Non-Gaussianity and its Evolution in Multi-field Inflation

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Recent Interest

Primordial non-Gaussianity has attracted a lot of interest over the past few years.

- Precision Cosmology: Non-Linearities may be observable

- Gravity is non-linear. Some non-Gaussianity will always be present

- Potentially useful for further testing inflation
  - Consistency check - Identification of new physics beyond inflation (which may produce stronger NG signals)
  - Discriminant among models

One more handle on the physics of Inflation. Primordial NG has been approached via various angles in an effort to compute it and relate it to observables.

Observations

- Various detections of non-Gaussian signals have been reported in the CMB. However, none has been linked to a primordial source.

- Extracting any primordial NG from LSS will probably be very difficult since it will be masked by later non-linear evolution. However, it has been suggested that finding early objects forming at high redshifts is probably a good strategy since they will be tracing the tail of the distribution (you just have to find one).

- Thus observations focus on the CMB measuring $f_{NL} \sim \frac{\langle TTT \rangle}{\langle TT \rangle^2}$ or even $\tau_{NL} \sim \frac{\langle TTTT \rangle}{\langle TT \rangle^2}$. Current limits set $-54 < f_{NL} < 114$ (95%CL, WMAP). Planck is expected to reach $f_{NL} < 5$ at best, while an ideal experiment is limited to $f_{NL} < 3$.

However, ...

... a recent paper (astro-ph/0610257) has claimed that even $f_{NL} < 0.01$ is accessible via observations of the 21 cm radio background. If this analysis goes through it will set the whole discussion on inflationary NG under a totally different light.

Calculating non-Gaussianity

Focusing on Inflation, there are four regimes relevant for NG generation and evolution

- Effects before or during horizon crossing.
  Calculating non-linear corrections from the inflationary mechanism for the generation of perturbations.

- Long wavelength evolution during inflation.
  The subject of this talk...

- Long wavelength evolution after inflation.
  E.g. Reheating, Preheating & The Curvaton scenario.

- The relation of this primordial NG to the observed CMB sky.
  Solving the full system of Boltzman equations at second order.

Calculational Approaches

- **Straightforward second-order perturbation theory**:
  Follow the route of linear theory by extending the perturbative analysis of the Einstein equations to second order. This seems essential for studying scales smaller or close to the horizon $\rightarrow$ proliferation of terms in the equations.

- **Long wavelength approximations**:
  First focus on long wavelengths, particularly relevant during inflation, where the dynamics simplifies. This is the approach of the δN formalism as well as the one taken in this talk.

Long wavelength approximation

$$ds^2 = -N^2(t, \mathbf{x})dt^2 + e^{2\alpha(t,\mathbf{x})}h_{ij}(\mathbf{x})dx^i dx^j$$

On long wavelengths ($\Delta\mathbf{x} > (aH)^{-1}$, ∇^2 dropped):

- $\frac{d}{Ndt}H = -\frac{1}{2}\bar{K}^{ij}\bar{K}_{ij} - \frac{1}{2M_p^2}(\mathcal{E} + S/3)$

- $\frac{d}{Ndt}\bar{K}_j^i = -3H\bar{K}_j^i \quad \Rightarrow \quad \bar{K}_j^i = C_j^i(\mathbf{x})e^{-3\alpha}$

- $\frac{\mathcal{D}}{Ndt}\Pi^A + 3H\Pi^A + \partial^A V = 0$

- $H^2 = \frac{1}{3M_p^2}\mathcal{E} + \frac{1}{6}\bar{K}^{ij}\bar{K}_{ij}, \quad \partial_i H = -\frac{1}{2M_p^2}\Pi_A\partial_i\phi^A - \frac{1}{2}\nabla_j\bar{K}_i^j$

$$\left\{ H = \frac{d}{Ndt}\alpha, \quad \Pi^A = \frac{d}{Ndt}\phi^A, \quad \bar{K}_{ij} = -e^{2\alpha}\frac{d}{2Ndt}h_{ij} \right\}$$

