Chiral Symmetry Restoration, Eigenvalue Density of Dirac Operator and axial U(1) anomaly at Finite Temperature

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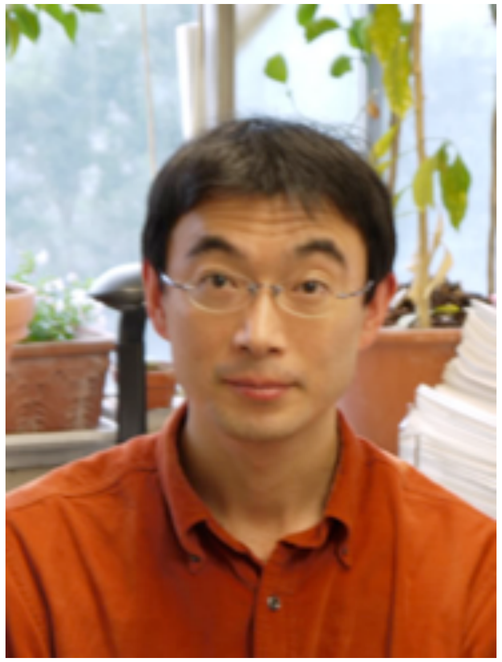


GGI workshop, “New Frontiers in Lattice Gauge Theory”,

GGI, Firenze, Italy, September 5, 2012

1. Introduction

Chiral symmetry of QCD

phase transition

low T $U(1)_B \otimes S(N_f)_V$ high T $U(1)_B \otimes S(N_f)_L \otimes SU(N_f)_R$

restoration of chiral symmetry

Some questions

1. Eigenvalue distribution of Dirac operator

related ?

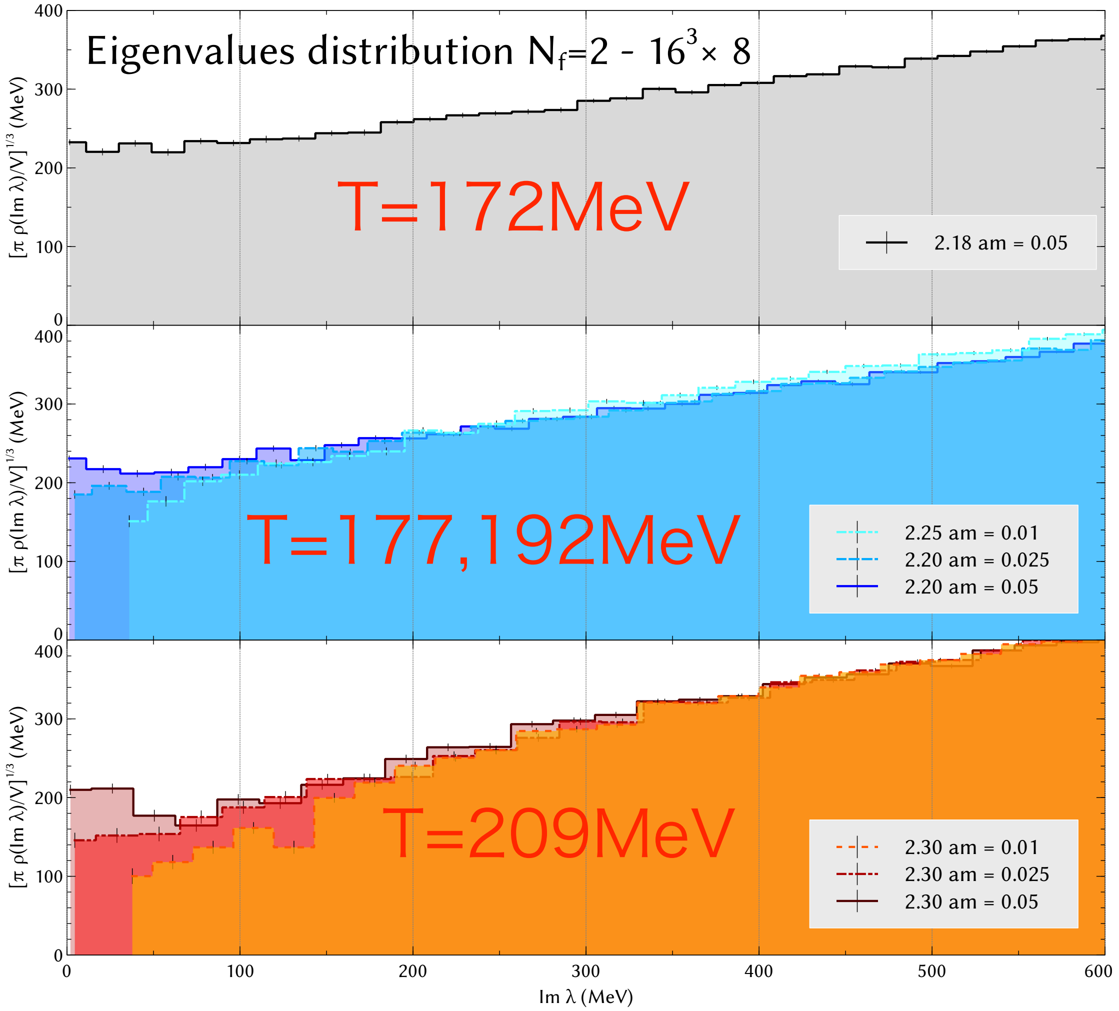
2. Recovery of U(1)_A symmetry at high T ?

