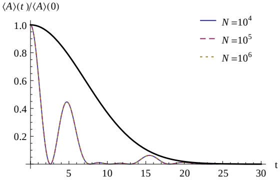
$$\langle A\rangle(t) = \lim_{N\to\infty} \langle A\rangle(0) \prod_{j=1}^{N/2} \cos^2\left(\frac{2\mathcal{N}_N t}{j^\alpha}\right)$$

$$\langle A\rangle(t) \leqslant \langle A\rangle(0) \exp\left(-cN^{-q}t^2\right)$$

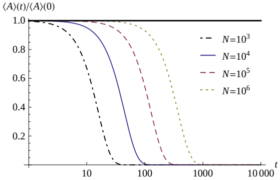
with

$$q = \begin{cases} 1 & \text{for } 0 \leqslant \alpha < 1/2, \\ 2-2\alpha & \text{for } 1/2 < \alpha < 1, \\ 0 & \text{for } \alpha > 1. \end{cases}$$

M. Kastner, Phys. Rev. Lett. **106**, 130601 (2011).



short-range



long-range

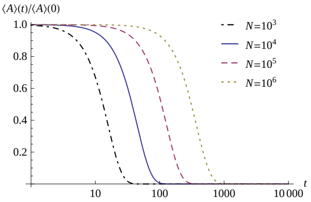
Lower bound on $\langle A\rangle(t)$ for $\alpha < 1$ (long-range)

$$\langle A\rangle(t) = \langle A\rangle(0) \prod_{j=1}^{N/2} \cos^2\left(\frac{2\mathcal{N}_N t}{j^\alpha}\right)$$

Proposition: For any fixed time τ and some small $\delta > 0$, there is a finite $N_0(\tau)$ such that

$$|\langle A\rangle(t) - \langle A\rangle(0)| < \delta \qquad \forall t < \tau, N > N_0(\tau).$$

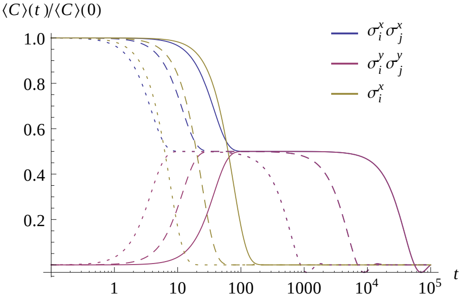
M. Kastner, Phys. Rev. Lett. **106**, 130601 (2011).



- δ: experimental resolution for measurement of A,
- τ: duration of the experiment.
- $\Rightarrow$ Within experimental resolution and for large enough system size, no deviation of $\langle A\rangle(t)$ from its initial value can be observed for times $t \leqslant \tau$.

Spin–spin correlators

Goal: Study time evolution of expectation values of spin–spin correlators
$$\langle \sigma_i^a \sigma_j^b \rangle \qquad \text{where } a, b \in \{x, y, z\},$$
with respect to initial state operators ϱ_0 which are diagonal in the σ_i^x-eigenbasis.



- Two-step process
- Second time scale involved
- N-scaling different for first and second step

Generalizations

- Higher-dimensional lattices ✓
- More general couplings,

$$H_N = \mathcal{N}_N \sum_{\langle i,j \rangle} \epsilon(|i-j|)\sigma_i^z \sigma_j^z \; - \; h \sum_{i=1}^{N} \sigma_i^z$$

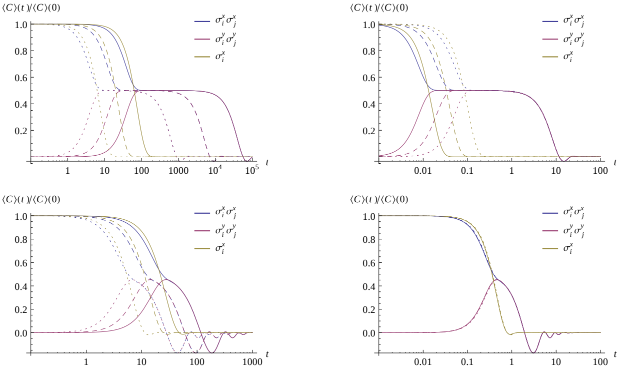
  with $|\epsilon(j)| \sim cj^{-\alpha}$ and some $c > 0$ ✓
- General observables ?
- More general (non-integrable) models ?