'Separate universe' evolution $\quad \Rightarrow \quad \Delta N$ formalism.
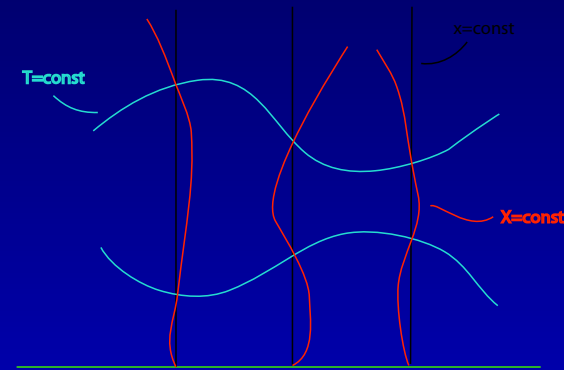
Long Wavelength Coordinate Transformations

Consider coordinate transformations which preserves $g_{0i} = 0$:

$$ds^2 = -\tilde{N}^2(T, \mathbf{X})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{X})}\tilde{h}_{ij}(T, \mathbf{X})dX^i dX^j$$

$$T = T(t, x^l), \quad X^i = X^i(t, x^l)$$

Then, up to $\mathcal{O}(\nabla^2)$ the transformation matrix is:

- $\Lambda^\mu_T \equiv \dfrac{\partial x^\mu}{\partial T} = N\dfrac{\partial^\mu T}{(\partial_t T)^2}$

- $\Lambda^i_{\tilde{j}} \equiv \dfrac{\partial x^i}{\partial X^j} = \delta^i_{\tilde{j}} + \mathcal{O}\left[(\partial_i)^2\right]$

- $\Lambda^0_{\tilde{j}} \equiv \dfrac{\partial t}{\partial X^j} = -\delta^i_{\tilde{j}}\dfrac{\partial_i T}{\partial_t T} + \mathcal{O}\left[(\partial_i)^2\right]$

Using these we learn: $\qquad\qquad\qquad x^i = X^i + \int dT\,\dfrac{N\partial^i T}{(\partial_t T)^2}$

- $ds^2 = -\tilde{N}^2(T, \mathbf{x})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{x})}h_{ij}(\mathbf{x})dx^i dx^j$

- $\dfrac{1}{N}\dfrac{\partial}{\partial t} = \dfrac{1}{\tilde{N}}\dfrac{\partial}{\partial T} + \mathcal{O}(\nabla^2), \quad \dfrac{\partial}{\partial x^i} = \dfrac{\partial}{\partial X^i} + \partial_i T\dfrac{\partial}{\partial T}$

Inhomogeneity

Separate inhomogeneous evolution from the homogeneous background:

- Given a spacetime scalars $A(t, \mathbf{x}) = A(t) + \Delta A(t, \mathbf{x})$ one can always set $\Delta A = 0$ by a suitable choice of time slicing - no coordinate invariant meaning for ΔA

- However, given two scalars $A(t, \mathbf{x})$ and $B(t, \mathbf{x})$ one can construct a fully non-linear variable which encodes the inhomogeneity and is a scalar (invariant) under long wavelength transformations:

$$C_i(t, \mathbf{x}) \equiv \partial_i A - \frac{\partial_t A}{\partial_t B} \partial_i B = \tilde{C}_i(T, \mathbf{x})$$

For example:

$$\zeta_i = \partial_i \alpha - \frac{H}{\dot{\rho}} \partial_i \rho \;\; \Rightarrow \;\; \frac{d}{N dt} \zeta_i = -\frac{H}{\rho + P} \left(\partial_i P - \frac{\dot{P}}{\dot{\rho}} \partial_i \rho \right)$$

This is a fully non-linear statement formally similar to that of linear theory.

For Inflation...

