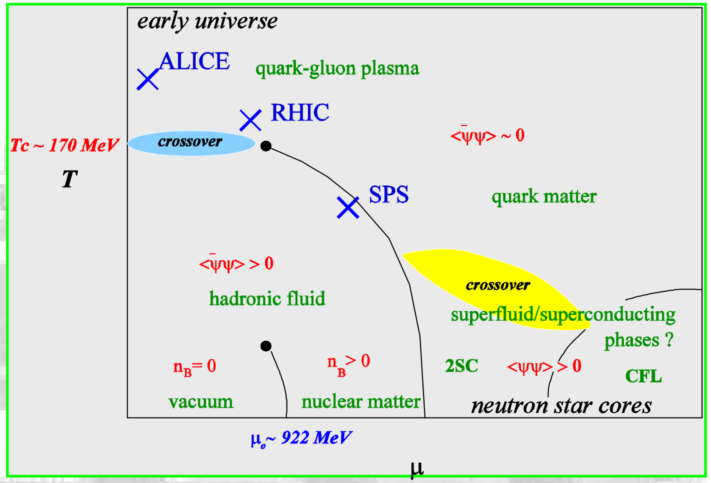
Schwinger-Dyson equations, nonlinear random processes and diagrammatic algorithms

Pavel Buividovich
(Regensburg University)

GGI Workshop “New Frontiers in Lattice Gauge Theory”, 27.08.2012

Motivation: Lattice QCD at finite baryon density

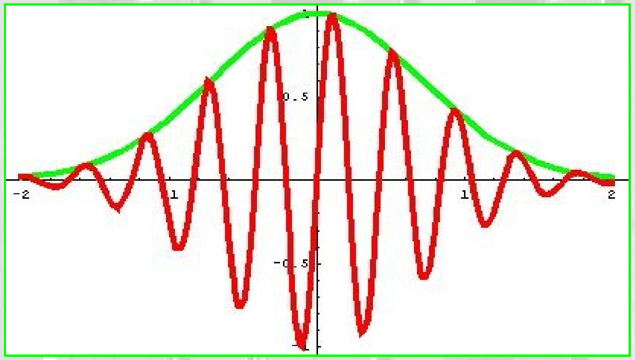
- Lattice QCD is one of the main tools to study quark-gluon plasma
- Interpretation of heavy-ion collision experiments: RHIC, LHC, FAIR,...



But: baryon density is finite in experiment !!!

Dirac operator is not Hermitean anymore

exp(-S) is complex!!! **Sign problem**



Monte-Carlo methods are not applicable !!!

Try to look for alternative numerical simulation strategies

Lattice QCD at finite baryon density: some approaches

- Taylor expansion in powers of μ
- Imaginary chemical potential
- SU(2) or G_2 gauge theories
- Solution of truncated Schwinger-Dyson equations in a fixed gauge
- Complex Langevin dynamics
- Infinitely-strong coupling limit
- Chiral Matrix models …

"Reasonable" approximations with <u>unknown errors,</u>

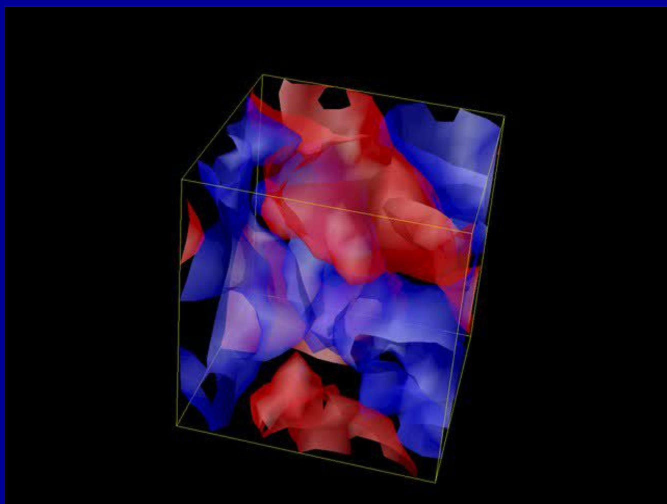
BUT

No systematically improvable methods!

Path integrals: sum over paths vs. sum over fields

Quantum field theory:

Sum over fields



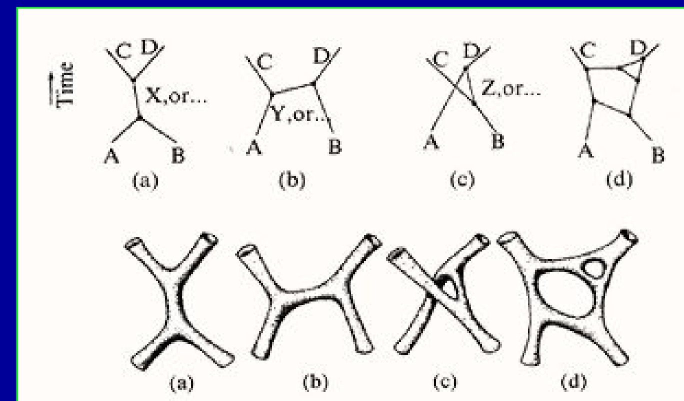
$$\mathcal{Z} = \mathrm{Tr}\, e^{-\hat{\mathcal{H}}/kT} =$$
$$= \int \mathcal{D}\phi\left(x^{\mu}\right)\exp\left(-S_E\left[\phi\left(x^{\mu}\right)\right]\right)$$

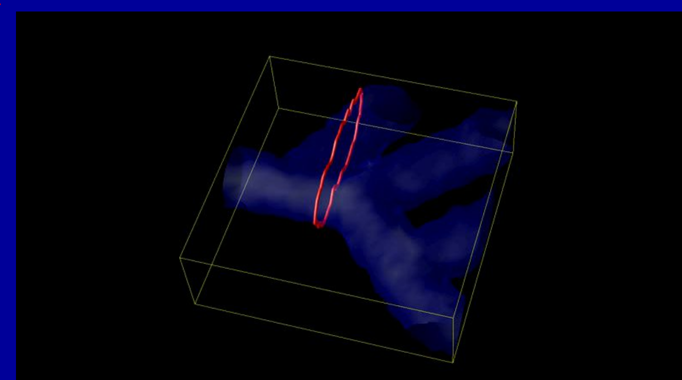
Euclidean action:

$$S_E = \int d^D x\left(\frac{1}{2}\,\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\,\phi^2 + V\left(\phi\right)\right)$$

Perturbative expansions

Sum over interacting paths

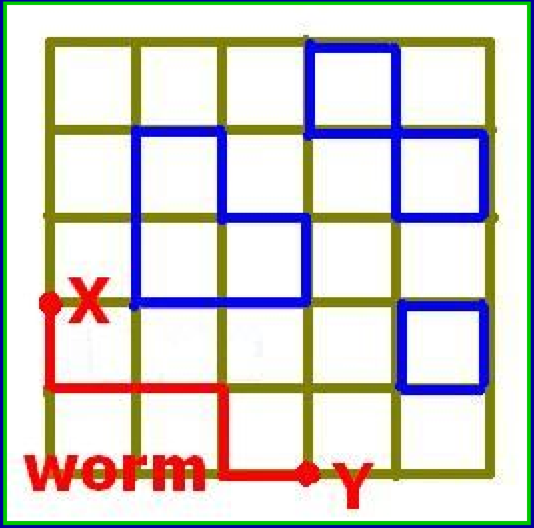




$$\mathcal{Z} = \sum_k \frac{\lambda^k}{k!}\exp\left(-L\left(\text{Paths connecting } k \text{ vertices}\right)\right)$$

Worm Algorithm [Prokof'ev, Svistunov]

- Monte-Carlo sampling of closed vacuum diagrams:
  nonlocal updates, closure constraint
- Worm Algorithm: sample closed diagrams + open diagram
- Local updates: open graphs ⟷ closed graphs
- Direct sampling of field correlators (dedicated simulations)



x, y – head and tail of the worm

$$\langle \sigma_x \sigma_y \rangle \sim p(x, y)$$

Correlator = probability distribution of head and tail

- Applications: systems with "simple" and convergent perturbative expansions (Ising, Hubbard, 2d fermions …)
- Very fast and efficient algorithm!!!

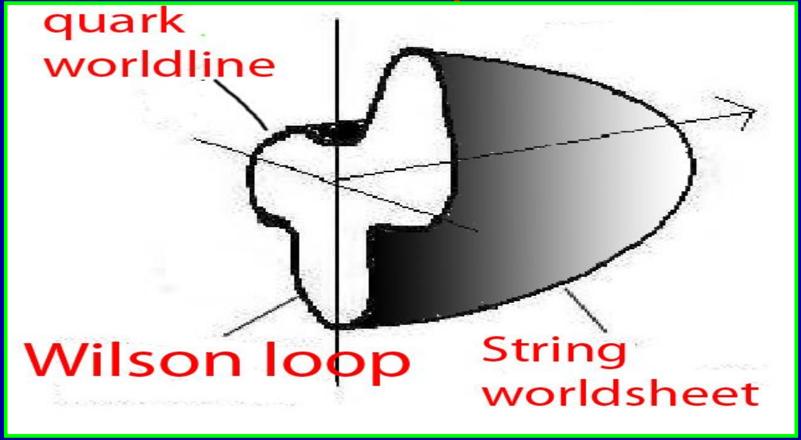
Worm algorithms for QCD?