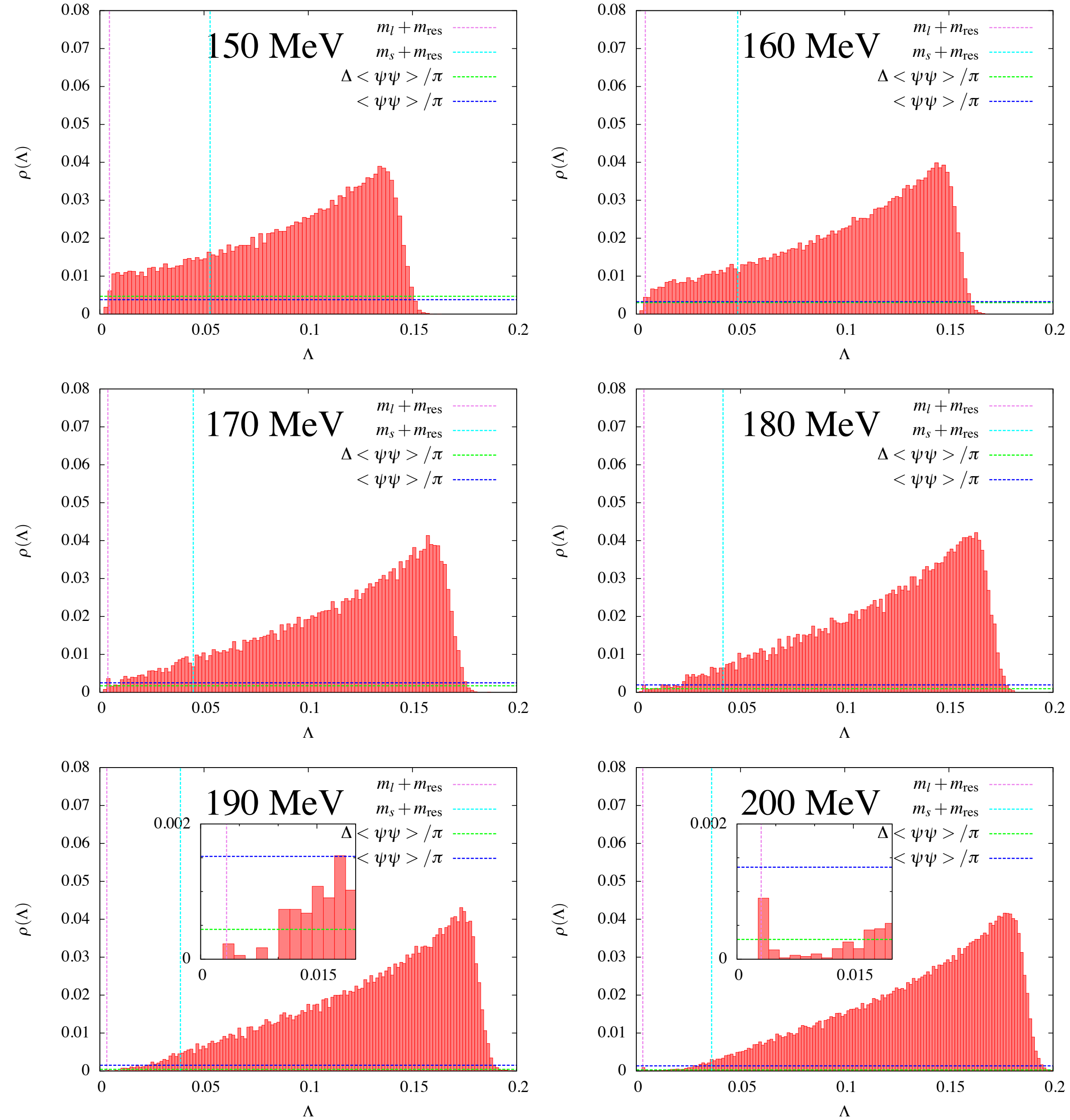
Previous studies on 1

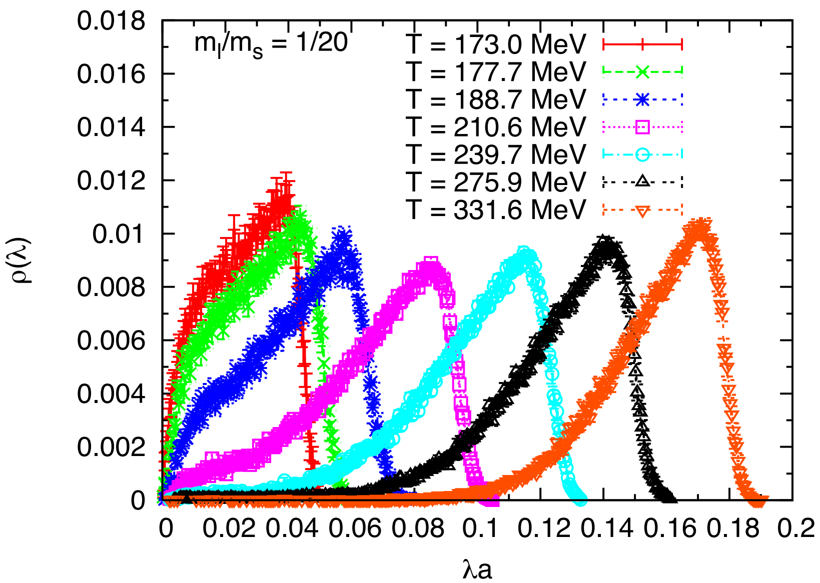
$$\rho(\lambda) = \lim_{V\to\infty} \frac{1}{V} \sum_n \delta(\lambda - \lambda_n)$$