Question: Is quasi-stationary behaviour generic
for long-range systems and arbitrary initial conditions

N-dependence of the pair interaction strength

$$H_N = \mathcal{N}_N \sum_{i=1}^{N} \sum_{j=1}^{N/2} \frac{\sigma_i^z \sigma_{i+j}^z}{j^\alpha} - h \sum_{i=1}^{N} \sigma_i^z \qquad \text{where } \mathcal{N}_N \sim \begin{cases} N^{\alpha-1} & \text{for } 0 \leqslant \alpha < 1, \\ \text{const.} & \text{for } \alpha > 1. \end{cases}$$

Is this N-dependent prefactor the sole cause of the N-scaling of relaxation times? No!



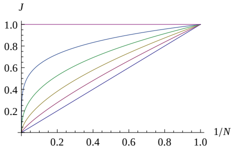
$0 < \alpha < 1/2$:

$1/2 < \alpha < 1$:

with prefactor $\mathcal{N}_N$ without prefactor $\mathcal{N}_N$

Extensivity in physics

$$H_N = \mathcal{N}_N \sum_{i=1}^{N} \sum_{j=1}^{N/2} \frac{\sigma_i^z \sigma_{i+j}^z}{j^\alpha} - h \sum_{i=1}^{N} \sigma_i^z$$



With or without N-dependent prefactor $\mathcal{N}_N$:
Which one is the physically relevant scenario?

Equilibrium properties:

Prefactor $\mathcal{N}_N$ necessary to have a well-defined and non-trivial thermodynamic limit.

Nonequilibrium properties:

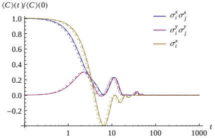
In general unclear. Some dynamical properties seem to have a well-defined limit in the absence of $\mathcal{N}_N$.

Summary / Take-home message

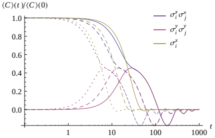
Equilibrium:

Prefactor $\mathcal{N}_N$ necessary

N-independent relaxation time scale for $\alpha > 1$



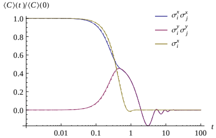
Diverging (with N) relaxation time scale $\tau \propto N^q$ with $q = \min\{1/2, 1-\alpha\}$ for $0 < \alpha < 1$



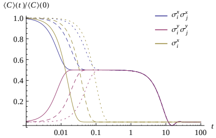
Nonequilibrium:

No general reason to include $\mathcal{N}_N$

N-independent relaxation time scale for $\alpha > 1/2$



Diverging (with N) relaxation time scale $\tau \propto N^{\alpha-1/2}$ for $0 < \alpha < 1/2$

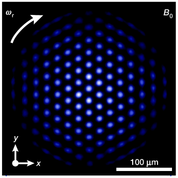


M. Kastner, Diverging equilibration times in long-range quantum spin models, Phys. Rev. Lett. **106**, 130601 (2011).

Experimental realization

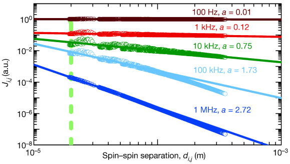
Beryllium ions in a Penning trap

J. W. Britton *et al.*, Engineered two-dimensional Ising interactions in a trapped-ion quantum simulator with hundreds of spins, Nature **484**, 489 (2012).



- 2d Coulomb crystal on a triangular lattice
- Valence-electron spin states as qubits (Ising spins)
- Spin–spin interactions mediated by crystal's transverse motional degrees of freedom

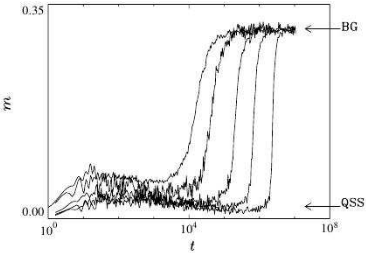
- Effective anti-ferromagnetic Ising Hamiltonian $H = \sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z - \sum_i \boldsymbol{B}\cdot\boldsymbol{\sigma}_i$
- $J_{ij} \approx |i-j|^{-\alpha}$ with $0.05 \lesssim \alpha \lesssim 1.4$



Classical Vlasov description of quasi-stationary behaviour

Vlasov equation: Time-evolution equation for 1-particle distribution function

- Like Boltzmann equation, but without collision integral
- Important in plasma physics
- Exact for Curie-Weiss ($\alpha = 0$) in the thermodynamic limit
- Quasi-stationary states correspond to stable stationary solutions of Vlasov equation



Campa, Dauxois, Ruffo, Phys. Rep. **480**, 57 (2009)

Quantum Vlasov equation for long-range spin systems?