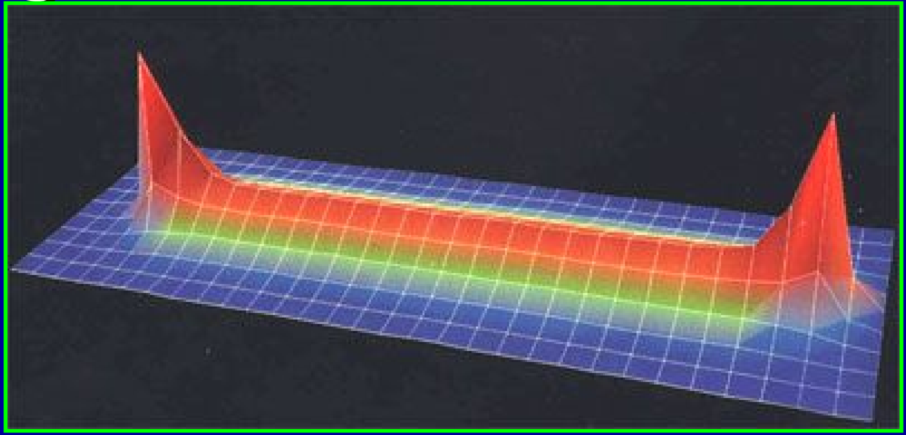
Attracted a lot of interest recently as a tool for QCD at finite density:

- Y. D. Mercado, H. G. Evertz, C. Gattringer, ArXiv:1102.3096 – Effective theory capturing center symmetry
- P. de Forcrand, M. Fromm, ArXiv:0907.1915 – Infinitely strong coupling
- W. Unger, P. de Forcrand, ArXiv:1107.1553 – Infinitely strong coupling, continuos time
- K. Miura et al., ArXiv:0907.4245 – Explicit strong-coupling series …

Worm algorithms for QCD?

- Strong-coupling expansion for lattice gauge theory: confining strings [Wilson 1974]
- Intuitively: basic d.o.f.'s in gauge theories = confining strings (also AdS/CFT etc.)
- Worm → something like "tube"





- BUT: complicated group-theoretical factors!!! Not known explicitly → Still no worm algorithm for non-Abelian LGT (Abelian version: [Korzec, Wolff' 2010])

Worm-like algorithms from Schwinger-Dyson equations

Basic idea:

- Schwinger-Dyson (SD) equations: infinite hierarchy of linear equations for field correlators $G(x_1, \ldots, x_n)$

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta\phi(x)} \left(\phi(x_1) \ldots \phi(x_n) \, \exp\left(-S[\phi]\right) \right) = 0$$

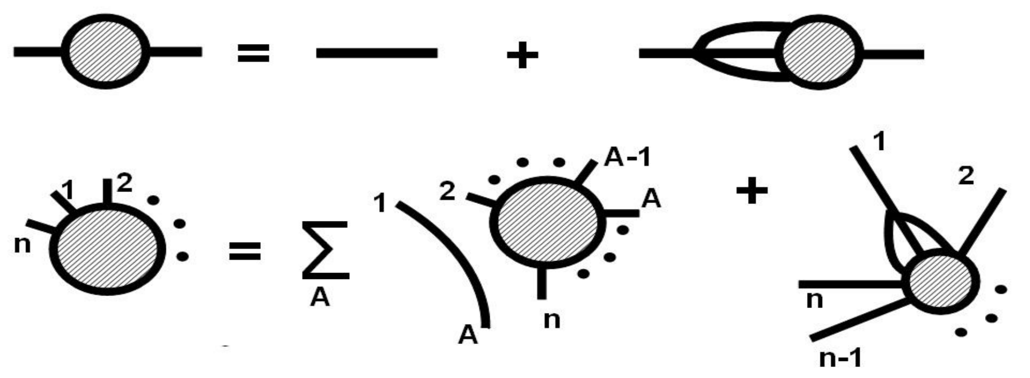
- Solve SD equations: interpret them as steady-state equations for some random process

$$w(A) = \sum_B P(B \to A) \, w(B)$$

- $G(x_1, \ldots, x_n)$: ~ probability to obtain $\{x_1, \ldots, x_n\}$ (Like in Worm algorithm, but for all correlators)

Example: Schwinger-Dyson equations in φ⁴ theory

$$S\left[\phi\left(x\right)\right] = \int d^D x \left(\frac{1}{2}\phi\left(x\right)\left(m^2 - \Delta\right)\phi\left(x\right) + \frac{\lambda}{4}\phi^4\left(x\right)\right)$$



$$G\left(x_1, x_2\right) = \delta\left(x_1, x_2\right) + \sum_{\pm\mu} \kappa\, G\left(x_1 \pm \hat{\mu}, x_2\right) - \lambda\, G\left(x_1, x_1, x_1, x_2\right)$$

$$G\left(x_1, x_2, \ldots, x_n\right) = \sum_{A=2}^{n} \delta\left(x_1, x_A\right) G\left(x_1, \ldots, x_{A-1}, x_{A+1}, \ldots, x_n\right) + $$

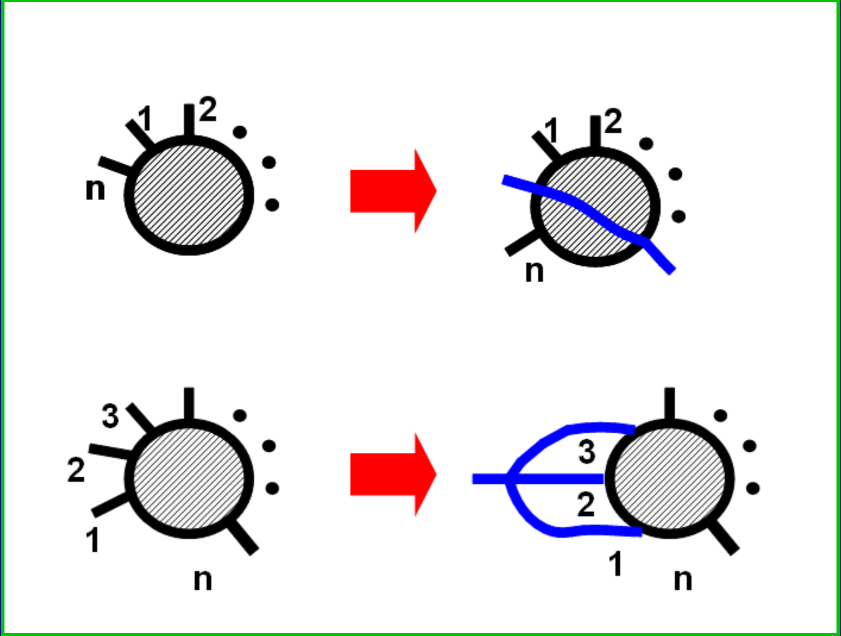
$$+ \sum_{\pm\mu} \kappa\, G\left(x_1 \pm \hat{\mu}, \ldots, x_n\right) - \lambda\, G\left(x_1, x_1, x_1, x_2, \ldots, x_n\right)$$

Schwinger-Dyson equations for φ^4 theory: stochastic interpretation

- Steady-state equations for Markov processes:

$$w(A) = \sum_B P(B \to A)\, w(B)$$

- Space of states:
  sequences of coordinates $\{x_1, \ldots, x_n\}$
- Possible transitions:
  - Add pair of points $\{x, x\}$ at random position $1 \ldots n+1$
  - Random walk for topmost coordinate
  - If three points meet – merge
  - Restart with two points $\{x, x\}$



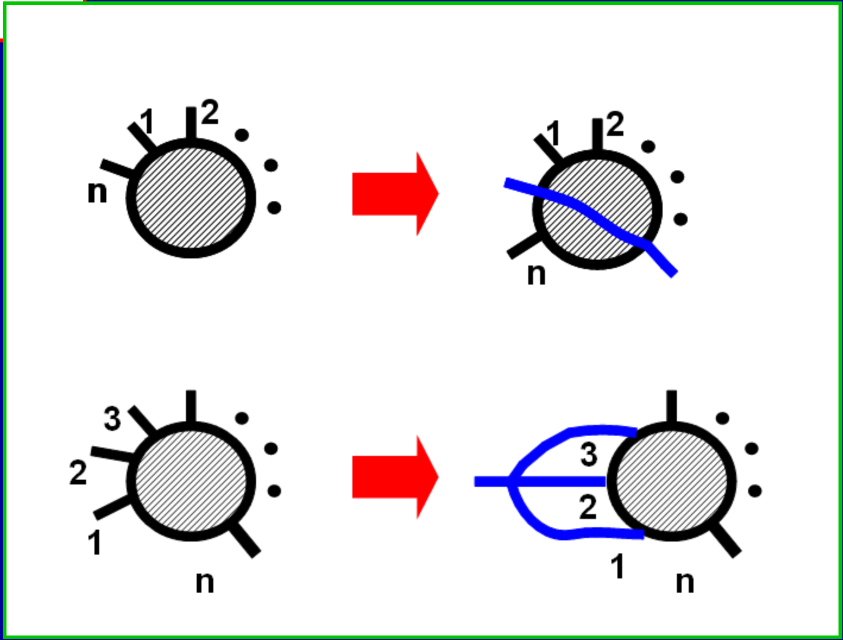
- No truncation of SD equations
- No explicit form of perturbative series