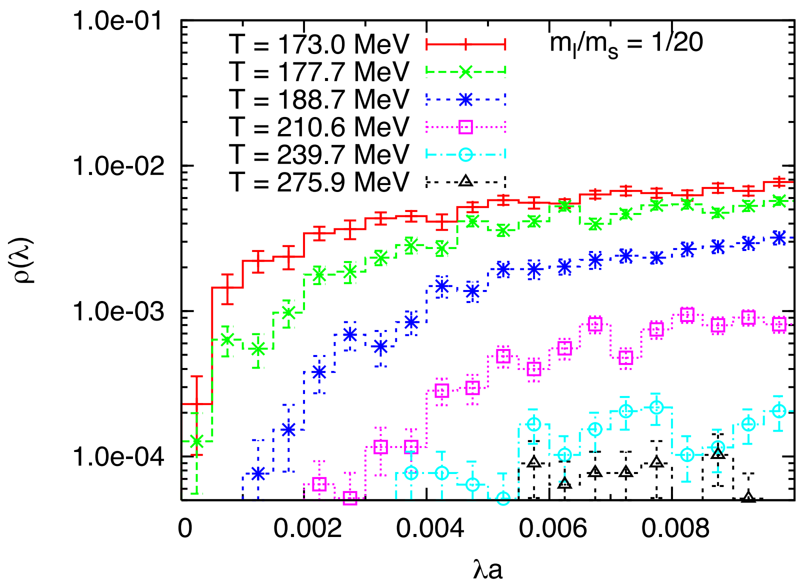
Lin (HotQCD11), DW

Cossu *et al.* (JLQCD11), Overlap









Ohno *et al.* (11), HISQ

Is small λ suppressed ?

Previous studies on 2

$$\chi_{U(1)_A} = \int d^4x \, \langle \sigma(x)\sigma(0) - \delta(x)\delta(0) \rangle$$

Cohen(96), Theory Yes !

$$\chi_{U(1)_A}/V = 0, \quad (m \to 0)$$

Lee-Hatsuda(96), Theory No !

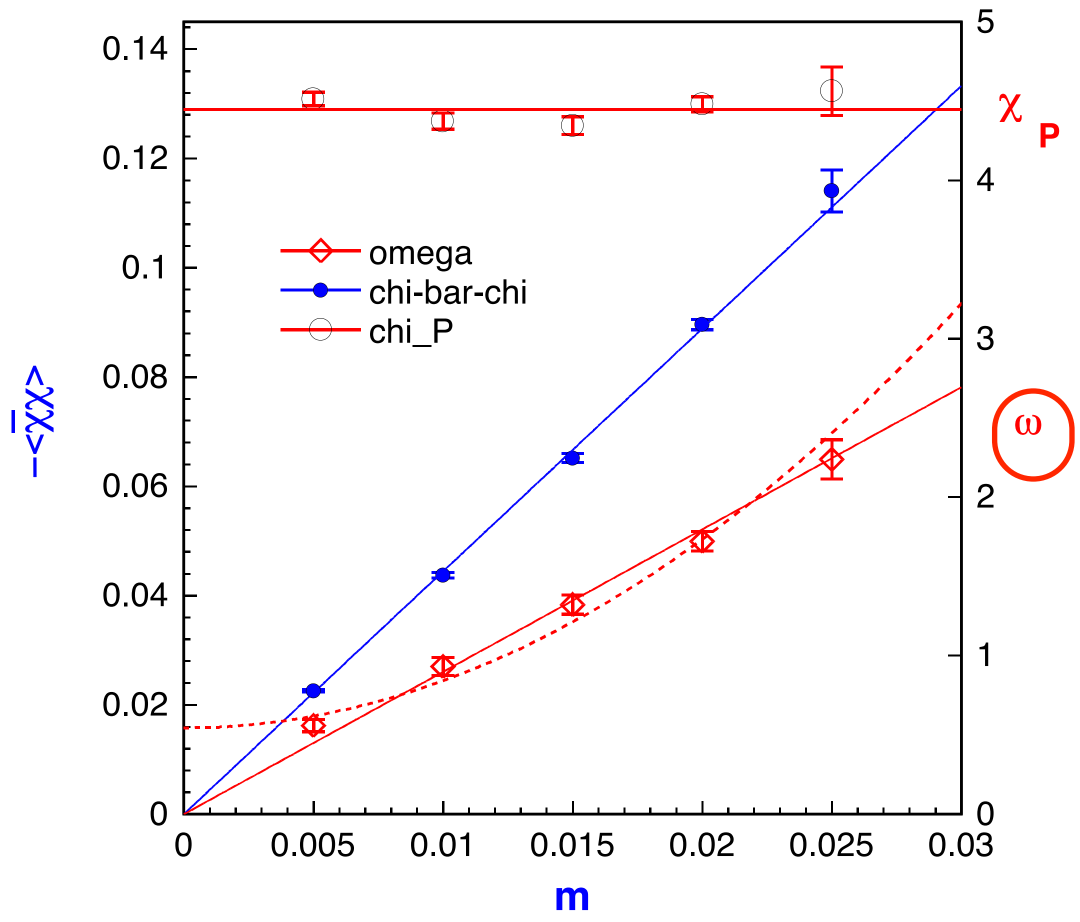
zero mode contributions are important.