Consider a set of scalar fields during inflation

$$T_{\mu\nu} = G_{AB}\partial_\mu\phi^A\partial_\nu\phi^B - g_{\mu\nu}\left(\tfrac{1}{2}\partial_\lambda\phi_A\partial^\lambda\phi^A + V(\phi)\right)$$

The following spatial vectors are scalar invariants:

$$\mathcal{Q}_i^A = e^\alpha\left(\partial_i\phi^A - \frac{\partial_t\phi^A}{NH}\partial_i\alpha\right), \qquad \left(H \equiv \frac{\partial_t\alpha}{N\alpha}\right)$$
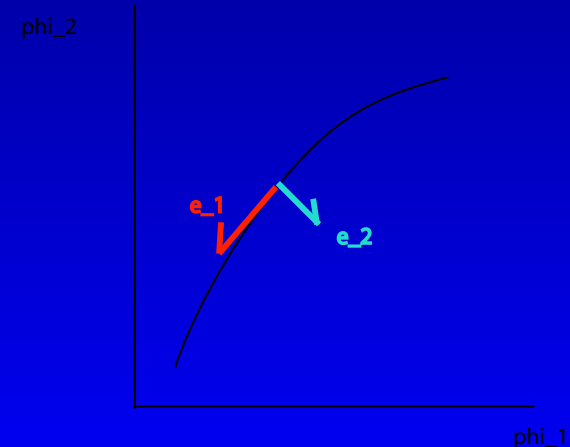
$$\zeta_i^A = -\frac{\kappa}{\sqrt{2\tilde{\epsilon}}}\partial_i\phi^A + \frac{\Pi^A}{\Pi}\partial_i\alpha, \quad \left(\Pi^A \equiv \frac{\partial_t\phi^A}{N}, \ \tilde{\epsilon} \equiv \frac{\kappa^2}{2}\frac{\Pi^2}{H^2}, \ \kappa^2 \equiv M_p^{-2}\right)$$

Note that: $\quad \mathcal{Q}_i^A = \partial_i q^A + \ldots \quad$ and $\quad \zeta_i^A = \partial_i \zeta^A + \ldots \quad$ where q^A

and ζ^A well known linear gauge-invariant variables.

Define isocurvature and adiabatic

directions $\quad \hat{e}_A^1 = \frac{\Pi_A}{\Pi}, \quad \hat{e}_A^2 \quad , \ldots$

with $\quad \hat{e}_A^2\hat{e}^{1A} = 0, \ldots$

... For more fields an iterative procedure will produce an orthonormal basis adapted to the trajectory with N-1 isocurvature directions.

$$\tilde{\eta}^A_{(n)} = \frac{\left(\frac{1}{N}\mathcal{D}_t\right)^{n-1}\Pi^A}{H^{n-1}\Pi} \qquad \hat{e}^A_n \equiv \frac{\tilde{\eta}^A_{(n)} - \sum_{B=1}^{n-1}\tilde{\eta}^B_{(n)}\hat{e}^A_B}{\tilde{\eta}^n_{(n)}}$$

where

$$\tilde{\eta}^n_{(n)} \equiv -\epsilon_{A_1\cdots A_n}\hat{e}^{A_1}_1 \ldots \hat{e}^{A_n-1}_{n-1}\tilde{\eta}^{A_n}_{(n)}$$

which gives the basis a definite handedness.

Non-linear Isocurvature and Adiabatic Variables

- $\zeta_i \equiv \hat{e}_A^1 \zeta_i^A = \partial_i \ln a - \frac{\kappa}{\sqrt{2\tilde{\epsilon}}} \hat{e}_{1A} \partial_i \phi^A = \partial_i \ln a - \frac{H}{\dot{\rho}} \partial_i \rho$

- $\sigma_i \equiv \hat{e}_A^2 \zeta_i^A = -\frac{\kappa}{\sqrt{2\tilde{\epsilon}}} \hat{e}_{2A} \partial_i \phi^A$

Define Slow Roll parameters:

$$\tilde{\epsilon} \equiv \frac{\kappa^2}{2} \frac{\Pi^2}{H^2}, \quad \tilde{\eta}^A \equiv -\frac{3H\Pi^A + \partial^A V}{H\Pi}, \quad \tilde{\xi}^A \equiv 3\tilde{\epsilon}\,\hat{e}^{1A} - 3\tilde{\eta}^A - \frac{\hat{e}^{1B} V_B^A}{H^2}$$

Project isocurvature and adiabatic parts:

$$\tilde{\eta}^{\parallel} \equiv \hat{e}_A^1 \tilde{\eta}^A, \quad \tilde{\eta}^{\perp} \equiv \hat{e}_A^2 \tilde{\eta}^A, \quad \tilde{\xi}^{\parallel} \equiv \hat{e}_A^1 \tilde{\xi}^A, \quad \tilde{\xi}^{\perp} \equiv \hat{e}_A^2 \tilde{\xi}^A$$