Stochastic interpretation in momentum space

- Steady-state equations for Markov processes:

$$w(A) = \sum_B P(B \to A)\, w(B)$$

- Space of states:
  sequences of momenta $\{p_1, \ldots, p_n\}$



- Possible transitions:
  - Add pair of momenta {p, -p} at positions 1, A = 2 … n + 1
  - Add up three first momenta (merge)
- Restart with {p, -p}
- Probability for new momenta:

$$\sim \frac{1}{p^2 + m_0^2}$$

Diagrammatic interpretation

History of such a random process: unique Feynman diagram
BUT: no need to remember intermediate states

Measurements of connected, 1PI, 2PI correlators are possible!!! In practice: label connected legs
Kinematical factor for each diagram:

$$\int d^D q_1 \ldots d^D q_{M_I} \prod_{i=1}^{M_I} \frac{1}{q_i^2 + m_0^2} \prod_{j=1}^{M_D} \frac{1}{Q_j^2 + m_0^2}$$

q_i are independent momenta, Q_j – depend on q_i

→ Monte-Carlo integration over independent momenta

Normalizing the transition probabilities

- Problem: probability of "Add momenta" grows as (n+1), rescaling $G(p_1, \ldots, p_n)$ – does not help.
- Manifestation of series divergence!!!
- Solution: explicitly count diagram order m. Transition probabilities depend on m
- Extended state space: $\{p_1, \ldots, p_n\}$ and m – diagram order
- Field correlators:

$$G\left(p_1, \ldots, p_n\right) = \sum_{m=0}^{+\infty} c_{n,m} \left(-\lambda\right)^m w_m\left(p_1, \ldots, p_n\right)$$

- $w_m(p_1, \ldots, p_n)$ – probability to encounter m-th order diagram with momenta $\{p_1, \ldots, p_n\}$ on external legs

Normalizing the transition probabilities

- Finite transition probabilities:

$$c_{n,m} = \Gamma\left(n/2 + m + 1/2\right)\, x^{-(n-2)}\, y^{-m}$$

- Factorial divergence of series is absorbed into the growth of $C_{n,m}$!!!

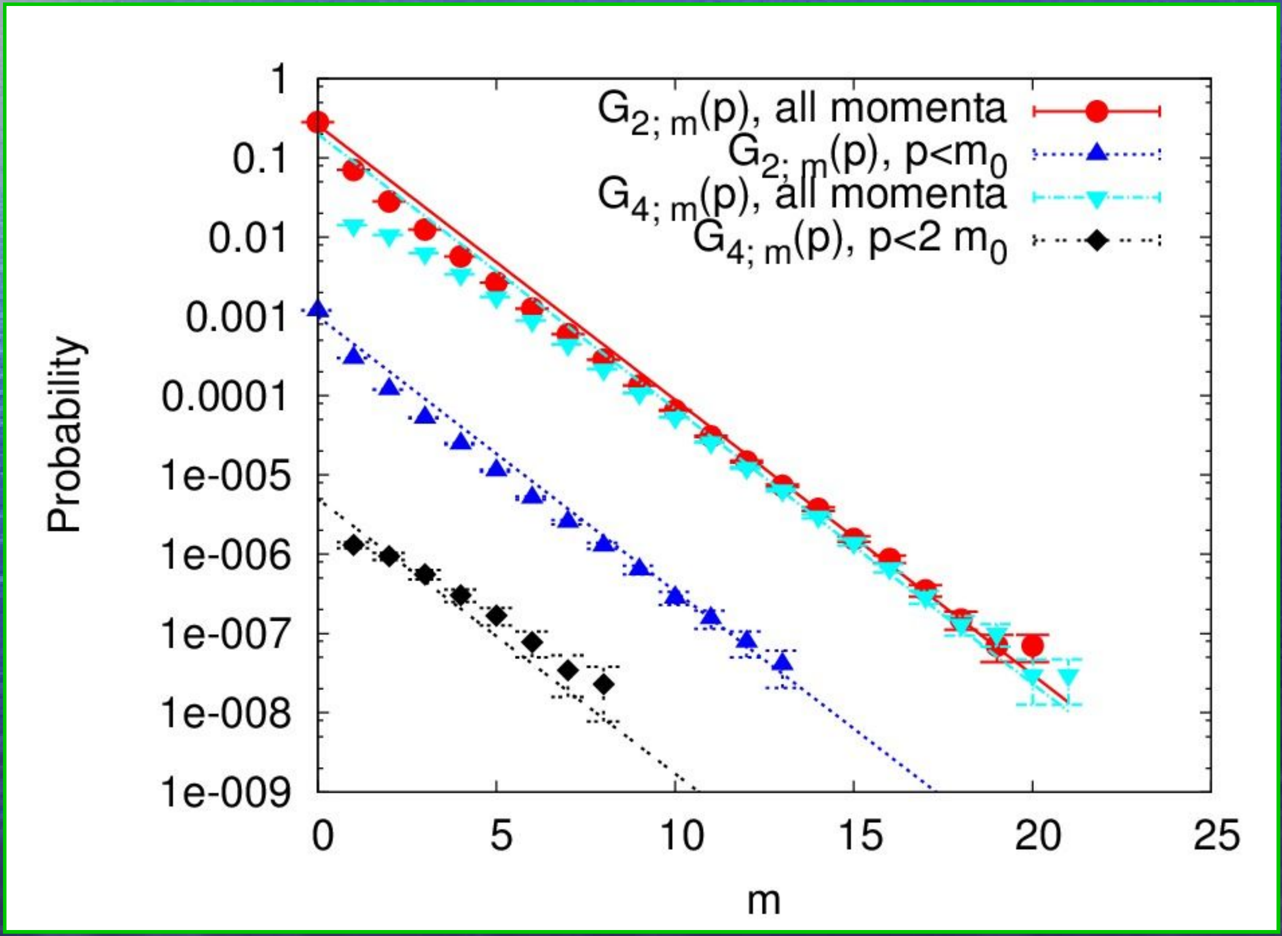
- Probabilities (for optimal x, y):
  - Add momenta:
  - Sum up momenta + increase the order:
- Otherwise restart

$$p_A = \frac{1}{2}\,\frac{n+1}{n+m+1}$$

$$p_V = \frac{1}{2}$$

Critical slowing down?

Transition probabilities do not depend on bare mass or coupling!!! (Unlike in the standard MC)
No free lunch: kinematical suppression of small-p region ($\sim \Lambda_{IR}^{D}$)



Resummation

- Integral representation of $C_{n,m} = \Gamma(n/2 + m + 1/2)\, x^{-(n-2)}\, y^{-m}$:

$$G_n = x^{-n+2} \left(\frac{y}{\lambda_0}\right)^{\frac{n+1}{2}} \int_0^{+\infty} dz\, \exp\left(-\frac{yz}{\lambda_0}\right) z^{\frac{n-1}{2}} \left(\sum_{m=0}^{+\infty} (-z)^m\, w_{n,m}\right)$$

Pade-Borel resummation. Borel image of correlators!!!

