Chiral Symmetry Restoration, Eigenvalue Density of Dirac Operator and axial U(1) anomaly at Finite Temperature

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1. Introduction

Chiral symmetry of QCD

phase transition

\[ U(1)_B \otimes S(N_f)_V \quad \rightarrow \quad U(1)_B \otimes S(N_f)_L \otimes SU(N_f)_R \]

restoration of chiral symmetry

Some questions

1. Eigenvalue distribution of Dirac operator

related ?

2. Recovery of U(1)\_A symmetry at high T?
Previous studies on $\rho(\lambda) = \lim_{V \to \infty} \frac{1}{V} \sum_n \delta(\lambda - \lambda_n)$

Cossu et al. (JLQCD11), Overlap

Eigenvalues distribution $N_f=2 - 16^3 \times 8$

- $T=172\text{MeV}$
- $T=177,192\text{MeV}$
- $T=209\text{MeV}$

Ohno et al. (11), HISQ

Is small $\lambda$ suppressed?
Previous studies on 2

Cohen(96), Theory  Yes!

Lee-Hatsuda(96), Theory  No!

Lattice results

Chandrasekharan et al., (98), KS  No!

Bernard, et al. (96), KS  No!

\[
\chi_{U(1)_A} = O(m^2) + \Delta
\]

\[
\Delta = O(1) \text{ at } N_f = 2: \text{ contributions from } Q = \pm 1
\]

Chiral symmetry is restored.  U(1)\textsubscript{A} is NOT.
Recent lattice results

Hegde (HotQCD11), DW

\[ \chi_{U(1)_A} = 0 \text{ or not ?} \]

Cossu et al. (JLQCD11), Overlap

\[ \beta = 2.30 \ (T \sim 208) \ am = 0.025 \]

\[ \beta = 2.30 \ (T \sim 208) \ am = 0.01 \]

\[ \chi_{\text{disc}}/T^2 \]

\[ \chi_{5, \text{disc}}/T^2 \]

\[ (\chi_{\nu} - \chi_{\delta})/T^2 \]
Our work
give constraints on eigenvalue densities of 2-flavor overlap fermions, if chiral symmetry in QCD is restored at finite temperature. discuss a behavior of singlet susceptibility using the constraints.

Content

1. Introduction

2. Overlap fermions

3. Constraints on eigenvalue densities

4. Discussions: singlet susceptibility
2. Overlap fermions

Action

\[ S = \bar{\psi} [D - mF(D)] \psi, \quad F(D) = 1 - \frac{Ra}{2} D \]

Ginsparg-Wilson relation

\[ D \gamma_5 + \gamma_5 D = a D R \gamma_5 D \]

Eigenvalue spectrum

\[ \lambda_n^A + \bar{\lambda}_n^A = a R \bar{\lambda}_n^A \lambda_n^A \]

zero modes (chiral)

doublers (chiral)

\[ \begin{align*}
D(A) \gamma_5 \phi_n^A &= \bar{\lambda}_n^A \gamma_5 \phi_n^A \\
D(A) \phi_n^A &= \lambda_n^A \phi_n^A
\end{align*} \]
Propagator

\[ S(x, y) = \sum_n \left[ \frac{\phi_n(x)\phi_n^+(y)}{f_m\lambda_n - m} + \gamma_5\phi_n(x)\phi_n^+(y)\gamma_5 \right] - \sum_{k=1}^{N_{R+L}} \frac{1}{m} \phi_k(x)\phi_k^+(y) + \sum_{K=1}^{N_D} \frac{Ra}{2} \phi_K(x)\phi_K^+(y) \]

bulk modes (non-chiral)  zero modes (chiral)  doublers (chiral)

Measure

\[ P_m(A) = e^{-S_{YM}(A)}(-m)^{N_fN^A_{R+L}} \left( \frac{2}{Ra} \right)^{N_fN^A_D} \prod_{\Re\lambda_n^A > 0} \left( Z_m^2 \bar{\lambda}_n^A\lambda_n^A + m^2 \right) \]

positive definite and even function of \( m \neq 0 \) for even \( N_f \)

\[ Z_m^2 = 1 - (ma)^2 \frac{R^2}{4} \]

\[ f_m = 1 + \frac{Rma}{2} \]

\[ N_f=2 \] in this talk.
Ward-Takahashi identities under “chiral” rotation

\[ \theta^a(x) \delta^a_x \psi(x) = i \theta^a(x) T^a \gamma_5 (1 - R a D) \]
\[ \theta^a(x) \delta^a_x \bar{\psi}(x) = i \bar{\psi}(x) \theta^a(x) T^a \gamma_5, \]