$$\chi_{U(1)_A} = O(m^2) + \Delta$$

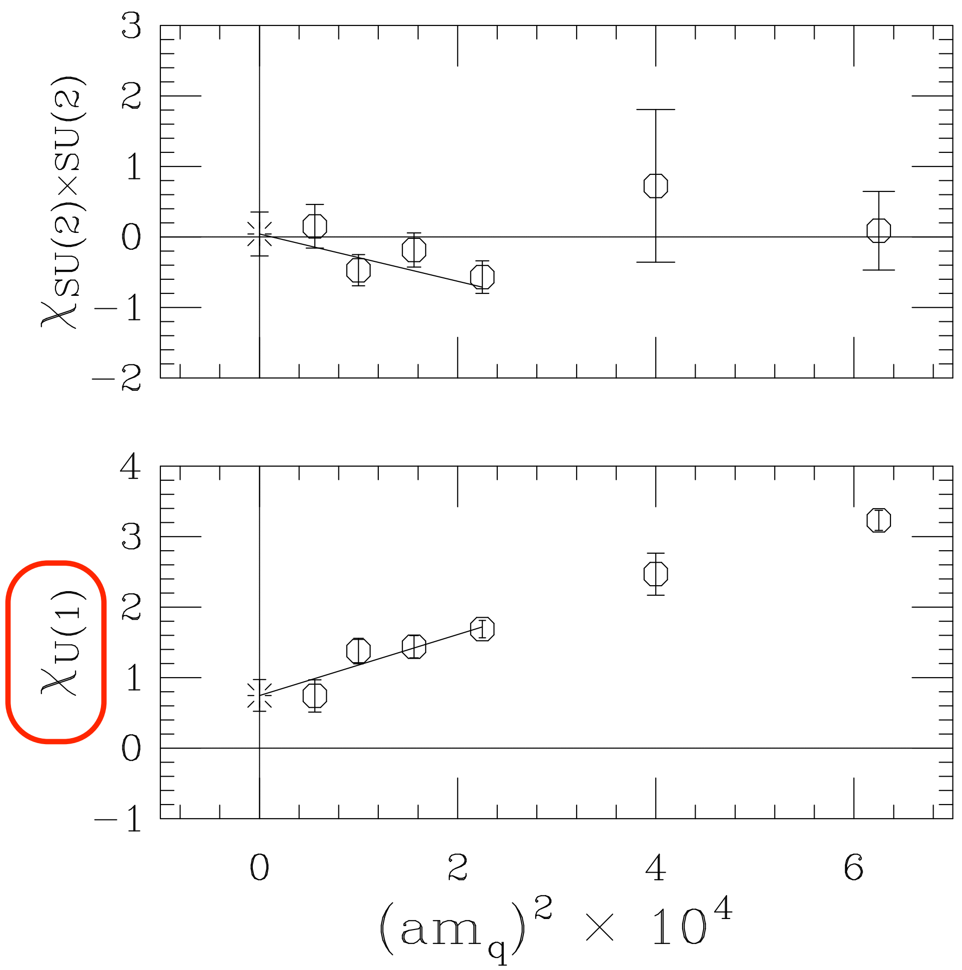
$\Delta = O(1)$ at $N_f = 2$: contributions from $Q = \pm 1$

Lattice results

Chandrasekharan *et al.*, (98), KS No !

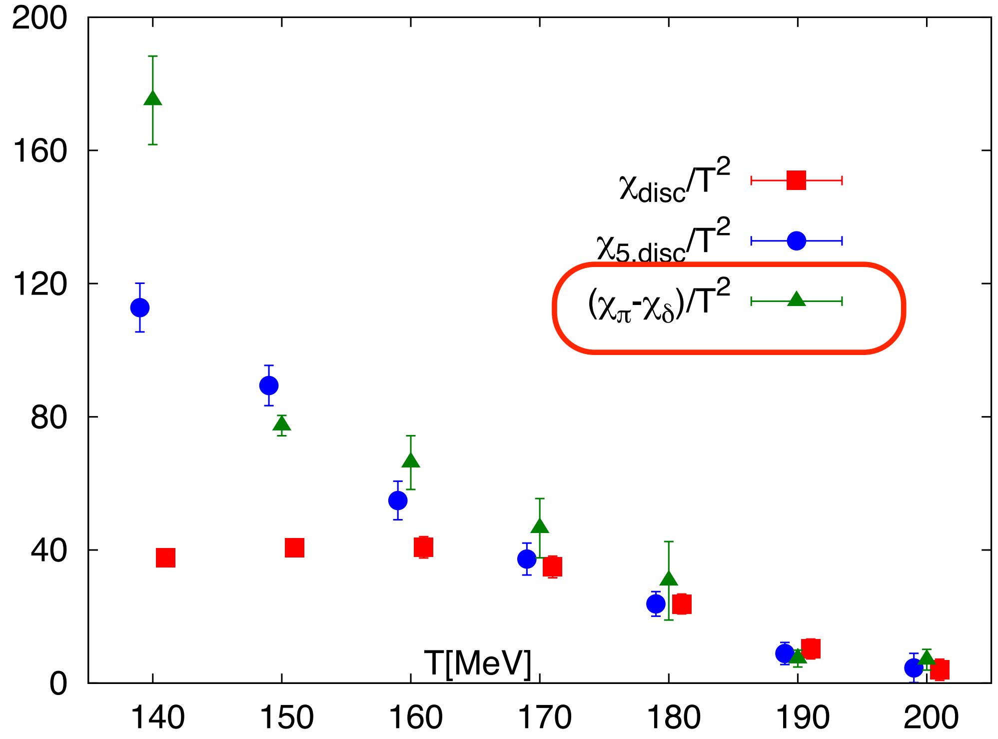


Bernard, *et al.* (96), KS No !



Chiral symmetry is restored. U(1)A is NOT.