The non-linear equations of motion are formally the same as those of linear perturbation theory with

$$\delta \to \partial_i$$
$$f(t) \to f(t, \mathbf{x})$$

Long wavelength equations of motion

Choose a gauge with $NH = 1$ ($\partial_i \alpha = 0$ - homogeneous expansion) to simplify expressions

$$\frac{d}{dt}\begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix} + \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}\begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix} = 0$$

where

$$\tilde{\kappa}(t,\mathbf{x}) = 3\left(\frac{V_{22}}{3H^2} + \tilde{\epsilon} + \tilde{\eta}^{\|}\right) + 2\tilde{\epsilon}^2 + 4\tilde{\epsilon}\tilde{\eta}^{\|} + 4(\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\|},$$

$$\tilde{\lambda}(t,\mathbf{x}) = 3 + \tilde{\epsilon} + 2\tilde{\eta}^{\|}$$

All local quantities are given by:

- $\partial_i \ln H = \tilde{\epsilon}\,\zeta_i, \quad e_{m\,A}\partial_i\phi^A = -\frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\,\zeta_i^m$

- $e_1^A\partial_i\Pi_A = -\frac{H\sqrt{2\tilde{\epsilon}}}{\kappa}\left(\tilde{\eta}^{\|}\zeta_i + \tilde{\eta}^{\perp}\sigma_i\right)$

- $e_2^A\partial_i\Pi_A = -\frac{H\sqrt{2\tilde{\epsilon}}}{\kappa}\left(\dot{\sigma}_i + \tilde{\eta}^{\perp}\zeta_i + \left(\tilde{\eta}^{\|} + \tilde{\epsilon}\right)\sigma_i\right)$

"Initial Conditions"

Assuming that non-linearities are not important on short scales, one can include in a straightforward manner perturbations from shorter wavelengths. This amounts to adding "sources" on the rhs:

$$\frac{d}{dt}\begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix} + \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}\begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix} = \partial_i \int \frac{d^3k}{(2\pi)^{3/2}} \begin{pmatrix} \zeta_l(k)\,\hat{\alpha}_{\mathbf{k}} \\ \sigma_l(k)\,\hat{\beta}_{\mathbf{k}} \\ \dot{\sigma}_l(k)\,\hat{\beta}_{\mathbf{k}} \end{pmatrix} \mathcal{W}(k) e^{i\mathbf{kx}}$$

where

$$\hat{\alpha}_{\mathbf{k}} = a^{\dagger}(\mathbf{k}) + a(-\mathbf{k})\,, \quad \hat{\beta}_{\mathbf{k}} = b^{\dagger}(\mathbf{k}) + b(-\mathbf{k})$$

with $\left[a(\mathbf{k}), a^{\dagger}(-\mathbf{k}')\right] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$, e.t.c.

$\zeta_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde{\epsilon}}}$ $\mathcal{W}(k)$ cuts off short wavelength modes. Simplest

$\sigma_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde{\epsilon}}}$ choice: $\mathcal{W}(k) = \Theta(caH - k)$. Final results are

$\sigma_l(k) = \frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde{\epsilon}}}\chi$ *independent* of the form of $\mathcal{W}(k)$.

When linearized, these equations are exact and valid to all scales, simply being linear perturbation theory.

Perturbation Theory

We can now perturb the e.o.m. to directly obtain solutions at second order

- Perturbing $A(t, \mathbf{x}) = \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}$ represents non-linearities in the long wavelength evolution

- Perturbing $\frac{H}{\sqrt{\tilde{\epsilon}}}$ represents "initial" non-linearities of the modes at horizon crossing

Write $(\Delta A)_{ab} = \bar{A}_{abc} \, v_c$, with $\boxed{v = \begin{pmatrix} \zeta \\ \tilde{\sigma} \\ \dot{\sigma} \end{pmatrix}}$.