Integrated operators

\[ S^a = \int d^4 x \, S^a(x), \quad P^a = \int d^4 x \, P^a(x) \]
\[ S^a(x) = \bar{\psi}(x) T^a F(D) \psi(x), \quad P^a(x) = \bar{\psi}(x) T^a i \gamma_5 F(D) \psi(x), \]

chiral rotation at \( N_f=2 \)

\[ \delta^a S^b = 2 \delta^{ab} P^0, \quad \delta^a P^b = -2 \delta^{ab} S^0 \]

If the chiral symmetry is restored,

\[ \lim_{m \to 0} \langle \delta^a \mathcal{O}_{n_1, n_2, n_3, n_4} \rangle_m = 0 \]

WT identities

\[ \mathcal{O}_{n_1, n_2, n_3, n_4} = (P^a)^{n_1} (S^a)^{n_2} (P^0)^{n_3} (S^0)^{n_4} \]
\[ N = \sum_i n_i, \quad n_1 + n_2 = \text{odd}, \quad n_1 + n_3 = \text{odd} \]

explicit from

\[ \frac{\delta^a}{2} \mathcal{O}_{n_1, n_2, n_3, n_4} = -n_1 \mathcal{O}_{n_1-1, n_2, n_3, n_4+1} + n_2 \mathcal{O}_{n_1, n_2-1, n_3+1, n_4} - n_3 \mathcal{O}_{n_1, n_2+1, n_3-1, n_4} + n_4 \mathcal{O}_{n_1+1, n_2, n_3, n_4-1} \]
3. Constraints on eigenvalue densities

**Assumption 1** non-singlet chiral symmetry is restored:

\[
\lim_{m \to 0} \lim_{V \to \infty} \langle \delta_a \mathcal{O} \rangle_m = 0 \quad \text{(for } a \neq 0),
\]

\[
\langle \mathcal{O}(A) \rangle_m = \frac{1}{Z} \int \mathcal{D}A P_m(A) \mathcal{O}(A), \quad Z = \int \mathcal{D}A P_m(A).
\]

\[P_m(A)\text{: even in } m\]

**Assumption 2** if \(\mathcal{O}(A)\) is \(m\)-independent

\[
\langle \mathcal{O}(A) \rangle_m = f(m^2)
\]

\[f(x)\text{ is analytic at } x = 0\]

Note that this does not hold if the chiral symmetry is spontaneously broken.

**Ex.**

\[
\lim_{V \to \infty} \frac{1}{V} \langle Q(A)^2 \rangle_m = m \frac{\Sigma}{N_f} + O(m^2)
\]
Assumption 3: if $\mathcal{O}(A)$ is $m$-independent and positive, and satisfies

$$\lim_{m \to 0} \frac{1}{m^{2k}} \langle \mathcal{O}(A) \rangle_m = 0$$

then

$$\langle \mathcal{O}(A) \rangle_m = m^{2(k+1)} \int \mathcal{D}A \hat{P}(m^2, A) \mathcal{O}(A)$$

is finite for $\hat{P}(0, A) \neq 0$ for $\exists A$.

Consequence: for $\forall l$ integer

$$\langle \mathcal{O}(A)^l \rangle_m = m^{2(k+1)} \int \mathcal{D}A \hat{P}(m^2, A) \mathcal{O}(A)^l = O(m^{2(k+1)})$$

since $\mathcal{O}(A)$ and $\mathcal{O}(A)^l$ are both positive and share the same support.
**Assumption 4**

Eigenvalues density can be expanded as

\[ \rho^A(\lambda) \equiv \lim_{V \to \infty} \frac{1}{V} \sum_n \delta \left( \lambda - \sqrt{\frac{\lambda}{\lambda_n} A_n A_n} \right) = \sum_{n=0}^{\infty} \rho^A_n \frac{\lambda^n}{n!} \quad \text{at } \lambda = 0 \quad (\lambda < \epsilon) \]

More precisely, configurations which can not be expanded at the origin are "measure zero" in the configuration space.
4. Constraints on eigenvalue densities