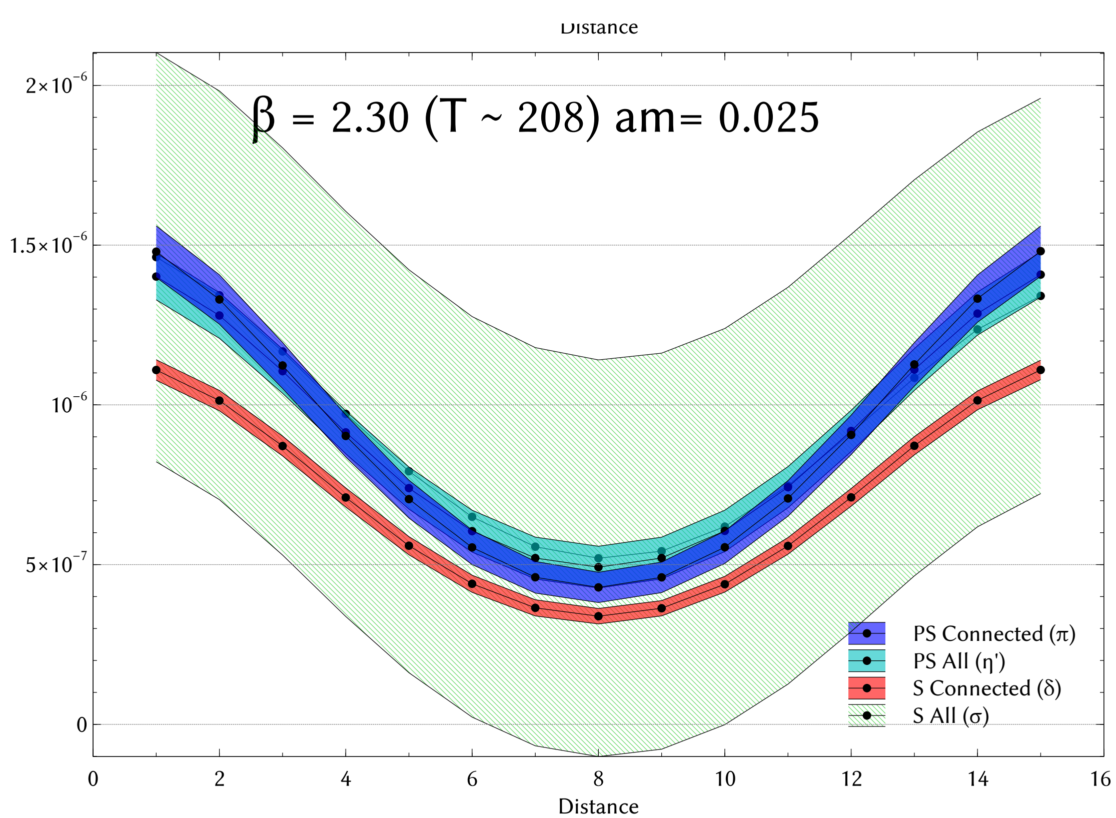
Recent lattice results

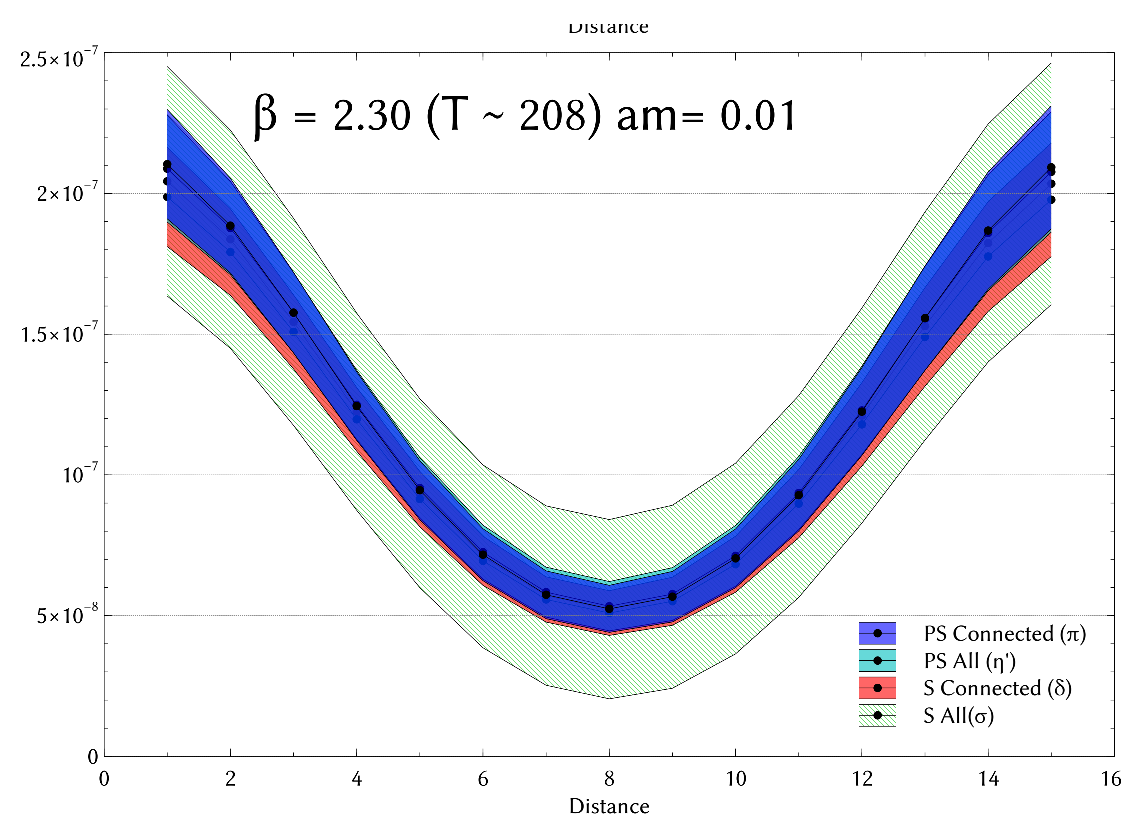


Hegde (HotQCD11), DW   No ?!

$$\chi_{U(1)_A} = 0 \text{ or not ?}$$

Cossu *et al.* (JLQCD11), Overlap   Yes ?!   meson correlators





Our work

give constraints on eigenvalue densities of 2-flavor overlap fermions, if chiral symmetry in QCD is restored at finite temperature.

discuss a behavior of singlet susceptibility using the constraints.