- Evolution terms:

$$\bar{A}_{12c} = 2 \begin{pmatrix} (\tilde{\epsilon} - 2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp} + \tilde{\xi}^{\perp} \\ -3\chi - (\tilde{\epsilon} + \tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} - (\tilde{\eta}^{\perp})^2 \\ -3 - \tilde{\eta}^{\parallel} \end{pmatrix}, \; \bar{A}_{33c} = \begin{pmatrix} -2\tilde{\epsilon}^2 + (2\tilde{\epsilon} - \tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} + (\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\parallel} \\ 2\tilde{\epsilon}\tilde{\eta}^{\perp} + \tilde{\xi}^{\perp} \\ \tilde{\eta}^{\perp} \end{pmatrix}$$

$$\bar{A}_{32c} = 2 \times$$

$$\left(\begin{array}{c} (-6\tilde{\epsilon}-10\tilde{\epsilon}^2-2\tilde{\epsilon}\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(6+6\tilde{\epsilon}-8\tilde{\eta}^{\parallel})(\tilde{\eta}^{\perp})^2-\tilde{\epsilon}(3\chi+4\tilde{\epsilon}^2+3\xi^{\parallel})-6\tilde{\eta}^{\perp}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{111}-V_{221}}{3H^2} \\[2mm] (-12\tilde{\epsilon}-6\tilde{\eta}^{\parallel}+12\chi-6\tilde{\epsilon}^2)\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\parallel})^2\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\perp})^3-4\tilde{\epsilon}\xi^{\perp}-2\tilde{\eta}^{\parallel}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{211}-V_{222}}{3H^2} \\[2mm] 6\tilde{\eta}^{\perp}-2\tilde{\epsilon}\tilde{\eta}^{\perp}+4\tilde{\eta}^{\parallel}\tilde{\eta}^{\perp}-2\tilde{\xi}^{\perp} \end{array} \right.$$

- "Initial Condition Terms": $\Delta(\frac{H}{\sqrt{\tilde{\epsilon}}}) = (2\tilde{\epsilon} + \tilde{\eta})\frac{H}{\sqrt{\tilde{\epsilon}}}\zeta$

We solve $\qquad \frac{d}{dt}v_{ai}^{(2)} + A(t)v_{ai}^{(2)} = \bar{A}_{abc}(t')\,v_{bi}^{(1)}\,v_c^{(1)} + \Delta\,b_{ai} \qquad \Rightarrow$

$$v_{ai}^{(2)} = \int\limits_{-\infty}^{t} G_{ab}(t,t')\left(\bar{A}_{bcd}\,v_{ci}^{(1)}\,v_d^{(1)} + \Delta\,b_{bi}\right)dt'$$

$$\frac{d}{dt}G_{ab}(t_\star,t) - G_{ac}(t_\star,t)A_{cb}(t) = 0, \qquad G_{ab}(t_\star,t_\star) = \delta_{ab}$$

The Bispectrum

$\vec{v}_a$ is a spatial vector but its curl depends on the time-slicing:

$$\nabla \times \vec{\zeta}(t, \mathbf{x}) = \nabla \times \vec{\zeta}(T, \mathbf{x}) + \nabla T \times \frac{\partial}{\partial T} \vec{\zeta}(T, \mathbf{x})$$

so focus on the scalar part v_a: $\qquad \vec{v}_a = \nabla v_a + \nabla \times \vec{w}_a$.

Focus on the adiabatic perturbation $v_1 = \zeta$ and define

$$v_{1m}(k, t) = \frac{\kappa}{2} \frac{H(t_k)}{k^{3/2}} \int_{-\infty}^{t} dt' G_{1b}(t, t') X_{bm}(t') \dot{\mathcal{W}}(k, t')$$

$X_{bm} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \chi(t') \end{pmatrix}$. Then:

$$\langle \zeta\zeta \rangle(k, t) = v_{1m}(k, t) v_{1m}(k, t)$$

$$\langle \zeta\zeta\zeta \rangle(k_i, t) = (2\pi)^3 \delta(\textstyle\sum \mathbf{k}_i) \left[f(k_1, k_2) + f(k_1, k_3) + f(k_2, k_3) \right]$$