**general N(odd)**

\[ \mathcal{O}_{1,0,0,N-1} \]

\[
\lim_{m \to 0} \lim_{V \to \infty} (-\langle \mathcal{O}_{0,0,0,N} \rangle_m + (N-1)\langle \mathcal{O}_{2,0,0,N-2} \rangle_m) = 0.
\]

large volume

\[
\frac{1}{VN} \langle (S_0)^N \rangle_m = N_f^N \left\langle \left\{ \frac{N_R^A + L}{mV} + I_1 \right\} \right\rangle^N_m + O(V^{-1}) \to 0 \quad m \to 0
\]

\[
I_1 = \frac{1}{Z_m} \int_{0}^{\Lambda_R} d\lambda \rho^A(\lambda) g_0(\lambda^2) \frac{2m_R}{\lambda^2 + m_R^2} = \pi \rho_0^A + O(m)
\]

\[ \Lambda_R = \frac{2}{R\alpha} : \text{cut-off} \]

\[ g_0(\lambda^2) = 1 - \frac{\lambda^2}{\Lambda_R^2}, \quad m_R = m/Z_m \]

Both \( \rho_0^A \) and \( N_R^{A+L} \) are positive.

\[
\langle \rho_0^A \rangle_m = O(m^2) \quad \text{1st constraint}
\]

\[
\lim_{V \to \infty} \left\langle \frac{N_{R+L}}{V} \right\rangle_m = O(m^{N+1})
\]

\[
\forall N \quad \lim_{V \to \infty} \left\langle \frac{N_{R+L}}{V} \right\rangle_m = 0 \quad \text{for small but non-zero } m
\]
\[ N=2 \]

\[
\chi^{\sigma-\pi} = \frac{1}{V} \langle S_0^2 - P_a^2 \rangle_m, \quad \chi^{\eta-\delta} = \frac{1}{V} \langle P_0^2 - S_a^2 \rangle_m
\]

\[ = 0 \]

\[
\chi^{\eta-\delta} = N_f \left\langle \frac{1}{m^2 V} \left\{ 2N_{R+L} - N_f Q(A)^2 \right\} + \frac{1}{Z_m} \left( \frac{I_1}{m_R} + I_2 \right) \right\rangle_m
\]

\[ = 0 \]

\[
I_2 = \frac{2}{Z_m} \int_0^{\Lambda_R} d\lambda \rho^A(\lambda) \frac{m_R^2 - \lambda^2 g_0 \langle \lambda^2 \rangle g_m}{(\lambda^2 + m_R^2)^2}, \quad g_m = \frac{1}{Z_m^2} \left( 1 + \frac{m^2}{2\Lambda_R^2} \right)
\]

\[
\frac{I_1}{m_R} + I_2 = \rho_0^A \left( \frac{\pi_m}{m} + \frac{2}{\Lambda_R} \right) + 2\rho_1^A + O(m).
\]

\[
\lim_{m \to 0} \chi^{\eta-\delta} = 0 \quad \Rightarrow \quad \lim_{m \to 0} \frac{N_f^2 \langle Q(A)^2 \rangle_m}{m^2 V} = 2 \lim_{m \to 0} \langle \rho_1^A \rangle_m
\]
WT identities

\[ \langle O_{2001} \rangle_m \to 0, \quad \langle -O_{0201} + 2O_{1110} \rangle_m \to 0, \quad \langle O_{0021} + 2O_{1110} \rangle_m = 0 \]
\[ \langle -O_{0003} + 2O_{2001} \rangle_m \to 0, \quad \langle O_{0021} - O_{0201} + O_{1110} \rangle_m \to 0, \]

\[ \langle \rho_0^A \rangle_m = -\frac{m^2}{2} \langle \rho_2^A \rangle_m + O(m^4) \]

\[ \lim_{V \to \infty} \frac{\langle Q(A)^2 \rho_0^A \rangle_m}{V} = O(m^4) \]
\[ \langle O_{4000} - O_{0004} \rangle_m \to 0, \quad \langle O_{4000} - 3O_{2002} \rangle_m \to 0, \]
\[ \langle O_{0400} - O_{0040} \rangle_m \to 0, \quad \langle O_{0400} - 3O_{0220} \rangle_m \to 0, \]
\[ \langle O_{2020} - O_{0202} \rangle_m \to 0, \quad \langle O_{2200} - O_{0022} \rangle_m \to 0, \]
\[ \langle 2O_{1111} - O_{0202} + O_{0022} \rangle_m \to 0. \]