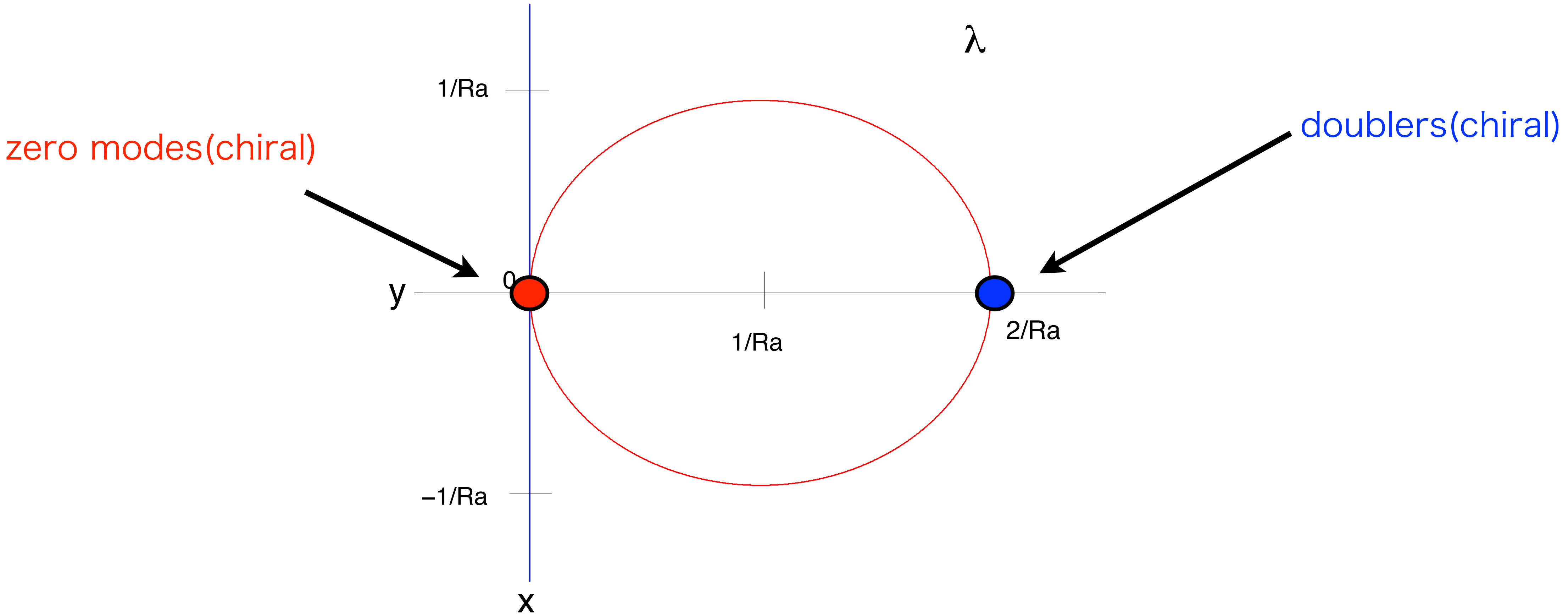
Content

2. Overlap fermions

Action $$S = \bar{\psi}[D - mF(D)]\psi, \quad F(D) = 1 - \frac{Ra}{2}D$$

Ginsparg-Wilson relation $$D\gamma_5 + \gamma_5 D = aDR\gamma_5 D$$

Eigenvalue spectrum $$\lambda_n^A + \bar{\lambda}_n^A = aR\bar{\lambda}_n^A \lambda_n^A$$



$$D(A)\gamma_5\phi_n^A = \bar{\lambda}_n^A \gamma_5 \phi_n^A \quad \Rightarrow \quad D(A)\phi_n^A = \lambda_n^A \phi_n^A$$

Propagator

$$S(x,y) = \sum_n \left[\frac{\phi_n(x)\phi_n^\dagger(y)}{f_m\lambda_n - m} + \frac{\gamma_5\phi_n(x)\phi_n^\dagger(y)\gamma_5}{f_m\bar{\lambda}_n - m} \right] - \sum_{k=1}^{N_{R+L}} \frac{1}{m}\phi_k(x)\phi_k^\dagger(y) + \sum_{K=1}^{N_D} \frac{Ra}{2}\phi_K(x)\phi_K^\dagger(y)$$

bulk modes(non-chiral) zero modes(chiral) doublers(chiral)

$$f_m = 1 + \frac{Rma}{2}$$

Measure

$$P_m(A) = e^{-S_{YM}(A)}(-m)^{\underbrace{N_f N_{R+L}^A}_{\text{\# of zero modes}}} \left(\frac{2}{Ra}\right)^{\overbrace{N_f N_D^A}^{\text{\# of doublers}}} \prod_{\Im\lambda_n^A > 0} \left(Z_m^2 \bar{\lambda}_n^A \lambda_n^A + m^2\right)$$

$$Z_m^2 = 1 - (ma)^2 \frac{R^2}{4}$$

positive definite and even function of $m \neq 0$ for even N_f

N_f=2 in this talk.

Ward-Takahashi identities under "chiral" rotation

$$
\begin{aligned}
\theta^a(x)\delta^a_x\psi(x) &= i\theta^a(x)T^a\gamma_5(1-RaD) \\
\theta^a(x)\delta^a_x\bar{\psi}(x) &= i\bar{\psi}(x)\theta^a(x)T^a\gamma_5,
\end{aligned}
$$

Integrated operators

$$S^a = \int d^4x\, S^a(x), \quad P^a = \int d^4x\, P^a(x)$$

$$
\begin{aligned}
S^a(x) &= \bar{\psi}(x)T^aF(D)\psi(x), \\
P^a(x) &= \bar{\psi}(x)T^a i\gamma_5 F(D)\psi(x),
\end{aligned}
$$

scalar

pseudo-scalar

chiral rotation at N_f=2 $\quad \delta^a S^b = 2\delta^{ab}P^0, \delta^a P^b = -2\delta^{ab}S^0$

If the chiral symmetry is restored,

$$\boxed{\lim_{m\to 0}\langle \delta^a \mathcal{O}_{n_1,n_2,n_3,n_4}\rangle_m = 0}$$

WT identities

$$\mathcal{O}_{n_1,n_2,n_3,n_4} = (P^a)^{n_1}(S^a)^{n_2}(P^0)^{n_3}(S^0)^{n_4} \qquad N = \sum_i n_i, \quad n_1+n_2 = \mathrm{odd},\ n_1+n_3=\mathrm{odd}$$

explicit from

$$\frac{\delta^a}{2}\mathcal{O}_{n_1,n_2,n_3,n_4} = -n_1\mathcal{O}_{n_1-1,n_2,n_3,n_4+1} + n_2\mathcal{O}_{n_1,n_2-1,n_3+1,n_4} - n_3\mathcal{O}_{n_1,n_2+1,n_3-1,n_4} + n_4\mathcal{O}_{n_1+1,n_2,n_3,n_4-1}$$