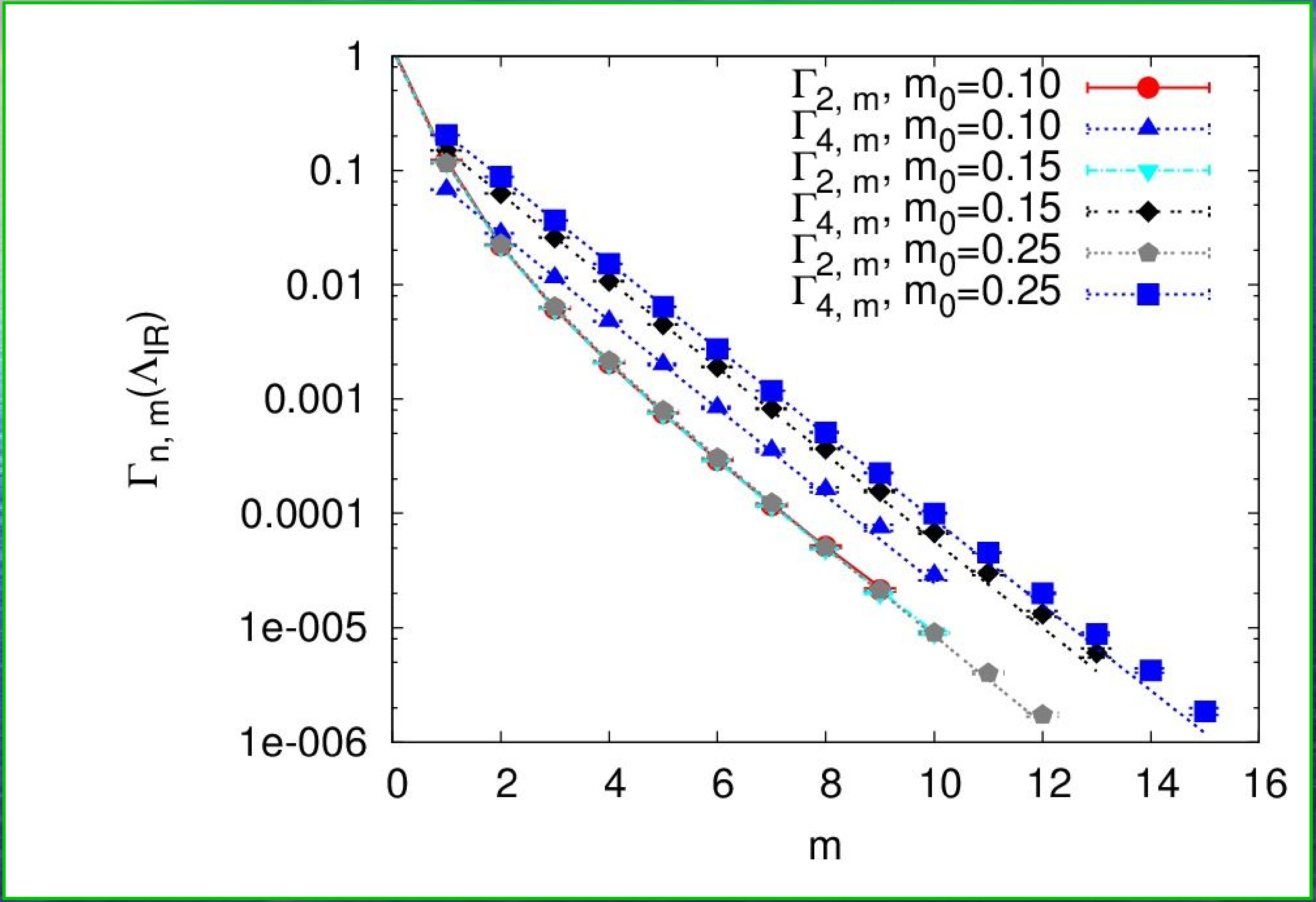
- Poles of Borel image: exponentials in $w_{n,m}$

$$w_{n,m} = \sum_k a_k\, b_k^m$$

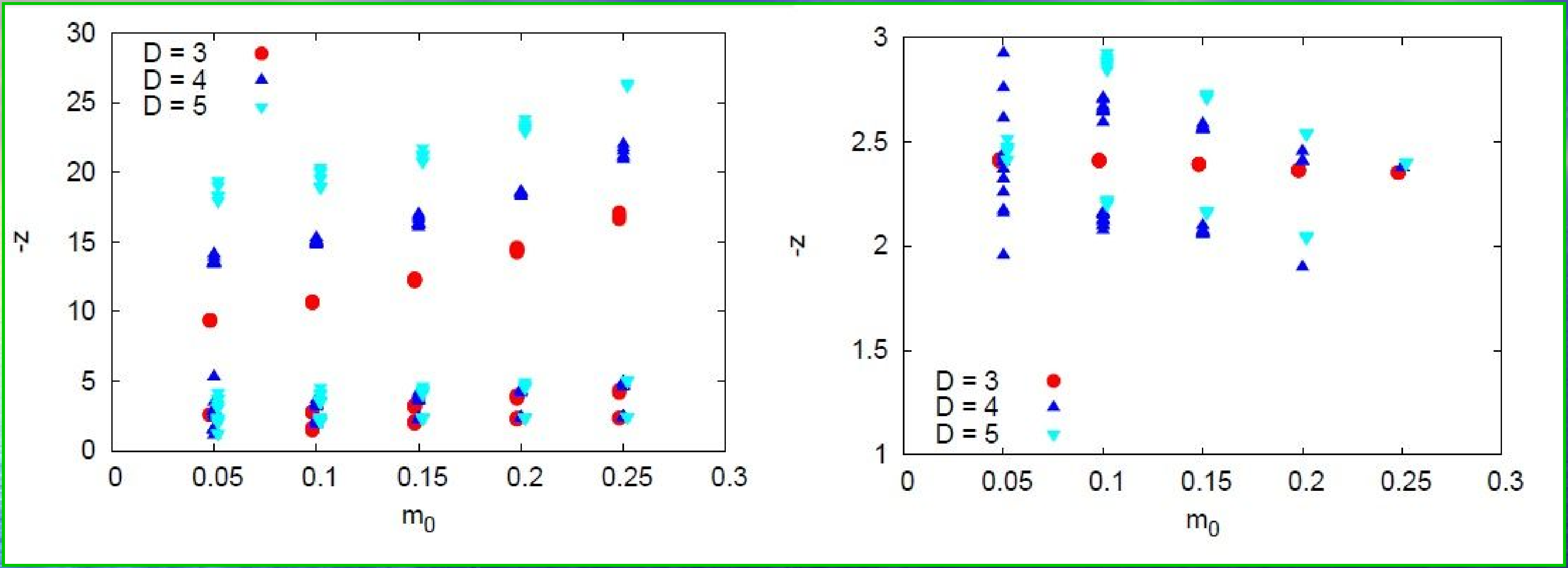
- Pade approximants are unstable
- Poles can be found by fitting
- Special fitting procedure using SVD of Hankel matrices

No need for resummation at large N!!!

Resummation: fits by multiple exponents



Resummation: positions of poles



Two-point function

Connected truncated four-point function

2-3 poles can be extracted with reasonable accuracy

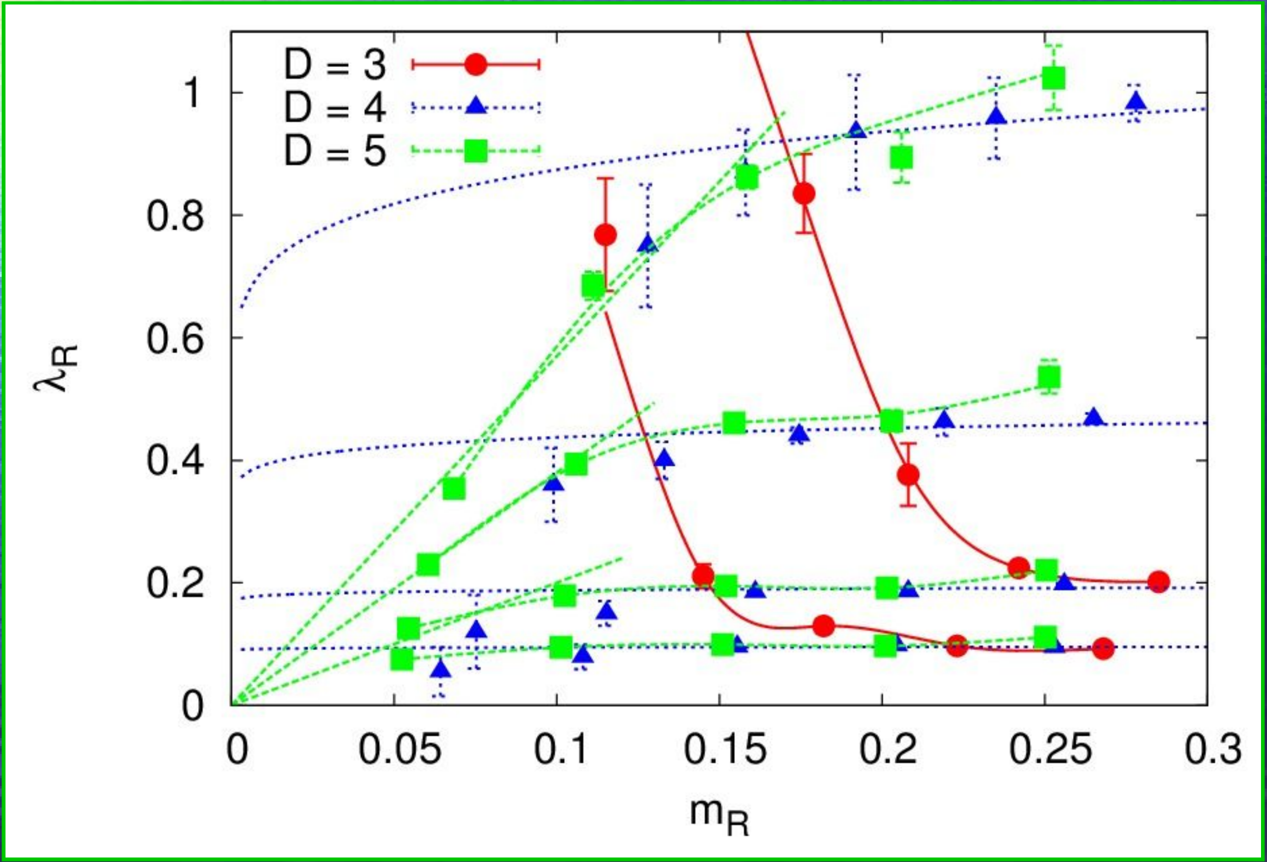
Test: triviality of φ^4 theory in D ≥ 4

Renormalized mass:

$$G(p) = \frac{Z_R}{m_R^2 + p^2 + O(p^4)}$$

Renormalized coupling:

$$\lambda_R = -1/6\, Z_R^2\, \Gamma(0,0,0,0)$$

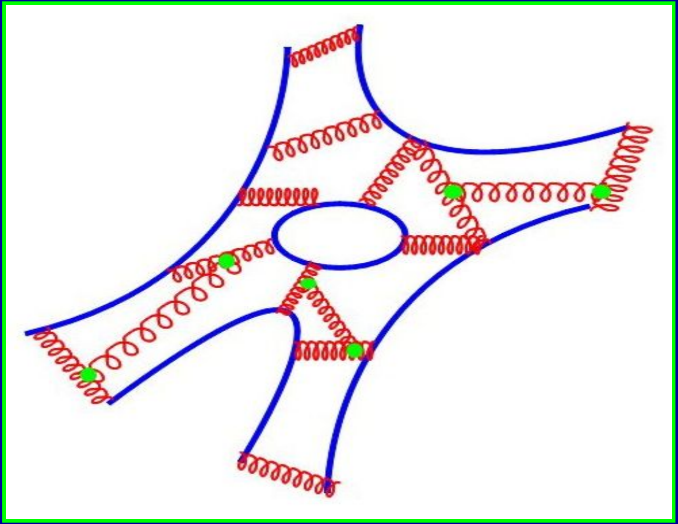


CPU time:
several hrs/point
(2GHz core)
[Buividovich, ArXiv:1104.3459]

Large-N gauge theory in the Veneziano limit

- Gauge theory with the action

$$L = -\frac{N}{\lambda}\,\mathrm{Tr}\,F_{\mu\nu}^2 + \sum_{f=1}^{N_f} \bar{\psi}_f \left(D+m\right)\psi_f$$



- t-Hooft-Veneziano limit:

  $N \to \infty$, $N_f \to \infty$, λ fixed, N_f/N fixed

- Only planar diagrams contribute! $\Rightarrow$ connection with strings
- Factorization of Wilson loops $W(C) = 1/N\ tr\ P\ exp(i\ \int dx^{\mu}\ A_{\mu})$:

$$\langle W\left[C_1\right] W\left[C_2\right]\rangle = \langle W\left[C_1\right]\rangle \langle W\left[C_2\right]\rangle + O\left(1/N\right)$$

- Better approximation for real QCD than pure large-N gauge theory: meson decays, deconfinement phase etc.

Large-N gauge theory in the Veneziano limit

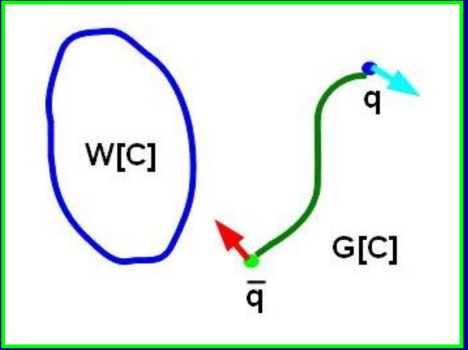
- Lattice action:

$$S = -N\beta \sum_p \mathrm{Tr}\, g_p + \sum_x \bar{\psi}^f \psi^f -$$

$$-\sum_x \sum_\mu \left(\kappa_\mu^{(+)} \bar{\psi}^f \left(x - \hat{\mu}\right) \left(\gamma_{+\mu}\right) g_{x-\hat{\mu},\mu} \psi^f \left(x\right) - \kappa_\mu^{(-)} \bar{\psi}^f \left(x + \hat{\mu}\right) \left(\gamma_{-\mu}\right) g^\dagger_{x,\mu} \psi^f \left(x\right) \right)$$

No EK reduction in the large-N limit! Center symmetry broken by fermions.
Naive Dirac fermions: N_f is infinite, no need to care about doublers!!!.

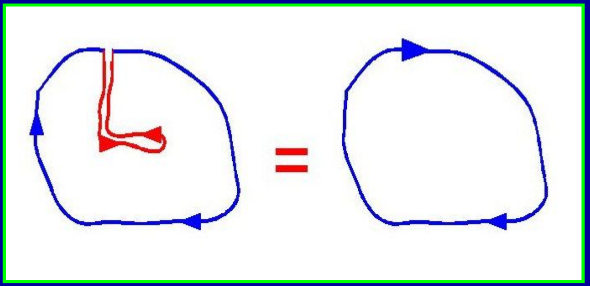
- Basic observables:
  - Wilson loops = closed string amplitudes
  - Wilson lines with quarks at the ends = open string amplitudes



$$W\left[l_1 \ldots l_n\right] = \langle \frac{1}{N} \mathrm{Tr}\left(g_{l_1} \ldots g_{l_n}\right) \rangle$$

$$G_{\alpha\beta}\left[l_1 \ldots l_n\right] = \langle \frac{1}{N N_f} \bar{\psi}^f_\beta \left(s_1\right) g_{l_1} \ldots g_{l_n} \psi^f_\alpha \left(s_n\right) \rangle$$

- Zigzag symmetry for QCD strings!!!



Migdal-Makeenko loop equations

Loop equations in the closed string sector:

$$
\begin{aligned}
W[l_1 \dots l_n] = \delta(l_1, -l_2)\, W[l_3 \dots l_n] + \delta(l_1, -l_n)\, W[l_2 \dots l_{n-1}] + \\
+ \sum_{A=3}^{n-1} \delta(l_1, -l_A)\, W[l_2 \dots l_{A-1}]\, W[l_{A+1} \dots l_n] - \\
- \sum_{A=2}^{n} \delta(l_1, l_A)\, W[l_1 \dots l_{A-1}]\, W[l_A \dots l_n] + \\
+\beta \sum_{staple(l_1)} W[st\, l_2 \dots l_n] - \beta \sum_{staple(l_1)} W[l_1\, (-st)\, l_1\, l_2 \dots l_n] + \\
+ \frac{N_f}{N} \kappa^{(-)}_{\mu(l_1)} \left(\gamma^{\beta\alpha}_{-\mu(l_1)}\right) G_{\alpha\beta}(l_2 \dots l_n) - \frac{N_f}{N} \kappa^{(+)}_{\mu(l_1)} \left(\gamma^{\beta\alpha}_{+\mu(l_1)}\right) G_{\alpha\beta}(l_1\, l_2 \dots l_n\, l_1)
\end{aligned}
$$

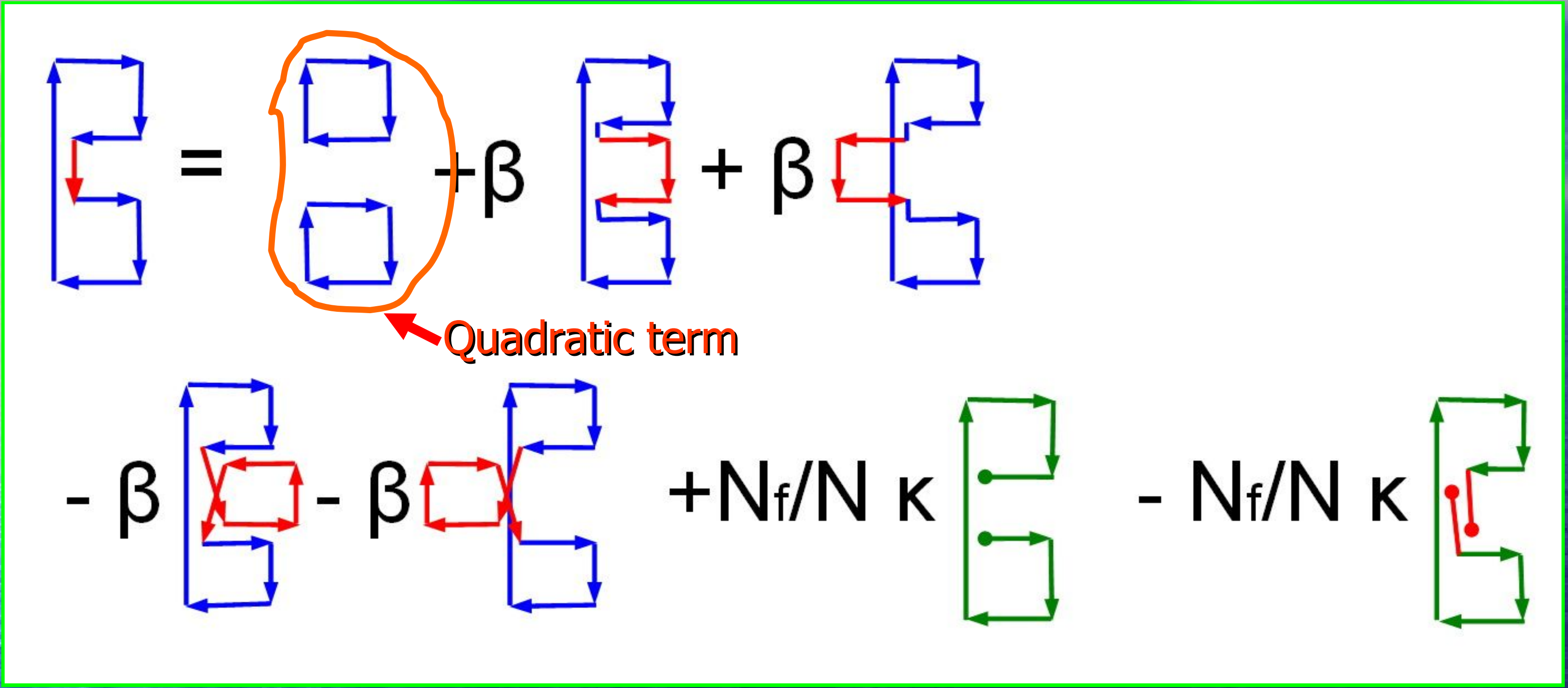
Loop equations in the open string sector:

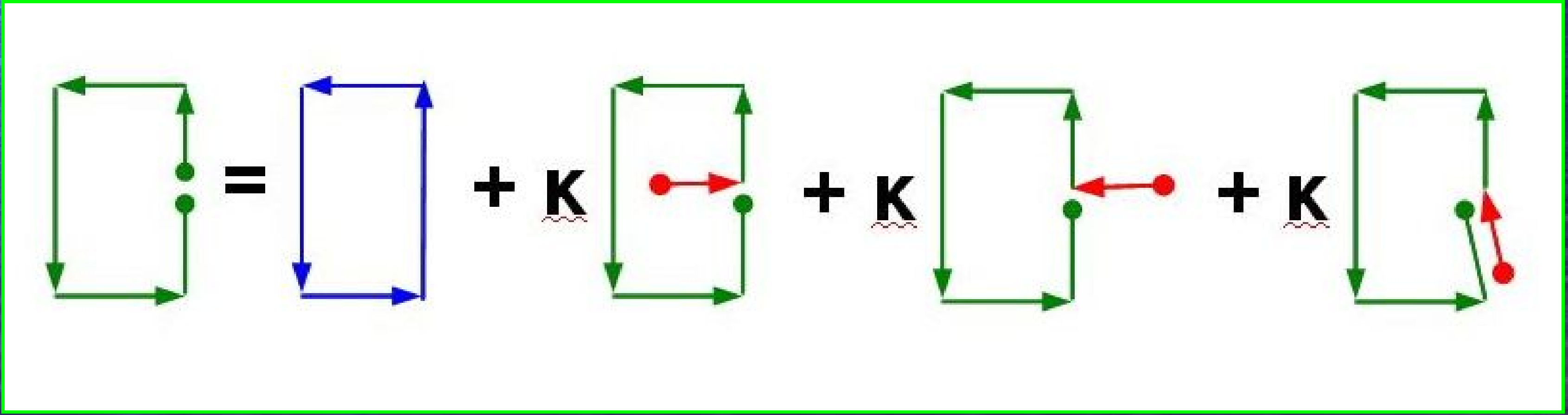
$$
\begin{aligned}
G_{\alpha\beta}[l_1 \dots l_n] = -\delta_{\alpha\beta}\, \delta(s_1, s_n)\, W[l_1 \dots l_n] + \\
+ \sum_{\mu} \kappa^{(+)}_{\mu} \left(\gamma^{\alpha\delta}_{+\mu}\right) G_{\delta\beta}(\mu l_1 \dots l_n) + \sum_{\mu} \kappa^{(-)}_{\mu} \left(\gamma^{\alpha\delta}_{-\mu}\right) G_{\delta\beta}((-\mu)\, l_1 \dots l_n)
\end{aligned}
$$

Infinite hierarchy of quadratic equations!

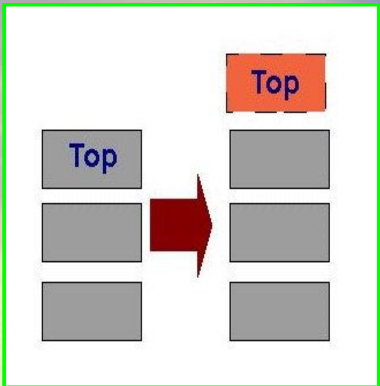
Markov-chain interpretation?

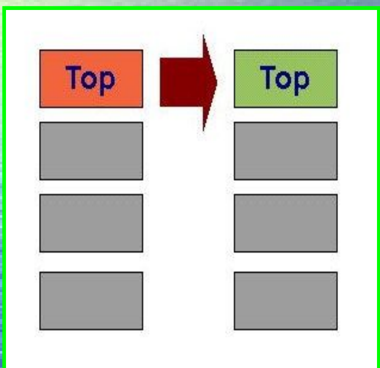
Loop equations illustrated

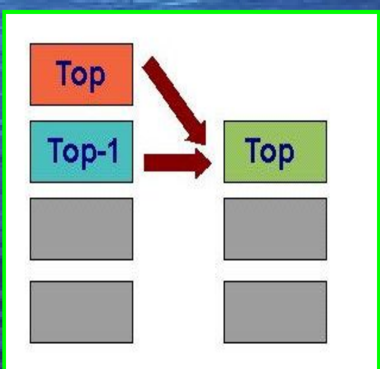




Nonlinear Random Processes







[Buividovich, ArXiv:1009.4033]
Also: Recursive Markov Chain
[Etessami,Yannakakis, 2005]

- Let X be some discrete set
- Consider stack of the elements of X

- At each process step:

➢ **Create:** with probability $P_c(x)$ create new x and push it to stack

➢ **Evolve:** with probability $P_e(x|y)$ replace y on the top of the stack with x

➢ **Merge:** with probability $P_m(x|y_1,y_2)$ pop two elements y_1, y_2 from the stack and push x into the stack

➢ Otherwise restart

Nonlinear Random Processes: Steady State and Propagation of Chaos

- Probability to find n elements $x_1 \dots x_n$ in the stack:

$$W(x_1, \dots, x_n)$$

- Propagation of chaos [McKean, 1966]
(= factorization at large-N [tHooft, Witten, 197x]):

$$W(x_1, \dots, x_n) = w_0(x_1)\, w(x_2) \dots w(x_n)$$

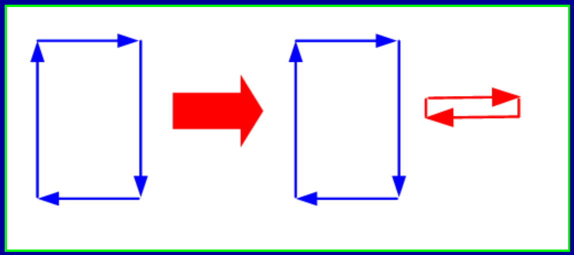
- Steady-state equation (sum over y, z):

$$w(x) = P_c(x) + P_e(x|y)\, w(y) + P_m(x|y,z)\, w(y)\, w(z)$$

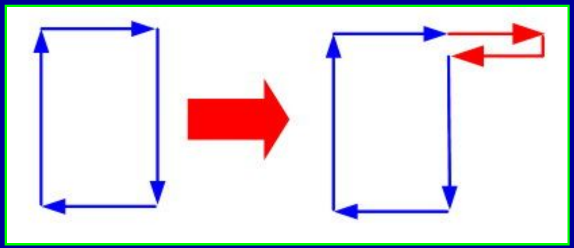
Loop equations: stochastic interpretation

Stack of strings (= open or closed loops)!
Wilson loop W[C] ~ Probabilty of generating loop C
Possible transitions (<u>closed string sector</u>):