$$f(k, k') \equiv -\frac{1}{2} v_{1m}(k, t) v_{1n}(k', t) \times$$

$$\int_{-\infty}^{t} dt' \, G_{1a}(t, t') \bar{A}_{abc}^{(0)}(t') v_{bm}^{(1)}(k, t') v_{cn}^{(1)}(k', t') + k \leftrightarrow k'$$
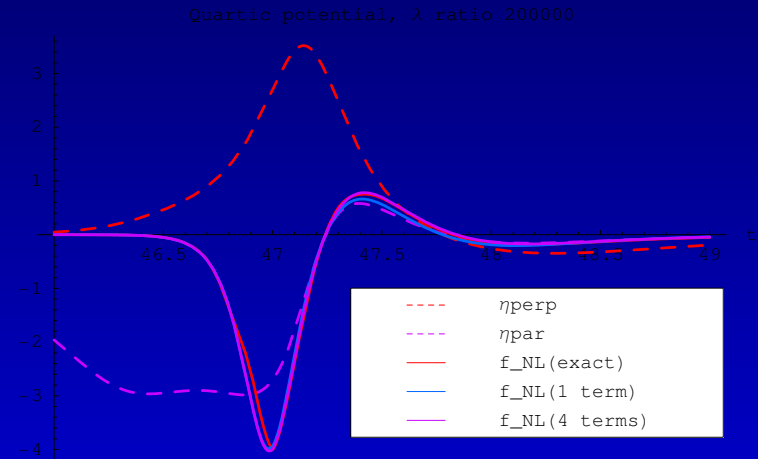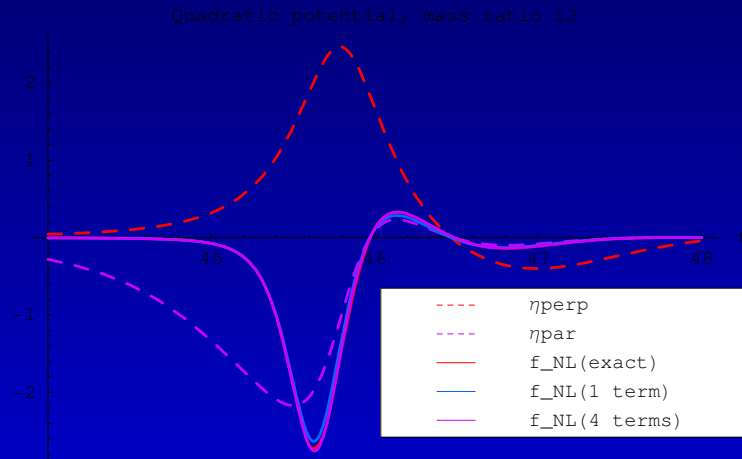
Note: No non-local terms

Non-Gaussianity

Parameterize non-Gaussianity by $f_{NL} \equiv \frac{\langle \zeta\zeta\zeta \rangle (k,k,k,t)}{\langle \zeta\zeta \rangle (k,t)}$

Two potentials:

- $V(\phi_1, \phi_2) = \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}m_2^2\phi_2^2$
- $V(\phi_1, \phi_2) = \frac{\lambda_1}{4!}\phi_1^4 + \frac{\lambda_2}{4!}\phi_2^4$

f_{NL} rises sharply when slow-roll breaks down and the fields make a turn. However, it reduces to a small value if inflation continues after the turn.

Large non-Gaussianity?

Simple inflationary models give $f_{NL} \sim \mathcal{O}(0.01) = \mathcal{O}(\tilde{\epsilon}, \tilde{\eta})$ for both single and multi field inflation.

- Less symmetric potentials?

- Non-separable potentials?

- Hybrid inflation

- Reheating, preheating

- Later dominance of another field (Curvaton)

We can expect however that if $\eta^{\perp}$ is large at the end of inflation, the next phase of evolution will be endowed with large NG. Of course this primordial input must be connected to the observable CMB sky and this connection adds one more NG component which may be dominant.

Conclusions

- Non-Gaussianity has been the focus of many studies over the past few years.

- It provides another observable that links the present to the early universe.

- Simple inflationary models, both single and multi field, predict it to be very small, $f_{NL} \sim \mathcal{O}(\tilde{\epsilon}, \tilde{\eta})$

- However, later processes and/or more complicated models may yield larger primordial non-Gaussianity.

- An interesting observable for the future.