3. Constraints on eigenvalue densities

Assumption 1 non-singlet chiral symmetry is restored:

$$\lim_{m\to 0}\lim_{V\to\infty}\langle\delta_a\mathcal{O}\rangle_m = 0 \quad (\text{for } a\neq 0),$$

$$\langle\mathcal{O}(A)\rangle_m = \frac{1}{Z}\int\mathcal{D}A\,P_m(A)\,\mathcal{O}(A), \quad Z=\int\mathcal{D}A\,P_m(A).$$

$P_m(A)$: even in m

Assumption 2 if $\mathcal{O}(A)$ is m-independent

$$\langle\mathcal{O}(A)\rangle_m = f(m^2)$$

$f(x)$ is analytic at $x=0$

Note that this does not hold if the chiral symmetry is spontaneously broken.

Ex.

$$\lim_{V\to\infty}\frac{1}{V}\langle Q(A)^2\rangle_m = m\frac{\Sigma}{N_f}+O(m^2)$$

Assumption 3 if $\mathcal{O}(A)$ is m-independent and positive, and satisfies

$$\lim_{m\to 0}\frac{1}{m^{2k}}\langle\mathcal{O}(A)\rangle_m = 0$$

$$\langle\mathcal{O}(A)\rangle_m = m^{2(k+1)}\underbrace{\int\mathcal{D}A\,\hat{P}(m^2,A)\mathcal{O}(A)}_{\text{finite}}$$

$\hat{P}(0,A)\neq 0$ for ${}^{\exists}A$

consequence for ${}^{\forall}l$ integer

$$\langle\mathcal{O}(A)^l\rangle_m = m^{2(k+1)}\int\mathcal{D}A\,\hat{P}(m^2,A)\mathcal{O}(A)^l = O(m^{2(k+1)})$$

since $\mathcal{O}(A)$ and $\mathcal{O}(A)^l$ are both positive and share the same support.

Assumption 4 eigenvalues density can be expanded as

$$\rho^A(\lambda) \equiv \lim_{V\to\infty} \frac{1}{V} \sum_n \delta\left(\lambda - \sqrt{\bar{\lambda}_n^A \lambda_n^A}\right) = \sum_{n=0}^{\infty} \rho_n^A \frac{\lambda^n}{n!} \qquad \text{at } \lambda = 0 \ (\lambda < \epsilon\ )$$

More precisely, configurations which can not be expanded at the origin are "measure zero" in the configuration space.

4. Constraints on eigenvalue densities

general N(odd) $\mathcal{O}_{1,0,0,N-1}$ $\lim_{m\to 0}\lim_{V\to\infty}\left(-\langle \mathcal{O}_{0,0,0,N}\rangle_m + (N-1)\langle \mathcal{O}_{2,0,0,N-2}\rangle_m\right) = 0.$

large volume

$m \longrightarrow 0$

$$\frac{1}{V^N}\langle (S_0)^N\rangle_m = N_f^N \left\langle \left\{ \frac{N^A_{R+L}}{mV} + I_1 \right\}^N \right\rangle_m + O(V^{-1}) \to 0$$

$$I_1 = \frac{1}{Z_m}\int_0^{\Lambda_R} d\lambda\, \rho^A(\lambda) g_0(\lambda^2)\frac{2m_R}{\lambda^2+m_R^2} = \pi\rho_0^A + O(m)$$

$\Lambda_R = \frac{2}{Ra}$: cut-off

$g_0(\lambda^2) = 1 - \frac{\lambda^2}{\Lambda_R^2},\ m_R = m/Z_m$

Both ρ_0^A and N^A_{R+L} are positive.

$$\langle \rho_0^A\rangle_m = O(m^2)$$

1st constraint

$$\lim_{V\to\infty}\left\langle \frac{N_{R+L}}{V}\right\rangle_m = O(m^{N+1}) \quad \Longrightarrow \quad \lim_{V\to\infty}\left\langle \frac{N_{R+L}}{V}\right\rangle_m = 0$$

$^\forall N$

for small but non-zero m

N=2

$$\chi^{\sigma-\pi} = \frac{1}{V^2}\langle S_0^2 - P_a^2\rangle_m, \qquad \chi^{\eta-\delta} = \frac{1}{V}\langle P_0^2 - S_a^2\rangle_m$$

=0

$$\chi^{\eta-\delta} = N_f \left\langle \frac{1}{m^2 V}\{\underbrace{2N_{R+L}}_{=0} - N_f Q(A)^2\} + \underline{\frac{1}{Z_m}\left(\frac{I_1}{m_R} + I_2\right)} \right\rangle_m$$

topological charge

$$Q(A) = N_R^A - N_L^A$$

$$I_2 = \frac{2}{Z_m}\int_0^{\Lambda_R} d\lambda\, \rho^A(\lambda)\, \frac{m_R^2 - \lambda^2 g_0(\lambda^2) g_m}{(\lambda^2 + m_R^2)^2}, \quad g_m = \frac{1}{Z_m^2}\left(1 + \frac{m^2}{2\Lambda_R^2}\right)$$

$$\frac{I_1}{m_R} + I_2 = \rho_0^A \left(\frac{\pi_m}{m} + \frac{2}{\Lambda_R}\right) + 2\rho_1^A + O(m),$$

$$\lim_{m\to 0} \chi^{\eta-\delta} = 0 \quad \Longrightarrow \quad \lim_{m\to 0} \frac{N_f^2 \langle Q(A)^2\rangle_m}{m^2 V} = 2 \lim_{m\to 0} \langle \rho_1^A \rangle_m$$