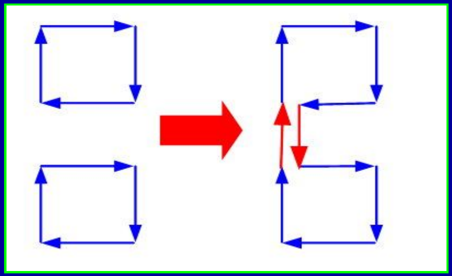
Create new string



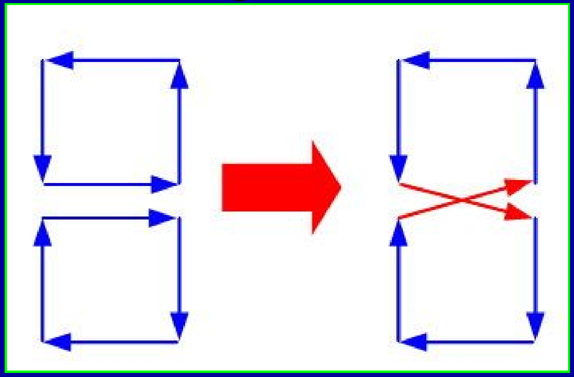
Append links to string



Join strings with links

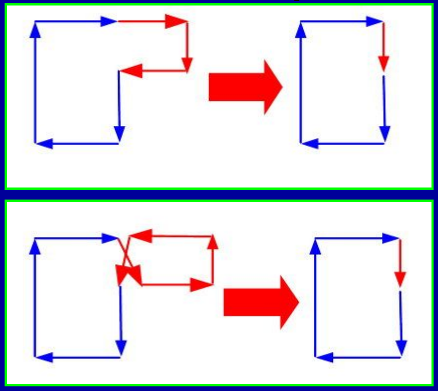


Join strings



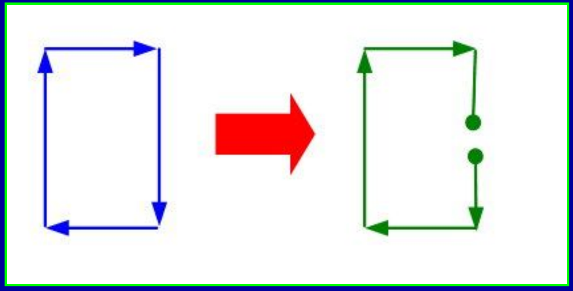
…if have collinear links

Remove staples



Probability ~ β

Create open string

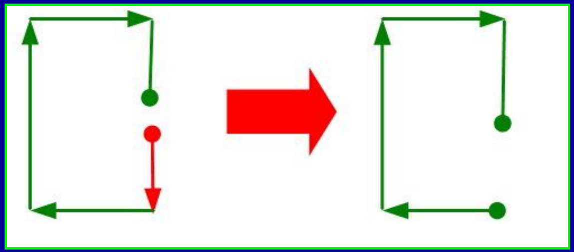


Identical spin states

Loop equations: stochastic interpretation

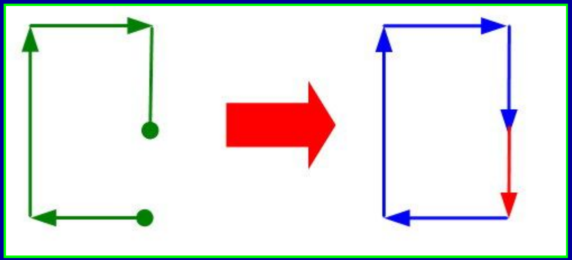
Stack of strings (= open or closed loops)!
Possible transitions (open string sector):

Truncate open string



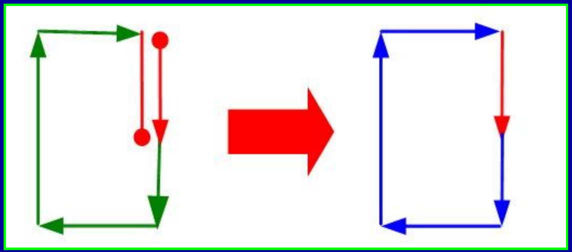
Probability ~ κ

Close by adding link



Probability ~ N_f/N κ

Close by removing link



Probability ~ N_f/N κ

- Hopping expansion for fermions (~20 orders)
- Strong-coupling expansion (series in β) for gauge fields (~ 5 orders)

Disclaimer: this work is in progress, so the algorithm is far from optimal...

Sign problem revisited

- Different terms in loop equations have different signs
- ⇒ Configurations should be additionally reweighted

- Each loop comes with a complex-valued phase
  (+/-1 in pure gauge, exp(i π k/4) with Dirac fermions)

- Sign problem is very mild (strong-coupling only)?

$$\frac{P_+ - P_-}{P_+ + P_-} \sim 0.7$$

for 1x1 Wilson loops

- For large β (close to the continuum):
  sign problem should be important

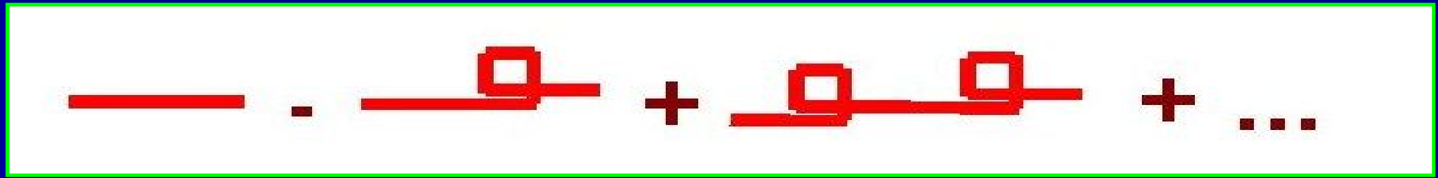
- Large terms ~β sum up to ~1

Chemical potential:
κ -> κ exp(±μ)
No additional phases

Sign problem revisited

Interacting fermions:

- Extremely severe sign problem in configuration space [U. Wolff, ArXiv:0812.0677]
- BUT: most time is spent on generating "free" random walks
- All worldlines can be summed up analytically
- ⇒ Manageable sign in momentum space [Prokof'ev, Svistunov]



Momentum space loops for QCD?

- Easy to construct in the continuum [Migdal, Makeenko, 198x]

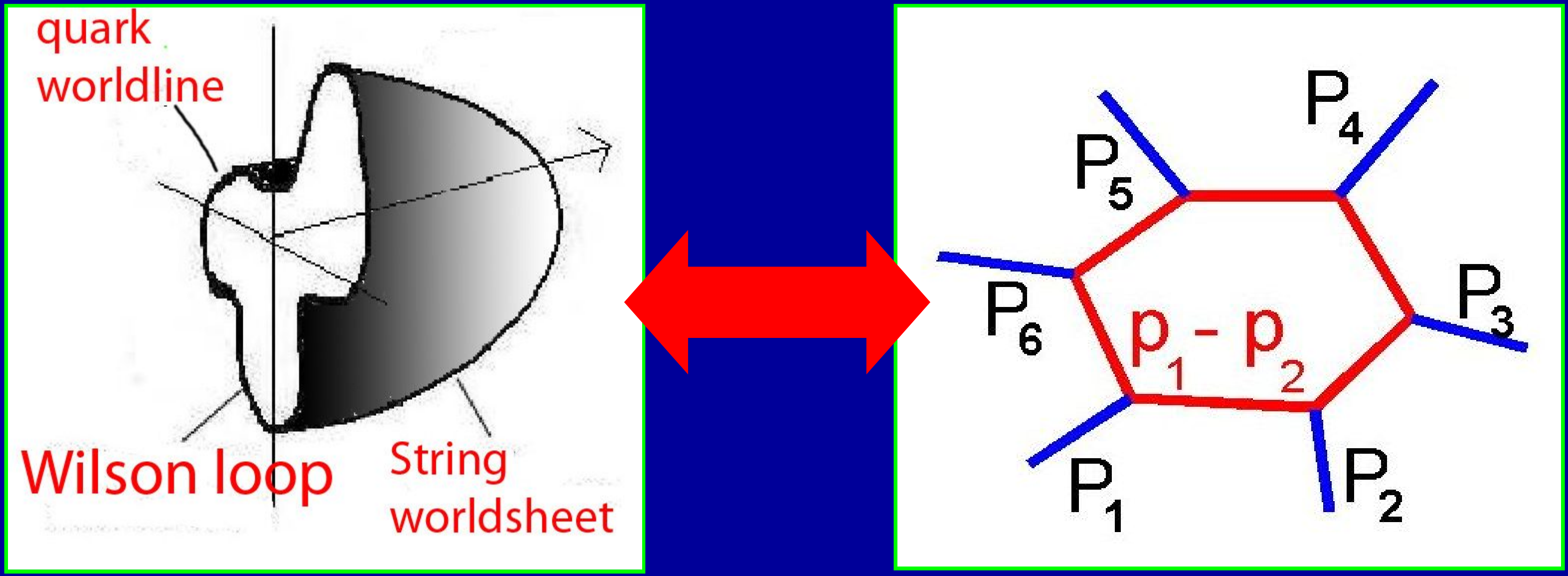
$$W\left[p_\mu(s)\right] = \int \mathcal{D}C \exp\left(i \int\limits_C dx^\mu p_\mu\right) W\left[C\right]$$

- BUT no obvious discretization suitable for numerics

Measurement procedure

- Measurement of string tension: probability to get a rectangular R x T Wilson loop - almost ZERO