\[ 3N_f^2 \langle (I_2 + I_1/m)(I_1 - I_2/m) \rangle_m + \frac{6N_f^3}{m^3V} \langle Q(A)^2 I_1 \rangle_m - \frac{N_f^4}{m^4V^2} \langle Q(A)^4 \rangle_m \to 0. \]
\[ \sim \log m \]
\[ \sim \frac{1}{m} \]
\[ \sim \frac{1}{m^2} \]

\[ \lim_{V \to \infty} \frac{\langle Q(A)^2 \rangle_m}{V} = O(m^4) \]
\[ \langle \rho_A^1 \rangle_m = O(m^2) \]

\[ \lim_{m \to 0} \frac{N_f^2 \langle Q(A)^2 \rangle_m}{m^2V} = 2 \lim_{m \to 0} \langle \rho_1^A \rangle_m \]
2nd constraint

\[ -3N_f^2 \frac{\pi^2}{m^2} \langle (\rho_0^A)^2 \rangle_m - \frac{N_f^4}{m^4V^2} \langle Q(A)^4 \rangle_m \to 0. \]

\[ \langle \rho_0^A \rangle_m = O(m^4) \],
\[ \langle \rho_2^A \rangle_m = O(m^2) \]

\[ \langle \rho_0^A \rangle_m = -\frac{m^2}{2} \langle \rho_2^A \rangle_m \]
3rd constraint

\[ \langle Q(A)^2 \rangle_m \to O(m^6) \]
Final results

\[ \lim_{m \to 0} \langle \rho^A(\lambda) \rangle_m = \lim_{m \to 0} \langle \rho^A_3 \rangle_m \frac{|\lambda|^3}{3!} + O(\lambda^4) \]

No constraints to higher \( \langle \rho^A_n \rangle_m \)

\( \langle \rho^A_3 \rangle_m \neq 0 \) even for ”free” theory.

\[ \langle \rho^A_0 \rangle_m = 0 \]

\[ \lim_{V \to \infty} \frac{1}{V^k} \langle (N^A_{R+L})^k \rangle_m = 0, \quad \lim_{V \to \infty} \frac{1}{V^k} \langle Q(A)^{2k} \rangle_m = 0 \]

+ result from N=4k (general)
5. Discussion: Singlet susceptibility

Singlet susceptibility at high T

\[
\lim_{V \to 0} \chi^{\pi - \eta} = \lim_{m \to 0} \lim_{V \to \infty} \frac{N_f^2}{m^2 V} \langle Q(A)^2 \rangle_m = 0
\]

Both Cohen and Lee-Hatsuda are inaccurate.

This, however, does not mean U(1)_A symmetry is recovered at high T.

\[
\lim_{m \to 0} \chi^{\pi - \eta} = 0
\]

is necessary but NOT “sufficient” for the recovery of U(1)_A.
More general Singlet WT identities

\[ \langle J^0 \mathcal{O} + \delta^0 \mathcal{O} \rangle_m = O(m) \]

anomaly(measure) singlet rotation

We can show for \( \mathcal{O} = \mathcal{O}_{n_1, n_2, n_3, n_4} = (P^a)^{n_1} (S^a)^{n_2} (P^0)^{n_3} (S^0)^{n_4} \)

\[
\lim_{V \to \infty} \frac{1}{V^k} \langle J^0 \mathcal{O} \rangle_m = \lim_{V \to \infty} \left\langle \frac{Q(A)^2}{mV} \times O(V^0) \right\rangle_m = 0
\]

where \( k \) is the smallest integer which makes the \( V \to \infty \) limit finite.

\[
S^0 \sim O(V), \ P^a, S^a, P^0 \sim O(V^{1/2})
\]

\[
\lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V^k} \langle \delta^0 \mathcal{O} \rangle_m = 0
\]

Breaking of \( U(1)_A \) symmetry is absent for these “bulk quantities”.
Important consequence

Effect of $U(1)_A$ anomaly is invisible in scalar and pseudo-scalar sector.

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Chiral phase transition in 2-flavor QCD is likely to be of first order !?

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Final Comments

1. Large volume limit is required for the correct result.

2. If the action breaks the chiral symmetry, the continuum limit is also required.

3. We only use a part of WT identities. Therefore, our constraints are necessary condition.

4. We can extend our analysis to the eigenvalue density with fractional power. The conclusion remains the same.