N=3

WT identities

$$\langle \mathcal{O}_{2001} \rangle_m \to 0, \quad \langle -\mathcal{O}_{0201} + 2\mathcal{O}_{1110} \rangle_m \to 0, \quad \langle \mathcal{O}_{0021} + 2\mathcal{O}_{1110} \rangle_m = 0$$

$$\langle -\mathcal{O}_{0003} + 2\mathcal{O}_{2001} \rangle_m \to 0, \quad \langle \mathcal{O}_{0021} - \mathcal{O}_{0201} + \mathcal{O}_{1110} \rangle_m \to 0,$$

$$\langle \rho_0^A \rangle_m = -\frac{m^2}{2} \langle \rho_2^A \rangle_m + O(m^4)$$

$\Rightarrow$

$$\lim_{V \to \infty} \frac{\langle Q(A)^2 \rho_0^A \rangle_m}{V} = O(m^4)$$

N=4

$$\langle \mathcal{O}_{4000} - \mathcal{O}_{0004} \rangle_m \to 0, \quad \langle \mathcal{O}_{4000} - 3\mathcal{O}_{2002} \rangle_m \to 0,$$
$$\langle \mathcal{O}_{0400} - \mathcal{O}_{0040} \rangle_m \to 0, \quad \langle \mathcal{O}_{0400} - 3\mathcal{O}_{0220} \rangle_m \to 0,$$
$$\langle \mathcal{O}_{2020} - \mathcal{O}_{0202} \rangle_m \to 0, \quad \langle \mathcal{O}_{2200} - \mathcal{O}_{0022} \rangle_m \to 0,$$
$$\langle 2\mathcal{O}_{1111} - \mathcal{O}_{0202} + \mathcal{O}_{0022} \rangle_m \to 0.$$

$$3N_f^2 \langle (I_2 + I_1/m)(I_1 - I_2/m) \rangle_m + \frac{6N_f^3}{m^3 V} \langle Q(A)^2 I_1 \rangle_m - \frac{N_f^4}{m^4 V^2} \langle Q(A)^4 \rangle_m \to 0.$$

$\sim \log m$ $\qquad \sim \frac{1}{m}$ $\qquad \sim \frac{1}{m^2}$

$$\lim_{V\to\infty} \frac{\langle Q(A)^2 \rangle_m}{V} = O(m^4)$$

$$\langle \rho_1^A \rangle_m = O(m^2)$$

$$\lim_{m\to 0} \frac{N_f^2 \langle Q(A)^2 \rangle_m}{m^2 V} = 2 \lim_{m\to 0} \langle \rho_1^A \rangle_m$$

2nd constraint

$$-3N_f^2 \frac{\pi^2}{m^2} \langle (\rho_0^A)^2 \rangle_m - \frac{N_f^4}{m^4 V^2} \langle Q(A)^4 \rangle_m \to 0.$$

negative semi-definite

$$\lim_{V\to\infty} \frac{\langle Q(A)^2 \rangle_m}{V} = O(m^6)$$

$$\langle \rho_0^A \rangle_m = O(m^4)$$

$$\langle \rho_2^A \rangle_m = O(m^2)$$

$$\langle \rho_0^A \rangle_m = -\frac{m^2}{2} \langle \rho_2^A \rangle_m$$

3rd constraint

+ result from N=4k (general)

Final results

$$\lim_{m\to 0}\langle \rho^A(\lambda)\rangle_m = \lim_{m\to 0}\langle \rho_3^A\rangle_m \frac{|\lambda|^3}{3!} + O(\lambda^4)$$

No constraints to higher $\langle \rho_n^A\rangle_m$

$\langle \rho_3^A\rangle_m \neq 0$ even for "free" theory.

$$\langle \rho_0^A\rangle_m = 0$$

$$\lim_{V\to\infty}\frac{1}{V^k}\langle (N_{R+L}^A)^k\rangle_m = 0, \quad \lim_{V\to\infty}\frac{1}{V^k}\langle Q(A)^{2k}\rangle_m = 0$$

5. Discussion: Singlet susceptibility

Singlet susceptibility at high T

$$\lim_{V\to 0} \chi^{\pi-\eta} = \lim_{m\to 0} \lim_{V\to\infty} \frac{N_f^2}{m^2 V} \langle Q(A)^2 \rangle_m = 0$$

Both Cohen and Lee-Hatsuda are inaccurate.

This, however, does not mean U(1)_A symmetry is recovered at high T.

$$\lim_{m\to 0} \chi^{\pi-\eta} = 0$$

is necessary but NOT "sufficient" for the recovery of U(1)_A .

More general Singlet WT identities

$$\langle \underline{J^0} \mathcal{O} + \underline{\delta^0} \mathcal{O} \rangle_m = O(m)$$

anomaly(measure) singlet rotation

We can show for $\mathcal{O} = \mathcal{O}_{n_1,n_2,n_3,n_4} = (P^a)^{n_1}(S^a)^{n_2}(P^0)^{n_3}(S^0)^{n_4}$

$$\lim_{V\to\infty} \frac{1}{V^k} \langle J^0 \mathcal{O} \rangle_m = \lim_{V\to\infty} \left\langle \frac{Q(A)^2}{mV} \times O(V^0) \right\rangle_m = 0$$

where k is the smallest integer which makes the $V \to \infty$ limit finite.

$S^0 \sim O(V),\ P^a, S^a, P^0 \sim O(V^{1/2})$

$$\Rightarrow \quad \lim_{m\to 0} \lim_{V\to\infty} \frac{1}{V^k} \langle \delta^0 \mathcal{O} \rangle_m = 0$$

Breaking of U(1)_A symmetry is absent for these “bulk quantities”.

Important consequence

Effect of U(1)_A anomaly is invisible in scalar and pseudo-scalar sector.

Pisarski-Wilczek argument

Chiral phase transition in 2-flavor QCD is likely to be of first order !?

Final Comments

1. Large volume limit is required for the correct result.

2. If the action breaks the chiral symmetry, the continuum limit is also required.

3. We only use a part of WT identities. Therefore, our constraints are necessary condition.

4. We can extend our analysis to the eigenvalue density with fractional power. The conclusion remains the same.