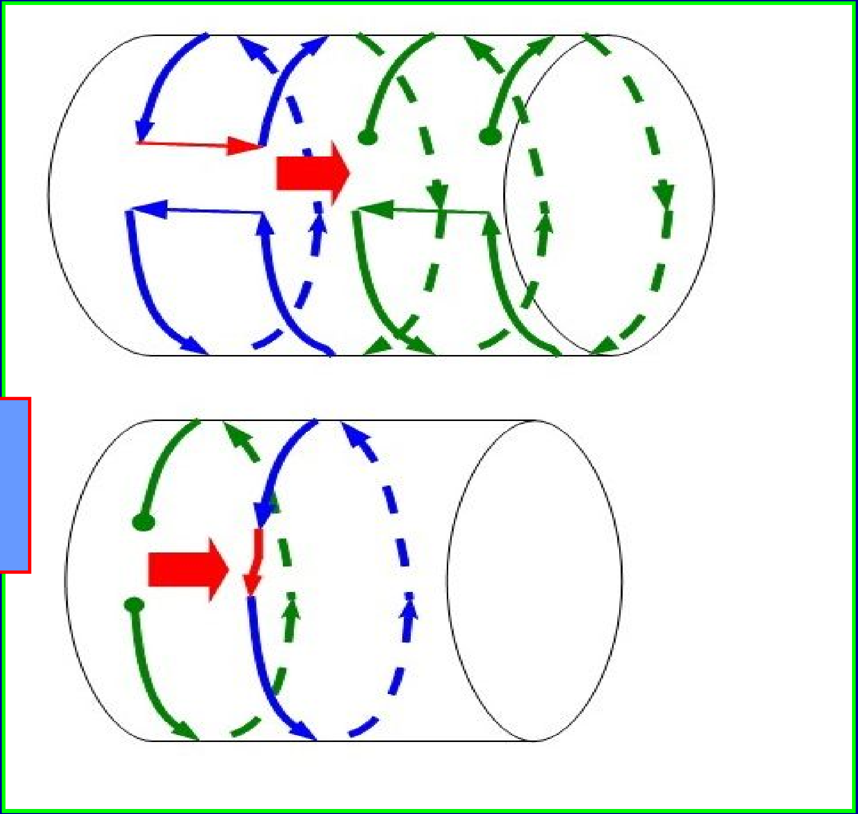
- Physical observables = Mesonic correlators = sums over all loops



- Mesonic correlators = Loops in momentum space [Makeenko, Olesen, ArXiv:0810.4778]

Temperature and chemical potential

- Finite temperature: strings on cylinder R ~1/T
- Winding strings = Polyakov loops ~ quark free energy
- No way to create winding string in pure gauge theory at large-N $\longrightarrow$ EK reduction



- Veneziano limit: open strings wrap and close
- Chemical potential:

$\kappa \to \kappa \exp(+/- \mu)$

No signs or phases!

- Strings oriented in the time direction are favoured

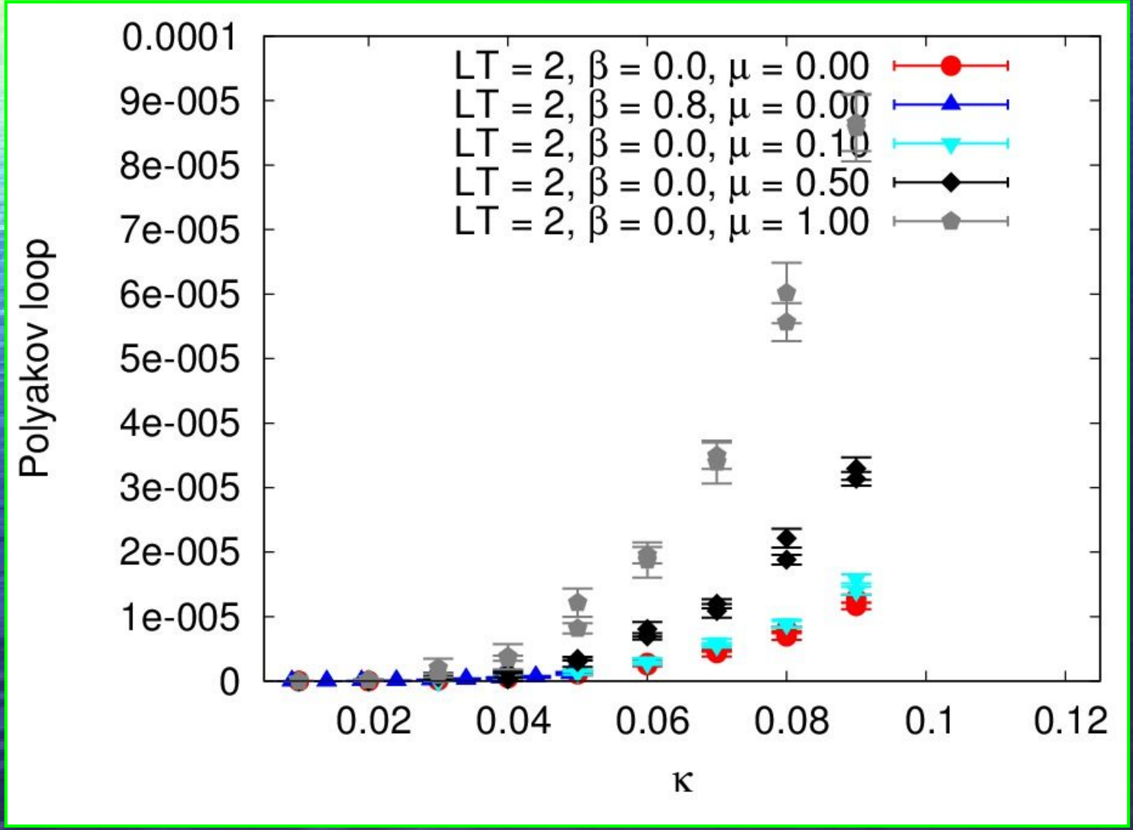
Phase diagram of the theory: a sketch

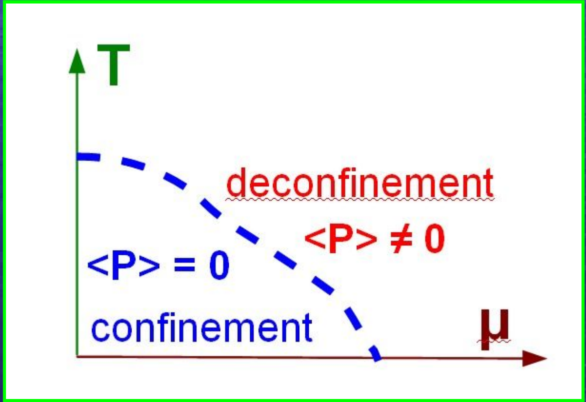
High temperature
(small cylinder radius)
OR
Large chemical potential

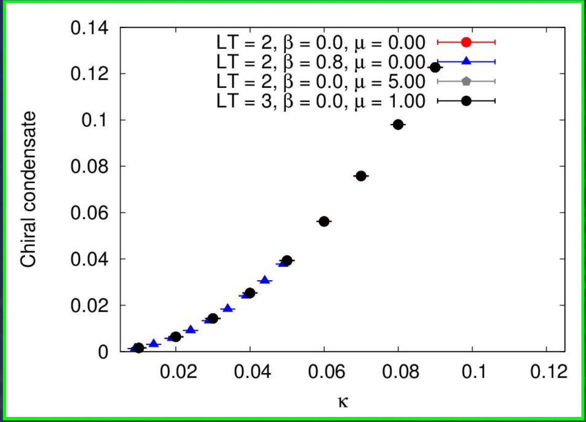
Numerous winding strings

Nonzero Polyakov loop

Deconfinement phase







Summary and outlook

- Diagrammatic Monte-Carlo and Worm algorithm:
  useful strategies complimentary to standard Monte-Carlo
- Stochastic interpretation of Schwinger-Dyson equations:
  a novel way to stochastically sum up perturbative series

Advantages:

- Implicit construction of perturbation theory
- No truncation of SD eq-s
- Large-N limit is very easy
- Naturally treats divergent series
- No sign problem at $\mu \neq 0$

Disadvantages:

- Limited to the "very strong-coupling" expansion (so far?)
- Requires large statistics in IR region

QCD in terms of strings without explicit "stringy" action!!!

Summary and outlook

Possible extensions:

- Weak-coupling theory: Wilson loops in momentum space?
  - Relation to meson scattering amplitudes
  - Possible reduction of the sign problem
- Introduction of condensates?
  - Long perturbative series ~ Short perturbative series + Condensates [Vainshtein, Zakharov]
  - Combination with Renormalization-Group techniques?

Thank you for your attention!!!

References:

- ArXiv:1104.3459 (φ^4 theory)
- ArXiv:1009.4033, 1011.2664 (large-N theories)

- Some sample codes are available at:

http://www.lattice.itep.ru/~pbaivid/codes.html

This work was supported by the S. Kowalewskaja award from the Alexander von Humboldt Foundation

Back-up slides

Some historical remarks

"Genetic" algorithm vs. branching random process

Probability to find some configuration of branches obeys nonlinear equation

Steady state due to creation and merging

Recursive Markov Chains
[Etessami, Yannakakis, 2005]

Also some modification of McKean-Vlasov-Kac models
[McKean, Vlasov, Kac, 196x]

"Extinction probability" obeys nonlinear equation
[Galton, Watson, 1974]
"Extinction of peerage"

Attempts to solve QCD loop equations
[Migdal, Marchesini, 1981]

"Loop extinction":
No importance sampling